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Topics in Analytic Number Theory

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Doctor of Philosophy

Mathematics

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B.S., Duke University, 2009

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An abstract of

A dissertation submitted to the Faculty of the

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Abstract

Topics in Analytic Number Theory

By Jesse Aaron Thorner

In this thesis, the author proves theorems on the distribution of primes by extending recent results in sieve theory and proving new results on the distribution of zeros of Rankin-Selberg L-functions. The author proves for any Galois extension of number fields $K/\mathbb{Q}$, there exist bounded gaps between primes with a given "splitting condition" in K, and the primes in question may be restricted to short intervals. Furthermore, we can count these gaps with the correct order of magnitude. The author also proves log-free zero density estimates for Rankin-Selberg L-functions with effective dependence on the key parameters. From this, the author proves an approximate short interval prime number theorem for Rankin-Selberg L-functions, an approximate short interval version of the Sato-Tate conjecture, and a bound on the least norm of a prime ideal counted by the Sato-Tate conjecture, all of which exhibit effective dependence on the key parameters.

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Contents

Chapter 1

Introduction

1.1 The distribution of primes

Let $\mathbb{N}$ denote the set of positive integers, and let $a, q \in \mathbb{N}$ satisfy $(a, q) = 1$. Let p be a rational prime, and define $\pi(x) = \#\{p \leq x\}$ and $\pi(x; q, a) = \#\{p \leq x : p \equiv a \pmod{q}\}$. The prime number theorem states that

$$\pi(x) \sim \mathrm{Li}(x),$$

where

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} = \frac{x}{\log x} + O\Big(\frac{x}{(\log x)^2}\Big).$$

The proof can be adjusted to give the prime number theorem for arithmetic progressions: if $q \leq (\log x)^D$ for some fixed $D > 0$, then

$$\pi(x; q, a) \sim \frac{\pi(x)}{\varphi(q)},$$

where φ denotes Euler's totient function. Understanding the error term in the prime number theorem for arithmetic progressions is important for many arithmetic problems. The generalized Riemann hypothesis (GRH) for Dirichlet L-functions implies if $\epsilon > 0$, then for any $q \leq x^{1/2-\epsilon}$, we have that

$$\pi(x;q,a) - \frac{\pi(x)}{\varphi(q)} \ll \sqrt{x}\log qx.$$

While this is currently out of reach, we know that the mean value of the error term in the prime number theorem for arithmetic progressions is about as small as predicted by GRH when we average over moduli q. Specifically, if $0 < \theta < \frac{1}{2}$ and $D > 0$ are constant, Bombieri and Vinogradov [Mon71, Theorem 15.1] proved that

$$\sum_{q \leq x^{\theta}} \max_{(a,q)=1} \max_{y \leq x} \left| \pi(y;q,a) - \frac{\pi(y)}{\varphi(q)} \right| \ll \frac{x}{(\log x)^D}.$$

To any value of $0 < \theta < 1$ for which the above inequality holds, we give the name *level of distribution* of the primes; thus Bombieri and Vinogradov showed that the primes have a level of distribution θ for any $0 < \theta < 1/2$. This provides significant evidence in favor of GRH, and in several arithmetic problems, the Bombieri-Vinogradov theorem may be used as a substitute for GRH. It is conjectured by Elliott and Halberstam [EH70] that the primes have a level of distribution θ for any $0 < \theta < 1$.

These results have been extended to a broader context. Let L/K be a Galois extension of number fields with Galois group G, let $a, q \in \mathbb{N}$ with $(a,q) = 1$, and let $\mathrm{N}_{K/\mathbb{Q}}$ denote the absolute field norm of K. For a prime ideal $\mathfrak{p}$ of K which is unramified in L, there corresponds a certain conjugacy class $C \subset G$ of Frobenius automorphisms attached to the prime ideals of L which lie over $\mathfrak{p}$. We denote this conjugacy class by the Artin symbol $[\frac{L/K}{\mathfrak{p}}]$. For a fixed conjugacy class C, define

$$\pi_C(x;q,a) = \#\left\{\mathrm{N}_{K/\mathbb{Q}}\mathfrak{p} \leq x : \mathrm{N}_{K/\mathbb{Q}}\mathfrak{p} \equiv a \pmod{q}, \mathfrak{p} \text{ unramified in } L, \left[\frac{L/K}{\mathfrak{p}}\right] = C\right\}.$$

The Chebotarev density theorem asserts that if $q \leq (\log x)^D$, then

$$\pi_C(x;q,a) \sim d(C;q,a)\pi(x)$$

for some rational density $d(C;q,a)\geq 0$. If $\zeta_q=e^{2\pi i/q}$ and $L\cap\mathbb{Q}(\zeta_q)=\mathbb{Q}$, then

$$d(C;q,a)=\frac{|C|}{|G|}\frac{1}{\varphi(q)}.$$

If $H\subset G$ is a largest abelian subgroup such that $H\cap C$ is nonempty and E is the fixed field of H, then M. R. Murty and V. K. Murty [MM87] proved that if $K=\mathbb{Q}$ and $\theta<1/\max\{[E:\mathbb{Q}]-2,2\}$, then

$$\sideset{}{'}\sum_{q\leq x^\theta}\max_{(a,q)=1}\max_{N\leq x}\left|\pi_C(N;q,a)-\frac{|C|}{|G|}\frac{\pi(N)}{\varphi(q)}\right|\ll\frac{x}{(\log x)^D},$$

where $\sum'$ denotes summing over moduli q satisfying $L\cap\mathbb{Q}(\zeta_q)=\mathbb{Q}$. (This was later generalized by M. R. Murty and Petersen [MP13] to encompass any Galois extension L/K.) This extends the Bombieri-Vinogradov estimate to a nonabelian setting. In fact, the Bombieri-Vinogradov estimate is recovered when $L=\mathbb{Q}$.

1.2 Gaps between primes

The elusive twin prime conjecture states that if p_n is the n-th prime, then

$$\liminf_{n\to\infty}(p_{n+1}-p_n)=2.$$

The fact that there is a large amount of numerical evidence supporting the twin prime conjecture is fascinating, considering that the prime number theorem tells us that on average, the gap between consecutive primes $p_{n+1}-p_n$ is about $\log p_n$. A resolution to the twin prime conjecture seems beyond the reach of current methods. The next best result for which one could hope is that there are bounded gaps between primes;

that is, there exist a constant $C \geq 2$ such that

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq C.$$

In [GPY09], Goldston, Pintz, and Yıldırım developed the "GPY method", which led to the proof that

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0.$$

This method relies heavily on the distribution of primes in arithmetic progressions; in particular, it relies on the fact that the primes have level of distribution θ for any $\theta < \frac{1}{2}$. The GPY method produces bounded gaps between primes assuming that $\theta > \frac{1}{2}$. In [Zha14], Zhang proved that

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq 70 \times 10^6$$

by finding a suitable modification for the Bombieri-Vinogradov estimate which is valid for $\theta > \frac{1}{2}$. Zhang's work is inspiring but seems difficult to adapt to other settings.

In [May15], Maynard improved Zhang's bound (using techniques independent of Zhang's proof) to

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq 600.$$

Furthermore, for any fixed positive integer m, Maynard proved that

$$\liminf_{n\to\infty}(p_{m+n} - p_n) \ll m^3 \exp(4m),$$

which does not follow from Zhang's work. (Tao developed the underlying sieve theory independently, but arrived at slightly different conclusions.) These results follow from a dramatic improvement to the GPY method arising from the use of more general sieve weights. Once we have this improvement, all that one must know in order to obtain bounded gaps between primes is the distribution of primes within the integers (which is given by the prime number theorem) and the fact that the level of distribution θ of the primes is positive (which is given by the Bombieri-Vinogradov theorem).

We exploit the flexibility in the methods presented in [May15] to obtain analogous results on bounded gaps between primes in Chebotarev sets $\mathcal{P}$ determined by a Galois extension $L/\mathbb{Q}$. Such a set takes the form

$$\mathcal{P} = \Big\{p : p \nmid d_L, \Big[\frac{L/\mathbb{Q}}{p}\Big] = C\Big\},$$

where d_L is the absolute discriminant of L. (A union of Chebotarev sets is also considered a Chebotarev set since the union of conjugacy classes is invariant under the action of conjugation.) The Chebotarev density theorem asserts that $\mathcal{P}$ has relative density within the primes that is both positive and rational, and the aforementioned work of Murty and Murty [MM87] tells us that we can extend the notion of a positive level of distribution to $\mathcal{P}$ if we omit certain "bad" arithmetic progressions (namely, those progressions with moduli q such that $L \cap \mathbb{Q}(\zeta_q) \neq \mathbb{Q}$). These two ingredients, in conjunction with the sieve developed in [May15], enable us to prove the existence of bounded gaps between primes in any Chebotarev set.

Theorem 1.1. *Let $L/\mathbb{Q}$ be a Galois extension of number fields with Galois group G and absolute discriminant d_L, and let C be a conjugacy class of G. Let $\mathcal{P}$ be the set of primes $p \nmid d_L$ for which $[\frac{L/\mathbb{Q}}{p}] = C$, and let q_n be the n-th prime of $\mathcal{P}$. Let H be a largest abelian subgroup of G such that $H \cap C$ is nonempty, and let E be the fixed field of H. If m is a fixed positive integer, then*

$$\liminf_{n\to\infty}(q_{m+n} - q_n) \ll (\eta m)^3 \exp(2\eta m),$$

where

$$\eta = \max\{[E:\mathbb{Q}]-2, 2\}\frac{|G|}{|C|}\frac{d_L}{\varphi(d_L)}.$$

The implied constant is absolute.

Remark. We note that if $L/\mathbb{Q}$ is Galois, then $[E:\mathbb{Q}]-2 \leq [L:\mathbb{Q}]/2$. The upper bound is noticeably larger when the abelian subgroup $H \subset G$ has small index in G, but it is more computationally tractable.

We use Theorem 1.1 to prove several results in algebraic number theory with an emphasis on applications to ranks of quadratic twists of elliptic curves, congruences for the Fourier coefficients of cuspidal modular forms, and representations of integers by binary quadratic forms. These are listed and proven in Chapter 2. One example of such a result is as follows.

Theorem 1.2. *Let $Q(x,y) = ax^2+bxy+cy^2 \in \mathbb{Z}[x,y]$ be a primitive, positive-definite quadratic form with $b^2-4ac<0$, and let q_n denote the n-th prime such that $Q(x,y)=q_n$ for some $(x,y)\in\mathbb{Z}^2$. There exists a positive constant c_Q (which depends only on Q) such that if $m\in\mathbb{N}$, then*

$$\liminf_{n\to\infty}(q_{m+n}-q_n) \ll m^3\exp(c_Q m)$$

with an implied constant depending on Q. In particular, if n is a positive integer, then there are bounded gaps between primes of the form x^2+ny^2.

1.3 The distribution of primes in short intervals

The Bombieri-Vinogradov estimate implies that the error term in the prime number theorem for arithmetic progressions for primes lying in intervals of length x is about as small as predicted by GRH when we average over moduli q. If GRH is true, then the error term in the prime number theorem for arithmetic progressions is so strong that one can easily prove that if $\delta < \frac{1}{2}$ and $h \geq x^{1-\delta}$, then

$$\pi(x+h;q,a) - \pi(x;q,a) \sim \frac{\pi(x+h)-\pi(x)}{\varphi(q)} \sim \frac{1}{\varphi(q)}\frac{h}{\log x}.$$

This predicts a rather high degree of regularity in the distribution of primes. In an attempt to find evidence in favor of the truth of GRH, one wants to find how small h can be while unconditionally maintaining the expected distribution of primes.

Depending on the quality of the error term in the prime number theorem for arithmetic progressions, it is possible to deduce a "short interval" prime number theorem, in the form

$$\pi(x+h) - \pi(x) \sim \int_x^{x+h} \frac{dt}{\log t} \sim \frac{h}{\log x},$$

provided that h is not too small. With the presently best known error terms, we may take h a bit smaller than $x/(\log x)^D$ for any fixed $D > 0$, but not as small as $x^{1-\delta}$ for any $\delta > 0$. Improving the error term in the prime number theorem to allow for h to be of size $x^{1-\delta}$ is a monumentally hard task, known as the quasi-Riemann hypothesis, and amounts to showing that there are no zeros of the Riemann zeta function $\zeta(s)$ in the region $\Re(s) > 1-\delta$.

Nevertheless, in 1930, Hoheisel [Hoh30] made the remarkable observation that, with Littlewood's improved zero-free region for $\zeta(s)$, if there are simply *not too many* zeros in the region $\Re(s) > 1-\delta$, then one can deduce a variant of the prime number theorem for intervals of length $h = x^{1-\delta}$. In particular, it turns out that if we define

$$N(\sigma,T) := \#\{\rho = \beta + i\gamma : \zeta(\rho) = 0, \beta \geq \sigma, |\gamma| \leq T\},$$

we have that

$$N(\sigma,T) \ll T^{c(1-\sigma)}(\log T)^{c'},$$

where $c > 2$ and $c' > 0$ are absolute constants; this is an example of a *zero density estimate*. Recall that there are about $\frac{T}{\pi}\log\frac{T}{2\pi e}$ nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $|\gamma| \le T$. Thus the above zero density estimate for $\zeta(s)$ implies that there is a vanishingly small proportion of zeros with $\beta > 1 - 1/c$. By the work of Huxley [Hux72], we may take c to be any number larger than $\frac{12}{5}$, which translates to a prime number theorem for intervals of length $h \ge x^{1-\delta}$ for any $\delta < \frac{5}{12}$.

Hoheisel's original observation was for the set of all primes, but his ideas are easily extended to arithmetic progressions. Let $L(s,\chi)$ be a Dirichlet L-function. Defining

$$N_\chi(\sigma,T) := \#\{\rho = \beta + i\gamma : L(\rho,\chi) = 0, \beta \ge \sigma, |\gamma| \le T\},$$

one can prove that

$$\sum_{\chi \bmod q} N_\chi(\sigma,T) \ll (qT)^{c(1-\sigma)}(\log qT)^{c'}$$

for any $c > \frac{12}{5}$ and a suitable absolute constant $c' > 0$. From this, one deduces a short interval prime number theorem for arithmetic progressions of the form

$$\pi(x+h;q,a) - \pi(x;q,a) \sim \frac{\pi(x+h) - \pi(x)}{\varphi(q)},$$

where $\delta < \frac{5}{12}$, $h \ge x^{1-\delta}$, and $q \le (\log x)^D$ for any fixed $D > 0$.

Building on the methods in Bombieri's original proof [Bom65] of the Bombieri-Vinogardov theorem, Jutila [Jut70] proved the "hybrid" density estimate

$$\sum_{q \le Q} \sideset{}{^\star}\sum_{\chi \bmod q} N_\chi(\sigma,T) \ll (Q^2T)^{c(1-\sigma)}(\log QT)^{c'},$$

where $\sum^\star$ denotes summation over primitive Dirichlet characters modulo q; Montgomery [Mon71] improved upon Jutila's work to show that one may take $c = \frac{5}{2}$. As a consequence of Jutila and Montgomery's estimate, one sees that the average value of $N_\chi(\sigma,T)$ is noticeably smaller that what was previously shown. Furthermore, Jutila and Montgomery's estimate can be used to prove a short interval version of the Bombieri-Vinogradov theorem in the form

$$\sum_{q\le x^\theta}\max_{(a,q)=1}\max_{\substack{y\le h\\ \frac12 x\le N\le x}}\left|\pi(N+y;q,a)-\pi(N;q,a)-\frac{\pi(N+y)-\pi(N)}{\varphi(q)}\right|\ll\frac{h}{(\log x)^D},$$

where $\delta>0$ and $\theta>0$ are constants, $D>0$, and $h\ge x^{1-\delta}$. It follows from GRH that the above estimate holds when $0\le\delta<\frac12$ and $0<\theta<\frac12-\delta$. Despite the fact that GRH remains unproven, there has been much progress toward this conjectured estimate; see [PPS84] and the sources contained therein. Currently, the sharpest version is due to Timofeev [Tim87], with

$$0\le\delta<\frac{5}{12},\qquad 0\le\theta<\begin{cases}\frac12-\delta & \text{if } 0\le\delta<\frac25,\\ \frac{9}{20}-\delta & \text{if } \frac25\le\delta<\frac{5}{12}.\end{cases}$$

Using a generalization of Montgomery's zero density estimate for Hecke L-functions, we prove a short interval variant of the Bombieri-Vinogradov estimate in the context of the Chebotarev density theorem for any Galois extension. This extends the work of M. R. Murty and V. K. Murty [MM87] and M. R. Murty and Petersen [MP13] to a short interval setting.

Theorem 1.3. *Let L/K be a Galois extension of number fields with Galois group G, and let $C\subset G$ be a fixed conjugacy class. Let $H\subset G$ be a largest abelian subgroup of G such that $H\cap C$ is nonempty, and let E be the fixed field of H. Let $0\le\delta<\frac{2}{5[E:\mathbb{Q}]}$ and $0<\theta<\frac13(\frac{2}{5[E:\mathbb{Q}]}-\delta)$. If $h\ge x^{1-\delta}$, then for any constant $D>0$,*

$$\sideset{}{'}\sum_{q\le x^\theta}\max_{(a,q)=1}\max_{\substack{y\le h\\ \frac12 x\le N\le x}}\left|\pi_C(N+y;q,a)-\pi_C(N;q,a)-\frac{|C|}{|G|}\frac{\pi(N+y)-\pi(N)}{\varphi(q)}\right|\ll\frac{h}{(\log x)^D},$$

where $\sum'$ denotes summing over moduli q satisfying $L\cap\mathbb{Q}(\zeta_q)=\mathbb{Q}$.

In Chapter 3, we use Theorem 1.3 to prove refinements in the bounded gaps results in Chapter 2 where the primes in question can be restricted to a short interval. We also consider additional applications to the study of Fourier coefficients of half-integer weight modular forms, central critical values of modular L-functions, and ranks of quadratic twists of elliptic curves.

1.4 The distribution of zeros of L-functions

Another classical problem in analytic number theory is to determine the least prime in an arithmetic progression $a \pmod q$ with $(a, q) = 1$. Linnik [Lin44] was able to show that the least such prime is no bigger than q^A, where A is an absolute constant; Xylouris [Xyl11] proved that one may take $A = 5.2$. (An improvement to $A = 5$ is in his Ph.D. thesis.) Modern treatments of Linnik's theorem typically use a simplification due to Fogels [Fog65], which involves proving a technically involved refinement of the zero density estimate for all Dirichlet L-functions $L(s, \chi)$ given in the previous section. Specifically, if we define

$$N_\chi(\sigma, T) := \#\{\rho = \beta + i\gamma : L(\rho, \chi) = 0, \beta \geq \sigma, \text{ and } |\gamma| \leq T\},$$

then Fogels showed that

$$\sum_{\chi \pmod q} N_\chi(\sigma, T) \ll T^{c(1-\sigma)}, \qquad T \geq q.$$

Due to the absence of a $\log T$ term, it is standard to call such a result a *log-free zero density estimate*. We are interested in log-free zero density estimates for automorphic L-functions and their arithmetic applications, specifically to analogues of the above theorems of Hoheisel and Linnik.

We consider the following general setup. Let $K/\mathbb{Q}$ be a number field with ring of adeles $\mathbb{A}_K$, and let π be a cuspidal automorphic representation of $\mathrm{GL}_d(\mathbb{A}_K)$ with unitary central character; we simply refer to such a representation π as an automorphic representation. There is an L-function $L(s,\pi,K)$ attached to π whose Dirichlet series and Euler product are given by

$$L(s,\pi,K)=\sum_{\mathfrak{a}}\frac{\lambda_\pi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s}=\prod_{\mathfrak{p}}\prod_{j=1}^{d}(1-\alpha_\pi(j,\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s})^{-1},$$

where the sum runs over the non-zero integral ideals of K, the product runs over the prime ideals, and $\mathrm{N}\mathfrak{a}=\mathrm{N}_{K/\mathbb{Q}}\mathfrak{a}$ denotes the norm of the ideal $\mathfrak{a}$.

Let π and π' be automorphic representations of $\mathrm{GL}_d(\mathbb{A}_K)$ and $\mathrm{GL}_{d'}(\mathbb{A}_K)$, respectively. The Rankin-Selberg convolution

$$L(s,\pi\otimes\pi',K)=\sum_{\mathfrak{a}}\frac{\lambda_{\pi\otimes\pi'}(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s}=\prod_{\mathfrak{p}}\prod_{j_1=1}^{d}\prod_{j_2=1}^{d'}(1-\alpha_\pi(j_1,\mathfrak{p})\alpha_{\pi'}(j_2,\mathfrak{p})(\mathrm{N}\mathfrak{p})^{-s})^{-1}$$

is itself an L-function with an analytic continuation and a functional equation. Define

$$N_{\pi\otimes\pi'}(\sigma,T):=\#\{\rho=\beta+i\gamma : L(\rho,\pi\otimes\pi',K)=0,\beta\geq\sigma,|\gamma|\leq T\}.$$

In joint work with Lemke Oliver, we prove a log-free zero density estimate for $L(s,\pi\otimes\pi',K)$ which is effective in its dependence on π, π', and K. This dependence is most naturally stated in terms of the analytic conductors $\mathfrak{q}(\pi)$ and $\mathfrak{q}(\pi')$ of π and π', respectively. We prove the following.

Theorem 1.4. *Let K be a number field with absolute discriminant D_K. Let π and π' be automorphic representations of $\mathrm{GL}_d(\mathbb{A}_K)$ and $\mathrm{GL}_{d'}(\mathbb{A}_K)$, respectively. Suppose that either both $d\leq 2$ and $d'\leq 2$ or that at least one of π and π' is self-dual, and* *suppose that the generalized Ramanujan conjecture (GRC) holds for* $L(s,\pi,K)$*. Let*

$$\mathcal{Q}=\mathcal{Q}(\pi,\pi')=\begin{cases}\mathfrak{q}(\pi)\mathfrak{q}(\pi') & \text{if } \pi \text{ is nontrivial,}\\ \mathfrak{q}(\pi') & \text{if } \pi \text{ is trivial,}\end{cases}$$

let

$$\mathcal{D}=\mathcal{D}(\pi,\pi')=\begin{cases}d^2 & \text{if } d=d' \text{ and both } \pi \text{ and } \pi' \text{ are self-dual,}\\ (d')^4 & \text{if } \pi \text{ is trivial,}\\ (d+d')^4 & \text{otherwise,}\end{cases}$$

and let $T\gg[K:\mathbb{Q}]\mathcal{Q}^{1/[K:\mathbb{Q}]}$*. There exists an absolute constant* $c>0$ *such that if* $\frac{1}{2}\leq\sigma\leq 1$*, then*

$$N_{\pi\otimes\pi'}(\sigma,T)\ll d^2T^{c\mathcal{D}[K:\mathbb{Q}](1-\sigma)}.$$

All of the $\ll$ *implied constants are absolute.*

In Chapter 4, we prove several log-free zero density estimates for automorphic L-functions, including Theorem 1.4, and explore several arithmetic consequences, including applications to the Sato-Tate conjecture for certain automorphic representations of $\mathrm{GL}_2(\mathbb{A}_K)$ when K is totally real.

Chapter 2

Bounded gaps between primes in Chebotarev sets

The Bombieri-Vinogradov theorem, which may be thought of as an average form of the generalized Riemann hypothesis for Dirichlet L-functions, states that

$$\sum_{q \leq x^{\theta}} \max_{(a,q)=1} \left| \pi(x; q, a) - \frac{\pi(x)}{\varphi(q)} \right| \ll \frac{x}{(\log x)^A}$$

for any $\theta < \frac{1}{2}$ and any fixed $A > 0$. In other words, the primes have level of distribution θ for any $\theta < \frac{1}{2}$. In [May15], Maynard uses new sieve techniques in conjunction with the Bombieri-Vinogradov theorem to prove the existence of many primes in infinitely many intervals of bounded length. In this chapter, we use the work of M. R. Murty and V. K. Murty [MM87] and the work of Maynard to extend this progress toward the twin prime conjecture and study some arithmetic consequences.

Theorem 2.1. *Let $L/\mathbb{Q}$ be a Galois extension of number fields with Galois group G and absolute discriminant d_L, and let C be a conjugacy class of G. Let $H \subset G$ be a largest abelian subgroup such that $H \cap C$ is nonempty, and let E be the fixed field of H. Let a and q be fixed integers satisfy $(a, q) = 1$ and $(q, d_L) = 1$. Let q_n be the*

n-th prime not dividing d_L for which $[\frac{L/\mathbb{Q}}{p}] = C$ and $q_n \equiv a \,(\mathrm{mod}\, q)$. If m is a fixed positive integer, then

$$\liminf_{n\to\infty}(q_{m+n} - q_n) \ll q(\eta m)^3 \exp(2\eta m),$$

where

$$\eta = \max\{[E:\mathbb{Q}] - 2, 2\}\frac{|G|}{|C|}\frac{d_L}{\varphi(d_L)}.$$

The implied constant is absolute.

2.1 Notation

We will let k be a fixed positive integer. A set $\mathcal{H}_k$ of nonnegative integers $\{h_1, \ldots, h_k\}$ is said to be *admissible* if the polynomial $\prod_{i=1}^{k}(n + h_i)$ has no fixed prime divisors. The functions φ, $\tau_r(n)$, and μ refer to the Euler totient function, the number of representations of n as a product of r positive integers, and the Möbius function, respectively. We let p be a rational prime; given a set $\mathcal{P} \subset \mathbb{P}$, q_n will denote the n-th prime of $\mathcal{P}$. We let $\#S$ or $|S|$ denote the cardinality of a finite set S. If $x \in \mathbb{R}$, we write $\lfloor x \rfloor = \min\{a \in \mathbb{Z} : a \leq x\}$ and $\lceil x \rceil = \max\{a \in \mathbb{Z} : a \geq x\}$. For a number field F, we let d_L be the absolute discriminant of L and $n_F = [F : \mathbb{Q}]$.

