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Ramsey and Turán-Type Theorems for Hypergraphs

By

Vindya V. Bhat
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Vojtěch Rödl, Ph.D.
Advisor

Dwight Duffus, Ph.D.
Committee Member

Ronald Gould, Ph.D.
Committee Member

Accepted:

Lisa A. Tedesco, Ph.D.
Dean of the James T. Laney School of Graduate Studies

Date

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Vindya V. Bhat
M.A. Statistics, Columbia University, 2007
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A dissertation submitted to the Faculty of the Graduate School
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Abstract

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This dissertation includes Ramsey and Turán-type results. Both topics involve finding substructures within hypergraphs under certain conditions.

Ramsey-type results:
The Induced Ramsey Theorem (1975) states that for $c, r \geq 2$ and every r-graph G, there exists an r-graph H such that every c-coloring of the edges of H contains a monochromatic induced copy of G. A natural question to ask is what other subgraphs F (besides edges) of G can be partitioned and have the F-Ramsey property. We give results on the F-Ramsey property of two types of objects: hypergraphs or partial Steiner systems. We find that while the restrictions on the Ramsey properties of hypergraphs are lifted by any linear ordering of the vertex set, the Ramsey properties for partial Steiner systems (with vertex set linearly ordered or unordered) are quite restricted.

Turán-type results:
Turán's Theorem (1941) states that for $1 < k \leq n$, every graph G on n vertices not containing a K_{k+1} has at most $|E(T_k(n))|$ edges, where $T_k(n)$ is the graph on n vertices obtained by partitioning n vertices into k classes of each size $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$ and joining two vertices if and only if they are in two different classes. In 1946, Erdős and Stone showed that any sufficiently large dense graph will contain $T_k(n)$. Nearly 75 years later, in spite of considerable interest and effort, no generalization of Turán's Theorem or Erdős-Stone Theorem for hypergraphs is known. Instead, we consider a variant of this question where we restrict to quasi-random hypergraphs and prove some partial results in this direction.

All results presented in this dissertation are joint work with Vojtěch Rödl and some results are joint work with Jaroslav Nešetřil.

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To Daana

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Chapter 1

Introduction

This dissertation includes Ramsey and Turán-type results, two vibrant topics in the discussion of combinatorics. Both topics involve finding substructures within hypergraphs under certain conditions.

1.1 Ramsey theory and the Ramsey property

Ramsey theory spans many branches of mathematics. Named after British mathematician and philosopher Frank P. Ramsey, Ramsey theory is the study of partitions of discrete objects. Such objects may include graphs, hypergraphs, integers, vector spaces, partially ordered sets, or points in Euclidean space. For a summary of such results, see [15]. It was founded on three theorems from various mathematics disciplines: Ramsey's theorem (1930) was a mere lemma for a result of a logician, Schur's theorem (1916) arose from a number theorist's attempt to prove Fermat's Last Theorem and van der Waerden's theorem (1927) was a result of an algebraic geometer. Paul Erdős was the first to systematically study Ramsey theory beginning in the mid 20th century, and since then it has been an active area of research in combinatorics.

Ramsey theory is also applied to various disciplines of mathematics. In 1931, logician Kurt Gödel proved theorems that were true for arithmetic in the natural numbers, but not provable in Peano arithmetic, using Ramsey theory. Nearly 50 years later, Jeff Paris and Leo Harrington used a statement from Ramsey theory to give the first concrete example in which Gödel's First Incompleteness theorem holds [25]. While Ramsey theory is intellectually interesting, it also has real-world applications in communications, information retrieval and decision making [30].

An underlying idea of many results in Ramsey theory is to find a well organized subobject in a chaotic combinatorial structure. Pick any discrete object G. Let F and H be such that $F \subseteq G \subseteq H$. A classical statement in Ramsey theory may read as follows: "If H is rich enough, then no matter how its subobjects F are partitioned into c parts, it is possible to find the given object G all of whose F-subobjects are completely contained in one of the parts." We now consider what is likely the simplest example to illustrate this abstract concept:

Example 1.1.1. *At a party of six people, you will always find either three people who are acquaintances or three people who are strangers.*

In the example above, the structure is the six people at the party and their relations to one another. The subobjects are pairs of people in which each pair is classified as acquaintance or stranger, and thus there are two partition classes. The object, what we are assured to find, is the three people such that all three possible pairings are in the same partition class.

We now consider an easy proof from graph theory to show that Example 1.1.1 is, in fact, true. Consider a complete graph on six vertices. We will add colors red and blue to the edges to depict the relation of acquaintance or stranger, respectively. By the Pigeon Hole Principle, at least three edges from one vertex have the same color, say red. If any pair of these three incident vertices are acquaintances with each other then we get a red triangle. If not, then we get a blue triangle. The monochromatic triangle represents three people in which all three pairs between them are either acquaintances (red) or strangers (blue). See Figure 1.1.

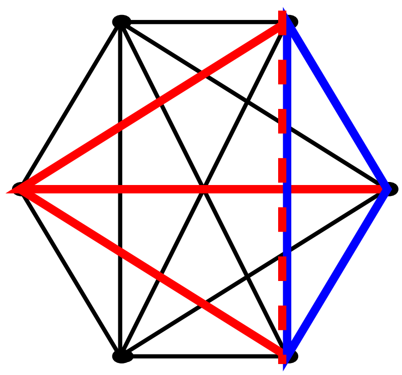


Figure 1.1: Monochromatic triangle

For a positive integer n, let K_n denote the complete graph on n vertices. Example 1.1.1 was generalized in Ramsey's theorem, restated in a graph theoretic context as follows:

Theorem 1.1.2 (Ramsey's Theorem, [28])**.** *For all integers c and ℓ, there exists an integer $n = n(c, \ell)$ such that every c-coloring of the edges of K_n yields a monochromatic copy of K_ℓ.*

In reference to Example 1.1.1, F is an edge, G is a graph triangle, and H is the complete graph on six vertices colored with $c = 2$ colors.

1.2 Ramsey-type results

Theorem 1.1.2 may be further generalized to partition subobjects other than edges, which we will now discuss.

Let $\mathscr{C}$ be a class of objects with specified definitions of both subobjects and isomorphism between objects. For $\mathcal{F}, \mathcal{G} \in \mathscr{C}$ we denote by $\binom{\mathcal{G}}{\mathcal{F}}$ the set of all subobjects of $\mathcal{G}$ isomorphic to $\mathcal{F}$. We say that the class $\mathscr{C}$ has the *$\mathcal{F}$-Ramsey property* if for every $\mathcal{G} \in \mathscr{C}$ there exists $\mathcal{H} \in \mathscr{C}$ such that any red-blue coloring of $\binom{\mathcal{H}}{\mathcal{F}}$ yields $\widetilde{\mathcal{G}} \in \binom{\mathcal{H}}{\mathcal{G}}$ such that $\binom{\widetilde{\mathcal{G}}}{\mathcal{F}}$ is monochromatic.

It is a natural question to ask for what types of objects $\mathcal{G}$ (other than graphs) and which subobjects $\mathcal{F}$ (other than edges) of $\mathcal{G}$ can be partitioned and have the $\mathcal{F}$-Ramsey property.

We shall assume the reader is familiar with the basic definitions and concepts of graph theory [14] and the probabilistic method [3]. Definitions of central importance will be introduced here and in the relevant chapter. Theorems that are stated here but are proved in subsequent chapters will appear with numbering from the subsequent chapter.

Necessary and sufficient conditions of if and only if statements will be presented as positive and negative results, respectively.

1.2.1 Objects and results

We study the $\mathcal{F}$-Ramsey property of two types of objects: hypergraphs and Steiner systems.

1.2.1.1 Hypergraphs

For fixed $r \geq 2$, an *r-graph* $\mathcal{G} = (V, \mathcal{E})$ is an r-uniform hypergraph with vertex set V and edge set $\mathcal{E} \subseteq \binom{V}{r}$. Let $\mathcal{G}$ and $\mathcal{H}$ be r-graphs. We say that $\widetilde{\mathcal{G}}$ is an *induced copy* of $\mathcal{G}$ in $\mathcal{H}$ if $V(\widetilde{\mathcal{G}}) \subset V(\mathcal{H})$, $\mathcal{E}(\widetilde{\mathcal{G}}) = \mathcal{E}(\mathcal{H}) \cap \binom{V(\widetilde{\mathcal{G}})}{r}$, and $\widetilde{\mathcal{G}} \cong \mathcal{G}$. An injective mapping $\phi \colon V(\mathcal{G}) \longrightarrow V(\mathcal{H})$ is an *induced embedding* of $\mathcal{G}$ in $\mathcal{H}$ if ϕ is an isomorphism between $\mathcal{G}$ and $\widetilde{\mathcal{G}} = (\phi(V(\mathcal{G})), \mathcal{E}(\mathcal{H}) \cap \binom{\phi(V(\mathcal{G}))}{r})$, an induced copy of $\mathcal{G}$ in $\mathcal{H}$.

In [1, 23] the following extension of Ramsey's theorem was proved:

Theorem 1.2.1 (Induced Ramsey Theorem,[1, 23])**.** *Let $c, r \geq 2$. For any r-graph $\mathcal{G}$ there exists an r-graph $\mathcal{H}$ with the property that any c-coloring of the edges of $\mathcal{H}$ yields a monochromatic induced copy of $\mathcal{G}$.*

In other words, Theorem 1.2.1 states that the class of all r-graphs and induced embeddings has the *edge-Ramsey property* [22]. For $k \geq r \geq 2$, let $\mathcal{K}_k$ denote the complete r-graph on k vertices and let $\overline{\mathcal{K}_k}$ denote its complement, an independent set on k vertices. The $\mathcal{F}$-Ramsey property of r-graphs was generalized in [1, 23] and is restated as follows:

Theorem 2.1.1. *For any $k \geq r \geq 2$, the class of r-graphs has the $\mathcal{F}$-Ramsey property if and only if $\mathcal{F} = \mathcal{K}_k$ or $\mathcal{F} = \overline{\mathcal{K}_k}$.*

In Chapter 2, we give a proof for Theorem 2.1.1. We give a short proof of the positive result of Theorem 2.1.1 based on a direct application of the Hales-Jewett theorem. A previous proof of the positive result of Theorem 2.1.1 was given based on a partite construction and a partite lemma (see [26], section 2). The proof we present in this dissertation follows directly by an appropriate adaptation of this lemma and avoids the use of the partite construction. The negative result for Theorem 2.1.1 was proved for graphs ($r = 2$) in [27] and we extend the result to hypergraphs via methods in [24, 27]. The proof of the negative result for Theorem 2.1.1 is based on two facts regarding uniform hypergraphs and a probabilistic counting argument first considered in [24].

Define ordered r-graphs as the class of all r-graphs with linearly ordered vertex sets where isomorphism is also order preserving. A class $\mathscr{C}$ is called *Ramsey* if it has the $\mathcal{F}$-Ramsey property for any $\mathcal{F} \in \mathscr{C}$. The following result was also proved in [24, 27]:

Theorem 1.2.2. *For $k \geq r \geq 2$, the class of ordered r-graphs is a Ramsey class.*

A proof of Theorem 1.2.2 is not included in this dissertation. For a detailed proof of this result, see [24, 27].

1.2.1.2 Steiner Systems

For fixed $r > t$, a *partial* (or *incomplete*) *Steiner* (r,t)*-system* $\mathcal{G} = (V, \mathcal{E})$ is an r-graph that has the property that every t-element set is contained in at most one edge of $\mathcal{G}$. If every t-element set is contained in precisely one r-element set we will refer to the Steiner (r,t)-system as *complete*. In fact, the solution to a 160 year old conjecture of Steiner regarding the characterization of the existence of complete Steiner systems was recently established in [18].

Let $\mathcal{S}(r,t)$ be the class of all partial Steiner (r,t)-systems with subobjects being induced copies of a subgraph $\mathcal{G} \in \mathcal{S}(r,t)$. A result of Nešetřil and Rödl in [26] showed that a partial Steiner system has the edge-Ramsey property. Here is a restatement:

Theorem 3.1.1([33])**.** *Let $c \geq 2$, $r > t$. For any $\mathcal{G} \in \mathcal{S}(r,t)$ there exists an $\mathcal{H} \in \mathcal{S}(r,t)$ with the property that any c-coloring of the edges of $\mathcal{H}$ yields a monochromatic induced copy of $\mathcal{G}$.*

In Chapter 3, we generalize Theorem 3.1.1 as follows:

Theorem 3.1.3. *The class $\mathcal{S}(r,t)$ has the $\mathcal{F}$-Ramsey property if and only if $\mathcal{F}$ is an edge or $|V(\mathcal{F})| < t$.*

An r-graph $\mathcal{G}$ has the $\mathcal{F}$-*union property* if and only if every edge of $\mathcal{G}$ is in precisely one copy of $\mathcal{F}$. Let $\mathcal{S}_<(r,t)$ be the class of all partial Steiner (r,t)-systems with linearly ordered vertex sets where isomorphism is also order preserving. For ordered partial Steiner (r,t)-systems $\mathcal{G}, \mathcal{H} \in \mathcal{S}_<(r,t)$, let $\binom{\mathcal{H}}{\mathcal{G}}$ denote the set of subobjects of $\mathcal{H}$ isomorphic to $\mathcal{G}$, that is, induced copies of $\mathcal{G}$ in $\mathcal{H}$. If we consider $\mathcal{G} \in \mathcal{S}_<(r,t)$ with the $\mathcal{F}$-*union* property, we may slightly generalize the result in [26] as follows:

Theorem 3.1.5. *For integers $c \geq 2$ and $r > t$ let $\mathcal{F} \in \mathcal{S}_<(r,t)$ and let $\mathcal{G} \in \mathcal{S}_<(r,t)$ have the $\mathcal{F}$-union property. Then there exists $\mathcal{H}$ which is $\mathcal{F}$-Ramsey for $\mathcal{G}$ if and only if $|V(\mathcal{F})| < t$, $\mathcal{F}$ is an edge or $\mathcal{F}$ is any (other) complete Steiner (r,t)-system.*

We give a proof of the positive results of Theorem 3.1.3 and Theorem 3.1.5 in Chapter 3 based on a Ramsey family result of J. Spencer [33]. A direct proof using an adaptation of the partite lemma and the partite construction described in [24, 26] is detailed in Chapter 4. A proof of the negative results of Theorem 3.1.3 and Theorem 3.1.5 are also included in Chapter 3.

We conjecture that the characterization of Theorem 3.1.5 holds without assuming the $\mathcal{F}$-union property. Further, a conjecture for a Ramsey class for Steiner systems is discussed in Section 3.4 of Chapter 3.

1.3 Turán-type results

Motivated by Ramsey's theorem, P. Turán posed the following question in 1940:

What is the maximum number of edges a graph G on n vertices can have without containing a K_{k+1}?

If we partition n vertices into k classes each of size $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$ and join two vertices if and only if they are in different classes, we obtain a k-chromatic graph not containing K_{k+1}. This graph is called the *Turán graph* on n vertices and is denoted by $T_k(n)$. Turán proved the following:

Theorem 1.3.1 (Turán's Theorem, [34]). *Given $1 < k \leq n$, every graph G on n vertices not containing a K_{k+1} has at most $|E(T_k(n))|$ edges.*

The *density* of an n-vertex graph G is defined as

$$d(G) = \frac{|E(G)|}{\binom{n}{2}}.$$

Clearly, $d(T_k(n)) = 1 - \frac{1}{k} + o(1)$ where $o(1) \to 0$ as $n \to \infty$. In 1946, Erdős and Stone showed that any sufficiently large dense graph will contain $T_k(n)$:

Theorem 1.3.2 (Erdős-Stone Theorem, [11])**.** *Let $\epsilon > 0$, $k \geq 2$ and $n \geq k$. Let G be a graph on N vertices with $d(G) \geq 1 - \frac{1}{k} + \epsilon$. If N is sufficiently large, then G contains a subgraph isomorphic to $T_k(n)$.*

1.3.1 Results

Nearly 75 years after Turán's theorem and 60 years after the Erdős-Stone theorem, there are no corresponding results for hypergraphs.

The following result of Erdős describing the behavior of r-graphs near density 0 was an early attempt to find a corresponding Erdős-Stone theorem for hypergraphs. In 1964, he proved that for any $\alpha > 0$, an r-graph $\mathcal{G}$ with $n \geq n_0(\alpha, r)$ vertices and $\alpha\binom{n}{r}$ edges contains a large complete r-equipartite subgraph. This implies that any sufficiently large $\mathcal{G}$ with density $\alpha > 0$ contains a large subgraph with density at least $r!/r^r$.

In Chapter 5 we present a similar problem for r-graphs $\mathcal{Q}$ with a weak quasi-random property (i.e. with edges uniformly distributed over the sufficiently large subsets of vertices). We prove that any sufficiently large quasi-random r-graph $\mathcal{Q}$ with density $\alpha > 0$ contains a large subgraph with density at least $\frac{(r-1)!}{r^{r-1}-1}$. We define this density of the largest subgraph of $\mathcal{Q}$ as the upper density of $\mathcal{Q}$, denoted by $\overline{d}(\mathcal{Q})$. In particular, for $r = 3$, any sufficiently large such $\mathcal{Q}$ has upper density at least $\frac{1}{4}$, which is the best possible lower bound.

We now state the result formally:

Theorem 4.1.3. *For a sequence $\mathcal{Q}$ of quasi-random r-graphs with $d(\mathcal{Q}) > 0$,*

(i) $\overline{d}(\mathcal{Q}) \geq \frac{(r-1)!}{r^{r-1}-1}$ *and*

(ii) when $r = 3$ there exists a quasi-random sequence of 3-graphs with $\overline{d}(\mathcal{Q}) = \frac{1}{4}$.

A consequence of Theorem 4.1.3 relates to jumps. We define jumps for quasi-random sequences of r-graphs and our result implies that every number between 0 and $\frac{(r-1)!}{r^{r-1}-1}$ is a jump for quasi-random r-graphs. For $r = 3$ this interval can be improved based on a recent result of Glebov, Král' and Volec [13]. We prove that every number in [0, 0.3192) is a jump for quasi-random 3-graphs.

Chapter 2

Ramsey Properties for Hypergraphs

2.1 Introduction

In this chapter, we prove Theorem 2.1.1, restated as follows:

Theorem 2.1.1. *For $k \geq r \geq 2$, the class of r-graphs has the $\mathcal{F}$-Ramsey property if and only if $\mathcal{F} = \mathcal{K}_k$ or $\mathcal{F} = \overline{\mathcal{K}_k}$.*

The positive result of Theorem 2.1.1 was originally proved in [1] and [23]. In Section 2.2, we give another proof of this result. We note that if $\mathcal{K}_k$ is replaced by some other subgraphs $\mathcal{F}$, $\mathcal{F} \neq \mathcal{K}_k$ or $\mathcal{F} \neq \overline{\mathcal{K}_k}$, then this fails to be true. This was proved for graphs in [27] and can be extended to hypergraphs via the methods in [27] and [24] as shown in Section 2.3. We also note that setting $k = 2$ in Theorem 2.2.1, that is, $F = K_2$, an edge, one obtains the induced Ramsey theorem stated in Chapter 1. For a history of other similar results, see [4], [5], [7], and [31].

2.2 Positive result

The positive result of Theorem 2.1.1 may be restated as follows:

Theorem 2.2.1. *For $c, k, r \geq 2$ and every r-graph $\mathcal{G}$, there exists an r-graph $\mathcal{H}$ such that every c-coloring of $\binom{\mathcal{H}}{\mathcal{K}_k}$ contains an induced copy of $\mathcal{G}$ in which each member of $\binom{\mathcal{G}}{\mathcal{K}_k}$ is monochromatic.*

The proof of the positive part of Theorem 2.2.1 we present here shows that the result is directly implied by the Hales-Jewett theorem, which we will now recall.

Given n and q, the n-dimensional *Hales-Jewett cube* over the alphabet $[q] = \{1, 2, \ldots, q\}$ is defined as $HJC(n, q) = \{f : [n] \to [q]\}$. In particular, 1-dimensional subcubes of $HJC(n, q)$ are known as lines. Formally, the set $\mathcal{L} \subset HJC(n, q)$ is called a *line* if there exists a non-empty subset $M \subset [n]$ and $g : [n] - M \to [q]$ such that

$$\mathcal{L} = \{f = g \cup s_M : 1 \leq s \leq q\},$$

where $s_M : M \to [q]$ is defined by $s_M(x) = s$ for all $x \in M$. We call M the set of *moving coordinates.*

With this terminology the Hales-Jewett theorem [16] can be stated.

Theorem 2.2.2 (Hales-Jewett Theorem)**.** *For every c, q, there exists $n = n(c, q)$ such that for every c-coloring of $HJC(n, q)$ there exists a monochromatic line.*

A previous proof of the positive part of Theorem 2.2.1 was given based on a partite construction and a partite lemma (see [26], section 2), where the Hales-Jewett theorem was applied to edges. The proof we present in this thesis follows directly by an appropriate adaptation of this lemma and avoids the use of the partite construction. In our proof, the Hales-Jewett theorem is applied to color patterns of cliques which we artificially introduce. This allows us to obtain the result directly without repeated applications of the partite lemma.

Before we begin the proof, we shall introduce some notation. An r-graph $\mathcal{G}$ is *m-partite* if $V(\mathcal{G}) = V$ is divided into m classes $V^1, \ldots, V^m$ such that $V = V^1 \cup \cdots \cup V^m$ and all edges $E \in \mathcal{E}(\mathcal{G})$ are *crossing*, i.e. $|E \cap V^i| \leq 1$ for $1 \leq i \leq m$. $\mathcal{K}_m$ and m-clique are used interchangeably to denote the complete r-graph on m vertices.

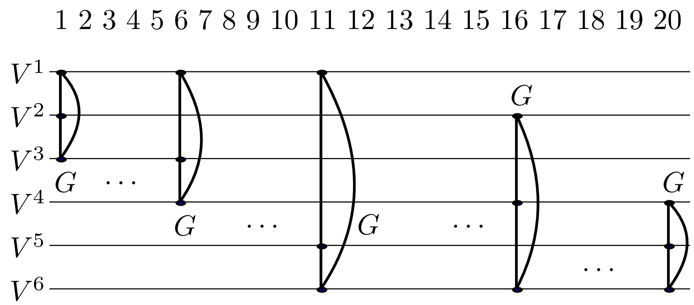
Proof of Theorem 2.2.1. Let $\mathcal{G}$ be the given r-graph on l vertices and let $m = R_r(l; c)$ be the r-graph-Ramsey number of $\mathcal{K}_l$ for c colors. Hence, any c-coloring of the edges of $\mathcal{K}_m$ yields a monochromatic copy of $\mathcal{K}_l$, and thus also of $\mathcal{G}$.