2.2 Bounded gaps between primes

The variant of the Selberg sieve developed by Maynard in [May15] eliminates the $\theta > \frac{1}{2}$ barrier to achieving bounded gaps between primes that the original GPY method encountered. By studying the proof of the following theorem, it is clear that we obtain bounded gaps between primes as long as $\theta > 0$, a condition which is guaranteed by the Bombieri-Vinogradov theorem.

Theorem 2.2 (Maynard-Tao)**.** *For any* $m \in \mathbb{N}$*,*

$$\lim_{n\to\infty}(p_{m+n} - p_n) \ll m^3 \exp(4m).$$

We provide a brief outline the important components of the proof given in [May15]. For a fixed admissible set $\mathcal{H}_k = \{h_1, \ldots, h_k\}$, we consider the sums

$$S_1(N) = \sum_{\substack{N\le n<2N \\ n\equiv v_0 (\mathrm{mod}\, W)}} w_n, \tag{2.1}$$

$$S_2(N) = \sum_{\substack{N\le n<2N \\ n\equiv v_0 (\mathrm{mod}\, W)}} \sum_{i=1}^{k} \chi_{\mathbb{P}}(n+h_i) w_n, \tag{2.2}$$

$$S(N,\rho) = S_2(N) - \rho S_1(N), \tag{2.3}$$

where w_n are nonnegative weights, $\rho > 0$, $\chi_{\mathbb{P}}$ is the indicator function of the primes, and

$$W = \prod_{p\le D_0} p, \qquad D_0 = \log\log\log N. \tag{2.4}$$

By the prime number theorem, $W \ll (\log\log N)^2$.

The goal is to show that $S(N,\rho) > 0$ for all sufficiently large N. This would imply that for infinitely many N, there exists $n \in [N, 2N)$ for which at least $\lfloor \rho + 1 \rfloor$ of the $n + h_i$ are prime, establishing an infinitude of intervals of bounded length containing $\lfloor \rho + 1 \rfloor$ primes.

Let $\mathcal{R}_k = \{\vec{x} \in [0,1]^k : \sum_{i=1}^{k} x_i \le 1\}$, let $F : [0,1]^k \to \mathbb{R}^k$ be a infinitely differentiable function supported on $\mathcal{R}_k$, and let $R = N^{\theta/2-\epsilon}$. The weights w_n are of the form

$$w_n = \Big(\sum_{d_i | n+h_i \forall i} \lambda_{d_1,\ldots,d_k} \Big)^2, \tag{2.5}$$

where

$$\lambda_{d_1,\ldots,d_k} = \Big(\prod_{i=1}^{k} \mu(d_i) d_i\Big) \sum_{\substack{r_1,\ldots,r_k \\ d_i | r_i \ \forall i \\ (r_i, W)=1 \ \forall i}} \frac{\mu(\prod_{i=1}^{k} r_i)^2}{\prod_{i=1}^{k} \varphi(r_i)} F\Big(\frac{\log(r_1)}{\log R}, \ldots, \frac{\log(r_k)}{\log R}\Big). \tag{2.6}$$

Setting $d = \prod_{i=1}^{k} d_i$, we choose $\lambda_{d_1,\ldots,d_k}$ to be supported when $d < R$, $(d, W) = 1$, and $\mu(d)^2 = 1$.

First, Maynard estimates the sums $S_1(N)$ and $S_2(N)$.

Proposition 2.3. *Let $\mathbb{P}$ have level of distribution $\theta > 0$. Let $F : [0,1]^k \to \mathbb{R}$ be a fixed infinitely differentiable function supported on $\mathcal{R}_k$. We have*

$$S_1(N) = (1 + o(1)) \frac{\varphi(W)^k N (\log R)^k}{W^{k+1}} I_k(F),$$
$$S_2(N) = (1 + o(1)) \frac{\varphi(W)^k N (\log R)^k}{W^{k+1}} \frac{\log R}{\log N} \sum_{i=1}^{k} J_k^{(i)}(F),$$

provided that $I_k(F) \neq 0$ and $J_k^{(i)}(F) \neq 0$ for each i, where

$$I_k(F) = \int_0^1 \cdots \int_0^1 F(t_1, \ldots, t_k)^2 \ dt_1 \cdots dt_k,$$
$$J_k^{(i)}(F) = \int_0^1 \cdots \int_0^1 \Big(\int_0^1 F(t_1, \ldots, t_k) \ dt_i\Big)^2 \ dt_1 \cdots dt_{i-1} dt_{i+1} \cdots dt_k.$$

Proof. This is proven in Sections 5 and 6 of [May15]. □

Following the GPY method, we want $S_2(N) - \rho S_1(N)$ to be positive for all sufficiently large N, ensuring that for infinitely many n, several of the $n + h_i$ are prime. The following proposition states this formally.

Proposition 2.4. *Let $\mathbb{P}$ have level of distribution $\theta > 0$, and let $\mathcal{H}_k = \{h_1, \ldots, h_k\}$ be an admissible set. Let $\mathcal{S}_k$ denote the set of smooth functions $F : [0,1]^k \to \mathbb{R}$ supported* *on $\mathcal{R}_k$ with $I_k(F) \neq 0$ and $J_k^{(i)}(F) \neq 0$ for each i. Define*

$$M_k = \sup_{F \in \mathcal{S}_k} \frac{\sum_{i=1}^k J_k^{(i)}(F)}{I_k(F)}, \qquad r_k = \left\lceil \frac{\theta M_k}{2} \right\rceil.$$

There are infinitely many n such that at least r_k of the $n+h_i$ are prime. Furthermore, if p_n is the n-th prime, then

$$\liminf_{n\to\infty}(p_{n+r_k-1} - p_n) \leq \max_{1\leq i<j\leq k}(h_i - h_j).$$

Proof. This is proven in Section 4 of [May15]. □

All that remains is to find a suitable lower bound for M_k.

Proposition 2.5. *If k is sufficiently large, then*

$$M_k > \log k - 2\log\log k - 2.$$

Proof. This is proven in Section 8 of [May15]. □

The exact manner in which these propositions are put together is outlined in Section 4 of [May15]. We emulate those arguments in the next section.

2.3 Proof of Theorem 2.1

One fascinating aspect of the proof of Theorem 2.2 is how adaptable it is to exploring bounded gaps between primes in special subsets of the primes. In this section, we will modify the proof to obtain a version applicable to sets of primes satisfying a Chebotarev condition.

Let $L/\mathbb{Q}$ be a Galois extension of number fields with Galois group G and discriminant d_L, and let C be a conjugacy class of G. Let $H \subset G$ be an abelian subgroup such that $H \cap C$ is nonempty, and let E be the fixed field of H. Let

$$\mathcal{P} = \left\{ p \text{ prime} : p \nmid d_L, \left[\frac{L/\mathbb{Q}}{p}\right] = C \right\}, \tag{2.7}$$

where $[\frac{L/\mathbb{Q}}{\cdot}]$ is the Artin symbol, and define

$$\pi_{\mathcal{P}}(N) = \#\{N \le p < 2N : p \in \mathcal{P}\}, \tag{2.8}$$

$$\pi_{\mathcal{P}}(N; q, a) = \#\{N \le p < 2N : p \in \mathcal{P} : p \equiv a \pmod{q}\}. \tag{2.9}$$

We say that $\mathcal{P}$ has level of distribution θ if there exists a fixed positive integer M such that for any fixed $A > 0$,

$$\sum_{\substack{q \le N^\theta \\ (q,M)=1}} \max_{(a,q)=1} \left| \pi_{\mathcal{P}}(y; q, a) - \frac{1}{\varphi(q)} \pi_{\mathcal{P}}(y) \right| \ll \frac{N}{(\log N)^A}. \tag{2.10}$$

Lemma 2.6. *Assume the above notation. Let* $\delta = |C|/|G|$.

1. *We have* $\pi_{\mathcal{P}}(N) = \delta N/\log N + O(N/(\log N)^2)$.

2. *Equation 2.10 holds when* $M = d_L$ *and* $0 < \theta < \frac{1}{\max\{[E:\mathbb{Q}]-2,2\}}$.

Proof. The first part is the Chebotarev density theorem with error term. The second part follows from the main result in [MM87]. □

In order to use the second part of Lemma 2.6, we must modify the work in the previous section. Let W be defined as in (2.4), and let $\mathcal{H}_k = \{h_1, \ldots, h_k\}$ be admissible. For a positive integer n, let

$$\text{rad}(n) = \prod_{p|n} p.$$

Define

$$\det(\mathcal{H}_k) = \prod_{i \ne j} (h_i - h_j), \qquad U = W/\text{rad}(d_L). \tag{2.11}$$

By the Chinese Remainder Theorem and the admissibility of $\mathcal{H}_k$, there exists an integer u_0 satisfying $(\prod_{i=1}^{k}(u_0+h_i), U) = 1$. Instead of the restriction $n \equiv v_0 \pmod{W}$, we use $n \equiv u_0 \pmod{U}$. We note that when N is sufficiently large, $\mathrm{rad}(d_L \det(\mathcal{H}_k))$ divides W. As in the previous section, $\lambda_{d_1,\ldots,d_k}$ will be supported when

$$d = \prod_{i=1}^{k} d_i < R, \quad (d, W) = 1, \quad \mu(d)^2 = 1, \quad (d_i, d_j) = 1 \text{ for all } i \neq j. \tag{2.12}$$

Therefore, if N is sufficiently large, then (2.4), (2.11), and (2.12) tell us

$$\lambda_{d_1,\ldots,d_k} \neq 0 \text{ implies that } \prod_{1 \le i < j \le k} (d_i, d_j) = (d, U \det(\mathcal{H}_k) d_L) = 1. \tag{2.13}$$

Define

$$S_1(N, \mathcal{P}) = \sum_{\substack{N \le n < 2N \\ n \equiv u_0 \pmod{U}}} \Big(\sum_{d_i | n+h_i \forall i} \lambda_{d_1,\ldots,d_k} \Big)^2, \tag{2.14}$$

$$S_2^{(m)}(N, \mathcal{P}) = \sum_{\substack{N \le n < 2N \\ n \equiv u_0 \pmod{U}}} \chi_{\mathcal{P}}(n + h_m) \Big(\sum_{d_i | n+h_i \forall i} \lambda_{d_1,\ldots,d_k} \Big)^2, \tag{2.15}$$

$$S_2(N, \mathcal{P}) = \sum_{i=1}^{k} S_2^{(i)}(N, \mathcal{P}), \tag{2.16}$$

$$S(N, \rho, \mathcal{P}) = S_2(N, \mathcal{P}) - \rho S_1(N, \mathcal{P}), \tag{2.17}$$

where $\rho > 0$. For a fixed $\theta > 0$ satisfying (2.10), let $R = N^{\theta/2-\epsilon}$. We have the following estimate $S_1(N, \mathcal{P})$.

Proposition 2.7. *Assume the above notation. If $\mathcal{P}$ has level of distribution $\theta > 0$, then*

$$S_1(N, \mathcal{P}) = (1 + o(1)) \mathrm{rad}(d_L) \frac{\varphi(W)^k N (\log R)^k}{W^{k+1}} I_k(F),$$

where $I_k(F)$ is defined in Proposition 2.3.

Proof. The only difference between S_1 from Proposition 2.3 and $S_1(\mathcal{P})$ is that instead of the condition $n \equiv v_0 \pmod W$, we have $n \equiv u_0 \pmod U$. Following the proof of Lemma 5.1 in [May15], we will alleviate $S_1(\mathcal{P})$ of any conditions in the sums that depend on U. Then the Selberg sieve manipulations and analysis from [May15] will give us the desired estimates.

Expanding the square gives us

$$S_1(N,\mathcal{P}) = \sum_{\substack{d_1,\ldots,d_k \\ e_1,\ldots,e_k}} \lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k} \sum_{\substack{N\le n<2N \\ n\equiv u_0 \pmod U \\ [d_i,e_i]|n+h_i\forall i}} \chi_{\mathcal{P}}(n+h_m).$$

We now show that we can write the conditions $n \equiv u_0 \pmod U$ and $[d_i, e_i] \mid n + h_i$ for all i as a single congruence condition. Let $d = \prod_{a=1}^k d_a$ and $e = \prod_{a=1}^k e_a$. If $\mu(d)^2 = 0$ or $\mu(e)^2 = 0$, then $\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k} = 0$ by (2.12). Thus we assume that $\mu(d)^2 = \mu(e)^2 = 1$, so each d_i and e_i is squarefree. With each d_i, e_i squarefree, we consider the following two cases:

1. If a prime p divides $(U, [d_i, e_i])$ for some i, then $p \mid (U, d_i)$ or $p \mid (U, e_i)$. Thus $p \mid (d, U)$ or $p \mid (e, U)$, and $\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k} = 0$ by (2.13).

2. If a prime p divides $([d_i, e_i], [d_j, e_j])$ for some $i \neq j$, then

$$p \mid d \ \text{ or } \ p \mid e, \qquad p \mid n + h_i, \qquad \text{and} \qquad p \mid n + h_j.$$

   Thus $p \mid (d, h_i - h_j)$ or $p \mid (e, h_i - h_j)$. Therefore, $p \mid (d, \det(\mathcal{H}_k))$ or $p \mid (e, \det(\mathcal{H}_k))$, and $\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k} = 0$ by (2.13).

Using the Chinese Remainder Theorem, we conclude that the inner sum can be written as a sum over a single residue class modulo $q = U \prod_{i=1}^k [d_i, e_i]$ when U and each $[d_i, e_i]$ are pairwise coprime, in which case the inner sum is $N/q + O(1)$. Otherwise,

$\lambda_{d_1\dots,d_k}\lambda_{e_1,\dots,e_k} = 0$. Using Lemma 5.1 of [May15] and (2.11), we have

$$\begin{aligned} S_1(N,\mathcal{P}) &= \frac{N}{U}\sideset{}{'}\sum_{\substack{d_1,\dots,d_k\\ e_1,\dots,e_k}} \frac{\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^k [d_i,e_i]} + O\Big(\sideset{}{'}\sum_{\substack{d_1,\dots,d_k\\ e_1,\dots,e_k}} |\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}|\Big) \\ &= \mathrm{rad}(d_L)\frac{N}{W}\sideset{}{'}\sum_{\substack{d_1,\dots,d_k\\ e_1,\dots,e_k}} \frac{\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^k [d_i,e_i]} + O(\lambda_{\max}^2 R^2 (\log R)^{2k}), \end{aligned}$$

where $\lambda_{\max} = \sup_{d_1,\dots,d_k} |\lambda_{d_1,\dots,d_k}|$ and $\sum'$ denotes the restriction that U and each $[d_i,e_i]$ are pairwise coprime and each d_i, e_i is squarefree. If a prime p divides $([d_i,e_i],U)$ for some i, then we have already shown that $\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k} = 0$. Therefore, we may take $\sum'$ to denote the condition that $\prod_{i\neq j}([d_i,e_i],[d_j,e_j]) = 1$, which is a condition that is independent of the arithmetic progression containing n. Therefore, the condition $\sum'$ is independent of our modulus U, as desired.

We now see that $S_1(N,\mathcal{P})$ is a multiple (depending only on d_L) of $S_1(N)$ in one of the intermediate steps in Lemma 5.1 of [May15]. Therefore, the proposition follows from Lemmata 5.1 and 6.2 of [May15]. □

We will use the reasoning from the above proof to estimate $S_2(N,\mathcal{P})$.

Proposition 2.8. *Assume the above notation. Let $L/\mathbb{Q}$ be a Galois extension of number fields with Galois group G and discriminant d_L, and let C be a conjugacy class of G. Let $\delta = |C|/|G|$. If the primes in $\mathcal{P}$ have level of distribution $\theta > 0$, then*

$$S_2(N,\mathcal{P}) = (1+o(1))\delta\varphi(\mathrm{rad}(d_L))\frac{\log R}{\log N}\frac{\varphi(W)^k N(\log R)^k}{W^{k+1}}\sum_{i=1}^k J_k^{(i)}(F),$$

where $J_k^{(i)}(F)$ is defined in Proposition 2.3.

Proof. The desired result follows from estimating each $S_2^{(m)}(N,\mathcal{P})$ for each $1 \le m \le k$.

Expanding the square gives us

$$S_2^{(m)}(N,\mathcal{P}) = \sum_{\substack{d_1,\dots,d_k\\ e_1,\dots,e_k}} \lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k} \sum_{\substack{N\le n<2N\\ n\equiv u_0 \,(\mathrm{mod}\, U)\\ [d_i,e_i]\mid n+h_i \forall i}} \chi_{\mathcal{P}}(n+h_m).$$

As with $S_1(N,\mathcal{P})$, the inner sum can be written as a sum over a single residue class a_m modulo $q = U\prod_{i=1}^{k}[d_i,e_i]$ when U and each $[d_i,e_i]$ are pairwise coprime, and $\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k} = 0$ otherwise.

Choose integers $d_1,\dots,d_k,e_1,\dots,e_k$ such that $\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k} \neq 0$. Clearly

$$a_m \equiv u_0 \,(\mathrm{mod}\, U) \quad \text{and} \quad [d_i,e_i] \mid a_m + h_i \text{ for all } i.$$

We conclude from the support of $\lambda_{d_1,\dots,d_k}$ and our choices of u_0 and a_m that

$$(u_0+h_m,U) = 1 \quad \text{and} \quad (h_m-h_i,[d_i,e_i]) = 1 \text{ for all } i \neq m,$$

so

$$(q/[d_m,e_m], a_m+h_m) = 1 \quad \text{and} \quad [d_m,e_m] \mid a_m+h_m.$$

Therefore, $(q,a_m+h_m) = 1$ if and only if $d_m = e_m = 1$. In this case, the inner sum will have size $\pi_{\mathcal{P}}(N)/\varphi(q) + O(E(N,q))$, where

$$E(N,q) = \max_{(a,q)=1}\left|\pi_{\mathcal{P}}(N;q,a) - \frac{1}{\varphi(q)}\pi_{\mathcal{P}}(N)\right|.$$

If $(q,a_m+h_m)\neq 1$, then the inner sum equals either 0 or 1. The inner sum equals 1 if and only if there exists a prime p satisfying $n+h_m = p$ for some $n\in[N,2N)$ with $p\mid q$. Since N is large, we have $N-|h_m| > \sqrt{N} > R$. Thus $n+h_m = p$ for some $n\in[N,2N)$ implies that $p>R$, so if $p\mid q$, then $\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k} = 0$. Thus the inner sum only contributes to $S_2^{(m)}(\mathcal{P})$ when $(q, a_m + h_m) = 1$. We conclude that

$$S_2^{(m)}(N, \mathcal{P}) = \frac{\pi_{\mathcal{P}}(N)}{\varphi(U)} \sideset{}{'}\sum_{\substack{d_1,\ldots,d_k \\ e_1,\ldots,e_k \\ d_m=e_m=1}} \frac{\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}}{\prod_{i=1}^k \varphi([d_i,e_i])} + O\Big(\sideset{}{'}\sum_{\substack{d_1,\ldots,d_k \\ e_1,\ldots,e_k}} |\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}| E(N,q)\Big),$$

where $q = U\prod_{i=1}^k [d_i, e_i]$ and $\sum'$ denotes the restriction that U and each $[d_i, e_i]$ be pairwise coprime.

We first analyze the error term. From the support of $\lambda_{d_1,\ldots,d_k}$, we only need to consider squarefree $q < R^2U \leq N^{\theta-\epsilon}$ satisfying $(q, d_L) = 1$, where $\epsilon > 0$ is sufficiently small. Given a squarefree integer r, there are at most $\tau_{3k}(r)$ choices of $d_1, \ldots, d_k, e_1, \ldots, e_k$ for which $r = U\prod_{i=1}^k [d_i, e_i]$. From Lemma 5.2 of [May15], the error term is now

$$\ll \lambda_{\max}^2 \sum_{\substack{r<N^{\theta-\epsilon} \\ (r,d_L)=1}} \mu(r)^2 \tau_{3k}(r) E(N, r),$$

Using the Cauchy-Schwarz inequality and the trivial bound $E(N, q) \ll N/\varphi(q)$, the error term is

$$\ll \lambda_{\max}^2 \Big(\sum_{\substack{r<N^{\theta-\epsilon} \\ (r,d_L)=1}} \mu(r)^2 \tau_{3k}(r)^2 \frac{N}{\varphi(r)}\Big)^{1/2} \Big(\sum_{\substack{r<N^{\theta-\epsilon} \\ (r,d_L)=1}} \mu(r)^2 E(N,r)\Big)^{1/2}.$$

It follows from elementary bounds on $\tau_{3k}(r)$ and Lemma 2.6 that the error is $\ll \lambda_{\max}^2 N/(\log N)^A$ for any fixed $A > 0$, which is also true of the error term in $S_2^{(m)}(N)$ in Lemma 5.2 of [May15].

Using (2.11), for any fixed $A > 0$, we have

$$S_2^{(m)}(N, \mathcal{P}) = \varphi(\mathrm{rad}(d_L)) \frac{\pi_{\mathcal{P}}(N)}{\varphi(W)} \sideset{}{'}\sum_{\substack{d_1,\ldots,d_k \\ e_1,\ldots,e_k \\ d_m=e_m=1}} \frac{\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}}{\prod_{i=1}^k \varphi([d_i,e_i])} + O(\lambda_{\max}^2 N/(\log N)^A),$$

where $\sum'$ denotes the restriction that U and each $[d_i, e_i]$ be pairwise coprime. As in the proof of Proposition 2.7, we can take $\sum'$ to denote the restriction that

$$\prod_{i\neq j}([d_i, e_i], [d_j, e_j]) = 1.$$

Therefore, up to the choice of prime counting function (which results in the factor of δ in the statement of the proposition), $S_2^{(m)}(N, \mathcal{P})$ is a multiple (depending only on δ and d_L) of $S_2^{(m)}(N)$ in one of the intermediate steps in Lemma 5.2 of [May15]. Thus the proposition follows from Lemmata 5.2 and 6.3 of [May15] and Lemma 2.6. □

We now modify Proposition 2.4 accordingly.

Proposition 2.9. *Let $\mathcal{H}_k = \{h_1, \ldots, h_k\}$ be an admissible set, let $\mathcal{P}$ have level of distribution $\theta > 0$, and let*

$$M_k = \sup_{F \in \mathcal{S}_K} \frac{\sum_{i=1}^k J_k^{(i)}(F)}{I_k(F)}, \qquad r_k = \left\lceil \frac{\delta\theta\varphi(d_L)M_k}{2d_L} \right\rceil.$$

Then there are infinitely many n such that at least r_k of the $n + h_i$ are in $\mathcal{P}$. Furthermore, if p_n is the n-th prime in $\mathcal{P}$, then

$$\liminf_{n\to\infty}(p_{n+r_k-1} - p_n) \leq \max_{1\leq i<j\leq k}(h_i - h_j).$$

Proof. We want to show that $S(N, \rho, \mathcal{P}) > 0$ for all sufficiently large N. Recall that $R = N^{\theta/2-\epsilon}$ for some small $\epsilon > 0$. By the definition of M_k, we can choose $F_0 \in \mathcal{S}_k$ such that

$$\sum_{i=1}^k J_k^{(i)}(F_0) > (M_k - \epsilon)I_k(F_0).$$

Using Propositions 2.7 and 2.8 and the identity $\frac{\varphi(\mathrm{rad}(d_L))}{\mathrm{rad}(d_L)} = \frac{\varphi(d_L)}{d_L}$, we have

$$\begin{aligned}
&S(N,\rho,\mathcal{P}) \\
&= \frac{\varphi(W)^k N(\log R)^k}{W^{k+1}}\Big(\frac{\log R}{\log N}\delta\varphi(\mathrm{rad}(d_L))\sum_{i=1}^{k} J_k^{(i)}(F_0) - \rho\mathrm{rad}(d_L)I_k(F_0) + o(1)\Big) \\
&\geq \frac{\varphi(W)^k N(\log R)^k I_k(F_0)}{W^{k+1}}\Big(\frac{\delta\varphi(d_L)}{d_L}\Big(\frac{\theta}{2} - \delta\Big)(M_k - 2\delta) - \rho + o(1)\Big).
\end{aligned}$$

Let

$$\rho = M_k\Big(\frac{\delta\theta\varphi(d_L)}{2d_L} - \epsilon\Big).$$

By choosing δ suitably small (depending on ϵ), we have $S(N,\rho,\mathcal{P}) > 0$ for all sufficiently large N. Thus there are infinitely many n for which at least $\lfloor \rho + 1 \rfloor$ of the $n + h_i$ are in $\mathcal{P}$. If ϵ is sufficiently small, then

$$\lfloor \rho + 1 \rfloor = \Big\lceil \frac{\delta\theta\varphi(d_L)M_k}{2d_L} \Big\rceil,$$

and we obtain the claimed result. □

Since a suitable lower bound for M_k is given by Proposition 2.5, we are ready to proceed with the proof.

Proof of Theorem 2.1. Lemma 2.6 tells us that $\mathcal{P}$ has level of distribution

$$\theta = \frac{1 - \frac{1}{k}}{\max\{[E:\mathbb{Q}] - 2, 2\}}.$$

Recall that $\delta = |C|/|G|$. By Proposition 2.5, if k is sufficiently large, then

$$\frac{\delta\theta\varphi(d_L)M_k}{2d_L} \geq \frac{|C|\max\{[E:\mathbb{Q}] - 2, 2\}\varphi(d_L)}{2|G|d_L}\Big(1 - \frac{1}{k}\Big)(\log k - 2\log\log k - 2). \quad (2.18)$$

It follows that (2.18) is greater than m if $k = c(2\eta m)^2\exp(2\eta m)$ for some suitably large positive absolute constant c, where

$$\eta = \max\{[E:\mathbb{Q}]-2,2\}\frac{|G|}{|C|}\frac{d_L}{\varphi(d_L)}.$$

Thus for any admissible set $\mathcal{H}_k = \{h_1,\dots,h_k\}$ with k as above, at least $m+1$ of the $n+h_i$ are in $\mathcal{P}$ for infinitely many integers n.