First we construct an auxiliary m-partite r-graph $\mathcal{P}$ as follows:

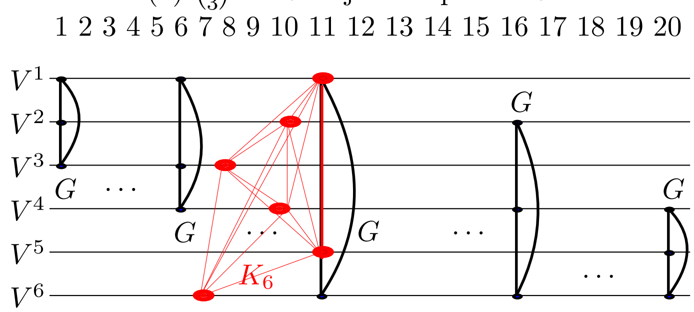
Consider $\binom{m}{l}$ vertex disjoint copies $\mathcal{G}^1, \ldots, \mathcal{G}^{\binom{m}{l}}$ of $\mathcal{G}$ on m-partite vertex sets V^i for $1 \leq i \leq m$ which are placed in such a way that the vertices of each copy of $\mathcal{G}$ are in distinct l-tuples of partite sets V^i and, further, the copies of $\mathcal{G}$ are disjoint. Now for each $\mathcal{G}^t$, $1 \leq t \leq \binom{m}{l}$, and every copy $\widetilde{\mathcal{K}_k}$ of $\mathcal{K}_k \subset \mathcal{G}^t$, we extend $\widetilde{\mathcal{K}_k}$ to a copy $\mathcal{K}_m$, which we denote by $\mathcal{K}_m(\widetilde{\mathcal{K}_k}, t)$, in such a way that if $\widetilde{\mathcal{K}_k}$ and $\widetilde{\widetilde{\mathcal{K}_k}}$ are copies of $\mathcal{G}^{t_1}$ and $\mathcal{G}^{t_2}$, respectively,

$$V(\mathcal{K}_m(\widetilde{\mathcal{K}_k}, t_1)) \cap V(\mathcal{K}_m(\widetilde{\widetilde{\mathcal{K}_k}}, t_2)) = V(\widetilde{\mathcal{K}_k}) \cap V(\widetilde{\widetilde{\mathcal{K}_k}}).$$

Thus, in particular, if $t_1 \neq t_2$, then two such copies of $\mathcal{K}_m$ are vertex disjoint. Note that we have $q = \binom{m}{l}|\binom{\mathcal{G}}{\mathcal{K}_k}|$ copies of $\mathcal{K}_m$. This completes the construction of m-partite r-graph $\mathcal{P}$. See Figure 2.1 for a display of the construction of P with G being a 2-graph triangle and $k = 2$.



(a) $\binom{6}{3} = 20$ disjoint copies of G



(b) Edge of a copy of G extended to K_6

Figure 2.1: Construction of P for $k, r = 2$, $G = K_3$, $l = 3$, $m = 6$ containing $q = 60$ copies of K_6

With c as in Theorem 2.2.1 and m, q given above, let $n = n(c^{\binom{m}{k}}, q)$ be the number ensured by Theorem 2.2.2. From $\mathcal{P}$, we define an m-partite r-graph $\mathcal{P}^n$ as follows:

(i) $V(\mathcal{P}^n) = V^1(\mathcal{P}^n) \cup \ldots V^m(\mathcal{P}^n)$ where $V^i(\mathcal{P}^n) = V^i \times V^i \times \ldots \times V^i = (V^i)^n$ is the Cartesian product of n copies of $V^i \subset V(\mathcal{P})$.

(ii) For an r-tuple of vertices $v^{i_1} \in V^{i_1}$, $v^{i_2} \in V^{i_2}$, $\ldots$, $v^{i_r} \in V^{i_r}$ with $v^{i_j} = (v_1^{i_j}, v_2^{i_j}, \ldots, v_n^{i_j})$, $1 \leq j \leq r$, we have $\{v^{i_1}, v^{i_2}, \ldots, v^{i_r}\} \in \mathcal{E}(\mathcal{P}^n)$ iff $\{v_h^{i_1}, v_h^{i_1}, \ldots, v_h^{i_r}\} \in \mathcal{E}(\mathcal{P})$ for all $1 \leq h \leq n$.

After proving the two claims that follow, we will then show that for our choice of n, $\mathcal{P}^n = \mathcal{H}$, the desired Ramsey r-graph.

Set $\mathcal{C} = \{\mathcal{K}_m(\widetilde{\mathcal{K}_k}, t) \colon \widetilde{\mathcal{K}_k} \subset \mathcal{G}(t), 1 \le t \le \binom{m}{l}\}$ to be the set of all m-cliques in $\mathcal{P}$. Set $|\mathcal{C}| = q$. We will also find it convenient to set $\mathcal{C} = \{C_1, \ldots, C_q\}$. Next we observe:

Claim 2.2.3. *A sequence $C_{\alpha_1}, \ldots, C_{\alpha_n}$, $C_{\alpha_h} \in \mathcal{C}$, $1 \le h \le n$, of not necessarily distinct m-cliques in $\mathcal{P}$, corresponds to an m-clique $\mathbf{C}$ in $\mathcal{P}^n$. More precisely, setting $V(C_{\alpha_h}) = \{v_h^i \colon 1 \le i \le m\}$, the set $\{(v_1^i, \ldots, v_n^i) \colon 1 \le i \le m\}$ induces an m-clique in $\mathcal{P}^n$.*

Proof. The claim follows from the definition of $\mathcal{P}^n$. $\square$

By Claim 2.2.3, each m-clique $\mathbf{C}$ in $\mathcal{P}^n$ corresponds to an element of a Hales-Jewett cube $(\alpha_1, \ldots, \alpha_n) \in HJC(n, q)$. See Figure 2.2a.

Consequently, a line $\mathcal{L}$ in $HJC(n, q)$ is a q-tuple $\mathbf{C}_1, \ldots, \mathbf{C}_q$ of m-cliques in $\mathcal{P}^n$. See Figure 2.2b. Viewing each $\mathbf{C}_s$ as the set of $\binom{m}{r}$ edges, we will now show that $\bigcup_{s=1}^{q} \mathbf{C}_s$ induces an r-graph $\mathcal{P}_L$ isomorphic to $\mathcal{P}$.

Claim 2.2.4. *A line in $HJC(n, q)$ induces a subgraph $\mathcal{P}_{\mathcal{L}} \subset \mathcal{P}^n$ such that $\mathcal{P}_{\mathcal{L}} \cong \mathcal{P}$.*

Proof. Let $\mathcal{L} = \{g \cup s_M \colon 1 \le s \le q\}$ be a line in $HJC(n, q)$ with moving coordinate set M. Define a mapping $\phi = \phi_{\mathcal{L}} \colon V(\mathcal{P}) \to V(\mathcal{P}_{\mathcal{L}})$ given by $\phi(v) = w = (w_h)_{h \in [n]}$ for $v \in V^i$, where

$$w_h = \begin{cases} v^i_{g(h)} & \text{if } h \in [n] - M \\ v & \text{for all } h \in M \end{cases}$$

and set $W^i = \phi(V^i)$.

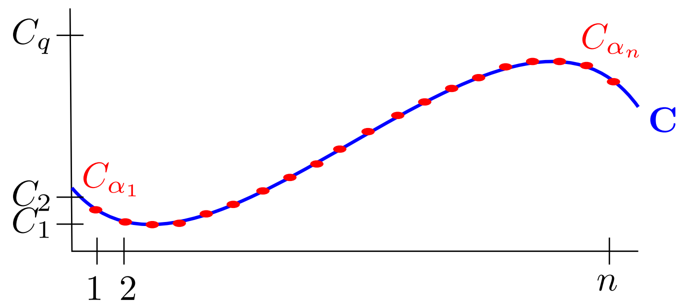
We will verify that ϕ is an isomorphism between $\mathcal{P}$ and an r-subgraph of $\mathcal{P}^n$, which we will refer to as $\mathcal{P}_{\mathcal{L}}$, induced on $W^1 \cup \cdots \cup W^m$. Indeed, for $v^{i_j} \in V^{i_j}, 1 \leq j \leq r$, let $\phi(v^{i_j}) = (w_h^{i_j})_{h \in [n]}$. We observe that for $h \in [n] - M$, the r-tuple $\{w_h^{i_1}, \ldots, w_h^{i_r}\}$ is a subset of clique C_{α_h} and hence an edge of $\mathcal{E}(\mathcal{P})$. On the other hand, for $h \in M$, $\{w_h^{i_1}, \ldots, w_h^{i_r}\} = \{v^{i_1}, \ldots, v^{i_r}\}$. Consequently, by part (ii) in the definition of $\mathcal{P}^n$, $\{v^{i_1}, \ldots, v^{i_r}\} \in \mathcal{E}(\mathcal{P})$ iff $\{\phi(v^{i_1}), \ldots, \phi(v^{i_r})\} \in \mathcal{E}(\mathcal{P}^n)$.

$\square$

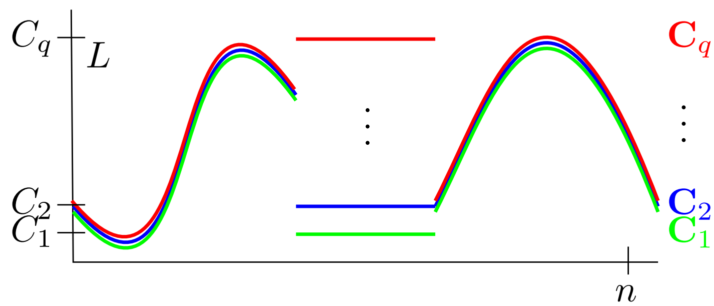
With affirmation of the claims above, we will now show that $\mathcal{P}^n = \mathcal{H}$ is the desired Ramsey r-graph using the Hales-Jewett Theorem to complete the proof of the positive part of Theorem 2.2.1.

Consider a c-coloring of $\mathcal{P}^n$. There are $c^{\binom{m}{k}}$ ways to c-color $\mathcal{K}_m$ and so each m-clique $\mathbf{C}$ of $\mathcal{P}^n$ is colored in one of these ways. By Claim 2.2.3, the m-cliques of $\mathcal{P}^n$ are in 1-1 correspondence with the sequences $C_{\alpha_1}, \ldots, C_{\alpha_n}, C_{\alpha_h} \in \mathcal{C}$, and thus also with the elements $(\alpha_1, \ldots, \alpha_n) \in HJC(n, q)$. By definition of $n = n(c^{\binom{m}{k}}, q)$ and Claim 2.2.4, there exists a monochromatic line which corresponds to a subgraph $\mathcal{P}_{\mathcal{L}} \subset \mathcal{P}^n, \mathcal{P}_{\mathcal{L}} \cong \mathcal{P}$, such that each m-clique in $\mathcal{P}_{\mathcal{L}}$ is colored in the same way. Consequently, we color $\mathcal{K}_k$ the color of each edge $\{w^{i_1}, \ldots, w^{i_r}\} \in \mathcal{E}(\mathcal{P}_{\mathcal{L}})$ which depends on $\{i_1, \ldots, i_r\}$ only. Since m is the r-graph-Ramsey number for $\mathcal{K}_l$ and $\mathcal{G}$ is an r-graph on l vertices, there exists a monochromatic copy of $\mathcal{G}$ in $\mathcal{P}_{\mathcal{L}}$. Thus, $\mathcal{P}^n = \mathcal{H}$ is the desired Ramsey r-graph.

$\square$



(a) $(C_{\alpha_1}, \ldots, C_{\alpha_n}) \iff \mathbf{C}, 1 \leq \alpha_h \leq q$



(b) $(\mathbf{C}_1, \ldots, \mathbf{C}_q) \iff \mathcal{L} \in HJC(n, q)$

Figure 2.2: Correspondence of sequences of m-cliques to elements and lines of $HJC(n, q)$

2.3 Negative result

In this subsection, we prove the negative result of Theorem 2.1.1.

Let the subscript $<$ denote a hypergraph with linearly ordered vertex set. Our method to confirm the negative result of Theorem 2.1.1 relies on the following lemma considered in [24] which we now restate and prove:

Lemma 2.3.1. *Given any hypergraph $\mathcal{K}_< = (V_<(\mathcal{K}), \mathcal{E}(\mathcal{K}))$ with linearly ordered vertex set, there exists an unordered hypergraph $\mathcal{G}$ such that every* *ordering of its vertex set $\mathcal{G}_<$ contains $\mathcal{K}_<$.*

Before we prove Lemma 2.3.1, we provide definitions required for the statement of a result due to Erdős and Hajnal [6]. A k-graph contains a *cycle* of length at most m if it contains $mk - m$ vertices inducing m edges. We denote by $g(\mathcal{G})$ the *girth* of a k-graph $\mathcal{G}$ defined as the number of edges in the shortest cycle.

Fact 2.3.2. *For all positive integers k and n, there exists a k-graph $\mathcal{G}'$ with $|V(\mathcal{G}')| = n$ and $g(\mathcal{G}') = 3$ such that $|\mathcal{E}(\mathcal{G}')| \geq n^{4/3}$ for n sufficiently large.*

We will now use Fact 2.3.2 to prove Lemma 2.3.1.

Proof. (of Lemma 2.3.1) We denote by $\mathcal{K}$ the hypergraph $\mathcal{K}_<$ with order of vertices surpressed. Suppose $\mathcal{K}$ is homogeneous, that is it is either a complete hypergraph or its complement. If any permutation $\alpha\colon V(\mathcal{K}) \longrightarrow V(\mathcal{K})$ such that for all $E \in \mathcal{E}(\mathcal{K}), \alpha(E) \in \mathcal{E}(\mathcal{K})$, also known as a hypergraph *automorphism*, then clearly $\mathcal{G} = \mathcal{K}$ does the job. Therefore we can assume that $\mathcal{K}$ is not homogeneous.

Set $|V(\mathcal{K})| = k$ and let $\mathcal{E}(\mathcal{K})$ be the set of edges of $\mathcal{K}$. Given k-graph $\mathcal{G}'$, guaranteed by Fact 2.3.2, for every $E \in \mathcal{E}(\mathcal{G}')$, we denote by $Bi(\mathcal{K}, E)$ the set of images of all distinct bijective mappings $\phi_E\colon V(\mathcal{K}) \longrightarrow E$. Let $\mathrm{Aut}(\mathcal{K})$ be the group of automorphisms of $\mathcal{K}$.

Let

$$b = |Bi(\mathcal{K}, E)| = \frac{k!}{|\mathrm{Aut}(\mathcal{K})|}.$$

For each $E \in \mathcal{E}(\mathcal{G}')$, fix $\phi_E \in Bi(\mathcal{K}, E)$ and consider $\Phi = \{\phi_{E(\mathcal{K})} \colon E \in \mathcal{E}(\mathcal{G}')\}$, the set of all images of distinct bijective mappings.

Since there are b ways to map $V(\mathcal{K})$ to $E \in \mathcal{E}(\mathcal{G}')$, and $|\mathcal{E}(\mathcal{G}')| = n^{4/3}$, there are $b^{n^{4/3}}$ distinct hypergraphs of the form:

$$\mathcal{G}_\Phi = ([n], \bigcup_{E \in \mathcal{E}(\mathcal{G})} \{\phi_E[E(\mathcal{K})]\}).$$

It remains to show that given a linear ordering on the vertex set of $\mathcal{G}$, $\mathcal{G}_<$ contains $\mathcal{K}_<$. Indeed, in the set of b distinct images of bijective mappings ϕ_E, there is one such image which is isomorphic to $\mathcal{K}_<$. Since there are $n!$ ways to linearly order the vertex set of $\mathcal{G}$, there are $n!(b-1)^{n^{4/3}}$ ways in which $\mathcal{G}_<$ will not contain $\mathcal{K}_<$. However,

$$n!(b-1)^{n^{4/3}} \ll b^{n^{4/3}}, \text{ if } l > l_0,$$

where $|V\mathcal{G}| = l$ and thus there is a choice of ϕ_E, $E \in \mathcal{E}(\mathcal{K})$, which for every ordering of its vertex set will contain $\mathcal{K}_<$. Set

$$\mathcal{G} = \bigcup_{E \in \mathcal{E}(\mathcal{K})} \phi_E(\mathcal{K}).$$

□

It may be shown that $\mathcal{G}$, given by Lemma 2.3.1, does not have a Ramsey r-graph. Given $\mathcal{F}$ with $|V(\mathcal{F})| = s \geq t$, we will construct an r-graph $\mathcal{G}$ that does not have the $\mathcal{F}$-Ramsey property. That is, any $\mathcal{H} \in \mathcal{S}(r, t)$ admits, say, a red-blue coloring of $\binom{\mathcal{H}}{\mathcal{F}}$ with no copy of $\mathcal{G}$ in which $\binom{\mathcal{G}}{\mathcal{F}}$ is monochromatic.

2.3.1 $V(\mathcal{F})$ is not independent

Suppose that $V(\mathcal{F})$ is a set of order $s \geq t$ that is not independent. We will first define a hypergraph $\mathcal{K}_<$ and then construct an r-graph $\mathcal{G}$ with

$V(\mathcal{G}) = \{1, \ldots, l\}$ in such a way that $\mathcal{G}$ satisfies Lemma 2.3.1. Finally we will show that $\mathcal{G}$ does not have a Ramsey r-graph when $V(\mathcal{F})$ is a (non-independent) set of order $s \geq t$.

Note that since $V(\mathcal{F})$ in not independent, $|V(\mathcal{F})| = s \geq t > r$. We may assume that $\mathcal{F} \neq \overline{\mathcal{K}_s}$ and thus, $\mathcal{F}$ contains at least one edge and a vertex.

Before we define the hypergraph $\mathcal{K}_<$, we define sets $R_1, R_2, S_1,$ and S_2 as follows: $R_1 = \{1, 2, \ldots, r\}$ and $R_2 = \{s+2, s+3, \ldots, s+r+1\}$ are sets of order r; $S_1 = \{1, \ldots, s\}$ and $S_2 = \{s+1, \ldots, 2s\}$ are (non-independent) sets of order s. Now we define a hypergraph $\mathcal{K}_< = (V(\mathcal{K}), \mathcal{E}(\mathcal{K}))$ with $V(\mathcal{K}) = [2s]$ and $\mathcal{E}(\mathcal{K}) = \{R_1, R_2\}$. In this case, hypergraph $\mathcal{K}_<$ is an r-graph. See Figure 2.3.

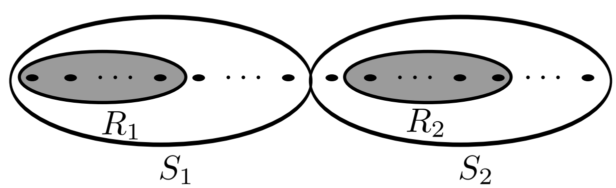


Figure 2.3: Edges (shaded) R_1 and R_2 in the hypergraph $\mathcal{K}_<$

We will now construct $\mathcal{G}$. Set $k = 2s = |V(\mathcal{K})|$ and $b = \frac{k!}{s!^2 r!^{2|\mathcal{E}(\mathcal{F})|}}$ in the proof of Lemma 2.3.1. Let $\mathcal{G} \in S(r, t)$ be an r-graph guaranteed by Lemma 2.3.1. Consider any linear ordering $<$ on $V(\mathcal{G})$ and let $\mathcal{G}_<$ denote the ordered hypergraph.

It can be shown that $\mathcal{G}$ does not have a Ramsey r-graph. Consider an r-graph $\mathcal{H}$ such that $\mathcal{G} \subset \mathcal{H}$. We will find, say, a red-blue coloring of the copies of $\mathcal{F}$ in $\binom{\mathcal{H}}{\mathcal{F}}$ which does *not* yield a copy of $\mathcal{G}$ in which $\binom{\mathcal{G}}{\mathcal{F}}$ is monochromatic.

Consider an arbitrary linear ordering $<$ of the vertex set of $\mathcal{H}$ denoted by $\mathcal{H}_<$. For each copy of $\mathcal{F}_<$ in $\binom{\mathcal{H}_<}{\mathcal{F}_<}$ with $|V(\mathcal{F})| = s \geq t$ and $V(\mathcal{F}) = S$ not independent in $V(\mathcal{H})$ we define the following coloring scheme:

(i) Color red the copies of $\mathcal{F}_<$ for which there exists an edge, say R, of $\mathcal{H}_<$ such that $R \cap S \prec S - R$ where $A \prec B$ if $\max(A) < \min(B)$ for $A, B \subset V(\mathcal{H})$.

(ii) Color blue all other copies of $\mathcal{F}_<$ of $\mathcal{H}_<$.

By Fact 2.3.2, each copy of $\mathcal{G}_<$ in $\binom{\mathcal{H}_<}{\mathcal{G}_<}$ contains a copy of $\mathcal{K}_<$. Therefore, $\mathcal{H}_<$ certainly contains edges R_1 and R_2. Edge R_1 is contained in a copy of $\mathcal{F}_<$ of color red according to our coloring scheme while edge R_2, distinguishable from edge R_1, is contained in a copy of $\mathcal{F}_<$ of color blue. Hence, if we remove the ordering of the vertices of $\mathcal{H}_<$, every copy of $\mathcal{G}$ contains $V(\mathcal{K})$, such a coloring defined by the scheme above does not allow for a copy of $\mathcal{G}$ in which $\binom{\mathcal{G}}{\mathcal{F}}$ is monochromatic.

Chapter 3

Ramsey Properties for Steiner Systems

3.1 Introduction

Recall that for fixed $r > t$, a *partial* (or *incomplete) Steiner (r,t)-system* $\mathcal{G} = (V, \mathcal{E})$ is an r-graph that has the property that every t-element set is contained in at most one edge of $\mathcal{G}$. If every t-element set is contained in precisely one r-element set we will refer to the Steiner (r,t)-system as *complete*. In fact, the solution to a 160 year old conjecture of Steiner regarding the characterization of the existence of complete Steiner systems was recently established in [18].

Let $\mathcal{S}(r,t)$ be the class of all partial Steiner (r,t)-systems with subobjects being induced subgraphs. A result of Nešetřil and Rödl in [26] states the following.

Theorem 3.1.1 ([26])**.** *Let $c \geq 2$, $r > t$. For any $\mathcal{G} \in \mathcal{S}(r,t)$ there exists an $\mathcal{H} \in \mathcal{S}(r,t)$ with the property that any c-coloring of the edges of $\mathcal{H}$ yields a monochromatic induced copy of $\mathcal{G}$.*

In other words, Theorem 3.1.1 states that the class of all partial Steiner systems and induced embeddings has the edge-Ramsey property [22]. A natural question to ask is what other partial Steiner system $\mathcal{F}$ can be partitioned instead of edges.