To construct an admissible set $\mathcal{H}_k$ which gives good bounds that are uniform over all choices of $L/\mathbb{Q}$ all conjugacy classes in the Galois group of $L/\mathbb{Q}$, we choose $h_j = q\cdot p_{\pi(k)+j}$, where $1\le j\le k$ and p_j is the j-th prime in $\mathbb{P}$; such a set is easily seen to be admissible. By the prime number theorem,

$$p_n = n\log n + n\log\log n + O(n) \qquad \text{and} \qquad \pi(n) = \frac{n}{\log n} + O\Big(\frac{n}{(\log n)^2}\Big).$$

Thus $p_{\pi(k)+k} - p_{\pi(k)+1} \ll k\log k$. Therefore, if q_n is the n-th prime of $\mathcal{P}$, then

$$\liminf_{n\to\infty}(q_{m+n}-q_n) \le \max_{1\le i<j\le k}|h_j-h_i| \ll k\log k \ll (\eta m)^3\exp(2\eta m)$$

with an absolute implied constant, as desired.

We now briefly describe why we may restrict the primes q_n to lie in an arithmetic progression $a \pmod q$ with $(a,q)=1$. Instead of considering the admissible set $\{p_{\pi(k)+1},\dots,p_{\pi(k)+k}\}$, consider the admissible set $\{q\cdot p_{\pi(k)+1},\dots,q\cdot p_{\pi(k)+k}\}$. By the Chinese remainder theorem, we may choose $v_0\equiv a \pmod q$. Thus the prime values attained by the linear forms $n+q\cdot p_{\pi(k)+1},\dots,n+q\cdot p_{\pi(k)+k}$ must be congruent to a modulo q. Now,

$$\liminf_{n\to\infty}(q_{m+n}-q_n) \le \max_{1\le i<j\le k}|h_j-h_i| \ll q\cdot k\log k \ll q(\eta m)^3\exp(2\eta m).$$

□

2.4 Applications to number fields and elliptic curves

We use Theorem 2.1 to prove several results in algebraic number theory. In order to make the results more explicit, we use the following result.

Proposition 2.10. *Let $L/\mathbb{Q}$ be a Galois extension of number fields with absolute discriminant d_L, and let $n_L = [L : \mathbb{Q}]$. Let C be a fixed conjugacy class of G, and let $H \subset G$ be an abelian subgroup such that $H \cap C$ is nonempty. Let E be the fixed field of H, and let*

$$\eta = \max\{n_E - 2, 2\}\frac{|G|}{|C|}\frac{d_L}{\varphi(d_L)}.$$

We have that $\eta = 2$ if and only if $L = \mathbb{Q}$. Otherwise,

$$\eta \leq \max\{n_L, 4\}n_L(\log\log d_L + 2) \leq \max\{n_L, 4\}n_L(\log(n_L \log(n_L \mathrm{rad}(d_L))) + 2).$$

Proof. The first part follows from Minkowski's inequality and basic Galois theory. For the second part, it follows from Theorem 8.8.7 of Bach and Shallit [BS96] that for $d_L \geq 2$,

$$\frac{d_L}{\varphi(d_L)} \leq e^{\gamma}\log\log d_L + 4 \leq 2(\log\log d_L + 2),$$

where

$$\gamma = \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n} - \log N\right) = 0.5772\ldots$$

By Serre [Ser81, Proposition 6], we have that since $L/\mathbb{Q}$ is Galois,

$$\frac{n_L}{2}\log\mathrm{rad}(d_L) \leq \log d_L \leq n_L\log n_L + (n_L - 1)\log\mathrm{rad}(d_L).$$

The fact that $|G| = n_L$ and $\max\{n_E - 2, 2\}\frac{|G|}{|C|} \leq \max\{n_L/2, 2\}n_L$ follows from basic Galois theory. □

Our first two applications are immediate.

Corollary 2.11. *Let $L/\mathbb{Q}$ be a Galois extension of number fields with ring of integers $\mathcal{O}_L$, and let $n_L = [L : \mathbb{Q}] \geq 2$. If $L/\mathbb{Q}$ is abelian, let* $\operatorname{cond}(L)$ *be the smallest positive integer q such that $L \subset \mathbb{Q}(e^{2\pi i/q})$. There exist infinitely $N \in \mathbb{N}$ such for any $m \in \mathbb{N}$, there are $m+1$ non-conjugate prime ideals in $\mathcal{O}_K$ whose norms lie in a subinterval of $[N, 2N]$ of length $\ll q(c_L m)^3 \exp(2c_L m)$, where*

$$q = \begin{cases} \operatorname{cond(L)} & \text{if } L/\mathbb{Q} \text{ is abelian,} \\ 1 & \text{otherwise} \end{cases}$$

and

$$c_L = \begin{cases} 1/2 & \text{if } L/\mathbb{Q} \text{ is abelian,} \\ \max\{n_L/2, 2\} n_L \cdot (e^{\gamma} \log\log d_L + 4) & \text{otherwise.} \end{cases}$$

The implied constant is absolute.

Proof. This follows from applying Theorem 2.1 to the set of primes $\mathcal{P}$ that are inert in L. In the special case that $L/\mathbb{Q}$ is abelian, the Kronecker-Weber theorem says that for some $q \geq 1$, $L \subset \mathbb{Q}(e^{2\pi i/q})$, and so the least prime that is inert in L is congruent to a modulo q for some $a \in \mathbb{Z}$ with $(a, q) = 1$ and $a \neq 1$. The computation of c_L follows from Proposition 2.10. □

Let $f \in \mathbb{Z}[x]$ be monic polynomial of degree d and discriminant d_L that is irreducible over $\mathbb{Q}$, and let G be the permutation representation of the Galois group of f. Let $p \nmid d_L$ be a prime, let $1 \leq r \leq d$, and suppose that $f \equiv \prod_{i=1}^{r} f_i \pmod{p}$ with the f_i distinct irreducible polynomials in $(\mathbb{Z}/p\mathbb{Z})[x]$ of degree n_i. Then G contains a permutation σ_p that is a product of disjoint cycles of length n_i; we call the cycle type of σ_p the *factorization type* of f mod p.

Corollary 2.12. *Assume the above notation. Let $f \in \mathbb{Z}[x]$ be an irreducible monic polynomial of degree $n_f \geq 3$ and discriminant d_f. Let q_n denote the n-th prime such* *that* $f \bmod q_n$ *has a given factorization type. For any* $m \in \mathbb{N}$*,*

$$\liminf_{n\to\infty}(q_{m+n} - q_n) \ll (c_f m)^3 \exp(2c_f m)$$

with an absolute implied constant. Here, $c_f \leq \frac{(n_f!)^2}{2} \cdot (e^\gamma \log(2n_f! \log d_f) + 4)$.

Proof. Consider the Galois group G of f as a permutation group, and let L be the fixed field of G. For all but finitely many primes p, $f \bmod p$ has a given factorization type if and only if the Artin symbol $[\frac{L/\mathbb{Q}}{p}]$ is the conjugacy class of automorphisms in G with the corresponding cycle type. Thus the primes p for which the factorization type of $f \bmod p$ is fixed is a Chebotarev set, and one may apply Theorem 2.1. Since $[L : \mathbb{Q}]$ divides $n_f!$ and L is unramified outside of the primes dividing d_f, we use Proposition 2.10 to compute c_f. □

Theorem 2.1 has many interesting applications to the theory of elliptic curves. Let $E/\mathbb{Q}$ be an elliptic curve with Weierstrass equation

$$E : y^2 = x^3 + ax^2 + bx + c,$$

and let $E_d/\mathbb{Q}$ denote the quadratic twist of E by d with Weierstrass equation

$$E_d : y^2 = x^3 + adx^2 + bd^2x + cd^3.$$

We denote the rank of the group of $\mathbb{Q}$-rational points $E(\mathbb{Q})$ by $\mathrm{rk}(E(\mathbb{Q}))$. Our applications are related to the following conjecture due to Silverman regarding $\mathrm{rk}(E_{\pm p}(\mathbb{Q}))$ when p is prime.

Conjecture. For a given elliptic curve $E/\mathbb{Q}$, there are infinitely many primes p for which $\mathrm{rk}(E_p(\mathbb{Q})) = 0$ or $\mathrm{rk}(E_{-p}(\mathbb{Q})) = 0$, and there are infinitely many primes ℓ for which $\mathrm{rk}(E_\ell(\mathbb{Q})) > 0$ or $\mathrm{rk}(E_{-\ell}(\mathbb{Q})) > 0$.

In light of Silverman's conjecture, we prove the following result for certain "good" elliptic curves, which is related to the rank zero component of Silverman's conjecture.

Theorem 2.13. *Let $E/\mathbb{Q}$ be a "good" elliptic curve (see Definition 2.14) with discriminant Δ. Let q_n denote the n-th prime for which $\mathrm{rk}(E_{\epsilon p}(\mathbb{Q})) = 0$, where $\epsilon \in \{-1, 1\}$ depends on E. For any $m \in \mathbb{N}$*

$$\liminf_{n\to\infty}(q_{m+n} - q_n) \ll (c_E m)^3 \exp(2c_E m)$$

with an absolute implied constant. Here, $c_E \leq 18(e^\gamma \log(12 \log |\Delta|) + 4)$.

To prove Theorem 2.13, let $E/\mathbb{Q}$ be an elliptic curve with Weierstrass form

$$E : y^2 = x^3 + ax^2 + bx + c, \quad a, b, c \in \mathbb{Z},$$

where the discriminant of the cubic is nonzero. We will assume that E and its points are $\mathbb{Q}$-rational. If d is a squarefree integer, we define E_d to be the d-quadratic twist of E given by

$$E_d : dy^2 = x^3 + ax^2 + bx + c.$$

Definition 2.14. *Let $E/\mathbb{Q}$ be an elliptic curve without $\mathbb{Q}$-rational 2-torsion. Following [BD10], we call E good if E satisfies the following criteria:*

1. *The 2-Selmer rank of E is zero.*
2. *The discriminant Δ of E is negative.*
3. *If p is any prime for which E has bad reduction, then E has multiplicative reduction at p, and $v_p(\Delta)$ is odd.*
4. *E has good reduction at 2 and the reduction of E modulo 2 has j-invariant zero.*

A prototypical example of a good elliptic curve is $E = X_0(11)$, which has Weierstrass form $E : y^2 = x^3 - 4x^2 - 160x - 1264$.

We define a squarefree integer d to be 2-trivial for E if E has no rational 2-torsion modulo p for every odd prime $p \mid d$. For good elliptic curves, the following is proven in [BD10].

Theorem 2.15. *Let $E/\mathbb{Q}$ be a good elliptic curve. If d is a squarefree 2-trivial integer for E with $(d, \Delta) = 1$, then*

$$\dim_{\mathbb{F}_2}(\mathrm{Sel}_2(E_d(\mathbb{Q})) = \begin{cases} 0 & \text{if } d \text{ is odd,} \\ 1 & \text{if } d \text{ is even.} \end{cases}$$

In particular, for such odd d, we have $\mathrm{rk}(E_d(\mathbb{Q})) = 0$.

We now prove Theorem 2.13.

Proof of Theorem 2.13. We write E in Weierstrass form $E : y^2 = f(x)$, where $f(x) = x^3 + ax^2 + bx + c \in \mathbb{Z}[x]$ has Galois group G and discriminant Δ. Since E is good, f is irreducible over $\mathbb{Z}$ and $G \cong S_3$. By the above discussion, the primes p satisfying the hypotheses of Theorem 2.15 are exactly the primes $p \nmid \Delta$ such that $f \bmod p$ is irreducible, that is, the factorization type of $f \bmod p$ corresponds to 3-cycles in S_3. The desired result now follows from Corollary 2.12. $\square$

We now consider an elliptic curve satisfying the rank one component of Silverman's conjecture. In light of recent results by Coates, Li, Ye, and Zhai [CLTZ15], we use our results to study ranks of twists of the elliptic curve $E = X_0(49)$, whose minimal Weierstrass equation is given by $E : y^2 + xy = x^3 - x^2 - 2x - 1$. Let $p > 7$ be a prime such that $p \equiv 3 \pmod 4$ and p is inert in the field $\mathbb{Q}(\sqrt{-7})$. For $k \geq 0$, let $q = \prod_{i=1}^{k} q_i$ be a product of distinct primes $q_i \neq p$, each of which splits completely in $\mathbb{Q}(E[4])$, where $E[4]$ denotes the torsion points of order 4. Suppose further that the ideal class group of $\mathbb{Q}(\sqrt{-pq})$ has no element of order 4. Under these hypotheses, Coates, Li, Yian, and Zhai prove that the Hasse-Weil L-function $L(E_{-pq}/\mathbb{Q}, s)$ has a simple zero at $s = 1$, $\mathrm{rk}(E_{-pq}(\mathbb{Q})) = 1$, and the Shafarevich-Tate group $\text{Ш}(E_{-pq}/\mathbb{Q})$ is finite of odd cardinality. They predict that every elliptic curve should satisfy a property similar to this. We prove the following.

Theorem 2.16. *Let $E = X_0(49)$. Let q_n be the n-th prime such that $L(E_{-q_n}/\mathbb{Q}, s)$ has a simple zero at $s = \frac{1}{2}$, $\mathrm{rk}(E_{-q_n}(\mathbb{Q})) = 1$, and $\text{Ш}(E_{-q_n}/\mathbb{Q})$ is finite of odd cardinality. For any $m \in \mathbb{N}$,*

$$\liminf_{n\to\infty}(q_{m+n} - q_n) \ll m^3 \exp(4m)$$

with an absolute implied constant.

We use the main result of [CLTZ15], which we now state, to prove Theorem 2.16.

Theorem 2.17 (Coates, Li, Ye, Zhai)**.** *Let $E = X_0(49)$. For $k \geq 0$, let $p, q_1, \ldots, q_k$ be distinct primes, and let $N = p\prod_{j\leq k} q_j$ satisfy*

1. *$p \equiv 3 \pmod 4$, $p \neq 7$, and p is a quadratic non-residue modulo 7.*
2. *$q_1, \ldots, q_k$ split completely in $\mathbb{Q}(E[4])$.*
3. *The ideal class group H_N of the field $\mathbb{Q}(\sqrt{-N})$ has no element of order 4.*

Then the Hasse-Weil L-function $L(E_{-N}, s)$ has a simple zero at $s = 1$, $E_{-N}(\mathbb{Q})$ has rank 1, and the Shafarevich-Tate group of E_{-N} is finite of odd order.

Proof of Theorem 2.16. We consider the case of Theorem 2.17 where $k = 0$. Using the theory of quadratic forms, Gauss proved that if $p \equiv 3 \pmod 4$, then $|H_p|$ is odd. Thus Theorem 2.17 holds when N is a prime such that $N \neq 7$ such that $N \equiv 3 \pmod 4$ and N is a quadratic non-residue modulo 7. Every prime p congruent to 3, 19, or 27 modulo 28 satisfies this condition, and the desired result follows from Theorem 2.1 by taking $L = \mathbb{Q}$ and choosing the arithmetic progression $3 \pmod{28}$. ∎

A specific elliptic curve for which the entirety of Silverman's conjecture is true is the congruent number elliptic curve $E : y^2 = x^3 - x$. We call a positive squarefree integer d a congruent number if d is the area of a right triangle with sides of rational length. It is well-known that d is a congruent number if and only if $E_d(\mathbb{Q})$ has positive rank. If p is prime, it is known [Mon90] that

- If $p \equiv 3 \pmod 8$, then $\mathrm{rk}(E_p(\mathbb{Q})) = 0$.
- If $p \equiv 5 \pmod 8$, then $\mathrm{rk}(E_{2p}(\mathbb{Q})) = 0$.
- If $p \equiv 5$ or $7 \pmod 8$, then $\mathrm{rk}(E_p(\mathbb{Q})) = 1$.
- If $p \equiv 3 \pmod 4$, then $\mathrm{rk}(E_{2p}(\mathbb{Q})) = 1$.

We obtain the following result for twists $E_p(\mathbb{Q})$, but one can easily adapt the statement to suit the twists $E_{2p}(\mathbb{Q})$.

Theorem 2.18. *Let $E/\mathbb{Q}$ denote the congruent number elliptic curve $y^2 = x^3 - x$. Let q_n denote the n-th prime for which $\mathrm{rk}(E'_{q_n}(\mathbb{Q})) = 0$. For any $m \in \mathbb{N}$,*

$$\liminf_{n\to\infty}(q_{m+n} - q_n) \ll m^3 \exp(4m)$$

with an absolute implied constant. The same is true if we replace the condition $\mathrm{rk}(E_{q_n}(\mathbb{Q})) = 0$ with the condition $\mathrm{rk}(E_{q_n}(\mathbb{Q})) = 1$. In particular, we have bounded gaps between primes which are congruent numbers, and we have bounded gaps between primes which are not congruent numbers.

Proof. This follows immediately from Theorem 2.1 with $L = \mathbb{Q}$ after choosing the appropriate arithmetic progression. □

2.5 Applications to modular forms and quadratic forms

We consider applications to the Fourier coefficients of holomorphic cuspidal normalized Hecke eigenforms, i.e. newforms, on congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

Theorem 2.19. *Let $f(z) = \sum_{n=1}^{\infty} a_f(n)e^{2\pi inz} \in \mathbb{Z}[[e^{2\pi inz}]]$ be a newform of even weight $k \geq 2$ and level $N \geq 1$, let d be a positive integer, and let $p_0 \nmid dN$ be a fixed prime. Let q_n be the n-th prime for which $a_f(q_n) \equiv a_f(p_0) \pmod{d}$. For any $m \in \mathbb{N}$,*

$$\liminf_{n\to\infty}(q_{m+n} - q_n) \ll (c_{d,N}m)^3 \exp(2c_{d,N}m)$$

with an absolute implied constant, where $c_{d,N} \leq \frac{d^8}{2}(e^{\gamma}\log(2d^4\log(dN)) + 4)$. In particular, we have bounded gaps between primes p satisfying $a_f(p) \equiv 0 \pmod{d}$.

Following Murty and Murty [MM84], let $q = e^{2\pi iz}$, and let

$$f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k^{\mathrm{new}}(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]]$$

be a newform of even weight $k \geq 2$ and character χ. (This forces χ to be real, and χ is nontrivial if and only if f has complex multiplication.) Let $G = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$, and let d be a positive integer. By the work of Deligne, there exists a representation $\rho_d : G \to \mathrm{GL}_2(\prod_{\ell|d}\mathbb{Z}_\ell)$ with the property that if $p \nmid dN$ is prime and σ_p is a Frobenius element at p in G, then ρ_d is unramified at p and $\mathrm{tr}\rho_d(\sigma_p) = a_f(p)$ and $\det\rho_d(\sigma_p) = \chi(p)p^{k-1}$. Let $\tilde{\rho}_d : G \to \mathrm{GL}_2(\mathbb{Z}/d\mathbb{Z})$ be the reduction modulo d of ρ_d. Let H_d be the kernel of $\tilde{\rho}_d$, let K_d be the subfield of $\bar{\mathbb{Q}}$ fixed by H_d, and let $G_d = \mathrm{Gal}(K_d/\mathbb{Q})$. If $q \nmid dN$ is prime, then the condition $a_f(q) \equiv 0 \pmod{d}$ means that for any Frobenius element σ_q of q, $\tilde{\rho}_d(\sigma_q) \in C_d$. Since C_d contains the image of complex conjugations, C_d is nonempty.

Proof of Theorem 2.19. By the preceding discussion, the set of primes p for which $a_f(p) \equiv 0 \pmod{d}$ is a Chebotarev set. By similar arguments, we find that for any fixed prime $p_0 \nmid dN$, the set of primes p for which $a_f(p) \equiv a_f(p_0) \pmod{d}$ is also a Chebotarev set. The desired result now follows from Theorem 2.1. In all cases, the ensuing Galois extension $L/\mathbb{Q}$ has degree at most d^4 and is unramified outside of dN, so the computation of $c_{d,N}$ follows from Proposition 2.10. □

As an application of Theorem 2.19, let τ be the Ramanujan tau function, so that

$$f(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n \in S_{12}^{\text{new}}(\Gamma_0(1), \chi_{\text{triv}}).$$

In this case, we have bounded gaps between primes p for which $\tau(p) \equiv 0 \pmod{d}$ for any positive integer d. If $k = 2$, then f is the newform associated to an elliptic curve $E/\mathbb{Q}$ with conductor N. In this case, $a_f(p) = p + 1 - \#E(\mathbb{F}_p)$, and we have bounded gaps between primes p for which $\#E(\mathbb{F}_p) \equiv p + 1 \pmod{d}$ for any $d \geq 1$.

Finally, we consider primes represented by binary quadratic forms.

Theorem 2.20. *Let $Q(x, y) = ax^2 + bxy + cy^2 \in \mathbb{Z}[x, y]$ be a primitive quadratic form with discriminant $D = b^2 - 4ac < 0$, let $h(D)$ be number of such quadratic forms of discriminant D, and let q_n be the n-th prime represented by Q. For any $m \in \mathbb{N}$,*

$$\liminf_{n\to\infty}(q_{m+n} - q_n) \ll (c_D m)^3 \exp(2c_D m)$$

with an absolute implied constant. Here, $c_D \leq 2h(D)^2(e^{\gamma}\log(4h(D)\log|D|) + 4)$. In particular, if $n \in \mathbb{N}$, then there are bounded gaps between primes of the form $x^2 + ny^2$.

Proof. Up to finitely many exceptions, the primes represented by Q form a Chebotarev se (cf. Theorem 9.12 of Cox [Cox89]). The proportion of primes that are represented by Q is either $1/h(D)$ or $1/2h(D)$ (depending on Q). If $K = \mathbb{Q}(\sqrt{D})$, $\mathcal{O}$ is the order of the discriminant D, and L is the ring class field of $\mathcal{O}$, then the Chebotarev condition satisfied by these primes is in the extension $L/\mathbb{Q}$, which is unramified outside of D. Thus the bound on c_D follows from Proposition 2.10. □

Chapter 3

A variant of the Bombieri-Vinogradov theorem for short intervals and some questions of Serre

Using deep analytic properties of Dirichlet L-functions, one can produce a short interval analogue of the Bombieri-Vinogradov estimate (2.6) in the form

$$\sum_{q\le x^\theta}\max_{(a,q)=1}\max_{y\le h}\max_{\frac{1}{2}x\le N\le x}\left|\pi(N+y;q,a)-\pi(N;q,a)-\frac{\pi(N+y)-\pi(y)}{\varphi(q)}\right| \ll \frac{h}{(\log x)^D}, \tag{3.1}$$

where $\delta>0$ and $\theta>0$ are certain constants, $D>0$, and $h\ge x^{1-\delta}$. In this chapter, we extend this result to the context of the Chebotarev density theorem and consider some arithmetic applications.

We now recall the Chebotarev density theorem for any Galois L/K, but we will introduce a standard and equivalent restatement that will make the ensuing analysis more convenient. Let L/K be a Galois extension of number fields with Galois group G, let $a,q\in\mathbb{N}$ with $(a,q)=1$, and let $\mathrm{N}_{K/\mathbb{Q}}$ denote the absolute field norm of K. For a prime ideal $\mathfrak{p}$ of K which is unramified in L, there corresponds a certain conjugacy class $C \subset G$ of Frobenius automorphisms attached to the prime ideals of L which lie over $\mathfrak{p}$. We denote this conjugacy class by the Artin symbol $[\frac{L/K}{\mathfrak{p}}]$. For a fixed conjugacy class C and an integral ideal $\mathfrak{a}$ of K, define

$$\Lambda_C(\mathfrak{a}) := \begin{cases} \log \mathrm{N}_{K/\mathbb{Q}}\mathfrak{p} & \text{if } \mathfrak{a} = \mathfrak{p}^m,\ m \geq 1,\ \mathfrak{p} \text{ unramified in } L,\ \left[\frac{L/K}{\mathfrak{p}}\right]^m = C, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\psi_C(x; q, a) = \psi_C(x, L/K; q, a) := \sum_{\substack{\mathrm{N}\mathfrak{a} \leq x \\ \mathrm{N}_{K/\mathbb{Q}}\mathfrak{a} \equiv a (\mathrm{mod}\, q)}} \Lambda_C(\mathfrak{a}). \tag{3.2}$$

The Chebotarev density theorem asserts that if $q \leq (\log x)^D$, then

$$\psi_C(2x; q, a) - \psi_C(x; q, a) \sim d(C; q, a)x \tag{3.3}$$

for some rational density $d(C; q, a) \geq 0$. If $\zeta_q = e^{2\pi i/q}$ and $L \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$, then

$$d(C; q, a) = \frac{|C|}{|G|}\frac{1}{\varphi(q)}.$$

Building on the work of M. R. Murty and V. K. Murty [MM87], M. R. Murty and Petersen [MP13] proved that if $H \subset G$ is a largest abelian subgroup of G such that $H \cap C$ is nonempty, E is the fixed field of H, and

$$0 < \theta < \frac{1}{\max\{[E : \mathbb{Q}] - 2, 2\}},$$

then

$$\sideset{}{'}\sum_{q \leq x^\theta} \max_{(a,q)=1} \max_{N \leq x} \left| \psi_C(N; q, a) - \frac{|C|}{|G|}\frac{N}{\varphi(q)} \right| \ll \frac{x}{(\log x)^D}, \tag{3.4}$$

where $\sum'$ denotes summing over moduli q satisfying $L \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. This extends

(2.6) to a nonabelian setting; in fact, (2.6) is recovered when $L = \mathbb{Q}$.

Balog and Ono [BO01] extended (3.3) to a short interval setting using Heath-Brown's zero density estimate for Dedekind zeta functions [HB77]. Specifically, if

$$0 < \delta < \begin{cases} 1/[L:\mathbb{Q}] & \text{if } [L:\mathbb{Q}] \geq 3, \\ 3/8 & \text{if } [L:\mathbb{Q}] = 2, \\ 5/12 & \text{if } [L:\mathbb{Q}] = 1 \end{cases} \tag{3.5}$$

and $h \geq x^{1-\delta}$, then

$$\psi_C(x+h;1,1) - \psi_C(x;1,1) \sim \frac{|C|}{|G|}h. \tag{3.6}$$

Our main result is a short interval variant of (3.4).

Theorem 3.1. *Let L/K be a Galois extension of number fields with Galois group G, and let $C \subset G$ be a fixed conjugacy class. Let $H \subset G$ be a largest abelian subgroup of G such that $H \cap C$ is nonempty, and let E be the fixed field of H. Let $0 < \delta < \frac{2}{5[E:\mathbb{Q}]}$ and $0 < \theta < \frac{1}{3}(\frac{2}{5[E:\mathbb{Q}]} - \delta)$. If $h \geq x^{1-\delta}$, then for any constant $D > 0$,*

$$\sum_{q \leq x^\theta}{}' \max_{(a,q)=1} \max_{y \leq h} \max_{\frac{1}{2}x \leq N \leq x} \left| \psi_C(N+y;q,a) - \psi_C(N;q,a) - \frac{|C|}{|G|}\frac{y}{\varphi(q)} \right| \ll \frac{h}{(\log x)^D}, \tag{3.7}$$

where $\sum'$ denotes summing over moduli q satisfying $L \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$.

When $|H| \geq 3$, Theorem 3.1 immediately yields an improvement to the range of δ in (3.5) for Balog and Ono's short interval variant of the Chebotarev density theorem. This improvement holds for the vast majority of choices of L/K and C. Examples of such situations include when C has an element of order at least 3 or when $[L:K]$ is odd.

Corollary 3.2. *Let L/K, G, C, H, and E be as in Theorem 3.1, and suppose that $|H| \geq 3$. Let $a, q \in \mathbb{N}$ satisfy $L \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$ and $(a,q) = 1$; furthermore, for any* *constant* $D > 0$, *let* $q \le (\log x)^D$. *If* $0 < \delta < \frac{2}{5[E:\mathbb{Q}]}$ *and* $h \ge x^{1-\delta}$, *then*

$$\psi_C(x+h;q,a) - \psi_C(x;q,a) \sim \frac{|C|}{|G|}\frac{h}{\varphi(q)}.$$

3.1 Preliminary Setup

For a number field F, we let $n_F = [F:\mathbb{Q}]$ and d_F equal the absolute discriminant of F. Let L/K be a Galois extension of number fields with Galois group G, and let $C \subset G$ be a fixed conjugacy class. Unless otherwise specified, all implied constants in the asymptotic notation $\ll$ or $O(\cdot)$ will depend in an effectively computable way on at most d_L.

The setup in this section is essentially the same as in [LO77, MP13]. To single out those $\mathfrak{p}^m$ in K such that both $[\frac{L/K}{\mathfrak{p}}]^m = C$ and $\mathrm{N}_{K/\mathbb{Q}}\mathfrak{p}^m \equiv a \pmod q$, we will use the characters $\eta = \phi \otimes \chi$ of the Galois group $\mathrm{Gal}(L(\zeta_q)/K) \cong G \times (\mathbb{Z}/q\mathbb{Z})^\times$, where ϕ is an irreducible character of G and $\chi \pmod q$ is a Dirichlet character. We work under the assumption that $L \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$ so that $\eta(\mathfrak{a}) = \phi(\mathfrak{a})\chi(\mathrm{N}_{K/\mathbb{Q}}\mathfrak{a})$ for any integral ideal $\mathfrak{a}$ of K. Let

$$F_C(s) = -\frac{|C|}{|G|}\frac{1}{\varphi(q)}\sum_\eta \bar{\eta}(g)\frac{L'}{L}(s,\eta,L(\zeta_q)/K),$$

where $L(s,\eta,L(\zeta_q)/K)$ is the Artin L-function associated to η. For $\mathrm{Re}(s) > 1$, we have the Dirichlet series expansion

$$F_C(s) = \sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\theta(\mathfrak{p}^m)(\log \mathrm{N}_{K/\mathbb{Q}}\mathfrak{p})\mathrm{N}_{K/\mathbb{Q}}\mathfrak{p}^{-ms}.$$

If $\mathfrak{p}$ is unramified in L, then $\theta(\mathfrak{p}^m) = 1$ if $[\frac{L/K}{\mathfrak{p}}]^m = C$ and $\mathrm{N}_{K/\mathbb{Q}}\mathfrak{p} \equiv a \pmod q$; otherwise, $\theta(\mathfrak{p}^m) = 0$. If $\mathfrak{p}$ ramifies in L or $\mathbb{Q}(\zeta_q)$, then $|\theta(\mathfrak{p}^m)| \le 1$. Thus apart from ramified primes, $\psi_C(x;q,a)$ is a sum of the coefficients of $F_C(s)$.

Unfortunately, $\theta(\mathfrak{p}^m)$ is expressed in terms of Artin L-functions corresponding to the (usually nonabelian) characters of $G \times (\mathbb{Z}/q\mathbb{Z})^\times$. Fortunately, $F_C(s)$ can be written in terms of L-functions associated to abelian characters. Let $H \subset G$ be a largest abelian subgroup such that $H \cap C$ is nonempty, let E be the fixed field of H, and let ξ denote the irreducible characters of H. Since H is abelian, the characters $\omega = \xi \otimes \chi$ of the Galois group $H_q = \mathrm{Gal}(L(\zeta_q)/E) \cong H \times (\mathbb{Z}/q\mathbb{Z})^\times$ are one-dimensional. Because $L \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$, we have that $\omega(\mathfrak{a}) = \xi(\mathfrak{a})\chi(\mathrm{N}\mathfrak{a})$; furthermore, if $\mathfrak{f}_\omega$ is the conductor of ω, then $\mathrm{N}_{E/\mathbb{Q}}\mathfrak{f}_\omega \ll q^{n_E}$. (We let $\mathrm{N} = \mathrm{N}_{E/\mathbb{Q}}$ for the rest of the chapter.)

Choose $g \in C \cap H$. By the same arguments as in [LO77, Lemma 4.1], we may write

$$F_C(s) = -\frac{|C|}{|G|}\frac{1}{\varphi(q)} \sum_{\omega \in \widehat{H}_q} \bar{\omega}(c) \frac{L'}{L}(s, \omega, L(\zeta_q)/E).$$

Repeating the analysis in [LO77], we find that if $2 \le T \le x$, then

$$\psi_C(x; q, a) - \frac{|C|}{|G|}\frac{1}{\varphi(q)}\Big(x - \sum_{\omega \in \widehat{H}_q} \bar{\omega}(c)\Big(\sum_{\substack{\rho = \beta + i\gamma \\ |\gamma| \le T}} \frac{x^\rho}{\rho} - \sum_{\substack{\rho \\ |\rho| \le 1/2}} \frac{1}{\rho}\Big)\Big) \ll \frac{x(\log x)^2}{T},$$

where ρ is a nontrivial zero of $L(s, \tilde{\omega}, L(\zeta_q)/E)$ and $\tilde{\omega}$ is the primitive character which induces ω. Now, suppose $y \le h$ and $\frac{1}{2}x \le N \le x$. Since

$$\Big| \frac{(N+y)^\rho - N^\rho}{\rho} \Big| = \Big| \int_N^{N+y} t^{\rho - 1} dt \Big| \le y N^{\mathrm{Re}(\rho) - 1} \ll h x^{\mathrm{Re}(\rho) - 1},$$

we find that

$$\max_{(a,q)=1} \max_{y \le h} \max_{\frac{1}{2}x \le N \le x} \Big| \psi_C(N + y; q, a) - \psi_C(N; q, a) - \frac{|C|}{|G|}\frac{y}{\varphi(q)} \Big| \\ \ll \frac{h}{\varphi(q)} \sum_{\omega \in \widehat{H}_q} \sum_{\substack{\rho = \beta + i\gamma \\ |\gamma| \le T}} x^{\beta - 1} + \frac{x(\log x)^2}{T}.$$

Therefore, the left hand side of (3.7) is

$$\ll h \sum_{q\le Q}{}' \frac{1}{\varphi(q)} \sum_{\omega\in\widehat{H}_q}{}^{*} \sum_{\substack{\rho=\beta+i\gamma\\ |\gamma_\omega|\le T}} x^{\beta-1} + \frac{Qx(\log x)^2}{T}, \tag{3.8}$$

where ρ is a nontrivial zero of $L(s,\omega,L(\zeta_q)/E)$ and $\sum^*$ denotes summing over primitive characters ω.

Theorem 3.1 will follow from proving that for any fixed $D>0$, we have

$$h \sum_{q\le Q}{}' \frac{1}{\varphi(q)} \sum_{\omega\in\widehat{H}_q}{}^{*} \sum_{\substack{\rho=\beta+i\gamma\\ |\gamma|\le T}} x^{\beta-1} + \frac{Qx(\log x)^2}{T} \ll \frac{h}{(\log x)^D}, \tag{3.9}$$

where $Q=x^\theta$, $h\ge x^{1-\delta}$, and δ and θ are given in Theorem 3.1.

3.2 Proof of Theorem 3.1

Decompose the interval $[1,Q]$ into the dyadic intervals $[2^n,2^{n+1})$, where $0\le n\le \lceil \log_2 Q\rceil$. Since $\varphi(q)^{-1}\ll q^{-1}\log\log q$, (3.8) is

$$\ll h(\log Q)(\log\log Q) \max_{1\le R\le Q} \frac{1}{R} \sum_{q\le R}{}' \sum_{\omega\in\widehat{H}_q}{}^{*} \sum_{\substack{\rho=\beta+i\gamma\\ |\gamma|\le T}} x^{\beta-1} + \frac{Qx(\log x)^2}{T}. \tag{3.10}$$

If $\omega=\eta\otimes\chi$ is primitive, then $\mathfrak{f}_\omega$ is also the modulus of ω. Since $\mathrm{N}\mathfrak{f}_\omega \ll q^{n_E}$, where q is the modulus of χ, (3.10) is

$$\ll h(\log Q)(\log\log Q) \max_{1\le R\le Q} \frac{1}{R} \sum_{\mathrm{N}\mathfrak{a}\le R^{n_E}} \sum_{\omega \bmod \mathfrak{a}}{}^{*} \sum_{\substack{\rho=\beta+i\gamma\\ |\gamma|\le T}} x^{\beta-1} + \frac{Qx(\log x)^2}{T}. \tag{3.11}$$

For $\frac{1}{2} \le \sigma \le 1$, let $N(\sigma, T) := \#\{\rho = \beta + i\gamma : L(\rho, \omega) = 0, \sigma \le \beta, |\gamma| \le T\}$ and

$$N(\sigma, R, T) := \sum_{\mathrm{N}\mathfrak{a} \le R} \sideset{}{^*}\sum_{\omega \bmod \mathfrak{a}} N(\sigma, T).$$

Proposition 3.3. *If $T \ge 2$, $R \ge 1$, and $\frac{1}{2} \le \sigma \le 1$, then*

$$N(\sigma, R, T) \ll (R^2 T^{n_E})^{\frac{5}{2}(1-\sigma)} (\log RT)^{9n_E+10}.$$

Proof. This follows directly from the work Montgomery [Mon71, Theorem 12.2] for $n_E = 1$ and Hinz [Hin76, Satz A and B] for $n_E \ge 2$. It is weaker than either Montgomery's or Hinz's results, but it is more convenient for our proof. □

Proof of Theorem 3.1. Let $D > 0$, let $0 \le \delta < \frac{2}{5n_E}$, and let $h \ge x^{1-\delta}$. Let

$$Q = x^{\frac{1}{3}(\frac{2}{5n_E} - \delta) - \frac{2}{15n_E}\epsilon} (\log x)^{-\frac{D+2}{3}} \qquad \text{and} \qquad T = x^{\frac{2}{15n_E}(1+5n_E\delta-\epsilon)} (\log x)^{\frac{2(D+2)}{3}},$$

where $0 < \epsilon < 1 - 5n_E\delta/2$ is fixed. With $1 \le R \le Q$, we have

$$\sum_{\mathrm{N}\mathfrak{a} \le R^{n_E}} \sideset{}{^*}\sum_{\omega \bmod \mathfrak{a}} \sum_{\substack{\rho = \beta + i\gamma \\ |\gamma| \le T}} x^{\beta - 1} \ll \log x \max_{\frac{1}{2} \le \sigma < 1} x^{\sigma - 1} N(\sigma, R^{n_E}, T). \tag{3.12}$$

By the zero-free region for Hecke L-functions proven by Bartz [Bar89] and the fact that we restrict q so that $L \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$, there exists a constant $b_L > 0$ such that if

$$1 - \alpha(R, x) < \sigma \le 1, \qquad \alpha(R, x) := \frac{b_L}{\max\{\log R, (\log x)^{3/4}\}}, \tag{3.13}$$

then $N(\sigma, R^{n_E}, T)$ is either 0 or 1. If $N(\sigma, R^{n_E}, T) = 1$, then the counted zero β_1 is a Landau-Siegel zero associated to an exceptional modulus q_1 and an exceptional real quadratic character in $\widehat{H}_{q_1}$. Just as in [MP13, Section 2], a field-uniform version of Siegel's theorem implies that $x^{\beta_1 - 1} \ll (\log x)^{-D-3}$ with an ineffective implied constant.

Since $(Q^2T)^{5n_E/2} = x^{1-\epsilon}$, it follows from Proposition 3.3 that

$$\max_{\frac{1}{2}\leq\sigma\leq 1-\alpha(R,T)} x^{\sigma-1}N(\sigma,R^{n_E},T) \ll (\log x)^{9n_E+10} \max_{\frac{1}{2}\leq\sigma\leq 1-\alpha(R,T)} ((Q^2T)^{5n_E/2}/x)^{1-\sigma}$$
$$\ll (\log x)^{9n_E+10}x^{-\epsilon\alpha(R,x)}.$$

By our definition of $\alpha(R,x)$, we have that $x^{-\epsilon\alpha(R,x)} \ll (\log x)^{-(9n_E+14+D)}$ when $1 \leq R \leq \exp((\log x)^{3/4})$, and $x^{-\epsilon\alpha(R,x)} \ll 1$ when $\exp((\log x)^{3/4}) < R \leq Q$. We have now bounded (3.12), and so (3.11) is bounded by

$$h(\log Q)^2(\log x)\max_{R\leq Q}\frac{1}{R}((\log x)^{-D-3} + (\log x)^{9n_E+11}x^{-\epsilon\alpha(R,x)}) + \frac{Qx(\log x)^2}{T}.$$

For our choice of h, Q, and T, this is $\ll h(\log x)^{-D}$, proving (3.9). □

Using the full strength of Montgomery and Hinz's work [Mon71, Hin76] instead of Proposition 3.3, one can improve the ranges of δ and θ in Theorem 3.1 by following Huxley and Iwaniec [HI75]. The improvement is very small, and the ensuing dependence of θ on δ and n_E is cumbersome. Ultimately, our proof of Theorem 3.1 cannot produce values of δ and θ comparable to those in [HI75] because we sum over primitive characters ω with $\mathrm{N}\mathfrak{f}_\omega \ll Q^{n_E}$ instead of $\ll Q$, which seems unavoidable at this time.

3.3 Bounded gaps between primes in Chebotarev sets: the short interval version

Much like the results of [MM87, MP13], nonabelian analogues of the Bombieri-Vinogradov theorem in short intervals can have interesting arithmetic consequences. In this paper, we will focus on consequences related to recent advances toward the

Hardy-Littlewood prime k-tuples conjecture. For these applications, we consider a Galois extension $L/\mathbb{Q}$ with Galois group G and absolute discriminant d_L, and we consider a fixed conjugacy class $C \subset G$. In this setting, (3.2) counts primes sets of the form

$$\mathcal{P} = \left\{ p : p \nmid d_L, \left[\frac{L/\mathbb{Q}}{p} \right] = C \right\} \tag{3.14}$$

Let $\mathbb{P}$ denote the set of all primes, and let h_i denote a nonnegative integer. Recall that a collection of nonnegative integers $\mathcal{H}_k = \{h_1, \ldots, h_k\}$ admissible if $\prod_{i=1}^{k}(n + h_i)$ has no fixed prime divisor. (We could consider more general admissible sets, but this sometimes hinders the applications we consider.)

Conjecture (Hardy-Littlewood). If $\mathcal{H}_k$ is admissible, then as $x \to \infty$, we have

$$\#\{n \in [x, 2x] : \#(\{n + h_1, \ldots, n + h_k\} \cap \mathbb{P}) = k\} \sim \mathfrak{S} \frac{x}{(\log x)^k},$$

where $\mathfrak{S}$ is a certain positive constant depending on $\mathcal{H}_k$.

Choosing $\mathcal{H}_2 = \{0, 2\}$, the Hardy-Littlewood conjecture implies the twin prime conjecture.

In [May], Maynard generalized his methods in [May15] to prove weak forms of the Hardy-Littlewood conjecture with specializations to primes in short intervals and primes in Chebotarev sets. More specifically, given $0 < \delta < \frac{5}{12}$ and $h \geq x^{1-\delta}$, Maynard proved that there exists an absolute constant $C > 0$ such that if $k \geq C$ and $\mathcal{H}_k$ is an admissible set, then

$$\#\{n \in [x, x + h] : \#(\{n + h_1, \ldots, n + h_k\} \cap \mathbb{P}) \geq C^{-1} \log k\} \gg \frac{h}{(\log x)^k} \tag{3.15}$$

Furthermore, if $\mathcal{P}$ is given by (3.14), then Maynard also proved that there exists a constant $C_L > 0$ such that if $k \geq C_L$ and $\mathcal{H}_k$ is admissible, then

$$\#\{n \in [x, 2x] : \#(\{n+h_1, \ldots, n+h_k\} \cap \mathcal{P}) \geq C_L^{-1} \log k\} \gg \frac{x}{(\log x)^k}. \qquad (3.16)$$

Using Theorem 3.1, we prove the following mutual refinement of (3.15) and (3.16), which extends applications in Chapter 2 to a short interval setting.

Theorem 3.4. *Let $L/\mathbb{Q}$ be a Galois extension of number fields, let $\mathcal{P}$ be as in* (3.14)*, and choose h as in Theorem 3.1. There exists a constant $C_L \in \mathbb{N}$ such that if $k \geq C_L$ and $\mathcal{H}_k = \{h_1, \ldots, h_k\}$ is admissible, then*

$$\#\{n \in [x, x+h] : \#(\{n+h_1, \ldots, n+h_k\} \cap \mathcal{P}) \geq C_L^{-1} \log k\} \gg \frac{h}{(\log x)^k}.$$

Remark. Some of the parameters in the statement of Theorem 3.4 can have some uniformity in x by appealing to the arguments in [May]. In what follows, we will assume that all parameters are constant with respect to x.

We will use Theorem 3.1 to prove Theorem 3.4. Given a set of integers $\mathfrak{A}$, a set of primes $\mathfrak{P} \subset \mathfrak{A}$, and a linear form $L(n) = n + h$, define

$$\mathfrak{A}(x) = \{n \in \mathfrak{A} : x < n \leq 2x\}, \qquad \mathfrak{A}(x; q, a) = \{n \in \mathfrak{A}(x) : n \equiv a \pmod q\},$$
$$L(\mathfrak{A}) = \{L(n) : n \in \mathfrak{A}\}, \qquad \varphi_L(q) = \varphi(hq)/\varphi(h),$$
$$\mathfrak{P}_{L,\mathfrak{A}}(x, y) = L(\mathfrak{A}(x)) \cap \mathfrak{P}, \qquad \mathfrak{P}_{L,\mathfrak{A}}(x; q, a) = L(\mathfrak{A}(x; q, a)) \cap \mathfrak{P}.$$

We consider the 6-tuple $(\mathfrak{A}, \mathcal{L}_k, \mathfrak{P}, B, x, \theta)$, where $\mathcal{H}_k$ is admissible, $\mathcal{L}_k = \{L_i(n) = n + h_i : h_i \in \mathcal{H}_k\}$, $B \in \mathbb{N}$ is constant, x is a large real number, and $0 \leq \theta < 1$. We present a very general hypothesis that Maynard states in Section 2 of [May].

Hypothesis 1. With the above notation, consider the 6-tuple $(\mathfrak{A}, \mathcal{H}_k, \mathfrak{P}, B, x, \theta)$.

1. We have

$$\sum_{q \le x^\theta} \max_a \left| \#\mathfrak{A}(x;q,a) - \frac{\#\mathfrak{A}(x)}{q} \right| \ll \frac{\#\mathfrak{A}(x)}{(\log x)^{100k^2}}.$$

2. For any $L \in \mathcal{H}_k$, we have

$$\sum_{q \le x^\theta,\ (q,B)=1} \max_{(L(a),q)=1} \left| \#\mathfrak{P}_{L,\mathfrak{A}}(x;q,a) - \frac{\#\mathfrak{P}_{L,\mathfrak{A}}(x)}{\varphi_L(q)} \right| \ll \frac{\#\mathfrak{P}_{L,\mathfrak{A}}(x)}{(\log x)^{100k^2}}.$$

3. For any $q \le x^\theta$, we have $\#\mathfrak{A}(x;q,a) \ll \#\mathfrak{A}(x)/q$.

For $(\mathfrak{A}, \mathcal{H}_k, \mathfrak{P}, B, x, \theta)$ satisfying Hypothesis 1, Maynard proves the following in [May].

Theorem 3.5. *Let $(\mathfrak{A}, \mathcal{H}_k, \mathfrak{P}, B, x, \theta)$ satisfy Hypothesis 1 with $0 \le \theta < 1$. There is a constant $C > 0$, depending only on θ, such that if $k \ge C$ and $\eta > (\log k)^{-1}$ satisfies*

$$\frac{1}{k}\frac{\varphi(B)}{B} \sum_{L \in \mathcal{H}_k} \#\mathfrak{P}_{L,\mathfrak{A}}(x) \ge \eta \frac{\#\mathfrak{A}(x)}{\log x},$$

then

$$\#\{n \in \mathfrak{A}(x) : \#(\mathcal{H}_k(n) \cap \mathfrak{P}) \ge C^{-1}\eta \log k\} \gg \frac{\#\mathfrak{A}(x)}{(\log x)^k \exp(Ck)}.$$

Proof of Theorem 3.4. Let δ, h, and θ be as in Theorem 3.1. Let $\mathfrak{A} = \mathbb{N} \cap [x, x+h]$, $B = d_L$, and $\mathfrak{P} = \mathcal{P}$ (as in (3.14)). The proof is the same as Theorems 3.4 and 3.5 in [May]: we show that the 6-tuple $(\mathbb{N} \cap [x, x+h], \mathcal{H}_k, \mathcal{P}, d_L, x, \frac{\theta}{2})$ satisfies Theorem 3.5.

Parts (i) and (iii) of Hypothesis 1 are trivial to check. For Part (ii), note that if $(d_L, q) = 1$, then $L \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Thus by Theorem 3.1 and partial summation, all of Hypothesis 1 holds when D and x are sufficiently large in terms of k and θ. Given a suitable constant $C_L > 0$ (computed as in Chapter 2), we let $k \ge C_L$. Then for all sufficiently large x,

$$\frac{1}{k}\frac{\varphi(d_L)}{d_L} \sum_{L \in \mathcal{H}_k} \#\mathfrak{P}_{L,\mathfrak{A}}(x) \ge (1+o(1))\frac{\varphi(d_L)}{d_L}\frac{|C|}{|G|}\frac{\#\mathfrak{A}(x)}{\log x},$$

where the $o(1)$ implied constant depends only on L. □

3.4 Arithmetic applications

We now consider arithmetic consequences of Theorem 3.4 in the theory of elliptic curves, modular forms, and modular L-functions. We consider the following questions of Serre [Ser81], which may be seen as an automorphic analogue of Bertrand's postulate that every interval $[x, 2x]$ contains a prime.

Serre's Questions. Let $q = e^{2\pi iz}$, and let $S_\ell(\Gamma_0(N), \chi)$ be the space of weight ℓ, level N cusp forms of nebentypus χ. For a nonzero cusp form $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_\ell(\Gamma_0(N), \chi)$, let

$$I_f(n) = \max\{i : a_f(n+j) = 0 \text{ for all } 0 \leq j \leq i\}.$$

1. Suppose that f is of weight $\ell \geq 2$ and is not a linear combination of forms with complex multiplication. Is $I_f(n) \ll n^\delta$ for some $0 \leq \delta < 1$?

2. More generally, are there analogous results for forms with non-integral weights, or forms with respect to other Fuchsian groups?

Motivated by Serre's questions, Balog and Ono [BO01] used (3.6) to prove that if $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_\ell(\Gamma_0(N), \chi)$ is a cusp form of weight $\ell \in \frac{1}{2}\mathbb{N} \setminus \{\frac{1}{2}\}$ which is not a linear combination of weight $\frac{3}{2}$ theta functions, then there exists $\nu_f \in \mathbb{N}$ such that if $0 \leq \delta < \frac{1}{\nu_f}$ and $h \geq x^{1-\delta}$, then

$$\#\{n \in [x, x+h] : a_f(n) \neq 0\} \gg h/\log x. \tag{3.17}$$

For such a cusp form f, it follows that $I_f(n) \ll n^{1-\frac{1}{\nu_f}+\epsilon}$ for any $\epsilon > 0$, affirmatively answering Serre's questions. By using Theorem 3.4 instead of (3.6) in Balog and

Ono's proof, we immediately conclude that the integers n for which $a_f(n) \neq 0$ exhibit dense clusters in short intervals.