Definition 3.1.2. *For an integer $c \geq 2$ and fixed hypergraphs $\mathcal{F}$ and $\mathcal{G}$ we say that the hypergraph $\mathcal{H}$ is $\mathcal{F}$-Ramsey for $\mathcal{G}$ (denoted $\mathcal{H} \longrightarrow (\mathcal{G})_c^{\mathcal{F}}$) if for any c-coloring of $\binom{\mathcal{H}}{\mathcal{F}}$ there exists a copy of $\mathcal{G}$, $\widetilde{\mathcal{G}} \in \binom{\mathcal{H}}{\mathcal{G}}$ with $\binom{\widetilde{\mathcal{G}}}{\mathcal{F}}$ monochromatic. Similarly, we say that a class $\mathcal{C}$ of hypergraphs has the $\mathcal{F}$-Ramsey property if for any $c \geq 2$ and $\mathcal{F}, \mathcal{G} \in \mathcal{C}$ there is $\mathcal{H} \in \mathcal{C}$ such that $\mathcal{H} \longrightarrow (\mathcal{G})_c^{\mathcal{F}}$.*

We consider induced subgraphs of partial Steiner systems and generalize the result in [26] restated as follows.

Theorem 3.1.3. *The class $\mathcal{S}(r,t)$ has the $\mathcal{F}$-Ramsey property if and only if $\mathcal{F}$ is an edge or $|V(\mathcal{F})| < t$.*

It turns out that considering the class of all partial Steiner systems with vertex set linearly ordered, this positive result is true for a broader class of Steiner systems. Let $\mathcal{S}_<(r,t)$ be the class of all partial Steiner (r,t)-systems with linearly ordered vertex sets where isomorphism is also order preserving. We say that $\widetilde{\mathcal{G}}_<$ is a copy of $\mathcal{G}_<$ in $\mathcal{H}_<$ if $\mathcal{G}_<$ is an induced subgraph of $\mathcal{H}_<$ which is isomorphic to $\mathcal{G}_<$. For ordered partial Steiner (r,t)-systems $\mathcal{G}_<, \mathcal{H}_< \in \mathcal{S}_<(r,t)$, let $\binom{\mathcal{H}_<}{\mathcal{G}_<}$ denote the set of subobjects of $\mathcal{H}_<$ isomorphic to $\mathcal{G}_<$, that is, induced copies of $\mathcal{G}_<$ in $\mathcal{H}_<$. For a fixed $\mathcal{F}_< \in \mathcal{S}_<(r,t)$, below we are going to define a special class of hypergraphs having the $\mathcal{F}$-union property, thereby slightly generalizing the result in [26].

Going forward, in the case there is no ambiguity, we may omit the linear order notation subscript "$<$" to objects in $S_<(r,t)$ but we will still assume that objects in $S_<(r,t)$ have linearly ordered vertex sets.

Definition 3.1.4. *For a fixed $\mathcal{F} \in \mathcal{S}_<(r,t)$, an r-graph $\mathcal{G}$ has the $\mathcal{F}$-union property if and only if every edge of $\mathcal{G}$ is in precisely one copy of $\mathcal{F}$.*

Theorem 3.1.5. *For integers $c \geq 2$ and $r > t$ let $\mathcal{F} \in \mathcal{S}_<(r,t)$ and let $\mathcal{G} \in \mathcal{S}_<(r,t)$ have the $\mathcal{F}$-union property. Then there exists $\mathcal{H}$ which is $\mathcal{F}$-Ramsey for $\mathcal{G}$ if and only if $|V(\mathcal{F})| < t$, $\mathcal{F}$ is an edge or $\mathcal{F}$ is any (other) complete Steiner (r,t)-system.*

We believe that the $\mathcal{F}$-union property condition on $\mathcal{G}$ in Theorem 3.1.5 may be lifted and we conjecture the following.

Conjecture 3.1.6. *The class $\mathcal{S}_<(r,t)$ has the $\mathcal{F}$-Ramsey property if and only if $\mathcal{F}$ is an edge, $|V(\mathcal{F})| < t$, or $\mathcal{F}$ is a complete Steiner (r,t)-system.*

We present all proofs of positive results in Section 3.2, followed by all proofs of negative results in Section 3.3.

3.2 Positive results

3.2.1 $\mathcal{F}$-Ramsey property for $\mathcal{S}(r,t)$ and $\mathcal{S}_<(r,t)$

First we observe that it is sufficient to prove the positive statement of Theorem 3.1.5 only. Indeed, if $\mathcal{F}$ is an edge, Theorem 3.1.3 was proved in [26] and for $\mathcal{F}$ with $|V(\mathcal{F})| < t$ it is implied by the somewhat stronger Theorem 3.1.5 which guarantees not only a monochromatic copy of $\mathcal{G}$ but a monochromatic copy of $\mathcal{G}$ with prescribed linear order of vertices.

We consider two cases: $\mathcal{F}$ an ordered complete Steiner (r,t)-system and $\mathcal{F}$ an independent set of size less than t.

Case I: Let $\mathcal{F}$ be an ordered complete Steiner (r,t)-system and $\mathcal{G} \in \mathcal{S}_<(r,t)$ have the $\mathcal{F}$-union property. As $\mathcal{F}$ is complete, it follows that any two copies of $\mathcal{F}$ in $\mathcal{G}$ intersect in at most $t-1$ vertices.

Set $|V(\mathcal{F})| = k$. Next, we define a k-graph $\mathcal{X}$ with $V(\mathcal{X}) = V(\mathcal{G})$ and $\mathcal{E}(\mathcal{X}) = \{V(\widetilde{\mathcal{F}}) \colon \widetilde{\mathcal{F}} \in \binom{\mathcal{G}}{\mathcal{F}}\}$. Note that $\mathcal{X} \in S_<(k,t)$. Indeed, by the above, any two edges in $\mathcal{X}$ share less than t vertices. Therefore, any t-element set of vertices of $\mathcal{X}$ is contained in at most one edge.

Consequently, we may apply Theorem 3.1.1 to $\mathcal{X}$. Thus there exists $\mathcal{Y} \in S_<(k,t)$ with the property that any c-coloring of the edges of $\mathcal{Y}$ yields a monochromatic copy of $\mathcal{X}$. Recall that we are interested in finding $\mathcal{H} \in S_<(r,t)$ with the property that any c-coloring of the copies of $\mathcal{F}$ in $\mathcal{H}$ yields a copy of $\mathcal{G}$ in which $\binom{\mathcal{G}}{\mathcal{F}}$ is monochromatic. We obtain such an $\mathcal{H}$ by replacing each edge (of size k) in $\mathcal{Y}$ by a copy of $\mathcal{F}$. Since both $\mathcal{F}$ and $\mathcal{Y}$ are linearly ordered there is a unique way to do such a replacement. Then a c-coloring of $\binom{\mathcal{H}}{\mathcal{F}} = \mathcal{E}(\mathcal{Y})$ yields a monochromatic copy of $\mathcal{X}$, or equivalently, a copy of $\mathcal{G}$ with $\binom{\mathcal{G}}{\mathcal{F}}$ monochromatic.

Case II: We will give a proof based on the result of J. Spencer in [33] (See [25] for a further generalization.)

Definition 3.2.1. *Let c,k,n be integers satisfying $c \geq 2, n > k \geq 2$ and let $\mathcal{N}$ be a family of n-element sets, $\bigcup \mathcal{N} = X$. We call $\mathcal{N}$ a c-Ramsey family if given any c-coloring of $\binom{X}{k}$, there is $N \in \mathcal{N}$ with the property that $\binom{N}{k}$ is monochromatic.*

The following was proved in [33].

Theorem 3.2.2. *For all $c \geq 2, n > k \geq 2$, there exists a c-Ramsey family $\mathcal{N}$ such that $|N| = n$ for all $N \in \mathcal{N}$ and for any distinct $N_1, N_2 \in \mathcal{N}, |N_1 \cap N_2| \leq k$.*

Let $\mathcal{F}$ be a set with $k < t < r$ vertices (thus, of course, $\mathcal{F}$ has no edges). We want to prove that $\mathcal{S}_<(r,t)$ has the $\mathcal{F}$-Ramsey property.

Let $\mathcal{G} \in \mathcal{S}_<(r,t), \mathcal{G} = (V, \mathcal{E})$ be given. Put $|V| = n$. In order to construct an $\mathcal{F}$-Ramsey family, we consider now a family $\mathcal{N}$ ensured by Theorem 3.2.2, and set $X = \bigcup \mathcal{N}$. We will fix a linear order $<_X$ of the vertices of X. For each $N \in \mathcal{N}$ consider a copy $\mathcal{G}_N$ of $\mathcal{G} \in \mathcal{S}_<(r,t)$ so that the monotone mapping from $\mathcal{G}$ to $\mathcal{X}$ is an isomorphism between $\mathcal{G}$ and $\mathcal{G}_N$. Let $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ where $\mathcal{E} = \bigcup\{\mathcal{E}(\mathcal{G}_N), N \in \mathcal{N}\}$. Since any two copies of $\mathcal{G}$ in $\mathcal{H}$ intersect in less than k ($k < t$) vertices, $\mathcal{H} \in \mathcal{S}_<(r,t)$. Moreover, the c-Ramseyness of $\mathcal{N}$ guarantees that any c-coloring of k-element subsets of X yields a monochromatic copy of $\mathcal{G}$. $\square$

3.3 Negative results

3.3.1 Negative results for $\mathcal{S}(r,t)$

Our method to confirm the negative results of Theorem 3.1.3 relies on the following fact proved in [24].

Theorem 3.3.1. *For every $r > t \geq 2$ and every $\mathcal{K} \in \mathcal{S}_<(r,t)$ with ordered vertex set there exists an r-graph $\mathcal{G} = (V, \mathcal{E}) \in \mathcal{S}(r,t)$ such that for every ordering $<$ of $V(\mathcal{G})$ there exists a monotone embedding of $\mathcal{K}_<$ into $\mathcal{G}_<$.*

Given $\mathcal{F}$ with $|V(\mathcal{F})| = s \geq t$, we will construct $\mathcal{G} \in \mathcal{S}(r,t)$ for which there is no $\mathcal{H}$ which is $\mathcal{F}$-Ramsey for $\mathcal{G}$. That is, any $\mathcal{H} \in \mathcal{S}(r,t)$ admits, say, a red-blue coloring of $\binom{\mathcal{H}}{\mathcal{F}}$ with no copy $\widetilde{\mathcal{G}}$ of $\mathcal{G}$ in which $\binom{\widetilde{\mathcal{G}}}{\mathcal{F}}$ is monochromatic. We will distinguish two cases depending on whether $\mathcal{F}$ is independent or not.

Case I: $\mathcal{F}$ contains no edge, $|V(\mathcal{F})| = s \geq t$

We will first define an auxilliary hypergraph $\mathcal{K}_<$ and then construct $\mathcal{G} \in S(r,t)$ for which $\mathcal{H}$ is not $\mathcal{F}$-Ramsey for $\mathcal{G}$.

We introduce hypergraphs $\mathcal{K}^a_<$ and $\mathcal{K}^b_<$ with vertex sets $V(\mathcal{K}^a_<) = \{a_1, a_2, \ldots, a_{r+s-t}\}$ and $V(\mathcal{K}^b_<) = \{b_1, b_2, \ldots, b_{r-t+s}\}$ such that $a_1 < a_2 < \ldots < a_{r-t+s}$ and $b_1 < b_2 < \ldots < b_{r-t+s}$, respectively. Each of these ordered hypergraphs contains precisely one edge.