Theorem 3.6. *Let $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_\ell(\Gamma_0(N), \chi)$ be a nonzero cusp form of weight $\ell \in \frac{1}{2}\mathbb{N} \setminus \{\frac{1}{2}\}$ which is not a linear combination of weight $\frac{3}{2}$ theta functions. There exist constants $C_f, \nu_f \in \mathbb{N}$ such that if $0 \leq \delta < \frac{1}{\nu_f}$, $h \geq x^{1-\delta}$, $k \geq C_f$ and $\mathcal{H}_k = \{h_1, \ldots, h_k\}$ is admissible, then*

$$\#\{n \in [x, x+h] : \#\{h_i \in \mathcal{H}_k : a_f(n+h_i) \neq 0\} \geq C_f^{-1} \log k\} \gg \frac{h}{(\log x)^k}.$$

We use Theorem 3.6 to study the central values of modular L-functions and ranks of elliptic curves. Let $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_\ell(\Gamma_0(N), \chi)$ be a cusp form such that $\ell \geq 2$ is an integer, $a_f(1) = 1$, and f is an eigenform of the Hecke operators T_p for $p \nmid N$ and U_p for $p \mid N$; we call such a cusp form f a newform. Let $\mathcal{D}$ be the set of all fundamental discriminants; given $d \in \mathcal{D}$, we consider the twisted L-function

$$L(s, f_d) = \sum_{n=1}^{\infty} \frac{a_f(n)\chi_d(n)}{n^{s+(\ell-1)/2}},$$

where χ_d is the Kronecker character for $\mathbb{Q}(\sqrt{d})$. For newforms of weight $\ell \in 2\mathbb{N}$ and trivial nebentypus χ, Goldfeld [Gol79] conjectured that the proportion of $d \in \mathcal{D}$ for which $L(\frac{1}{2}, f_d) \neq 0$ is $\frac{1}{2}$.

If g is a half-integer weight cusp form satisfying the hypotheses of Theorem 3.6, then by the work of Shimura [Shi73] and Waldspurger [Wal81], the Fourier coefficients $a_g(n)$ interpolate central critical values $L(\frac{1}{2}, f_d)$ as d varies, where f is the Shimura correspondent of g. Even though the Shimura correspondence is not surjective, Ono and Skinner [OS98] proved that such central critical values can be obtained in this fashion when f is a newform of even weight and trivial nebentypus χ. Using this along with (3.17), Balog and Ono [BO01] proved that there exists $\nu_f \in \mathbb{N}$ such that if $0 \leq \delta < \frac{1}{\nu_f}$ and $h \geq x^{1-\delta}$, then

$$\#\{|d| \in [x, x+h] : d \in \mathcal{D},\ L(1/2, f_d) \neq 0\} \gg h/\log x. \tag{3.18}$$

This is the sharpest result toward Goldfeld's conjecture which is valid for all newforms f of even weight and trivial nebentypus; for most such newforms, the power of $\log x$ can be improved [Ono01]. By using Theorem 3.6 instead of (3.17) in Balog and Ono's proof, we conclude that fundamental discriminants d for which $L(1/2, f_d) \neq 0$ exhibit dense clusters in short intervals.

Corollary 3.7. *Let $f \in S_{2\ell}(\Gamma_0(N), \chi_{\text{triv}})$ be a newform with $\ell \in \mathbb{N}$. There exists an arithmetic progression $a \bmod q$ (depending on f) and there exist constants $\nu_f, C_f \in \mathbb{N}$ such that if $0 \leq \delta < \frac{1}{\nu_f}$, $h \geq x^{1-\delta}$, $k \geq C_f$, $\mathcal{H}_k = \{h_1, \ldots, h_k\}$ is admissible, and $\mathcal{N}_f(k, n) = \{h_i \in \mathcal{H}_k : n + qh_i \in \mathcal{D}, L(1/2, f_{n+qh_i}) \neq 0\}$, then*

$$\#\{|n| \in [x, x+h] : n \equiv a \pmod{q}, \#\mathcal{N}_f(k, n) \geq C_f^{-1} \log k\} \gg h/(\log x)^k.$$

Remark. We need to restrict to the arithmetic progression a mod q for technical reasons; see [OS98] for details. We accomplish this by choosing $\mathcal{H}_k$ using the arguments at the very end of the proof of Theorem 2.1.

Let f be the newform associated to an elliptic curve $E/\mathbb{Q}$ of conductor N with Weierstrass equation $y^2 = x^3 + ax^2 + bx + c$. If $(d, 4N) = 1$, then $L(s, f_d)$ is the L-function of the d-quadratic twist $E_d/\mathbb{Q}$, whose Weierstrass equation is given by $y^2 = x^3 + adx^2 + bd^2x + cd^3$. By the work of Kolyvagin [Kol88], if $L(1/2, f_d) \neq 0$, then the rank $\text{rk}(E_d(\mathbb{Q}))$ of the Mordell-Weil group $E_d(\mathbb{Q})$ is zero. Thus Corollary 3.7 immediately implies the following result.

Corollary 3.8. *Let $E/\mathbb{Q}$ be an elliptic curve. There exists an arithmetic progression $a \bmod q$ (depending on E) and there exist constants $\nu_E, C_E \in \mathbb{N}$ such that if $0 \leq \delta <$*

$\frac{1}{\nu_E}$, $h \geq x^{1-\delta}$, $k \geq C_E$, $\mathcal{H}_k = \{h_1, \ldots, h_k\}$ *is admissible, and we define* $\mathcal{N}_E(k,n) = \{h_i \in \mathcal{H}_k : n + qh_i \in \mathcal{D}, \mathrm{rk}(E_{n+qh_i}(\mathbb{Q})) = 0\}$*, then*

$$\#\{|n| \in [x, x+h] : n \equiv a \pmod{q}, \#\mathcal{N}_E(k,n) \geq C_E^{-1} \log k\} \gg h/(\log x)^k.$$

Consider an elliptic curve $E/\mathbb{Q}$. The distribution of the quantity $a_E(p) := p + 1 - \#E(\mathbb{F}_p)$ is well-studied [Mur97, MMS88, Ser81]; we apply our results to study the distribution of $a_E(p) \pmod{m}$ in short intervals. It follows from the work of Shiu [Shi00] that if $E/\mathbb{Q}$ has a rational point of order m, then for every $j \in \mathbb{N}$ and every $i \not\equiv 1 \pmod{m}$, there exists an $n \in \mathbb{N}$ such that

$$a_E(p_n) \equiv a_E(p_{n+1}) \equiv a_E(p_{n+2}) \equiv \cdots \equiv a_E(p_{n+j}) \equiv i \pmod{m},$$

where p_n is the n-th prime. Using (3.6) and the action of Galois on the torsion points of E, Balog and Ono [BO01] proved that for any $m \in \mathbb{N}$ and any residue class i mod m for which there is a prime of good reduction p_0 with $a_E(p_0) \equiv i \pmod{m}$, there exists $\nu_{E,m} \in \mathbb{N}$ such that if $0 \leq \delta < \frac{1}{\nu_{E,m}}$ and $h \geq x^{1-\delta}$, then

$$\#\{p \in [x, x+h] : a_E(p) \equiv i \pmod{m}\} \gg h/\log x.$$

Using Theorem 3.4 instead of (3.6) in Balog and Ono's proof, we immediately have:

Corollary 3.9. *Let* $E/\mathbb{Q}$ *be an elliptic curve, let* $m \in \mathbb{N}$*, and let* i mod m *be a residue class for which there is a prime of good reduction* p_0 *with* $a_E(p_0) \equiv i \pmod{m}$*. There exist constants* $\nu_{E,m}, C_{E,m} \in \mathbb{N}$ *such that if* $0 \leq \delta < \frac{1}{\nu_{E,m}}$, $h \geq x^{1-\delta}$, $k \geq C_{E,m}$*, and* $\mathcal{H}_k = \{h_1, \ldots, h_k\}$ *is admissible, then*

$$\#\{n \in [x, x+h] : \#\{h_j \in \mathcal{H}_k : n + h_j \in \mathbb{P}, a_E(n+h_j) \equiv i \pmod{m}\} \geq C_{E,m}^{-1} \log k\} \gg \frac{h}{(\log x)^k}.$$

Chapter 4

Effective log-free zero density estimates for automorphic L-functions and the Sato-Tate conjecture

For a prime p, let $\mathbb{Z}_p$ denote the p-adic integers, and let $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ be the profinite completion of $\mathbb{Z}$. We define the *ring of integral adeles* $\mathbb{A}_{\mathbb{Z}}$ by the direct product $\mathbb{R} \times \widehat{\mathbb{Z}}$. We define the *ring of adeles* $\mathbb{A}_{\mathbb{Q}}$ of $\mathbb{Q}$ to be the tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{A}_{\mathbb{Z}}$, which can be endowed with a topology such that $\mathbb{A}_{\mathbb{Z}}$ is an open subring. More generally, for any number field K, we define $\mathbb{A}_K = K \otimes_{\mathbb{Z}} \mathbb{A}_{\mathbb{Z}}$, which can be topologized as a product of $[K : \mathbb{Q}]$ copies of $\mathbb{A}_{\mathbb{Q}}$.

We consider the following general setup. Let $K/\mathbb{Q}$ be a number field with ring of adeles $\mathbb{A}_K$, and let π be a cuspidal automorphic representation of $\mathrm{GL}_d(\mathbb{A}_K)$ with unitary central character. We simply refer to such a representation π as an *automorphic representation*. There is an L-function $L(s, \pi, K)$ attached to π whose Dirichlet series and Euler product are given by

$$L(s, \pi, K) = \sum_{\mathfrak{a}} \frac{\lambda_\pi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s} = \prod_{\mathfrak{p}} \prod_{j=1}^{d} (1 - \alpha_\pi(j, \mathfrak{p}) \mathrm{N}\mathfrak{p}^{-s})^{-1},$$

where the sum runs over the non-zero integral ideals of K, the product runs over the prime ideals, and $\mathrm{N}\mathfrak{a} = \mathrm{N}_{K/\mathbb{Q}}\mathfrak{a}$ denotes the norm of the ideal $\mathfrak{a}$.

Let π and π' be automorphic representations of $\mathrm{GL}_d(\mathbb{A}_K)$ and $\mathrm{GL}_{d'}(\mathbb{A}_K)$, respectively. The Rankin-Selberg convolution

$$L(s, \pi \otimes \pi', K) = \sum_{\mathfrak{a}} \frac{\lambda_{\pi\otimes\pi'}(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s} = \prod_{\mathfrak{p}} \prod_{j_1=1}^{d} \prod_{j_2=1}^{d'} (1 - \alpha_\pi(j_1, \mathfrak{p})\alpha_{\pi'}(j_2, \mathfrak{p})(\mathrm{N}\mathfrak{p})^{-s})^{-1}$$

is itself an L-function with an analytic continuation and a functional equation. Define $\Lambda_{\pi\otimes\pi'}(\mathfrak{a})$ by the Dirichlet series identity

$$-\frac{L'}{L}(s, \pi \otimes \pi', K) = \sum_{\mathfrak{a}} \frac{\Lambda_{\pi\otimes\pi'}(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s}.$$

If $\tilde{\pi}$ is the representation which is contragredient to π, then it follows from standard Rankin-Selberg theory and the Wiener-Ikehara Tauberian theorem that we have a prime number theorem for $L(s, \pi \otimes \tilde{\pi}, K)$ in the form

$$\sum_{\mathrm{N}\mathfrak{a} \le x} \Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a}) \sim x.$$

It is reasonable to expect (for example, it follows from the generalized Riemann hypothesis) that there is some small $\delta > 0$ such that for x sufficiently large and any $h \ge x^{1-\delta}$, we have

$$\sum_{x < \mathrm{N}\mathfrak{a} \le x+h} \Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a}) \sim h. \tag{4.1}$$

Unfortunately, a uniform analogue of Littlewood's improved zero-free region does not yet exist for all automorphic L-functions, so it seems that (4.1) is currently inaccessible except in special situations. Despite this setback, Moreno [Mor73] proved the following approximate short interval prime number theorem, which he called the **Hoheisel phenomenon**.

Theorem 4.1. *Suppose that $L(s,\pi\otimes\tilde{\pi},\mathbb{Q})$ has a "standard" zero-free region (one of a quality similar to Hadamard's and de la Vallée Poussin's for $\zeta(s)$). Suppose also that there is a log-free zero density estimate of the form*

$$N_{\pi\otimes\pi'}(\sigma,T):=\#\{\rho=\beta+i\gamma : L(\rho,\pi\otimes\pi',K)=0,\beta\geq\sigma,|\gamma|\leq T\}\ll T^{c_{\pi,\pi'}(1-\sigma)}$$

for $L(s,\pi\otimes\tilde{\pi},\mathbb{Q})$. For any $0<\delta<1/c_{\pi,\tilde{\pi}}$ and any $h\geq x^{1-\delta}$, one has

$$\sum_{x<\mathrm{N}\mathfrak{a}\leq x+h}\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a})\asymp h.$$

At the time of Moreno's work, such log-free zero density estimates only existed in special cases. Moreover, in general, it is only known that $L(s,\pi\otimes\tilde{\pi},K)$ has a standard zero-free region if π is self-dual.

Recall that π and π' are automorphic representations of $\mathrm{GL}_d(\mathbb{A}_K)$ and $\mathrm{GL}_{d'}(\mathbb{A}_K)$, respectively. Suppose that $K=\mathbb{Q}$ and that either both d and d' are at most 2 or that one of π and π' is self-dual. Building on the work of Fogels, Akbary and Trudgian [AT15] proved in this case that if one has a certain amount of control over the Dirichlet coefficients of $L(s,\pi,\mathbb{Q})$ and $L(s,\pi',\mathbb{Q})$ in short intervals and T is sufficiently large in terms of π and π', then

$$N_{\pi\otimes\pi'}(\sigma,T)\leq T^{c_{d,d'}(1-\sigma)},$$

where $c_{d,d'}>2$ is a constant depending on d and d'. This allowed them to prove a variant of the Hoheisel phenomenon for $L(s,\pi\otimes\tilde{\pi},\mathbb{Q})$ when π is self-dual. Unfortunately, the dependence of $c_{d,d'}$ on d and d' was not made effective, whence also the length of the interval in their variant of the Hoheisel phenomenon. This makes their result difficult to use in situations where uniformity in parameters over several L-functions is required, especially when the L-functions in question vary in degree.

Furthermore, the range of T for which their bound holds is also not made effective. This is necessary to obtain analogues of Linnik's theorem.

Effective log-free zero density estimates have been proven for certain natural families of L-functions. Weiss [Wei83] proved an effective analogue of Fogels' log-free density estimate for the Hecke L-functions of ray class characters, which enabled him to access prime ideals of K satisfying splitting conditions in a finite extension M/K. Additionally, Kowalski and Michel [KM02] obtained a log-free zero density estimate for L-functions associated to any family of automorphic representations of $\mathrm{GL}_d(\mathbb{A}_{\mathbb{Q}})$ satisfying certain conditions, including the generalized Ramanujan conjecture (see Hypothesis 2). Their result works best when T is essentially constant, which is useful for variants of Linnik's theorem but not for the Hoheisel phenomenon.

Our first result is a log-free zero density estimate for $L(s, \pi \otimes \pi', K)$ which is effective in its dependence on π, π', and K; it is useful for variants of both the Hoheisel phenomenon and Linnik's theorem. This dependence is most naturally stated in terms of the analytic conductors $\mathfrak{q}(\pi)$ and $\mathfrak{q}(\pi')$ of π and π', respectively, whose definition we postpone to Section 4.1.1. We prove the following.

Theorem 4.2. *Let K be a number field with absolute discriminant D_K. Let π and π' be automorphic representations of $\mathrm{GL}_d(\mathbb{A}_K)$ and $\mathrm{GL}_{d'}(\mathbb{A}_K)$, respectively. Suppose that either both $d \leq 2$ and $d' \leq 2$ or that at least one of π and π' is self-dual, and suppose that the generalized Ramanujan conjecture (GRC) holds for $L(s, \pi, K)$. Let*

$$\mathcal{Q} = \mathcal{Q}(\pi, \pi') = \begin{cases} \mathfrak{q}(\pi)\mathfrak{q}(\pi') & \text{if } \pi \text{ is nontrivial,} \\ \mathfrak{q}(\pi') & \text{if } \pi \text{ is trivial,} \end{cases}$$

let

$$\mathcal{D} = \mathcal{D}(\pi, \pi') = \begin{cases} d^2 & \text{if } d = d' \text{ and both } \pi \text{ and } \pi' \text{ are self-dual,} \\ (d')^4 & \text{if } \pi \text{ is trivial,} \\ (d+d')^4 & \text{otherwise,} \end{cases}$$

and let $T \gg [K:\mathbb{Q}]\mathcal{Q}^{1/[K:\mathbb{Q}]}$.[1] *There exists an absolute constant* $c_1 > 0$ *such that if* $\frac{1}{2} \leq \sigma \leq 1$, *then*

$$N_{\pi\otimes\pi'}(\sigma, T) \ll d^2 T^{c_1 \mathcal{D}[K:\mathbb{Q}](1-\sigma)}.$$

Remark. 1. We impose the self-duality condition in Theorem 4.2 in order to ensure that $L(s, \pi \otimes \pi', K)$ has a standard zero-free region; see Lemma 4.7.

2. In the case where π is trivial, Theorem 4.2 gives the first unconditional log-free zero density estimate for all automorphic L-functions $L(s, \pi', K)$. (Recall that Akbary and Trudgian's result is conditional on a hypothesis on the Dirichlet coefficients of $L(s, \pi, K)$ in short intervals.) In particular, Theorem 4.2 gives an unconditional log-free zero density estimate for $L(s, \pi', \mathbb{Q})$ when π' is an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ associated to a Hecke-Maass form, which was not previously known.

In addition to the density estimate of Fogels that helps simplify the proof of Linnik's bound on the least prime in an arithmetic progression, Jutila [Jut70] proved a "hybrid" density estimate of the form

$$\sum_{q \leq Q} \sideset{}{^\star}\sum_{\chi \bmod q} N_\chi(\sigma, T) \ll (Q^2 T)^{c(1-\sigma)} (\log QT)^{c'}, \tag{4.2}$$

where the $\star$ on the summation indicates it is to be taken over primitive characters; Montgomery [Mon71] improved upon Jutila's work to show that one may take $c = \frac{5}{2}$. This simultaneously generalizes Fogels' result and Bombieri's large sieve density estimate [Bom65]. As a consequence of (4.2), one sees that the average value of

[1] Unless mentioned otherwise, the implied constant in an asymptotic inequality is absolute and computable.

$N_\chi(\sigma, T)$ is noticeably smaller than what is given by Fogels' result. Furthermore, (4.2) can be used to prove versions of the Bombieri-Vinogradov theorem in both long and short intervals.

Gallagher [Gal70] proved that

$$\sum_{q \leq T} \sideset{}{^\star}\sum_{\chi \bmod q} N_\chi(\sigma, T) \ll T^{c(1-\sigma)}, \qquad T \geq q. \tag{4.3}$$

Gallagher's refinement of (4.2) can be used to prove both Linnik's bound on the least prime in an arithmetic progression and variants of the Bombieri-Vinogradov theorem for short intervals. Our second result generalizes (4.3) to consider twists of Rankin-Selberg L-functions associated to automorphic representations over $\mathbb{Q}$.

Theorem 4.3. *Assume the above notation. Under the hypotheses of Theorem 4.2 with $K = \mathbb{Q}$, there exists an absolute constant $c_2 > 0$ such that*

$$\sum_{q \leq T} \sideset{}{^\star}\sum_{\chi \bmod q} N_{(\pi \otimes \pi') \otimes \chi}(\sigma, T) \ll d^2 T^{c_2 \mathcal{D}(1-\sigma)}.$$

We now turn to the applications of Theorems 4.2 and 4.3. We begin by considering a version of the Hoheisel phenomenon for L-functions satisfying the generalized Ramanujan conjecture. In some cases, it is desirable to incorporate an auxiliary splitting condition on the prime ideals. To this end, we let M/K be a Galois extension with Galois group $\mathcal{G}$, let $\mathcal{C} \subseteq \mathcal{G}$ be a conjugacy class, and let $\left[\frac{M/K}{\cdot}\right]$ denote the Artin symbol. For an ideal $\mathfrak{a}$, define $\mathbf{1}_{\mathcal{C}}(\mathfrak{a})$ to be 1 if $\mathfrak{a} = \mathfrak{p}^k$ for some unramified prime $\mathfrak{p}$ with $\left[\frac{M/K}{\mathfrak{p}}\right]^k = \mathcal{C}$ and to be 0 otherwise. It is then possible to prove an analogue of the Chebotarev density theorem for $L(s, \pi \otimes \tilde{\pi}, K)$ in the form

$$\sum_{\mathrm{N}\mathfrak{a} \leq x} \mathbf{1}_{\mathcal{C}}(\mathfrak{a}) \Lambda_{\pi \otimes \tilde{\pi}}(\mathfrak{a}) \sim \frac{|\mathcal{C}|}{|\mathcal{G}|} x. \tag{4.4}$$

Our first application is a short interval version of (4.4), with effective bounds on the size of the intervals.

Theorem 4.4. *Assume the above notation. Let π be a self-dual automorphic representation of $\mathrm{GL}_d(\mathbb{A}_K)$ whose L-function $L(s,\pi,K)$ satisfies GRC. There exists an absolute constant $c_3 > 0$ such that if*

$$\delta \leq \frac{c_3}{d^4[M:\mathbb{Q}]\log(3d[M:K])},$$

x is sufficiently large, and $h \geq x^{1-\delta}$, then

$$\sum_{x < \mathrm{N}\mathfrak{a} \leq x+h} \mathbf{1}_{\mathcal{C}}(\mathfrak{a})\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a}) \asymp h,$$

where the implied constant depends on π and the extension M/K. If π is an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$ or the symmetric square of such a representation, then the assumption of GRC can be removed. If $M = K$, then d^4 can be replaced with d^2.

Remark. In certain instances, we can remove the assumption of GRC when we know how $L(s,\pi\otimes\tilde{\pi},K)$ factors. For example, let π be a self-dual cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$, and consider the factorization

$$L(s,\mathrm{Sym}^n\pi\otimes\mathrm{Sym}^n\pi,K) = L(s,\omega,K)\prod_{j=1}^{n} L(s,\mathrm{Sym}^{2j}\pi,K), \tag{4.5}$$

where ω is the central character; an analogous factorization holds when the representations are twisted by Hecke characters. Thus we see that the result is unconditional when the symmetric power L-functions are known to be automorphic and cuspidal. For $n = 1$, this follows from Kim and Shahidi [KS02b], and for $n = 2$, this follows from Kim [Kim03]. When $n = 3$ or 4 and π is associated to a classical modular form, this follows from the recent work of Clozel and Thorne [CT] when $K\cap\mathbb{Q}(e^{2\pi i/35}) = \mathbb{Q}$. A case where this is interesting is when π is associated to a Hecke-Maass form over

$\mathbb{Q}$, where GRC is not known. However, in this case, Motohashi [Mot15] recently established a log-free zero density estimate for $L(s, \mathrm{Sym}^2\pi, \mathbb{Q})$, thus obtaining Theorem 4.4.

It is of course somewhat unsatisfying that we cannot obtain an asymptotic formula in Theorem 4.4 to provide a true short interval analogue of (4.4), but, as remarked earlier, this is due to the lack of a strong zero-free region for general automorphic L-functions and seems unavoidable at present. Good zero-free regions of a quality better than Littlewood's exist for Dedekind zeta functions, which enabled Balog and Ono [BO01] to prove a prime number theorem for primes in Chebotarev sets lying in short intervals.

Even though versions of Theorem 4.4 with asymptotic equality are only known in special cases, we can use Theorem 4.3 to show that the predicted asymptotic holds on average. We prove the following generalization of [Gal70, Theorem 7]; to obtain unconditional and effective results, we restrict ourselves to consider cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$.

Theorem 4.5. *Assume the above notation. Let π be either a self-dual automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with trivial central character or the symmetric square of such a representation. There exist absolute constants $c_4, c_5 \in (0,1)$ such that if $\exp(\sqrt{\log x}) \leq Q \leq x^{c_4}$ and $x/Q \leq h \leq x$, then*

$$\sum_{q\leq Q}\ \sideset{}{^\star}\sum_{\chi \bmod q}\Big|\sum_{x<n\leq x+h}\Lambda_{\pi\otimes\tilde{\pi}}(n)\chi(n)-\delta(\chi)h+\delta_{q,*}(\chi)h\xi^{\beta_1-1}\Big| \ll h\exp\Big(-\frac{c_5\log x}{\log(Q\mathfrak{q}(\pi))}\Big)$$

for some $\xi \in [x, x+h]$. Here, $\delta(\chi) = 1$ if χ is the trivial character and is zero otherwise, and β_1 denotes the Landau-Siegel zero associated to an exceptional real Dirichlet character $\chi^* \pmod q$ if it exists. We set $\delta_{q,*}(\chi) = 1$ if $\chi = \chi^*$ and zero otherwise, including if the exceptional zero does not exist. The implied constant depends effectively on at most $\mathfrak{q}(\pi)$.

Unlike the previous log-free zero density estimates for general automorphic L-functions discussed earlier, Theorem 4.2 allows us to handle questions where maintaining uniformity in parameters is crucial. One famous example of such an application is the Sato-Tate conjecture, which concerns the distribution of the quantities $\lambda_\pi(\mathfrak{p})$ attached to a cuspidal automorphic representations π of $\mathrm{GL}_2(\mathbb{A}_K)$, where K is a totally real field; for generalizations to higher degree representations, see, for example, Serre [Ser12]. If π has trivial central character and is genuine in the sense of Shahidi [Sha94, Section 2] (in the case that $K = \mathbb{Q}$, this amounts to assuming that π is associated with a holomorphic cuspidal Hecke newform), then, by work of Deligne [Del74], it satisfies the generalized Ramanujan conjecture that $|\lambda_\pi(\mathfrak{p})| \leq 2$ at all unramified $\mathfrak{p}$. We may thus write $\lambda_\pi(\mathfrak{p}) = 2\cos\theta_\mathfrak{p}$ for some angle $\theta_\mathfrak{p} \in [0, \pi]$, and the Sato-Tate conjecture predicts that if $I = [a, b] \subset [-1, 1]$ is a fixed subinterval, then

$$\lim_{x\to\infty} \frac{1}{\pi_K(x)} \#\{\mathrm{N}\mathfrak{p} \leq x : \cos\theta_\mathfrak{p} \in I\} = \frac{2}{\pi}\int_I \sqrt{1-t^2}\, dt =: \mu_{\mathrm{ST}}(I),$$

where $\pi_K(x) := \#\{\mathfrak{p} : \mathrm{N}\mathfrak{p} \leq x\}$. The Sato-Tate conjecture is now a theorem, due to the remarkable work of Barnet-Lamb, Geraghty, Harris, and Taylor [BLGHT11]. The proof relies upon showing that the odd symmetric power L-functions $L(s, \mathrm{Sym}^n\pi, K)$ are all potentially automorphic, i.e., automorphic when restricted to some finite extension of K. It is expected that $L(s, \mathrm{Sym}^n\pi, K)$ is automorphic over K for each $n \geq 1$, but as of right now, this is known in general only for $n \leq 4$ (see [GJ78, Kim03, KS02a, KS02b]). By recent work of Clozel and Thorne [CT], if π is associated to a classical modular form, then $L(s, \mathrm{Sym}^n\pi, K)$ is automorphic for $n \leq 8$, provided that $K \cap \mathbb{Q}(e^{2\pi i/35}) = \mathbb{Q}$. Consequently, the number of symmetric powers needed to access the interval I is especially important in this problem.

Recall that the Chebyshev polynomials $U_n(t)$, defined by

$$\sum_{n=0}^{\infty} U_n(t)x^n = \frac{1}{1-2tx+x^2},$$

form an orthnormal basis for $L^2([-1,1],\mu_{\mathrm{ST}})$. If $\pi_{\mathfrak{p}}$ is unramified, then $U_n(\cos\theta_{\mathfrak{p}})$ is the Dirichlet coefficient of $L(s,\mathrm{Sym}^n\pi,K)$ at the prime $\mathfrak{p}$. We say that a subset $I\subseteq[-1,1]$ can be **Sym$^{\mathrm{n}}$-minorized** if there exist $b_0,\ldots,b_n\in\mathbb{R}$ with $b_0>0$ such that

$$\mathbf{1}_I(t)\geq\sum_{j=0}^{n} b_jU_i(t) \tag{4.6}$$

for all $t\in[-1,1]$, where $\mathbf{1}_I(\cdot)$ denotes the indicator function of I. Note that if I can be Sym^n-minorized, then it is the union of intervals which individually need not be Sym^n-minorizable. We have the following.

Theorem 4.6. *Assume the above notation. Let $K/\mathbb{Q}$ be a totally real field, and let π be a genuine automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$ with trivial central character. Suppose that a fixed subset $I\subseteq[-1,1]$ can be Sym^n-minorized and that the L-functions $L(s,\mathrm{Sym}^j\pi,K)$ are automorphic for each $j\leq n$. Let $B=\max_{0\leq j\leq n}|b_j|/b_0$, where $b_0,\ldots,b_n$ are as in (4.6).*

1. *There exists an absolute constant $c_6>0$ such that if*

$$\delta\leq\frac{c_6}{n^4[K:\mathbb{Q}]\log(3Bn)},$$

x is sufficiently large, and $h\geq x^{1-\delta}$, then

$$\sum_{\substack{x<\mathrm{N}\mathfrak{p}\leq x+h\\ \pi_{\mathfrak{p}}\text{ unramified}}}\mathbf{1}_I(\cos\theta_{\mathfrak{p}})\log\mathrm{N}\mathfrak{p}\asymp h,$$

where the implied constant depends on B, I, and K. In particular, if I can be Sym^4-minorized, then this is unconditional; the assumption of automorphy can *be dropped if I can be* Sym^8*-minorized and* $K \cap \mathbb{Q}(e^{2\pi i/35}) = \mathbb{Q}$.

2. *If I can be* Sym^n*-minorized and* $L(s, \mathrm{Sym}^j \pi, K)$ *is automorphic for each* $j \leq 2n$, *then there exists an absolute constant* $c_7 > 0$ *such that the statement "x is sufficiently large" can be replaced with*

$$x \geq ([K : \mathbb{Q}]^{[K:\mathbb{Q}]} \mathfrak{q}(\mathrm{Sym}^n \pi))^{c_7 n^4 \log(3Bn)}.$$

In particular, if I can be Sym^4*-minorized, then the assumption of automorphy can be dropped when* $K \cap \mathbb{Q}(e^{2\pi i/35}) = \mathbb{Q}$.

Remark. 1. For any fixed n, determining the subsets $I \subset [-1, 1]$ that can be Sym^n-minorized is an elementary combinatorial problem. We carry this out in Lemma 4.17 to determine the intervals that can be Sym^4-minorized, which we consider to be the most interesting case; it turns out that the proportion of subintervals of $[-1, 1]$ which can be Sym^4-minorized is roughly 0.388. If one is not concerned with obtaining the optimal minorization or if n is large, it is likely more convenient to apply a standard minorant for I instead. For the Beurling-Selberg minorant (see Montgomery [Mon94, Lecture 1]), a tedious calculation shows that if $n \geq 4(1 + \delta)/\mu_{\mathrm{ST}}(I) - 1$ for some $\delta > 0$, then I can be Sym^n-minorized and

$$B \leq \frac{2 + 3/\delta}{\mu_{\mathrm{ST}}(I)}.$$

It follows that any interval can be Sym^n-minorized for n sufficiently large, and thus every interval is at least conditionally covered by Theorem 4.6; at the end of this chapter, we will show that this minorant might be far from optimal. With the Beurling-Selberg minorant, we have unconditional results for intervals I satisfying $\mu_{\mathrm{ST}}(I) > 4/5$. By contrast, Lemma 4.17 implies unconditional results for all intervals satisfying $\mu_{\mathrm{ST}}(I) \geq 0.534$, and for some with measure as small as 0.139.

2. It is tempting to ask whether one can exploit existing results on potential automorphy for symmetric power L-functions and the explicit dependence on the base field in Theorem 4.2 to obtain unconditional, albeit weaker, results for all intervals. The proof of the Sato-Tate conjecture uses crucially work of Moret-Bailly [MB89] establishing the existence of number fields over which certain varieties have points. The proof of this result unfortunately only permits control over the ramification at finitely many places, so it is not possible to even obtain bounds on the discriminants of the fields over which the symmetric power L-functions are automorphic. Thus, the authors do not believe it is possible to obtain an unconditional analogue of Theorem 4.6 for all I at this time.

3. When $K = \mathbb{Q}$ and the arithmetic conductor of π is squarefree, Cogdell and Michel [CM04] use the local Langlands correspondence to predict what $\mathfrak{q}(\mathrm{Sym}^n\pi)$ should be when $\mathrm{Sym}^n\pi$ is an automorphic representation satisfying Langlands functoriality. Under these assumptions, they prove that $\log \mathfrak{q}(\mathrm{Sym}^n\pi) \ll n \log \mathfrak{q}(\pi)$. In all other cases, we have that $\log \mathfrak{q}(\mathrm{Sym}^n\pi) \ll n^3[K:\mathbb{Q}] \log \mathfrak{q}(\pi)$ by Rouse [Rou07] under the assumption of automorphy alone.

4. Part 2 gives an upper bound on the least norm of an unramified prime $\mathfrak{p}$ such that $\cos\theta_{\mathfrak{p}} \in I$. When I is fixed and π varies, this upper bound has the shape $\mathrm{N}\mathfrak{p} \leq \mathfrak{q}(\pi)^A$ for some absolute constant A, and so is comparable to Linnik's theorem. However, if π is fixed and I is varying, the dependence is much worse. This comes partially from the constants in the zero-free region for $L(s, \mathrm{Sym}^n\pi, K)$, where the n dependence in particular is of the form n^4 (see Lemma 4.7). Without improving the quality of these constants, it seems likely that only minor improvements can be made to the lower bound on x.

In Section 4.1, we discuss the basic properties of automorphic L-functions that we will use in the proofs of the theorems and we prove a few useful lemmas. In Section 4.2, we prove Theorems 4.2 and 4.3. In Section 4.3, we prove Theorems 4.4-4.6.

4.1 Preliminaries

4.1.1 Definitions and notation

We follow the account of Rankin-Selberg L-functions given by Brumley [Bru06, Section 1] and Moreno [Mor85, Sections 0 and 1]. Let $K/\mathbb{Q}$ be a number field of absolute discriminant D_K, and let $n_K := [K : \mathbb{Q}]$. Let $\mathbb{A}_K$ denote the ring of adeles over K, and let π be a cuspidal automorphic representation of $\mathrm{GL}_d(\mathbb{A}_K)$ with unitary central character. For brevity, we will say that π is an automorphic representation.

We have the factorization $\pi = \otimes_{\mathfrak{v}} \pi_{\mathfrak{v}}$ over the places of K. For each nonarchimedean $\mathfrak{p}$, we have the Euler factor

$$L_{\mathfrak{p}}(s, \pi, K) = \prod_{j=1}^{d} (1 - \alpha_\pi(j, \mathfrak{p}) \mathrm{N}\mathfrak{p}^{-s})^{-1}$$

associated with $\pi_{\mathfrak{p}}$. Let R_π be the set of primes ideals $\mathfrak{p}$ for which $\pi_{\mathfrak{p}}$ is ramified. We call $\alpha_\pi(j, \mathfrak{p})$ the local roots of $L(s, \pi, K)$ at $\mathfrak{p}$, and if $\mathfrak{p} \notin R_\pi$, then $\alpha_\pi(j, \mathfrak{p}) \neq 0$ for all $1 \leq j \leq d$. The representation π has an associated automorphic L-function whose Euler product and Dirichlet series are given by

$$L(s, \pi, K) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(s, \pi, K) = \sum_{\mathfrak{a}} \frac{\lambda_\pi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s},$$

where $\mathfrak{p}$ runs through the finite primes and $\mathfrak{a}$ runs through the non-zero integral ideals of K. This Euler product converges absolutely for $\mathrm{Re}(s) > 1$, which implies that $|\alpha_\pi(j, \mathfrak{p})| < \mathrm{N}\mathfrak{p}$. Luo, Rudnick, and Sarnak [LRS99] showed that this may be improved to

$$|\alpha_\pi(j, \mathfrak{p})| \leq \mathrm{N}\mathfrak{p}^{\frac{1}{2} - \frac{1}{d^2+1}},$$

and the generalized Ramanujan conjecture asserts a further improvement.

Hypothesis 2 (GRC). Assume the above notation. For each prime $\mathfrak{p} \notin R_\pi$, we have

$|\alpha_\pi(j,\mathfrak{p})| = 1$, and for each prime $\mathfrak{p} \in R_\pi$, we have $|\alpha_\pi(j,\mathfrak{p})| \le 1$.

Remark. It is expected that all automorphic L-functions $L(s,\pi,K)$ satisfy GRC. Indeed, it is already known for many of the most commonly used automorphic L-functions. Such L-functions include Hecke L-functions and the L-function of a cuspidal normalized Hecke eigenform of positive even integer weight k on the congruence subgroup $\Gamma_0(N)$.

At each archimedean place $\mathfrak{v}$, we associate to $\pi_\mathfrak{v}$ a set of n complex numbers $\{\mu_\pi(j,\mathfrak{v})\}_{j=1}^d$(the *Langlands parameters*), which are known to satisfy $\mathrm{Re}(\mu_\pi(j,\mathfrak{v})) > -1/2$ by the work of Luo, Rudnick, and Sarnak [LRS99]. The local factor at $\mathfrak{v}$ is defined to be

$$L_\mathfrak{v}(s,\pi,K) = \prod_{j=1}^{d} \Gamma_{K_\mathfrak{v}}(s+\mu_\pi(j,\mathfrak{v})),$$

where $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$ and $\Gamma_\mathbb{C}(s) = \Gamma_\mathbb{R}(s)\Gamma_\mathbb{R}(s+1)$. Letting S_∞ denote the set of archimedean places, we define the gamma factor of $L(s,\pi,K)$ by

$$\gamma(s,\pi,K) = \prod_{\mathfrak{v}\in S_\infty} L_\mathfrak{v}(s,\pi,K).$$

For notational convenience, we will define the complex numbers $\kappa_\pi(j)$ by

$$\gamma(s,\pi,K) = \prod_{j=1}^{dn_K} \Gamma_\mathbb{R}(s+\kappa_\pi(j)).$$

Let $q(\pi)$ be the arithmetic conductor of π; for a certain integral ideal $\mathfrak{f}$ of K depending on π, we have that

$$q(\pi) = D_K^d \mathrm{N}\mathfrak{f}.$$

Any automorphic L-function $L(s,\pi,K)$ admits a meromorphic continuation to $\mathbb{C}$ with poles possible only at $s=0$ and 1. Letting $r(\pi)$ denote the order of the pole at $s=1$, and defining the completed L-function

$$\Lambda(s,\pi,K) = (s(1-s))^{r(\pi)} q(\pi)^{s/2}\gamma(s,\pi,K)L(s,\pi,K),$$

it is well-known that $\Lambda(s,\pi,K)$ is an entire function of order 1 and that there exists a complex number $\varepsilon(\pi)$ of modulus 1 such that $\Lambda(s,\pi,K)$ satisfies the functional equation

$$\Lambda(s,\pi,K) = \varepsilon(\pi)\Lambda(1-s,\tilde{\pi},K),$$

where $\tilde{\pi}$ is the representation contragredient to π. We have the relations

$$\alpha_{\tilde{\pi}}(j,\mathfrak{p}) = \overline{\alpha_{\pi}(j,\mathfrak{p})}, \quad \gamma(s,\tilde{\pi},K) = \gamma(s,\pi,K), \text{ and } q(\tilde{\pi}) = q(\pi).$$

To maintain uniform estimates for the analytic quantities associated to $L(s,\pi,K)$, we define the **analytic conductor** of $L(s,\pi,K)$ by

$$\mathfrak{q}(s,\pi,K) = q(\pi)\prod_{j=1}^{dn_K}(|s+\kappa_{\pi}(j)|+3). \tag{4.7}$$

We will frequently make use of the quantity $\mathfrak{q}(0,\pi,K)$, which we simply write as $\mathfrak{q}(\pi)$.

As in the introduction, define the von Mangoldt function $\Lambda_{\pi}(\mathfrak{a})$ by

$$-\frac{L'}{L}(s,\pi,K) = \sum_{\mathfrak{a}} \frac{\Lambda_{\pi}(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s},$$

and let $\Lambda_K(\mathfrak{a})$ be that associated to the Dedekind zeta function $\zeta_K(s)$. Using the bounds for $|\alpha_{\pi}(j,\mathfrak{p})|$ from Luo, Rudnick, and Sarnak [LRS99], we have that

$$|\Lambda_{\pi}(\mathfrak{a})| \le d\Lambda_K(\mathfrak{a})\mathrm{N}\mathfrak{a}^{\frac{1}{2}-\frac{1}{d^2+1}}, \tag{4.8}$$

and under GRC, we have

$$|\Lambda_\pi(\mathfrak{a})| \leq d\Lambda_K(\mathfrak{a}).$$

Consider two cuspidal automorphic representations π and π' of $\mathrm{GL}_d(\mathbb{A}_K)$ and $\mathrm{GL}_{d'}(\mathbb{A}_K)$, respectively. We are interested in the Rankin-Selberg product $\pi \otimes \pi'$ of π and π', which, at primes $\mathfrak{p} \notin R_\pi \cup R_{\pi'}$, has a local factor given by

$$L_{\mathfrak{p}}(s, \pi \otimes \pi', K) = \prod_{j_1=1}^{d} \prod_{j_2=1}^{d'} (1 - \alpha_\pi(j_1, \mathfrak{p})\alpha_{\pi'}(j_2, \mathfrak{p})\mathrm{N}\mathfrak{p}^{-s})^{-1}.$$

For $\mathfrak{p} \in R_\pi \cup R_{\pi'}$, there exist local roots $\beta_{\pi\otimes\pi'}(j, \mathfrak{p})$ which satisfy $|\beta_{\pi\otimes\pi'}(j, \mathfrak{p})| < \mathrm{N}\mathfrak{p}^{1-\frac{1}{d^2+1}-\frac{1}{(d')^2-1}}$ for all $1 \leq j \leq d'd$, and we define for such $\mathfrak{p}$

$$L_{\mathfrak{p}}(s, \pi \otimes \pi', K) = \prod_{j=1}^{d'd} (1 - \beta_{\pi\otimes\pi'}(j, \mathfrak{p})\mathrm{N}\mathfrak{p}^{-s})^{-1}.$$

This gives rise to the L-function $L(s, \pi \otimes \pi', K)$, which we call the Rankin-Selberg convolution of π and π', whose Euler product and gamma factor are given by

$$L(s, \pi \otimes \pi', K) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(s, \pi \otimes \pi', K)$$

and

$$\gamma(s, \pi \otimes \pi', K) = \prod_{\mathfrak{v} \in S_\infty} \prod_{j_1=1}^{d} \prod_{j_2=1}^{d'} \Gamma_{K_{\mathfrak{v}}}(s + \mu_{\pi\otimes\pi'}(j_1, j_2, \mathfrak{v})) = \prod_{j=1}^{d'dn_K} \Gamma_{\mathbb{R}}(s + \kappa_{\pi\otimes\pi'}(j)),$$

where $\mathrm{Re}(\mu_{\pi\otimes\pi'}(j_1, j_2, \mathfrak{v})) > -1$ and $\mathrm{Re}(\kappa_{\pi\otimes\pi'}(j)) > -1$. By Equation 8 of Brumley [Bru06], we have

$$\mathfrak{q}(s, \pi \otimes \pi') \leq \mathfrak{q}(\pi)^{d'} \mathfrak{q}(\pi')^{d} (|s| + 3)^{d'dn_K}.$$

Finally, we note that if $\pi' = \tilde{\pi}$, then $L(s, \pi \otimes \pi', K)$ has a pole at $s = 1$ of order 1, so that $r(\pi \otimes \pi') = 1$.

4.1.2 Preliminary lemmata

We begin with a zero-free region for $L(s,\pi\otimes\pi',K)$, obtained by adapting Theorem 5.10 of Iwaniec and Kowalski [IK04] to L-functions over number fields.

Lemma 4.7. *Suppose that either both d and d' are at most 2 or that at least one of π and π' is self-dual. Let $T\geq 3$, and let*

$$\mathcal{L}=\mathcal{L}(T,\pi\otimes\pi',K)=\mathcal{D}\log(\mathcal{Q}T^{n_K}),$$

where $\mathcal{D}$ and $\mathcal{Q}$ are defined in Theorem 4.2. There is a positive absolute constant c_8[2] *such that the region*

$$\{s=\sigma+it:\sigma\geq 1-c_8\mathcal{L}^{-1},|t|\leq T\}$$

contains at most one zero of $L(s,\pi\otimes\pi',K)$. If such an exceptional zero β_1 exists, then it is real and simple, and $L(s,\pi\otimes\pi',K)$ must be self-dual.

Proof. If π is trivial, then the result follows from the proof of [IK04, Theorem 5.10], but we bound the ensuing analytic conductors using (4.1.1). Now, assume that π is nontrivial. If one of π and π' is not self-dual or $d\neq d'$, then this follows from the proof of Lemma 5.9 and Exercise 4 in [IK04, Chapter 5] by considering the auxiliary L-function

$$L(s+it/2,\pi\otimes\pi',K)L(s,\pi\otimes\tilde{\pi}',K)L(s,\tilde{\pi}\otimes\pi',K)L(s-it/2,\pi\otimes\pi',K).$$

Again, we bound the ensuing analytic conductors using (4.1.1). In the remaining cases, we have that $d=d'$ and both π and π' are self-dual; thus one may use Moreno's zero-free region [Mor85, Theorem 3.3]. □

[2] We denote by $c_1,c_2,\ldots$ a sequence of constants, each of which is absolute, positive, and effectively computable. We do not recall this convention in future statements, as we find it to be notationally cumbersome.

Lemma 4.8. *Let $T \gg 1$, and let $\tau \in \mathbb{R}$ satisfy $|\tau| \leq T$.*

1. *Uniformly on the disk $|s-(1+i\tau)| \leq 1/4$, we have that*

$$\frac{L'}{L}(s, \pi \otimes \pi', K) + \frac{r(\pi \otimes \pi')}{s} + \frac{r(\pi \otimes \pi')}{s-1} - \sum_{|\rho-(1+i\tau)| \leq 1/2} \frac{1}{s-\rho} \ll \mathcal{L},$$

where the sum runs over zeros ρ of $L(s, \pi \otimes \pi', K)$.

2. *For $1 \geq \eta \gg \mathcal{L}^{-1}$, we have that*

$$\sum_{|\rho-(1+i\tau)| \leq \eta} 1 \ll \eta \mathcal{L}.$$

Proof. Part 1 is Lemma 2.4 of Akbary and Trudgian [AT15]. Part 2 follows from combining Theorem 5.6 of [IK04] and Proposition 5.8 of [IK04]. □

Lemma 4.9. *If $0 < \eta \ll 1$ and $y \gg 1$, then*

1. $\displaystyle\sum_{\mathfrak{a}} \frac{|\Lambda_{\pi \otimes \pi'}(\mathfrak{a})|}{\mathrm{N}\mathfrak{a}^{1+\eta}} \ll \frac{1}{\eta} + d'd \log(\mathcal{Q}).$
2. $\displaystyle\sum_{\mathrm{N}\mathfrak{a} \leq y} \frac{|\Lambda_{\pi \otimes \pi'}(\mathfrak{a})|}{\mathrm{N}\mathfrak{a}} \ll \log y + d'd \log(\mathcal{Q}).$

Proof. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum_{\mathfrak{a}} \frac{|\Lambda_{\pi \otimes \pi'}(\mathfrak{a})|}{\mathrm{N}\mathfrak{a}^{1+\eta}} &\ll \Big(\sum_{\mathfrak{a}} \frac{\Lambda_{\pi \otimes \tilde{\pi}}(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^{1+\eta}}\Big)^{1/2} \Big(\sum_{\mathfrak{a}} \frac{\Lambda_{\pi' \otimes \tilde{\pi}'}(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^{1+\eta}}\Big)^{1/2} \\ &= \Big(-\frac{L'}{L}(1+\eta, \pi \otimes \tilde{\pi}, K)\Big)^{1/2} \Big(-\frac{L'}{L}(1+\eta, \pi' \otimes \tilde{\pi}', K)\Big)^{1/2}.\end{aligned}$$

We first estimate $-\frac{L'}{L}(1+\eta, \pi \otimes \tilde{\pi}, K)$, which is a positive quantity because $\eta > 0$ is real and the Dirichlet coefficients of $-\frac{L'}{L}(s, \pi \otimes \tilde{\pi}, K)$ are real and nonnegative. By

Theorem 5.6 of [IK04] and part 3 of Proposition 5.7 of [IK04], we have that

$$-\mathrm{Re}\Big(\frac{L'}{L}(1+\eta,\pi\otimes\tilde{\pi},K)\Big) =\frac{1}{2}\log q(\pi\otimes\tilde{\pi}) + \mathrm{Re}\Big(\frac{\gamma'}{\gamma}(1+\eta,\pi\otimes\tilde{\pi},K)\Big) + \frac{1}{1+\eta} + \frac{1}{\eta} - \sum_{\rho\neq 0,1}\mathrm{Re}\Big(\frac{1}{1+\eta-\rho}\Big).$$

Since

$$\mathrm{Re}\Big(\frac{1}{1+\eta-\rho}\Big) \geq \frac{\eta}{(1+\eta)^2+\gamma^2} > 0,$$

we have that

$$-\mathrm{Re}\Big(\frac{L'}{L}(1+\eta,\pi\otimes\tilde{\pi},K)\Big) \leq \frac{1}{2}\log q(\pi\otimes\tilde{\pi}) + \mathrm{Re}\Big(\frac{\gamma'}{\gamma}(1+\eta,\pi\otimes\tilde{\pi},K)\Big) + \frac{1}{1+\eta} + \frac{1}{\eta}.$$

By the proof of part 2 in Proposition 5.7 in [IK04], we have that

$$\mathrm{Re}\Big(\frac{\gamma'}{\gamma}(s,\pi\otimes\tilde{\pi},K)\Big) = -\sum_{|s+\kappa_{\pi\otimes\tilde{\pi}}(j)|<1}\mathrm{Re}\Big(\frac{1}{s+\kappa_{\pi\otimes\tilde{\pi}}(j)}\Big) + O(\log\mathfrak{q}(\pi\otimes\tilde{\pi})).$$

Since $\mathrm{Re}(\kappa_{\pi\otimes\tilde{\pi}}(j)) > -1$ for all $1 \leq j \leq d'dn_K$, we find

$$\mathrm{Re}\Big(\frac{1}{s+\kappa_{\pi\otimes\tilde{\pi}}(j)}\Big) \geq \frac{\eta}{(1+\eta+\mathrm{Re}(\kappa_{\pi\otimes\tilde{\pi}}(j)))^2+\mathrm{Im}(\kappa_{\pi\otimes\tilde{\pi}}(j))^2} > 0.$$

Now, using (4.1.1), we find that

$$-\frac{L'}{L}(1+\eta,\pi\otimes\tilde{\pi},K) \ll \frac{1}{\eta} + \log(\mathfrak{q}(\pi\otimes\tilde{\pi})) \ll \frac{1}{\eta} + d\log\mathfrak{q}(\pi).$$

Since the analogue must hold for π', part 1 follows. Part 2 follows by choosing $\eta = \frac{1}{\log y}$. □

We conclude this section with a bound on the mean value of a Dirichlet polynomial.