$\mathcal{E}(\mathcal{K}^a_<) = \{R^a\}$ where $R^a = \{a_1, a_2, \ldots, a_r\}$ and

$\mathcal{E}(\mathcal{K}^b_<) = \{R^b\}$ where $R^b = \{b_1, \ldots, b_t, \ldots, b_{s+1}, \ldots, b_{r+s-t}\}$

as shown in Figure 3.1.

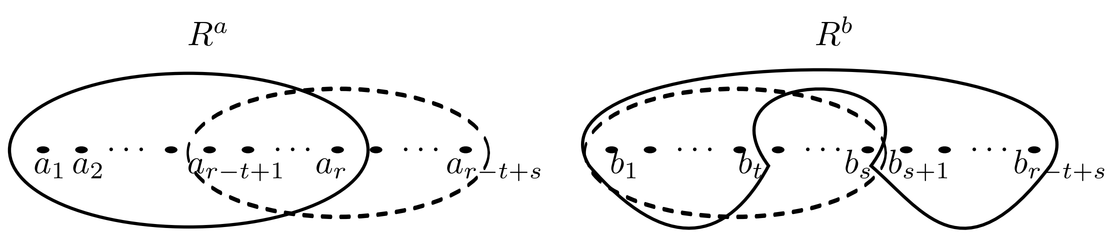


Figure 3.1: Edges $R^a \in \mathcal{K}^a_<$ and $R^b \in \mathcal{K}^b_<$ in hypergraph $\mathcal{K}_<$

Let $\mathcal{K}_< = \mathcal{K}^a_< \cup \mathcal{K}^b_<$ be an ordered hypergraph, where (for definiteness) $V(\mathcal{K}^a_<) < V(\mathcal{K}^b_<)$. Clearly, $\mathcal{K}_< \in \mathcal{S}(r,t)$ for any t, $2 \leq t < r$ and thus there is $\mathcal{G} \in \mathcal{S}(r,t)$ with the property of Theorem 3.3.1. We claim that there is no $\mathcal{H} \in \mathcal{S}(r,t)$ with $\mathcal{H} \longrightarrow (\mathcal{G})_2^{\mathcal{F}}$ with $\mathcal{F}$ independent, $|V(\mathcal{F})| = s \geq t$.

Indeed, suppose that such a hypergraph $\mathcal{H}$ exists. Fix one vertex order $<_{\mathcal{H}}$ of $\mathcal{H}$. We will consider all independent sets $\widetilde{\mathcal{F}} \in \binom{\mathcal{H}}{\mathcal{F}}$ with $V(\widetilde{\mathcal{F}}) = \{x_1, x_2, \ldots, x_s\}$ where $x_1 < x_2 < \ldots < x_s$ for which there exists $E \in \mathcal{E}(\mathcal{H})$ with $E \cap V(\widetilde{\mathcal{F}}) = \{x_1, x_2, \ldots, x_t\}$. We will color by red those for which $E - V(\widetilde{\mathcal{F}}) <_{\mathcal{H}} V(\widetilde{\mathcal{F}})$, i.e. we color by red those for which the remaining vertices of the edge E precede the vertices of $V(\widetilde{\mathcal{F}})$ in the order $<_{\mathcal{H}}$. We color blue all other independent sets of size s in $\mathcal{H}$. We claim that there is no $\mathcal{F}$-monochromatic copy of $\mathcal{G}$. Indeed, any $\widetilde{\mathcal{G}} \in \binom{\mathcal{H}}{\mathcal{G}}$, with order inherited by $<_{\mathcal{H}}$, contains a copy $\widetilde{\mathcal{K}}_<$of $\mathcal{K}_<$ and thus also copies $\widetilde{\mathcal{K}_<}^a$ and $\widetilde{\mathcal{K}_<}^b$ of $\mathcal{K}_<^a$ and $\mathcal{K}_<^b$, respectively. Set $V(\widetilde{\mathcal{K}}_<^a) = \{y_1, y_2, \ldots, y_{r+s-t}\}$ and $V(\widetilde{\mathcal{K}}_<^b) = \{z_1, z_2, \ldots, z_{r-t+s}\}$. While the s-set $\{y_{r-t+1}, \ldots, y_{r-t+s}\}$ is colored red by definition of the coloring (see $\mathcal{K}_<^a$ in Figure 3.1), we claim that $\{z_1, \ldots, z_s\}$ must be colored blue. This is because, due to the fact that $\mathcal{H} \in \mathcal{S}(r,t)$, the only edge containing $\{z_1, \ldots, z_t\}$ is $\{z_1, \ldots, z_t, \ldots, z_{s+1}, \ldots, z_{r+s-t}\}$ (see $\mathcal{K}_<^b$ in Figure 3.1). Consequently, no copy of $\mathcal{G}$ in $\mathcal{H}$ is $\mathcal{F}$-monochromatic.

$\square$

Case II: $\mathcal{F} \neq \emptyset$ and $\mathcal{F}$ is not an edge, hence $|V(\mathcal{F})| = s > r > t$

Consider an edge $R \in \mathcal{F}$ and fix $T \subset R$ and $v \in V(\mathcal{F}) - R$. Let $R' \subset V(\mathcal{F})$ be an r-subset of $V(\mathcal{F})$ containing $T \cup v$. Since $\mathcal{F} \in \mathcal{S}(r,t)$, we infer that $R' \notin \mathcal{F}$. Consider now two linear orders $<_1$ and $<_2$ of $V(\mathcal{F})$ in which the first r vertices of $<_1$ is R and the first r vertices of $<_2$ is R'. Similarly as in Case I let $\mathcal{K}_<$ be the ordered partial Steiner (r,t)-system containing both $\mathcal{F}_{<_1}$ and $\mathcal{F}_{<_2}$.

Let $\mathcal{G} \in \mathcal{S}(r,t)$ be the hypergraph from Theorem 3.3.1. Similarly as in Case I we will show that there is no $\mathcal{H} \in \mathcal{S}(r,t)$ with $\mathcal{H} \longrightarrow (\mathcal{G})_2^{\mathcal{F}}$. Indeed, fixing a vertex order $<_{\mathcal{H}}$ of $\mathcal{H}$ and coloring all copies of $\widetilde{\mathcal{F}} \in \binom{\mathcal{H}}{\mathcal{F}}$ by red and blue depending on whether the first r elements of $V(\widetilde{\mathcal{F}})$ are an edge of $\mathcal{H}$ or not yields the desired coloring in which no copy of $\mathcal{G}$ is $\widetilde{\mathcal{F}}$-monochromatic.

$\square$

3.3.2 Negative results for $\mathcal{S}_<(r,t)$

Given $\mathcal{F} \in \mathcal{S}_<(r,t)$ which is not a complete Steiner (r,t)-system, we will construct $\mathcal{G} \in \mathcal{S}_<(r,t)$ for which there is no $\mathcal{H}$ which is $\mathcal{F}$-Ramsey for $\mathcal{G}$.

Without loss of generality, assume that $V(\mathcal{F}) = \{1,2,\dots,s\}$. Since $\mathcal{F}$ is not a complete Steiner (r,t)-system, there is a t-set $T \subset V(\mathcal{F})$ which is not contained in any edge of $\mathcal{F}$. Consider two vertex disjoint copies $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{F}$ and let $T_1 \subset V(\mathcal{F}_1)$ and $T_2 \subset V(\mathcal{F}_2)$ be the corresponding copies of T. Set $V(\mathcal{F}_1) = \{a_1, a_2, \dots, a_s\}$ with $a_1 < a_2 < \dots < a_s$ and $V(\mathcal{F}_2) = \{b_{r-t+1}, \dots, b_{r-t+s}\}$ with $b_{r-t+1} < \dots < b_{r-t+s}$. Similarly as before, we will now define r-graph $\mathcal{G}$ as follows.

$V(\mathcal{G}) = \{a_1, \dots, a_{r+s-t}, b_1, \dots, b_{r+s-t}\}$ with $a_1 < \dots < a_{r+s-t} < b_1 < \dots < b_{r+s-t}$ and

$\mathcal{E}(\mathcal{G}) = \mathcal{E}(\mathcal{F}_1) \cup \mathcal{E}(\mathcal{F}_2) \cup \{R_1, R_2\}$ where $R_1 = T \cup \{a_{s+1}, \dots, a_{r+s-t}\}$ and $R_2 = T \cup \{b_1, \dots, b_s\}$.

We claim that there is no $\mathcal{H} \in \mathcal{S}_<(r,t)$ with $\mathcal{H} \longrightarrow (\mathcal{G})_2^{\mathcal{F}}$. Indeed, for any $\mathcal{H} \in \mathcal{S}_<(r,t)$ we define the desired coloring as follows. If $\widetilde{\mathcal{F}} \in \binom{\mathcal{H}}{\mathcal{F}}$ with $V(\widetilde{\mathcal{F}}) = \{x_1, \dots, x_s\}$ we will color $\widetilde{\mathcal{F}}$ red if for some $\{x_{s+1}, \dots, x_{r+t-s}\}$ with $x_s < x_{s+1} < \dots < x_{r+t-s}$ the r-set $\{x_i, i \in T\} \cup \{x_{s+1}, \dots, x_{r+t-s}\} \in \mathcal{H}$. Otherwise, we color $\{x_1, \dots, x_s\}$ blue.

Similarly as in the proof of the negative part of Theorem 3.1.3, we infer that there is no $\widetilde{\mathcal{F}}$-monochromatic copy of $\mathcal{G}$.

$\square$

3.4 Concluding remarks

A class $\mathcal{C}$ is called *Ramsey* if it has the $\mathcal{F}$-Ramsey property for any $\mathcal{F} \in \mathcal{C}$. Perhaps the most well known examples of Ramsey classes include finite sets, finite vector spaces (over a fixed field F), and finite partially ordered sets (with fixed linear extension). The study of Ramsey classes found its revival after several decades due to its connection with topological dynamics (see [17]).

Suppose, rather than induced subgraphs of partial Steiner systems as subobjects when studying the Ramsey property, we considered *strongly* induced subgraphs, which we define next.

Definition 3.4.1. *Let $\mathcal{G}$ and $\mathcal{H}$ be partial Steiner (r,t)-systems. We say that $\mathcal{G}$ is strongly induced in $\mathcal{H}$ if*

(i) $\mathcal{G}$ is induced in $\mathcal{H}$, i.e $\mathcal{E}(\mathcal{G}) = \mathcal{E}(\mathcal{H}) \cap \binom{V(\mathcal{G})}{r}$, and, moreover,

(ii) for all $E \in \mathcal{E}(\mathcal{H}), |E \cap V(\mathcal{G})| < t$.

We denote by $\binom{\mathcal{H}}{\mathcal{G}}^*$ the set of all strongly induced copies of $\mathcal{G}$ in $\mathcal{H}$. For $\mathcal{F} \subset \mathcal{G}$, we say that $V(\mathcal{F})$ is strongly independent if for all $E \subset V(\mathcal{G})$,

$$|E \cap V(\mathcal{F})| < t.$$

Let $\mathcal{S}^*(r,t)$ be the class of all partial Steiner (r,t)-systems with subobjects being strongly induced subgraphs. We conjecture the following.

Conjecture 3.4.2. *The class $\mathcal{S}^*(r,t)$ has the $\mathcal{F}$-Ramsey property if and only if $\mathcal{F}$ is an edge or a strongly independent set.*

Next we modify Definition 3.4.1 to the class of all partial Steiner (r,t)-systems with linearly ordered vertex sets where isomorphism is also order preserving and subobjects of $\mathcal{H}_<$ *strongly* isomorphic to $\mathcal{G}_<$, that is, strongly induced copies of $\mathcal{G}_<$ in $\mathcal{H}_<$. We denote this class by $\mathcal{S}^*_<(r,t)$.

We conjecture the following Ramsey class.

Conjecture 3.4.3. *$\mathcal{S}^*_<(r,t)$ is a Ramsey class.*

Chapter 4

Alternative Positive Direction Proof of Theorem 3.1.5

In this section we give an alternate proof of the positive result of Theorem 3.1.5 presented in Chapter 3. Recall that Theorem 3.1.3 is implied by Theorem 3.1.5. Here are restatements.

Theorem 3.1.3. *The class $\mathcal{S}(r,t)$ has the $\mathcal{F}$-Ramsey property if and only if $\mathcal{F}$ is an edge or $|V(\mathcal{F})| < t$.*

Theorem 3.1.5. *For integers $c \geq 2$ and $r \geq t$ let $\mathcal{F} \in \mathcal{S}_<(r,t)$ and let $\mathcal{G} \in \mathcal{S}_<(r,t)$ have the $\mathcal{F}$-union property. Then there exists $\mathcal{H}$ which is $\mathcal{F}$-Ramsey for $\mathcal{G}$ if and only if $|V(\mathcal{F})| < t$, $\mathcal{F}$ is an edge or $\mathcal{F}$ is any (other) complete Steiner (r,t)-system.*

The direct proof technique of the positive results of Theorem 3.1.3 and Theorem 3.1.5 presented here relies on the Hales-Jewett theorem [16] and is based on an appropriate adaptation of the partite lemma and partite construction described in [24, 26].

4.1 Positive results for $\mathcal{S}_<(r,t)$ (direct proof)

In this section we explore the $\mathcal{F}$-Ramsey property for the class $\mathcal{S}_<(r,t)$ asserted by Theorem 3.1.5. The positive part of Theorem 3.1.5 asserts that the class $\mathcal{S}_<(r,t)$ has the $\mathcal{F}$-Ramsey property if $\mathcal{F}$ is an edge, $|V(\mathcal{F})| < t$, or $\mathcal{F}$ is a complete Steiner (r,t)-system. Below we discuss the proof of the case when $\mathcal{F}$ is a complete Steiner (r,t)-system in full detail. The proof technique involves a partite lemma and a partite construction, discussed in Sections 4.1.1 and 4.1.2, respectively. The other cases are similar and we will discuss them briefly in Section 4.1.3.

4.1.1 The Partite Lemma

We will adapt the partite lemma from [26] to show that $\mathcal{S}_<(r,t)$ has the $\mathcal{F}$-Ramsey property when $\mathcal{F}$ is a complete Steiner (r,t)-system. Let $\mathcal{F}$ be a complete Steiner (r,t)-system with $|V(\mathcal{F})| = k$. An r-graph $\mathcal{G}$ is k-partite if $V(\mathcal{G}) = V$ is divided into k classes $V^1, \dots, V^k$ such that $V = V^1 \cup \dots \cup V^k$ and all edges $E \in \mathcal{E}(\mathcal{G})$ are *crossing*, i.e. $|E \cap V^i| \le 1$ for $1 \le i \le k$. Let $Part(k,r,t)$ denote the class of all k-partite r-graphs $\mathcal{G} = ((V^i)_{i=1}^k, \mathcal{E})$ such that $|E_1 \cap E_2| < t$ for all $E_1, E_2 \in \mathcal{E}, E_1 \ne E_2$.

With this framework, we state, and then prove, the following main lemma:

Lemma 4.1.1 (The Partite Lemma). *Let $c \ge 2$, $k \ge r > t$. For any complete Steiner (r,t)-system $\mathcal{F}$ and $\mathcal{X} \in Part(k,r,t)$ with the $\mathcal{F}$-union property there exists $\mathcal{Y} \in Part(k,r,t)$ and a set of copies $\mathcal{C} \in \binom{\mathcal{Y}}{\mathcal{X}}$ with the following properties.*

(i) Any c-coloring of $\binom{\mathcal{Y}}{\mathcal{F}}$ yields a copy of $\mathcal{X} \in \mathcal{C}$ in which $\binom{\mathcal{X}}{\mathcal{F}}$ is monochromatic.

(ii) Any $\mathcal{X}_1, \mathcal{X}_2 \in \mathcal{C}$ such that $\mathcal{X}_1 \neq \mathcal{X}_2$ satisfy the following:

if there exists t distinct vertices

$$v^{i_1}, v^{i_2}, \ldots, v^{i_t} \in \bigcup_{i=1}^{k} V^i(\mathcal{X}_1) \cap \bigcup_{i=1}^{k} V^i(\mathcal{X}_2)$$

with the property that

$$|\{v^{i_1}, v^{i_2}, \ldots, v^{i_t}\} \cap V^i(\mathcal{X}_1) \cap V^i(\mathcal{X}_2)| \leq 1 \text{ for all } 1 \leq i \leq k, 1 \leq j \leq t$$

then there exists

$$\widetilde{\mathcal{F}} \in \binom{\mathcal{X}_1}{\mathcal{F}} \cap \binom{\mathcal{X}_2}{\mathcal{F}} \text{ such that } \{v^{i_1}, v^{i_2}, \ldots, v^{i_t}\} \subset E \text{ for some } E \in \mathcal{E}(\widetilde{\mathcal{F}}).$$

Before we prove Lemma 4.1.1, we shall introduce some additional notation and prove some auxiliary lemmas and propositions.

For any $\mathcal{X} \in \mathrm{Part}(k, r, t)$ and positive integer n, we define the k-partite *product* r-graph $\mathcal{X}^n$ as follows:

(i) $V(\mathcal{X}^n) = V^1(\mathcal{X}^n) \cup \ldots \cup V^k(\mathcal{X}^n)$ where $V^i(\mathcal{X}^n) = V^i \times V^i \times \ldots \times V^i = (V^i)^n$ is the Cartesian product of n copies of $V^i \subset V(\mathcal{X})$.

(ii) For $\mathbf{v}^i = (v_1^i, v_2^i, \ldots, v_n^i) \in V^i(\mathcal{X}^n)$ for $1 \leq i \leq k$, we have $\{\mathbf{v}^{i_1}, \mathbf{v}^{i_2}, \ldots, \mathbf{v}^{i_r}\} \in \mathcal{E}(\mathcal{X}^n)$ if and only if $\{v_h^{i_1}, v_h^{i_2}, \ldots, v_h^{i_r}\} \in \mathcal{E}(\mathcal{X})$ for all $1 \leq h \leq n$.

For clarity, we will use E and F (possibly with subscripts) to denote edges of $\mathcal{X}$ and $\mathcal{X}^n$, respectively.

For $1 \leq h \leq n$ and $1 \leq i \leq k$, we will consider a mapping

$$\pi_h \colon V^i(\mathcal{X}^n) \longrightarrow V^i(\mathcal{X}) \text{ defined by } \pi_h(\mathbf{v}^i) = \pi_h((v_1^i, v_2^i, \ldots, v_n^i)) = v_h^i.$$

We will eventually show that $\mathcal{Y}$ from Lemma 4.1.1 may be chosen to be $\mathcal{X}^n$ for sufficiently large n.

Proposition 4.1.2. *Let $k \geq r > t$ and $\mathcal{X} \in \mathrm{Part}(k, r, t)$. Then for any positive integer n, $\mathcal{X}^n \in \mathrm{Part}(k, r, t)$.*

Proof. Assume for contradiction that there exist $F_1, F_2 \in \mathcal{E}(\mathcal{X}^n)$ such that $|F_1 \cap F_2| \geq t$ for $F_1 \neq F_2$. Consequently, there exist distinct $\mathbf{v}^{i_1}, \mathbf{v}^{i_2}, \ldots, \mathbf{v}^{i_t} \in F_1 \cap F_2$. Then for any h, $1 \leq h \leq n$, $\pi_h(\mathbf{v}^{i_1}), \pi_h(\mathbf{v}^{i_2}), \ldots, \pi_h(\mathbf{v}^{i_t})$ are also distinct, and moreover, all are elements of $\pi_h(F_1) \cap \pi_h(F_2)$. However, $F_1 \neq F_2$ implies that there exists $1 \leq h_0 \leq n$ such that $\pi_{h_0}(F_1), \pi_{h_0}(F_2)$ are distinct edges of $\mathcal{X}$. Since $\mathcal{X} \in \mathrm{Part}(k, r, t)$, $|\pi_{h_0}(F_1) \cap \pi_{h_0}(F_2)| < t$, a contradiction. $\square$

Let $\lambda \colon \binom{\mathcal{X}^n}{\mathcal{F}} \longrightarrow (\binom{\mathcal{X}}{\mathcal{F}})^n$ be a mapping defined by $\lambda(\widetilde{\mathcal{F}}) = (\pi_1(\widetilde{\mathcal{F}}), \pi_2(\widetilde{\mathcal{F}}), \ldots, \pi_n(\widetilde{\mathcal{F}}))$ where $\pi_h(\widetilde{\mathcal{F}}) \in \binom{\mathcal{X}}{\mathcal{F}}$, $1 \leq h \leq n$, and $\widetilde{\mathcal{F}} \in \binom{\mathcal{X}^n}{\mathcal{F}}$. Thus, the mapping λ establishes a 1-1 correspondence between any copy of $\mathcal{F}$ in $\mathcal{X}^n$ and a sequence $\pi_1(\widetilde{\mathcal{F}}), \pi_2(\widetilde{\mathcal{F}}), ..., \pi_n(\widetilde{\mathcal{F}})$ of copies of $\mathcal{F}$ in $\mathcal{X}$. The r-graph $\mathcal{Z}$ is a *canonical subgraph* of $\mathcal{X}^n$ if the set $\{\lambda(\widetilde{\mathcal{F}}) \colon \widetilde{\mathcal{F}} \in \binom{\mathcal{Z}}{\mathcal{F}}\}$ is a line in $HJC(n, \mathcal{Q})$, where $\mathcal{Q} = \binom{\mathcal{X}}{\mathcal{F}}$. In other words, the edge set of a canonical subgraph $\mathcal{Z}$ corresponds to a set of functions $f \colon [n] \to \mathcal{Q}$ which form a line in $HJC(n, \mathcal{Q})$. For clarity, we will use G (possibly with subscripts) to denote edges of $\mathcal{Z}$.

Set $|\binom{\mathcal{X}}{\mathcal{F}}| = q$. Note that

$$|\mathcal{E}(\mathcal{X})| = q|\mathcal{E}(\mathcal{F})| = q\binom{k}{t} / \binom{r}{t}, \tag{1}$$

since $\mathcal{X}$ has $\mathcal{F}$-union property and $\mathcal{F}$ is a complete Steiner (r, t)-system on k vertices.

Next, we verify that $\mathcal{X}$ and $\mathcal{Z}$ are isomorphic in the following lemma.

Lemma 4.1.3. *Let $\mathcal{X} \in \mathrm{Part}(k, r, t)$, n be a positive integer, and $\mathcal{Z}$ be a canonical subgraph of $\mathcal{X}^n$. Then $\mathcal{X} \cong \mathcal{Z}$.*

Proof. Let $\mathcal{Z}$ be a line in $HJC(n, \mathcal{Q})$ with moving coordinate set M. Let $g\colon [n] - M \longrightarrow \mathcal{Q} = \binom{\mathcal{X}}{\mathcal{F}}$ and let $g(h) = \{v^1_{g(h)}, v^2_{g(h)}, \ldots, v^k_{g(h)}\}$ be the vertex set of $\pi_h(\mathcal{Z})$ which is a copy of $\mathcal{F}$ in $\mathcal{X}$. Define a mapping $\phi\colon V(\mathcal{X}) \to V(\mathcal{Z})$ given by $\phi(v^i) = (w^i_h)_{h\in[n]}$ for $v^i \in V^i(\mathcal{X})$, where

$$w^i_h = \begin{cases} v^i_{g(h)} & \text{if } h \in [n] - M \\ v^i & \text{if } h \in M \end{cases}$$

and set $W^i(\mathcal{Z}) = \phi(V^i(\mathcal{X}))$ for all $1 \le i \le k$.