Lemma 4.10. *Let $T \gg n_K \mathcal{Q}^{1/n_K}$ and $u > y > T^{16 n_K}$. Define*

$$S_{y,u}(\tau, \pi \otimes \pi') := \sum_{y < \mathrm{N}\mathfrak{p} \leq u} \frac{\Lambda_{\pi\otimes\pi'}(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^{1+i\tau}}.$$

1. If $L(s, \pi, K)$ satisfies GRC, then

$$\log y \int_{-T}^{T} |S_{y,u}(\tau, \pi \otimes \pi')|^2 d\tau \ll d^2((\log u)^2 + (d')^2(\log \mathfrak{q}(\pi'))(\log u)).$$

2. If $K = \mathbb{Q}$ and $L(s, \pi, \mathbb{Q})$ satisfies GRC, then

$$\sum_{q \leq T^2} \log \frac{T^2}{q} \sideset{}{^\star}\sum_{\chi \bmod q} \int_{-T}^{T} \Big| S_{y,u}(\tau, (\pi \otimes \pi') \otimes \chi) \Big|^2 dt \ll d^2((\log u)^2 + (d')^2(\log \mathfrak{q}(\pi'))(\log u)).$$

Proof. 1. Let $b(\mathfrak{p})$ be a complex-valued function on the prime ideals of K such that $\sum_{\mathfrak{p}} |b(\mathfrak{p})| < \infty$ and $b(\mathfrak{p}) = 0$ whenever $\mathrm{N}\mathfrak{p} \leq y$. By [Wei83, Corollary 3.8], if $T \gg D_K^{1/n_K} n_K$ and $y \geq T^{16 n_K}$, then

$$\int_{-T}^{T} \Big| \sum_{\mathfrak{p}} b(\mathfrak{p}) \mathrm{N}\mathfrak{p}^{-i\tau} \Big|^2 d\tau \ll \frac{1}{\log y} \sum_{\mathfrak{p}} |b(\mathfrak{p})|^2 \mathrm{N}\mathfrak{p}.$$

If we define $b(\mathfrak{p})$ by

$$b(\mathfrak{p}) = \begin{cases} \dfrac{\Lambda_{\pi\otimes\pi'}(\mathfrak{p})}{\mathrm{N}\mathfrak{p}} & \text{if } y < \mathrm{N}\mathfrak{p} \leq u, \\ 0 & \text{otherwise}, \end{cases} \tag{4.9}$$

and recall the definition of $S_{y,u}(\tau, \pi \otimes \pi')$, then it follows immediately that

$$\int_{-T}^{T} \Big| S_{y,u}(\tau, \pi \otimes \pi') \Big|^2 d\tau \ll \frac{1}{\log y} \sum_{y < \mathrm{N}\mathfrak{p} \leq u} \frac{|\Lambda_{\pi\otimes\pi'}(\mathfrak{p})|^2}{\mathrm{N}\mathfrak{p}}.$$

Since $q(\pi) = D_K^d \mathrm{N}\mathfrak{f}$ for a certain integral ideal $\mathfrak{f}$ of K, the condition $T \gg n_K \mathcal{Q}^{1/n_K}$ ensures that y is larger than any norm of a prime $\mathfrak{p} \in R_\pi \cup R_{\pi'}$ and $T \gg D_K^{1/n_K} n_K$.

By assuming GRC for $L(s,\pi,K)$, we conclude that

$$\sum_{y<\mathrm{N}\mathfrak{p}\le u}\frac{|\Lambda_{\pi\otimes\pi'}(\mathfrak{p})|^2}{\mathrm{N}\mathfrak{p}}=\sum_{y<\mathrm{N}\mathfrak{p}\le u}\frac{|\lambda_{\pi}(\mathfrak{p})|^2|\lambda_{\pi'}(\mathfrak{p})|^2(\log\mathrm{N}\mathfrak{p})^2}{\mathrm{N}\mathfrak{p}}\ll d^2\sum_{y<\mathrm{N}\mathfrak{p}\le u}\frac{(\log\mathrm{N}\mathfrak{p})\Lambda_{\pi'\otimes\tilde{\pi}'}(\mathfrak{p})}{\mathrm{N}\mathfrak{p}}.$$

The claimed result now follows by partial summation using Part 2 of Lemma 4.9.

2. Let $K=\mathbb{Q}$. Suppose that $b(p)$ is a function on primes such that $b(p)=0$ if $p\le Q$ and $\sum_p|b(p)|^2p<\infty$. By [Gal70, Theorem 4], we have that for $T\ge 2$ and $Q\ge 3$,

$$\sum_{q\le Q}\log\frac{Q}{q}\sideset{}{^\star}\sum_{\chi\bmod q}\int_{-T}^{T}\Big|\sum_p b(p)\chi(p)p^{-it}\Big|^2dt\ll\sum_p(Q^2T+p)|b(p)|^2.$$

If we define $b(p)$ as in (4.9) and let $Q=T^2$, then

$$\sum_{q\le T^2}\log\frac{T^2}{q}\sideset{}{^\star}\sum_{\chi\bmod q}\int_{-T}^{T}\Big|S_{y,u}(\tau,(\pi\otimes\pi')\otimes\chi)\Big|^2dt\ll\sum_{y<p\le u}(T^5+p)\frac{|\Lambda_{\pi\otimes\pi'}(p)|^2}{p^2}.$$

By assuming GRC for $L(s,\pi,\mathbb{Q})$, it follows that

$$\sum_{y<p\le u}(T^5+p)\frac{|\Lambda_{\pi\otimes\pi'}(p)|^2}{p^2}\ll d^2\sum_{y<p\le u}\frac{(\log p)\Lambda_{\pi'\otimes\tilde{\pi}'}(p)}{p}.$$

The claimed result now follows by partial summation using Lemma 4.9. □

4.2 The zero density estimate

In this section, we prove Theorem 4.2 by generalizing Gallagher's [Gal70] and Weiss's [Wei83] treatment of Turán's method for detecting zeros of L-functions, obtaining a result that is uniform in K, π, and π'. The key result is the following technical proposition, whose proof we defer to the end of the section.

Proposition 4.11. *Let $T\gg n_K\mathcal{Q}^{1/n_K}$, $\mathcal{L}=\mathcal{D}\log(\mathcal{Q}T^{n_K})$, and $y=e^{c_9\mathcal{L}}$. Suppose* *that η satisfies $\mathcal{L}^{-1} \ll \eta \ll 1$. Let*

$$S_{y,u}(\tau, \pi \otimes \pi') := \sum_{y < \mathrm{N}\mathfrak{p} \leq u} \frac{\Lambda_{\pi \otimes \pi'}(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^{1+i\tau}}.$$

If $L(s, \pi \otimes \pi')$ has a non-exceptional zero ρ_0 satisfying $|\rho_0 - (1 + i\tau)| \leq \eta$, then

$$\frac{y^{c_{10}\eta}}{(\log y)^3} \int_y^{y^{c_{11}}} |S_{y,u}(\tau, \pi \otimes \pi')|^2 \frac{du}{u} \gg 1.$$

We first deduce Theorem 4.2 from Proposition 4.11. The proof of Proposition 4.11 relies on certain upper and lower bounds on the derivatives of $\frac{L'}{L}(s, \pi \otimes \pi')$, which are proven and assembled subsequently.

4.2.1 Proof of Theorems 4.2 and 4.3

By Theorem 5.8 of [IK04], we have

$$N_{\pi \otimes \pi'}(0, T) = \frac{T}{\pi} \log\Big(\frac{q(\pi \otimes \pi') T^{d' d n_K}}{(2\pi e)^{d' d n_K}}\Big) + O(\log \mathfrak{q}(iT, \pi \otimes \pi')). \tag{4.10}$$

Thus it suffices to prove the theorem for $1 - \sigma$ sufficiently small. Since the left side of Theorem 4.2 is a decreasing function of σ and the right side of Theorem 4.2 is essentially constant for $1 - \sigma \ll \mathcal{L}^{-1}$, it suffices to prove the theorem for $1 - \sigma \gg \mathcal{L}^{-1}$. Therefore, we may assume that $c_{12} \leq \sigma \leq 1 - c_8 \mathcal{L}^{-1}$, where $\frac{1}{2} < c_{12} < 1$ and $c_8 > 0$ are chosen such that we may take $\eta = \sqrt{2}(1 - \sigma)$ in Proposition 4.11.

It follows from the same reasoning as in the proof of Theorem 4.3 in [Wei83] (with Proposition 4.11 replacing Lemma 4.2 of [Wei83]) that

$$N_{\pi \otimes \pi'}(\sigma, T) \ll \mathcal{L} \frac{y^{c_{10}\eta}}{(\log y)^3} \int_y^{y^{c_{11}}} \Big(\int_{-T}^{T} |S_{y,u}(\tau, \pi \otimes \pi')|^2 d\tau\Big) \frac{du}{u}.$$

Suppose that $L(s, \pi', K)$ satisfies GRC. By Part 1 of Lemma 4.10, the definition of y, and the definition of $S_{y,u}(\tau, \pi \otimes \pi')$,

$$N_{\pi\otimes\pi'}(\sigma, T) \ll d^2 \mathcal{L} \frac{y^{c_{10}\eta}}{(\log y)^4} \int_y^{y^{c_{11}}} \frac{(\log u)^2 + (d')^2 (\log u) \log \mathfrak{q}(\pi')}{u} du \ll d^2 y^{c_{10}\eta}.$$

Since we may take $\eta = \sqrt{2}(1-\sigma)$, recalling the definition of y, we have

$$N_{\pi\otimes\pi'}(\sigma, T) \ll d^2 y^{\sqrt{2}c_{10}(1-\sigma)} \ll d^2 (\mathcal{Q}T^{n_K})^{\mathcal{D}\sqrt{2}c_{10}(1-\sigma)}.$$

To conclude the proof of Theorem 4.2, let c_{10} be sufficiently larger than c_9 and set $c_1 = \sqrt{2}c_{10}$.

Theorem 4.3 is proven in almost exactly the same way as Theorem 4.2, except that it requires Part 2 of Lemma 4.10; we omit the proof.

4.2.2 Bounds on derivatives

We begin by introducing notation which we will use throughout this section and the next. First, let $r = r(\pi \otimes \pi')$ be the order of the possible pole of $L(s, \pi \otimes \pi', K)$ at $s = 1$. We suppose that $L(s, \pi \otimes \pi', K)$ has a non-exceptional zero ρ_0 satisfying $|\rho_0 - (1 + i\tau)| \leq \eta$, and we set

$$F(s) = \frac{L'}{L}(s, \pi \otimes \pi', K).$$

Suppose that $|\tau| \leq T$, where $T \geq 2$, as in the statement of Proposition 4.11. On the disk $|s - (1 + i\tau)| < 1/4$, by part 1 of Lemma 4.8, we have

$$F(s) + \frac{r}{s} + \frac{r}{s-1} = \sum_{|\rho-(1+i\tau)|\leq 1/2} \frac{1}{s-\rho} + G(s),$$

where $G(s)$ is analytic and $|G(s)| \ll \mathcal{L}$. Setting $\xi = 1 + \eta + i\tau$, we have

$$\frac{(-1)^k}{k!}\frac{d^kF}{ds^k}(\xi) + r(\xi-1)^{-(k+1)} = \sum_{|\rho-(1+i\tau)|\le 1/2} (\xi-\rho)^{-(k+1)} + O(8^k\mathcal{L}), \tag{4.11}$$

where the error term absorbs the contribution from integrating $G(s)$ over a circle of radius 1/8 centered at ξ and the term coming from differentiating $\frac{r}{s}$. We begin by obtaining a lower bound on the derivatives of $F(s)$.

Lemma 4.12. *Assume the notation above. For any $M \gg \eta\mathcal{L}$, there is some $k \in [M, 2M]$ such that*

$$\frac{\eta^{k+1}}{k!}\left|\frac{d^kF}{ds^k}(\xi)\right| \ge \frac{1}{2}(100)^{-(k+1)},$$

where $\xi = 1 + \eta + i\tau$.

We prove Lemma 4.12 using a version of Turán's [ST55] power-sum estimate.

Lemma 4.13 (Turán)**.** *Let $z_1, \dots, z_m \in \mathbb{C}$. If $M \ge m$, then there exists $k \in \mathbb{Z} \cap [M, 2M]$ such that $|z_1^k + \cdots + z_m^k| \ge (\frac{1}{50}|z_1|)^k$.*

Proof of Lemma 4.12. We begin by considering the contribution to (4.11) from those zeros ρ satisfying $200\eta < |\rho - (1 + i\tau)| \le 1/2$. In particular, by decomposing the sum dyadically and applying part 2 of Lemma 4.8, we have

$$\sum_{200\eta<|\rho-(1+i\tau)|\le 1/2} |\rho-\xi|^{-(k+1)} \ll \sum_{j=0}^{\infty}(2^j 200\eta)^{-(k+1)}2^{j+1}r\mathcal{L} \ll (200\eta)^{-k}\mathcal{L},$$

This shows that it suffices to consider the zeros ρ whose distance from $1 + i\tau$ is less than 200η.

Since $\eta \ll 1$, we have

$$\frac{1}{k!}\frac{d^kF}{ds^k}(\xi) + r(\xi-1)^{-(k+1)} \ge \left|\sum_{|\rho-(1+i\tau)|\le 200\eta} (\xi-\rho)^{-(k+1)}\right| - O((200\eta)^{-k}\mathcal{L}). \tag{4.12}$$

By Lemma 4.8 (part 2), the sum over zeros has $\ll \eta\mathcal{L}$ terms. Choosing $M \gg \eta\mathcal{L}$, Lemma 4.13 tells us that for some $k \in [M, 2M]$, the sum over zeros on the right side of (4.12) is bounded below by $(50|\xi - \rho_0|)^{-(k+1)}$, where ρ_0 is the nontrivial zero which is being detected.

Since $|\xi - \rho_0| \leq 2\eta$, the right side of the above inequality is

$$\geq (100\eta)^{-(k+1)}(1 - O(2^{-k}\eta\mathcal{L})).$$

Since $k \geq M \gg \eta\mathcal{L}$ and $\mathcal{L}^{-1} \ll \eta \ll 1$, there is a constant $0 < \theta < 1$ so that

$$O(2^{-k}\eta\mathcal{L}) = O(\theta^{\eta\mathcal{L}}\eta\mathcal{L}) \leq 1/4.$$

Therefore, for some $k \in [M, 2M]$ with $M \gg \eta\mathcal{L}$, we have

$$\frac{\eta^{k+1}}{k!}\left|\frac{d^k F}{ds^k}(\xi)\right| + r\eta^{k+1}|(\xi - 1)^{-(k+1)}| \geq \frac{3}{4}(100)^{-(k+1)}.$$

During the proof of Theorem 4.2 in [Wei83], Weiss proves that

$$r\eta^{k+1}|(\xi - 1)^{-(k+1)}| \leq \frac{1}{4}(100)^{-(k+1)}.$$

The desired result now follows. □

We now turn to obtaining an upper bound on the derivatives of $F(s)$, for which we have the following.

Lemma 4.14. *Assume the notation preceeding Lemma 4.12. Set $M = 300\eta \log y$, and let k be determined by Lemma 4.12. Then*

$$\frac{\eta^{k+1}}{k!}\left|\frac{d^k F}{ds^k}(\xi)\right| \leq \eta^2 \int_y^{y^{c_{11}}} |S_{y,u}(\tau, \pi \otimes \pi')|\frac{du}{u} + \frac{1}{4}(100)^{-(k+1)},$$

where $S_{y,u}(\tau, \pi \otimes \pi')$ is as in Proposition 4.11.

Proof. Let $M = 300\eta \log y$ and recall that $y = e^{c_9 \mathcal{L}}$ for some c_9, which we will take to be sufficiently large. For $u > 0$, define $j_k(u) = \frac{u^k e^{-u}}{k!}$, which satisfies

$$j_k(u) \leq \begin{cases} (100)^{-k} & \text{if } u \leq k/300, \\ (110)^{-k} e^{-u/2} & \text{if } u \geq 20k. \end{cases}$$

Letting $c_{11} \geq 12000$ be sufficiently large, we thus have

$$j_k(\eta \log(\mathrm{N}\mathfrak{a})) \leq \begin{cases} (110)^{-k} & \text{if } \mathrm{N}\mathfrak{a} \leq y, \\ (100)^{-k} (\mathrm{N}\mathfrak{a})^{-\eta/2} & \text{if } \mathrm{N}\mathfrak{a} \geq y^{c_{11}}. \end{cases} \tag{4.13}$$

Differentiating the Dirichlet series for $F(s)$ directly, we obtain

$$\frac{(-1)^{k+1}\eta^{k+1}}{k!} \frac{d^k F}{ds^k}(\xi) = \eta \sum_{\mathfrak{a}} \frac{\Lambda_{\pi \otimes \pi'}(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^{1+i\tau}} j_k(\eta \log(\mathrm{N}\mathfrak{a}))$$

Splitting the above sum $\sum$ in concert with the inequality (4.13) and suppressing the summands, we write

$$\sum = \sum_{\mathrm{N}\mathfrak{p} \in (0,y] \cup (y^{c_{11}}, \infty)} + \sum_{\mathfrak{a} \text{ not prime}} + \sum_{y < \mathrm{N}\mathfrak{p} \leq y^{c_{11}}}.$$

We will estimate these three sums separately.

First, note that

$$1 \ll \eta \mathcal{L} \ll \eta \log y \ll M \ll k. \tag{4.14}$$

We use Lemma 4.9 and (4.14) to obtain

$$\begin{aligned}\Big|\eta\sum_{\mathrm{N}\mathfrak{p}\in(0,y]\cup(y^{c_{11}},\infty)}\Big| &\ll \eta(110)^{-k}\Big(\sum_{\mathrm{N}\mathfrak{a}\le y}\frac{\Lambda_{\pi\otimes\pi'}(\mathfrak{a})}{\mathrm{N}\mathfrak{a}}+\sum_{\mathfrak{a}}\frac{\Lambda_{\pi\otimes\pi'}(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^{1+\eta/2}}\Big)\\ &\ll \eta(110)^{-k}\Big(\frac{1}{\eta}+\log y+d'd\log\mathcal{Q}\Big)\\ &\ll (110)^{-k}\Big(1+\eta\log y+\eta\mathcal{L}\Big)\\ &\ll k(110)^{-k}.\end{aligned}$$

If $\eta\le 1/55$, which we may assume, then the identity $\sum_{m\ge 0} j_m(u)=1$ implies that

$$\mathrm{N}\mathfrak{a}^{-1/2}j_k(\eta\log(\mathrm{N}\mathfrak{a}))=(2\eta)^k\mathrm{N}\mathfrak{a}^{-\eta}j_k(\log(\mathrm{N}\mathfrak{a})/2)\le(110)^{-k}\mathrm{N}\mathfrak{a}^{-\eta}.$$

Thus, as above,

$$\Big|\eta\sum_{\mathfrak{a}\text{ not prime}}\Big|\ll\eta(110)^{-k}\sum_{\substack{\mathfrak{a}=\mathfrak{p}^m\\ m\ge 2}}\frac{\Lambda_{\pi\otimes\pi'}(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^{1/2+\eta}}\ll\eta(110)^{-k}\sum_{\mathfrak{a}}\frac{\Lambda_{\pi\otimes\pi'}(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^{1+2\eta}}\ll k(110)^{-k}.$$

as well. Finally, recall that $S_{y,u}(\tau,\pi\otimes\pi')=\sum_{y<\mathrm{N}\mathfrak{p}\le u}\frac{\Lambda_{\pi\otimes\pi'}(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^{1+i\tau}}$. Summation by parts gives us

$$\sum_{y<\mathrm{N}\mathfrak{p}\le y^{c_{11}}}=S_{y,y^{c_{11}}}(\tau,\pi\otimes\pi')j_k(\eta\log y^{c_{11}})-\eta\int_y^{y^{c_{11}}}S_{y,u}(\tau,\pi\otimes\pi')j_k'(\eta\log u)\frac{du}{u}$$

since $S_{y,y}(\tau,\pi\otimes\pi')=0$. Much like above,

$$|\eta S_{y,y^{c_{11}}}(\tau,\pi\otimes\pi')j_k(\eta\log y^{c_{11}})|\ll\eta(110)^{-k}y^{-c_{11}\eta/2}\sum_{\mathrm{N}\mathfrak{p}\le y^{c_{11}}}\frac{\Lambda_{\pi\otimes\pi'}(\mathfrak{p})}{\mathrm{N}\mathfrak{p}}\ll k(110)^{-k}.$$

Therefore, since $|j_k'(u)|=|j_{k-1}(u)-j_k(u)|\le j_{k-1}(u)+j_k(u)\le 1$, we have

$$\Big|\eta\sum_{y<\mathrm{N}\mathfrak{p}\le y^{c_{11}}}\Big|\le\eta^2\int_y^{y^{c_{11}}}|S_{y,u}(\tau,\pi\otimes\pi')|\frac{du}{u}+O(k(110)^{-k}).$$

However, by (4.14) and $\eta \gg \mathcal{L}^{-1}$, we have that if k is sufficiently large, then each term of size $O(k(110)^{-k})$ is at most $\frac{1}{16}(100)^{-(k+1)}$. The lemma follows. □

4.2.3 Zero detection: The proof of Proposition 4.11

We now combine our upper and lower bounds on the derivatives of F to prove Proposition 4.11. Thus, we wish to show that if ρ_0 is a zero satisfying $|\rho_0 - (1 + i\tau)| \leq \eta$, then

$$\frac{y^{c_{10}\eta}}{(\log y)^3} \int_y^{y^{c_{11}}} |S_{y,u}(\tau, \pi \otimes \pi')|^2 \frac{du}{u} \gg 1.$$

Combining Lemmas 4.12 and 4.14, we find that

$$\eta^2 \int_y^{y^{c_{11}}} |S_{y,u}(\tau, \pi \otimes \pi')| \frac{du}{u} \geq \frac{1}{4}(100)^{-(k+1)}.$$

Using (4.14), we have

$$\eta^2 \int_y^{y^{c_{11}}} |S_{y,u}(\tau, \pi \otimes \pi')| \frac{du}{u} \gg y^{-c_{10}\eta/4},$$

where c_{10} is sufficiently large. Multiplying both sides by $y^{-c_{10}\eta/4}$ yields

$$y^{-c_{10}\eta/4}\eta^2 \int_y^{y^{c_{11}}} |S_{y,u}(\tau, \pi \otimes \pi')| \frac{du}{u} \gg y^{-c_{10}\eta/2}.$$

Since $y^{-c_{10}\eta/4}\eta^2 \ll (\log y)^{-2}$, we have

$$\frac{1}{(\log y)^2} \int_y^{y^{c_{11}}} |S_{y,u}(\tau, \pi \otimes \pi')| \frac{du}{u} \gg y^{-c_{10}\eta/2}.$$

Squaring both sides and using the Cauchy-Schwarz inequality yields the proposition.

4.3 Arithmetic consequences

For this section, we assume that π satisfies GRC. Ultimately, we will take π' to either be trivial or $\tilde{\pi}$.

4.3.1 Setup and proof of Theorem 4.4

Let $T > 0$, and define

$$\eta_1(x) = \begin{cases} \sqrt{T}/2 & \text{if } |x| < 1/\sqrt{T}, \\ \sqrt{T}/4 & \text{if } |x| = 1/\sqrt{T}, \\ 0 & \text{if } |x| > 1/\sqrt{T}. \end{cases}$$

Let $\eta_k(x) = (\eta_1 * \eta_{k-1})(x)$ for all $k \geq 2$, where $(f * g)(x) = \int_{\mathbb{R}} f(x-t)g(t)dt$, and

$$\hat{\eta}_k(s) = \int_{-\infty}^{\infty} \eta_k(x) e^{-sx} dx, \qquad s \in \mathbb{C}.$$

Lemma 4.15. *Let $k \geq 3$.*

1. *The function $\eta_k(t)$ is even and is supported on $[-k/\sqrt{T}, k/\sqrt{T}]$. Moreover, $0 \leq \eta_k(t) \leq \sqrt{T}/2$ for all $t \in \mathbb{R}$.*
2. *The function $\hat{\eta}_k(s)$ is entire.*
3. *Uniformly for $|\sigma| \leq \sqrt{T/k}$, $|\hat{\eta}_k(s)| \ll 1$ with an absolute implied constant.*
4. *For every $c \in \mathbb{R}$, we have that if $y > 0$, then $\eta_k(\log y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{\eta}_k(s) y^s ds$. The integral converges absolutely.*

Proof. These classical results are proven in Lemma 3.2 of [Wei83]. □

We use Lemma 4.15 to bound sums of Dirichlet coefficients of $L(s, \pi \otimes \pi', K)$.

Lemma 4.16. *Assume the above notation. If*

$$0 < \Delta < c_{13} \mathcal{Q}^{-2/n_K}, \quad T = \frac{16(2n_K+3)^2}{\Delta^2}, \quad x \geq \Big(\mathcal{Q}\Big(\frac{30 n_K}{\Delta}\Big)^{2n_K}\Big)^{4c_1\mathcal{D}},$$

then

$$\sum_{\mathfrak{p}} \frac{\Lambda_{\pi\otimes\pi'}(\mathfrak{p})}{\mathrm{N}\mathfrak{p}} \eta_{2n_K+3}\Big(\log \frac{xe^{\Delta/4}}{\mathrm{N}\mathfrak{p}}\Big) = r(\pi\otimes\pi') - \sum_{\substack{\rho=\beta+i\gamma \\ \beta\geq 1-\frac{5}{4c_1} \\ |\gamma|\leq T}} \hat{\eta}_{2n_K+3}(\rho-1)(xe^{\Delta/4})^{\rho-1} + O(\mathcal{D}\Delta),$$

where $r(\pi\otimes\pi')$ is the order of the pole at $s=1$ of $L(s,\pi\otimes\pi',K)$ and ρ runs through the nontrivial zeros of $L(s,\pi\otimes\pi',K)$.

Proof. With Δ as in the theorem statement and $xe^{\Delta/4}\geq \mathcal{Q}^{2\mathcal{D}}$, we essentially repeat the proof of [Wei83, Lemma 5.1], *mutatis mutandis*, to find that

$$\sum_{\mathfrak{p}} \frac{\Lambda_{\pi\otimes\pi'}(\mathfrak{p})}{\mathrm{N}\mathfrak{p}} \eta_{2n_K+3}\Big(\log \frac{xe^{\Delta/4}}{\mathrm{N}\mathfrak{p}}\Big) = r(\pi\otimes\pi') - \sum_{\substack{\rho=\beta+i\gamma \\ \beta\geq 1-\frac{5}{4c_1},\ |\gamma|\leq T}} \hat{\eta}_{2n_K+3}(\rho-1)(xe^{\Delta/4})^{\rho-1}$$
$$+ O(((xe^{\Delta/4})^{-\frac{5}{4c_1}} + T^{-(2n_K+3)/2})T\mathcal{L}), \tag{4.15}$$

where $r(\pi\otimes\pi')$ is the order of the pole at $s=1$ of $L(s,\pi\otimes\pi',K)$ and ρ runs through the nontrivial zeros of $L(s,\pi\otimes\pi',K)$. (The proof of [Wei83, Lemma 5.1] proceeds along classical lines.) Let c_{13} be sufficiently small so that $T\geq\Delta^{-2}$. We note that

$$\mathcal{L} = \mathcal{D}\log(\mathcal{Q}T^{n_K}) = \mathcal{D}\log\Big(\mathcal{Q}\Big(\frac{4(2n_K+3)}{\Delta}\Big)^{2n_K}\Big) \leq \frac{1}{4c_1}\log x. \tag{4.16}$$

By our parameters choices, $xe^{\Delta/4}\geq(\mathcal{Q}T^{n_K})^{2c_1\mathcal{D}}$. Thus $(xe^{\Delta/4})^{-\frac{5}{4c_1}} \leq T^{-5\mathcal{D}n_K/2} \leq T^{-\frac{5n_K}{2}} \leq T^{-\frac{2n_K+3}{2}}$. Thus the error in (4.15) is $\ll T^{1-(2n_K+3)/2}\mathcal{D}\mathcal{Q}^{1/2}T^{n_K/2} \ll \mathcal{D}\Delta$. □

We now use Theorem 4.2 and Lemma 4.16 to prove Theorem 4.4.

Proof of Theorem 4.4. The upper bound easily follows from GRC and the prime number theorem, so we prove only the lower bound. Choose $g\in\mathcal{C}$, and let $\mathcal{H}=\langle g\rangle$ be the cyclic group generated by g. Regarding $\mathbf{1}_{\mathcal{C}}(\cdot)$ as a class function on $\mathcal{G}$, we have

$$\mathbf{1}_{\mathcal{C}} = \frac{|\mathcal{C}|}{|\mathcal{G}|}\sum_{\chi\in\widehat{\mathcal{H}}} \bar{\chi}(g)\mathrm{Ind}_{\mathcal{G}}^{\mathcal{H}}\chi.$$

Thus, if we let E be the fixed field of $\mathcal{H}$ and set $\psi = \pi \otimes \tilde{\pi}$, by applying Frobenius reciprocity, we find that, as class functions of the absolute Galois group,

$$\mathrm{tr}(\psi) \cdot \mathbf{1}_{\mathcal{C}} = \frac{|\mathcal{C}|}{|\mathcal{G}|} \sum_{\chi \in \widehat{\mathcal{H}}} \bar{\chi}(g) \cdot \mathrm{tr}(\psi \otimes \mathrm{Ind}_{\mathcal{G}}^{\mathcal{H}} \chi) = \frac{|\mathcal{C}|}{|\mathcal{G}|} \sum_{\chi \in \widehat{\mathcal{H}}} \bar{\chi}(g) \cdot \mathrm{tr}(\psi|_E \otimes \chi),$$

where $\psi|_E$ denotes the restriction of ψ to E. At the level of primes, this implies

$$\sum_{x < \mathrm{N}\mathfrak{a} \leq x+h} \mathbf{1}_{\mathcal{C}}(\mathfrak{a}) \Lambda_{\pi \otimes \tilde{\pi}}(\mathfrak{a}) = \frac{|\mathcal{C}|}{|\mathcal{G}|} \sum_{\chi \in \widehat{\mathcal{H}}} \bar{\chi}(g) \sum_{x < \mathrm{N}\mathfrak{p} \leq x+h} \Lambda_{\psi|_E \otimes \chi}(\mathfrak{p}) + O(\sqrt{x} \log x),$$

where the implied constant depends on M/K and π effectively.

Recall the notation of Theorem 4.2 and Lemma 4.16, with $\Delta = h/x$. We have that $h\sqrt{T} = 4(2n_K + 3)x$, so

$$\begin{aligned} \frac{|\mathcal{C}|}{|\mathcal{G}|} \sum_{\chi \in \widehat{\mathcal{H}}} \bar{\chi}(g) \sum_{x < \mathrm{N}\mathfrak{p} \leq x+h} \Lambda_{\psi|_E \otimes \chi}(\mathfrak{p}) &= \frac{h\sqrt{T}}{4(2n_K+3)x} \frac{|\mathcal{C}|}{|\mathcal{G}|} \sum_{\chi \in \widehat{\mathcal{H}}} \bar{\chi}(g) \sum_{x < \mathrm{N}\mathfrak{p} \leq x+h} \Lambda_{\psi|_E \otimes \chi}(\mathfrak{p}) \\ &\geq \frac{h\sqrt{T}}{4(2n_K+3)} \frac{|\mathcal{C}|}{|\mathcal{G}|} \sum_{\chi \in \widehat{\mathcal{H}}} \bar{\chi}(g) \sum_{x < \mathrm{N}\mathfrak{p} \leq x+h} \frac{\Lambda_{\psi|_E \otimes \chi}(\mathfrak{p})}{\mathrm{N}\mathfrak{p}}. \end{aligned}$$

By Lemma 4.15 Part 1, $0 \leq \eta_{2n_E+3}(t) \leq \sqrt{T}/2$ for all $t \in \mathbb{R}$. Furthermore, when $h \leq x$, the support of $\eta_{2n_E+3}(\log(xe^{\frac{h}{4x}}/\mathrm{N}\mathfrak{p}))$ is $[x, xe^{\frac{h}{2x}}] \subset [x, x+h]$. Thus

$$\mathbf{1}_{[x,x+h]}(\mathrm{N}\mathfrak{p}) \geq \frac{2}{\sqrt{T}} \eta_{2n_E+3}\Big(\log \frac{xe^{\frac{h}{4x}}}{\mathrm{N}\mathfrak{p}} \Big),$$

so by Lemma 4.16,

$$\begin{aligned} &\frac{|\mathcal{C}|}{|\mathcal{G}|} \frac{h\sqrt{T}}{4(2n_E+3)} \sum_{\chi \in \widehat{\mathcal{H}}} \bar{\chi}(g) \sum_{x < \mathrm{N}\mathfrak{p} \leq x+h} \frac{\Lambda_{\psi|_E \otimes \chi}(\mathfrak{p})}{\mathrm{N}\mathfrak{p}} \\ &\geq \frac{|\mathcal{C}|}{|\mathcal{G}|} \frac{h}{2(2n_E+3)} \sum_{\chi \in \widehat{\mathcal{H}}} \bar{\chi}(g) \sum_{\mathrm{N}\mathfrak{p}} \frac{\Lambda_{\psi|_E \otimes \chi}(\mathfrak{p})}{\mathrm{N}\mathfrak{p}} \eta_{2n_E+3}\Big(\log \frac{xe^{\frac{h}{4x}}}{\mathrm{N}\mathfrak{p}} \Big) \\ &= \frac{|\mathcal{C}|}{|\mathcal{G}|} \frac{h}{2(2n_E+3)} \Big(1 - \sum_{\chi \in \widehat{\mathcal{H}}} \bar{\chi}(g) \sum_{\substack{\rho_\chi = \beta_\chi + i\gamma_\chi \neq \beta_1 \\ \beta_\chi \geq 1 - \frac{5}{4c_1},\ |\gamma_\chi| \leq T}} \hat{\eta}_{2n_E+3}(xe^{\frac{h}{4x}})^{\rho_\chi - 1} \Big) + O\Big(\mathcal{D}[M:K] \frac{h^2}{x} \Big) + o(h), \end{aligned}$$

where ρ_χ denotes a nontrivial zero of $L(s,\psi|_E\otimes\chi,E)$ and the $o(1)$ contribution arises from a potential Landau-Siegel zero β_1.

Lemma 4.15 Part 3 and Lemma 4.7 imply that

$$\sum_{\substack{\rho_\chi=\beta_\chi+i\gamma_\chi\neq\beta_1\\ \beta_\chi\geq 1-\frac{5}{4c_1},\ |\gamma_\chi|\leq T}}\hat{\eta}_{2n_E+3}(\rho_\chi-1)(xe^{\frac{h}{4x}})^{\rho_\chi-1}\ll\sum_{\substack{\rho_\chi=\beta_\chi+i\gamma_\chi\neq\beta_1\\ \beta_\chi\geq 1-\frac{5}{4c_1},\ |\gamma_\chi|\leq T}}(xe^{\frac{h}{4x}})^{\beta_\chi-1}$$
$$=\int_{1-\frac{5}{4c_1}}^{1-c_8/\mathcal{L}}(xe^{\frac{h}{4x}})^{\sigma-1}dN_{\pi\otimes\tilde{\pi}}(\sigma,T). \tag{4.17}$$

By Theorem 4.2 and our choice of parameters in Lemma 4.16,

$$N_{(\pi\otimes\tilde{\pi})\otimes\chi}(\sigma,T)\ll d^2T^{c_1\mathcal{D}n_E(1-\sigma)}\leq d^2\Big(\Big(\frac{2(2n_E+3)}{15n_E}\mathcal{Q}^{-\frac{1}{2n_E}}\Big)^{2c_1\mathcal{D}n_E}x^{1/4}\Big)^{1-\sigma}. \tag{4.18}$$

Now, let c_1 be sufficiently large (so that x is sufficiently large), and let $h\geq x^{1-\frac{1}{16c_1\mathcal{D}n_E}}$. Using integration by parts, (4.16), and (4.18), the integral in (4.17) equals

$$(xe^{\frac{h}{4x}})^{-5/4c_1}N_{(\pi\otimes\tilde{\pi})\otimes\chi}\Big(1-\frac{5}{4c_1},T\Big)+\log(xe^{\frac{h}{4x}})\int_{1-\frac{5}{4c_1}}^{1-c_8/\mathcal{L}}(xe^{\frac{h}{4x}})^{\sigma-1}N_{(\pi\otimes\tilde{\pi})\otimes\chi}(\sigma,T)d\sigma$$
$$\ll d^2\log(xe^{\frac{h}{4x}})\int_{1-\frac{5}{4c_1}}^{1-4c_1c_8/\log x}\Big(\frac{xe^{\frac{h}{4x}}}{(\frac{2(2n_E+3)}{15n_E}\mathcal{Q}^{-\frac{1}{2n_E}})^{2c_1\mathcal{D}n_E}x^{1/4}}\Big)^{\sigma-1}d\sigma\ll d^2e^{-3c_1c_8}.$$

After combining the contribution from each of the $[M:E]$ characters χ of $\widehat{\mathcal{H}}$, we conclude that there exists an absolute constant $c_{14}>0$ such that

$$\sum_{x<\mathrm{N}\mathfrak{p}\leq x+h}\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{p})\geq\frac{h}{2(2n_K+3)}(1-c_{14}[M:E]d^2e^{-3c_1c_8}-o(1)).$$

Since $[M:E]\leq[M:K]$ and $n_E\leq n_M$, we increase c_1 so that $c_{14}d^2[M:K]e^{-3c_1c_8}<1$ to obtain desired result. □

4.3.2 The Sato-Tate conjecture

We now address applications to the Sato-Tate conjecture. Thus, we assume that K is a totally real field and that π is a genuine cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$ with trivial central character. Recall that the Sato-Tate conjecture concerns the distribution of the quantities $\lambda_\pi(\mathfrak{p}) = 2\cos\theta_\mathfrak{p}$ as $\mathfrak{p}$ ranges over primes for which $\pi_\mathfrak{p}$ is unramified, where $\theta_\mathfrak{p} \in [0,\pi]$. At each such prime $\mathfrak{p}$, the local factor of the n-th symmetric power L-function is given by

$$L_\mathfrak{p}(s, \mathrm{Sym}^n\pi, K) = \prod_{j=0}^{n}(1 - e^{i\theta_\mathfrak{p}(n-2j)}\mathrm{N}\mathfrak{p}^{-s})^{-1} = \sum_{k=0}^{\infty}\frac{U_n(\cos(k\theta_\mathfrak{p}))}{\mathrm{N}\mathfrak{p}^s},$$

where U_n is the n-th Chebyshev polynomial of the second kind. We observe that $L(s, \mathrm{Sym}^1\pi, K) = L(s,\pi,K)$ and $L(s, \mathrm{Sym}^0\pi, K) = \zeta_K(s)$.

Langlands functoriality implies that $\mathrm{Sym}^n\pi$ is a cuspidal automorphic representation of $\mathrm{GL}_{n+1}(\mathbb{A}_K)$ with trivial central character for all $n \geq 1$, in which case $L(s, \mathrm{Sym}^n\pi, K)$ would have an analytic continuation to the entire complex plane and satisfy a functional equation of the type described in Section 4.1. Unfortunately, the analytic continuation is only known for $n \leq 4$ for all totally real number fields K. If one restricts to the case where $K \cap \mathbb{Q}(e^{2\pi i/35}) = \mathbb{Q}$, then the analytic continuation is known for $n \leq 8$. This poses problems if one wants finer distributional information about the sequence $\{\cos\theta_\mathfrak{p}\}$ than the ineffective equidistribution result of Barnet-Lamb, Geraghty, Harris, and Taylor [BLGHT11].

In Theorem 4.6, our goal is to estimate for $I \subseteq [-1,1]$ the summation

$$\sum_{\substack{x<\mathrm{N}\mathfrak{p}\leq x+h \\ \pi_\mathfrak{p}\text{ unramified}}} \mathbf{1}_I(\cos\theta_\mathfrak{p})\log \mathrm{N}\mathfrak{p} \tag{4.19}$$

where $h \geq x^{1-\delta}$ for some $\delta > 0$. We recall from the introduction that I can be

Sym^n-minorized if there exist $b_0, \ldots, b_n$ with $b_0 > 0$ such that

$$\mathbf{1}_I(t) \geq \sum_{j=0}^{n} b_j U_j(t) \tag{4.20}$$

for all $t \in [-1,1]$. Thus, if I can be Sym^n-minorized, we can obtain a non-trivial lower bound for (4.19) by considering an appropriate linear combination of the logarithmic derivatives of $L(s, \mathrm{Sym}^j\pi, K)$ for $j \leq n$. We now prove Theorem 4.6.

Proof of Theorem 4.6. It suffices to prove the second part. Let K be a totally real number field, and let π be a genuine automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$ with trivial central character. Suppose that $I \subset [-1,1]$ can be Sym^n-minorized and that $\mathrm{Sym}^j\pi$ is an automorphic representation of $\mathrm{GL}_{j+1}(\mathbb{A}_K)$ with trivial central character for all $j \leq 2n$. In this setting, $\mathcal{Q} = \mathfrak{q}(\mathrm{Sym}^n\pi)$ and $\mathcal{D} = (n+1)^4$.

Recall the notation and setup of Lemma 4.15, Lemma 4.16, and the proof of Theorem 4.4; with $\mathrm{Sym}^j\pi$ in place of $\pi \otimes \pi'$, choose T, x, and $\Delta = h/x$ as in Lemma 4.16. We have that

$$\begin{aligned}\sum_{x<\mathrm{N}\mathfrak{p}\leq x+h} \mathbf{1}_I(\cos\theta_{\mathfrak{p}})\log \mathrm{N}\mathfrak{p} &= \frac{h\sqrt{T}}{4(2n_K+3)} \sum_{x<\mathrm{N}\mathfrak{p}\leq x+h} \frac{\mathbf{1}_I(\cos\theta_{\mathfrak{p}})\log\mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} \\ &\geq \frac{h}{2(2n_K+3)} \sum_{j=0}^{n} b_j \sum_{\mathfrak{p}} \frac{U_j(\cos\theta_{\mathfrak{p}})\log\mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} \eta_{2n_K+3}\Big(\log\frac{xe^{\frac{h}{4x}}}{\mathrm{N}\mathfrak{p}}\Big).\end{aligned}$$

Since $x \geq \mathcal{Q}$ is larger than the norm of any ramified prime, $U_j(\cos\theta_{\mathfrak{p}})\log\mathrm{N}\mathfrak{p} = \Lambda_{\mathrm{Sym}^j\pi}(\mathfrak{p})$ for all $x < \mathrm{N}\mathfrak{p} \leq x+h$. Thus by Lemma 4.16,

$$\begin{aligned}&\frac{h}{2(2n_K+3)} \sum_{j=0}^{n} b_j \sum_{\mathfrak{p}} \frac{U_j(\cos\theta_{\mathfrak{p}})\log\mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} \eta_{2n_K+3}\Big(\log\frac{xe^{\frac{h}{4x}}}{\mathrm{N}\mathfrak{p}}\Big) \\ &= \frac{h}{2(2n_K+3)}\Big(b_0 - \sum_{j=0}^{n} b_j \Big(\sum_{\substack{\rho_j=\beta_j+i\gamma_j \\ \beta_j \geq 1-\frac{5}{4c_1},\ |\gamma_j|\leq T}} \hat{\eta}_{2n_K+3}(\rho_j-1)(xe^{\frac{h}{4x}})^{\rho_j-1} + O(n^4\Delta)\Big)\Big),\end{aligned}$$

where ρ_j is a nontrivial zero of $L(s, \mathrm{Sym}^j\pi, K)$ and $B = \max_{j\leq n}|b_j|/|b_0|$. Recall that since we are assuming that $L(s, \mathrm{Sym}^j\pi, K)$ is automorphic for all $j \leq 2n$, it follows from Section 4 of [HR95] that there are no Landau-Siegel zeros in any of the sums over nontrivial zeros.

By choosing $h \geq x^{1-\frac{1}{16c_1(n+1)^4 n_K}}$ and repeating the proof of Theorem 4.4, *mutatis mutandis*, there exists $c_{15} > 0$ such that

$$\sum_{x < \mathrm{N}\mathfrak{p} \leq x+h} \mathbf{1}_I(\cos\theta_{\mathfrak{p}}) \log \mathrm{N}\mathfrak{p} \geq b_0 \frac{h}{2(2n_K+3)}(1 - Bc_{15}n^6(e^{-3c_1c_8} + h/x)).$$

After increasing c_1 so that $Bn^6 c_{15} e^{-3c_1c_8} < 1$, we obtain the claimed lower bound. Furthermore, choosing h as small as we are permitted, our lower bound holds when $x \geq (\mathcal{Q}(\frac{30n_K}{h/x})^{2n_K})^{4c_1(n+1)^4} = (30n_K \mathcal{Q}^{\frac{1}{2n_K}})^{8c_1(n+1)^4 n_K}\sqrt{x}$. Solving for x gives the claimed upper bound for the least value of x for which our lower bound holds. □

4.3.3 Proof of Theorem 4.5

In what follows, all implied constants depend on $\mathfrak{q}(\pi)$.

Proof of Theorem 4.5. Our proof will handle the case where π is a self-dual cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with trivial central character; the case where it is the symmetric square of such a representation is proven similarly.

Let $Q^5 = T \leq x^{\frac{1}{512c_2}}$, and suppose that $x \leq hQ$ and $\log x \leq (\log Q)^2$. Let χ be a primitive Dirichlet character modulo $q \leq Q$. By (4.5) and the assumption that π has trivial central character, we have that

$$L(s, (\pi \otimes \tilde{\pi}) \otimes \chi, \mathbb{Q}) = L(s, \chi, \mathbb{Q}) L(s, \mathrm{Sym}^2\pi \otimes \chi, \mathbb{Q}). \tag{4.21}$$

Furthermore, $L(s, \mathrm{Sym}^2\pi \otimes \chi, \mathbb{Q})$ has no Siegel zero (cf. Ramakrishnan and Wang [RW03, Theorem A]), so $L(s, (\pi \otimes \tilde{\pi}) \otimes \chi, \mathbb{Q})$ has a Siegel zero if and only if it is inherited from $L(s, \chi, \mathbb{Q})$.

By arguments similar to those in [Gal70, Section 4], we have that

$$\sum_{x<n\le x+h} \Lambda_{\pi\otimes\tilde{\pi}}(n)\chi(n) - \delta(\chi)h + h\xi^{\beta_1-1} \ll h\Big(\sum_{|\gamma|\le T} x^{\beta-1} + Q^2/T\Big),$$

where the summation on the right-hand side is over the nontrivial, non-exceptional zeros of $L(s,(\pi\otimes\tilde{\pi})\otimes\chi,\mathbb{Q})$. Thus

$$\sum_{q\le Q}\ \sideset{}{^\star}\sum_{\chi \bmod q}\Big|\sum_{x<n\le x+h}\Lambda_{\pi\otimes\tilde{\pi}}(n)\chi(n) - \delta(\chi)h + \delta_{q,*}(\chi)h\xi^{\beta_1-1}\Big|$$
$$\ll h\Big(\sum_{q\le Q}\ \sideset{}{^\star}\sum_{\chi \bmod q}\ \sum_{|\gamma|\le T} x^{\beta-1} + Q^4/T\Big). \tag{4.22}$$

Using the factorization (4.21), the triple sum in (4.22) is bounded by

$$\log x\int_{\frac{1}{2}}^{1} x^{\sigma-1}\sum_{q\le Q}\ \sideset{}{^\star}\sum_{\chi \bmod q} N_\chi(\sigma,T)d\sigma + x^{-1/2}\sum_{q\le Q}\ \sideset{}{^\star}\sum_{\chi \bmod q} N_\chi(1/2,T)$$
$$+\log x\int_{\frac{1}{2}}^{1} x^{\sigma-1}\sum_{q\le Q}\ \sideset{}{^\star}\sum_{\chi \bmod q} N_{\mathrm{Sym}^2\pi\otimes\chi}(\sigma,T)d\sigma + x^{-1/2}\sum_{q\le Q}\ \sideset{}{^\star}\sum_{\chi \bmod q} N_{\mathrm{Sym}^2\pi\otimes\chi}(1/2,T),$$

Using Theorem 4.3 and recalling our choice of T, the triple sum is now bounded by

$$\log x\int_{1/2}^{1-c_8/\mathcal{L}} x^{(\sigma-1)/2}d\sigma + x^{-1/2} \ll x^{-c_8/2\mathcal{L}'} + x^{-1/4},$$

where $\mathcal{L}' = 256\log(\mathfrak{q}(\mathrm{Sym}^2\pi)QT)$. Since $T = Q^5$ and $\log\mathfrak{q}(\mathrm{Sym}^2\pi)\asymp\log\mathfrak{q}(\pi)$ (with an absolute implied constant), the right-hand side of (4.22) is bounded by the quantity claimed in the statement of the theorem. □

4.4 Sym^n-minorants

We close with an easy lemma on Sym^n-minorants which explicitly classifies the intervals which can be Sym^4-minorized.

Lemma 4.17. *Let* $\beta_0 = \frac{1+\sqrt{7}}{6} = 0.6076\ldots$ *and* $\beta_1 = \frac{-1+\sqrt{7}}{6} = 0.2742\ldots$. *The* *interval* $[a,b] \subseteq [-1,1]$ *can be* Sym^4*-minorized if and only if it satisfies one of the following conditions:*

1. $a = -1$ *and* $b > -\beta_0$*,*

2. $-1 < a \le -\beta_0$ *and* $b > \frac{a+\sqrt{16a^4-11a^2+2}}{2(1-4a^2)}$*,*

3. $-\beta_0 \le a \le -\beta_1$ *and* $b > \frac{-1}{6a}$*,*

4. $-\beta_1 \le a < \beta_1$ *and* $b > \frac{a+\sqrt{16a^4-11a^2+2}}{2(1-4a^2)}$*, and*

5. $\beta_1 \le a < \beta_0$ *and* $b = 1$*.*

Proof. We begin with sufficiency. For each case, we list a polynomial $F(x)$ which, for $x \in [-1,1]$, is positive only if $x \in [a,b]$. We then compute $b_0(F) := \int_{-1}^{1} F d\mu_{ST}$ and verify that it is positive. This is sufficient, since any such $F(x)$ can be scaled to minorize the indicator function.

1. $F(x) = (x-1)(x-b)(x-\beta_1)^2$ and $b_0(F) = (b+\beta_0)(\frac{14+\sqrt{7}}{36})$.

2. $F(x) = -(x-a)(x-b)\left(x + \frac{a+b}{4ab+1}\right)^2$ and $b_0(F) = \frac{(1-4a^2)b^2-ab+a^2-1/2}{4(4ab+1)}$.

3. $F(x) = (x-1)(x+1)(x-a)(x-b)$ and $b_0(F) = -\frac{3}{4}(ab+\frac{1}{6})$.

4. $F(x) = -(x-a)(x-b)\left(x + \frac{a+b}{4ab+1}\right)^2$.

5. $F(x) = (x+1)(x-a)(x+\beta_1)^2$ and $b_0(F) = (\beta_0 - a)(\frac{14+\sqrt{7}}{36})$.

The proof of necessity necessarily involves tedious casework, which we omit. Let us say only that we consider polynomials $F(x)$, ordered by degree, the number of real roots, and the placement of those roots relative to $a, b, 1$, and -1, and in each case we determine conditions under which $b_0(F) > 0$. $\square$

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