We will verify that ϕ is an isomorphism between $\mathcal{X}$ and $\mathcal{Z} \subset \mathcal{X}^n$, induced on $W^1 \cup \cdots \cup W^k$. Consider a crossing r-tuple with vertices $\{v^{i_1}, v^{i_2}, \ldots, v^{i_r}\}$ for $v^{i_j} \in V^{i_j}, 1 \le j \le r$. Recall that $\phi(v^{i_j}) = (w^{i_j}_h)_{h\in[n]}$. Also recall that for $h \in [n] - M$, the r-tuple $\{w^{i_1}_h, w^{i_2}_h, \ldots, w^{i_r}_h\}$ is an edge of $\mathcal{E}(\mathcal{X})$. On the other hand, for $h \in M$, $w^{i_j}_h = v^{i_j}$ for $1 \le j \le r$, and consequently, $\{v^{i_1}, v^{i_2}, \ldots, v^{i_r}\} \in \mathcal{E}(\mathcal{X})$ if and only if $\{w^{i_1}_h, w^{i_2}_h, \ldots, w^{i_r}_h\} \in \mathcal{E}(\mathcal{X})$ for every h, $1 \le h \le n$, which holds if and only if $\{\phi(v^{i_1}), \phi(v^{i_2}), \ldots, \phi(v^{i_r})\} \in \mathcal{E}(\mathcal{Z}) \subset \mathcal{E}(\mathcal{X}^n)$. Therefore, ϕ is an isomorphism.

□

In our final auxiliary lemma, we show that if two different canonical subgraphs $\mathcal{Z}_1$ and $\mathcal{Z}_2$ share t vertices each in a different set $(V^i)^n$, then they must share an a copy of $\mathcal{F}$.

Before we state and prove this lemma, we claim the following.

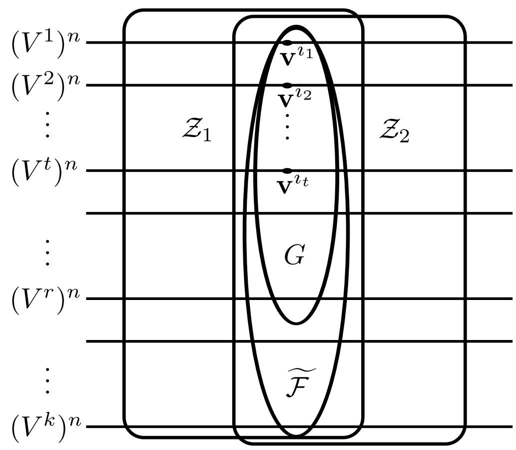


Figure 4.1: If $\mathcal{Z}_1$ and $\mathcal{Z}_2$ share t vertices each in a different set $(V^i)^n$, then $\mathcal{Z}_1$ and $\mathcal{Z}_2$ share a copy of $\mathcal{F}$ (Lemma 4.1.5)

Claim 4.1.4. *Let $\mathcal{F}$ be a complete Steiner (r, t)-system and $\mathcal{X} \in S_<(r, t)$ be with $\mathcal{F}$-union property. Then for any two copies of $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$, in $\binom{\mathcal{X}}{\mathcal{F}}$, $|V(\mathcal{F}_1) \cap V(\mathcal{F}_2)| < t$.*

Proof. Assume there exists t distinct vertices $v^{i_1}, v^{i_2}, \ldots, v^{i_t} \in V(\mathcal{F}_1) \cap V(\mathcal{F}_2)$. Since both $\mathcal{F}_1$ and $\mathcal{F}_2$ are complete, $v^{i_1}, v^{i_2}, \ldots, v^{i_t}$ are contained in edges $E_1 \in \mathcal{E}(\mathcal{F}_1)$ and $E_2 \in \mathcal{E}(\mathcal{F}_2)$. Because $\mathcal{X}$ is Steiner, $E_1 = E_2$. Consequently $\mathcal{F}_1$ and $\mathcal{F}_2$ would share an edge which contradicts the assumption that $\mathcal{X}$ has $\mathcal{F}$-union property. Hence, $|V(\mathcal{F}_1) \cap V(\mathcal{F}_2)| < t$.

□

Lemma 4.1.5. *Let $\mathcal{F}$ be a complete Steiner (r, t)-system, $\mathcal{X} \in \mathrm{Part}(k, r, t)$ be with $\mathcal{F}$-union property, and n be a positive integer. If $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are two distinct canonical subgraphs of $\mathcal{X}^n$ such that there exist distinct vertices*

$$\mathbf{v}^{i_1}, \mathbf{v}^{i_2}, \ldots, \mathbf{v}^{i_t} \in \bigcup_{i=1}^{k} V^i(\mathcal{Z}_1) \cap \bigcup_{i=1}^{k} V^i(\mathcal{Z}_2) \tag{1}$$

with the property that

$$|\{\mathbf{v}^{i_1}, \mathbf{v}^{i_2}, \ldots, \mathbf{v}^{i_t}\} \cap V^i(\mathcal{Z}_1) \cap V^i(\mathcal{Z}_2)| \leq 1 \textit{ for all } 1 \leq i \leq k, 1 \leq j \leq t \tag{2}$$

then there exists

$$\widetilde{\mathcal{F}} \in \binom{\mathcal{Z}_1}{\mathcal{F}} \cap \binom{\mathcal{Z}_2}{\mathcal{F}} \text{ such that } \{v^{i_1}, v^{i_2}, \ldots, v^{i_t}\} \subset G \text{ for some } G \in \mathcal{E}(\widetilde{\mathcal{F}}). \tag{3}$$

Proof. Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be lines in $HJC(n, \mathcal{Q})$ with moving coordinate sets M_1 and M_2, respectively. Note that if $M_1 = M_2$, then condition (1) of Lemma 4.1.5 does not hold. Indeed, since $\mathcal{Z}_1 \neq \mathcal{Z}_2$, the lines differ on some fixed coordinate, say h_0. Thus the set $\pi_{h_0}(\mathcal{Z}_1)$ consists of precisely one copy of $\mathcal{F}$ in $\mathcal{X}$ which we denote by $\mathcal{F}_1 \in \binom{\mathcal{X}}{\mathcal{F}}$. Similarly, $\pi_{h_0}(\mathcal{Z}_2) = \mathcal{F}_2 \in \binom{\mathcal{X}}{\mathcal{F}}$. By 4.1.4, $|V(\mathcal{F}_1) \cap V(\mathcal{F}_2)| < t$. Hence, condition (1) fails.

Therefore we will assume that $M_1 \neq M_2$. We need to prove that for each $h \in [n]$,

$$\pi_h(\mathbf{v}^{i_1}), \pi_h(\mathbf{v}^{i_2}), \ldots, \pi_h(\mathbf{v}^{i_t}) \in V(\mathcal{F}_h) \tag{4}$$

for some $\mathcal{F}_h \in \binom{\mathcal{X}}{\mathcal{F}}$ for the following cases:

First we consider $h \notin M_1 \cup M_2$. For such h observe that by Claim 4.1.4 and condition (1), we infer that $\pi_h(\mathcal{Z}_1) = \pi_h(\mathcal{Z}_2) = \mathcal{F}_h \in \binom{\mathcal{X}}{\mathcal{F}}$. For each $h \notin M_1 \cup M_2$, set $\mathcal{F}_h = \pi_h(\mathcal{Z}_1) = \pi_h(\mathcal{Z}_2)$. Consequently, $\pi_h(\mathbf{v}^{i_1}), \pi_h(\mathbf{v}^{i_2}), \ldots, \pi_h(\mathbf{v}^{i_t}) \in V(\mathcal{F}_h)$ and so (4) holds for all $h \notin M_1 \cup M_2$.

Next we consider $h \in M_1 \cup M_2$. Since $M_1 \neq M_2$, we may assume there exists $h_2 \in M_2 - M_1$. Set $\pi_{h_2}(\mathcal{Z}_1) = \mathcal{F}_1 \in \binom{\mathcal{X}}{\mathcal{F}}$. Since $\pi_{h_2}(\mathcal{Z}_2) = \binom{\mathcal{X}}{\mathcal{F}}$, we infer that $\pi_{h_2}(\mathcal{Z}_1 \cap \mathcal{Z}_2) = \mathcal{F}_1 \in \binom{\mathcal{X}}{\mathcal{F}}$. On the other hand, $\pi_h(\mathcal{Z}_1 \cap \mathcal{Z}_2) = \pi_{h_2}(\mathcal{Z}_1 \cap \mathcal{Z}_2)$ for all $h \in M_2$ and hence $\pi_h(\mathcal{Z}_1 \cap \mathcal{Z}_2) = \mathcal{F}_1 \in \binom{\mathcal{X}}{\mathcal{F}}$ for all such h as well. We conclude that $\pi_h(\mathbf{v}^{i_1}), \pi_h(\mathbf{v}^{i_2}), \ldots, \pi_h(\mathbf{v}^{i_t}) \in V(\mathcal{F}_1)$ and thus (4) holds for all $h \in M_2$.

If $M_1 \subset M_2$ we are done since (4) has been verified for all $h \in [n]$. If there exists $h_1 \in M_1 - M_2$, then similarly as before, we infer that for some $\mathcal{F}_2 \in \binom{\mathcal{X}}{\mathcal{F}}$ (which is equal to $\mathcal{F}_1$ if $M_1 \cap M_2 \neq \emptyset$) $\pi_h(\mathbf{v}^{i_1}), \pi_h(\mathbf{v}^{i_2}), \ldots, \pi_h(\mathbf{v}^{i_t}) \in V(\mathcal{F}_2)$ for all $h \in M_1$ and hence (4) holds for all $h \in M_1$ as well.

□

Now we are ready for the proof of Lemma 4.1.1 stated at the start of this section.

Proof. (of Lemma 4.1.1) For sufficiently large n, we will show that $\mathcal{Y} = \mathcal{X}^n \in Part(k, r, t)$, together with a system $\mathcal{C}$ of all canonical subgraphs of $\mathcal{X}^n$ that satisfy the conditions of Lemma 4.1.1, is the desired Ramsey r-graph. Let $c \geq 2$ and $\mathcal{Q} = \binom{\mathcal{X}}{\mathcal{F}}$. Set $|\mathcal{Q}| = q$ and $n = n(c, q)$, and consider a c-coloring of the copies of $\mathcal{F}$ in $\mathcal{X}^n$. By definition of the mapping λ, there is a 1-1 correspondence between sequences $\mathcal{F}_{h_1}, \mathcal{F}_{h_2}, \ldots, \mathcal{F}_{h_n}$ of copies of $\mathcal{F}$ in $\mathcal{X}$ and copies of $\mathcal{F}$ in $\mathcal{X}^n$. By definition of $n = n(c, q)$, Lemma 4.1.3 and Theorem **??**, there exists a monochromatic line which corresponds to a copy of canonical subgraph $\mathcal{Z} \subset \mathcal{X}^n$, $\mathcal{Z} \cong \mathcal{X}$, such that $\binom{\mathcal{Z}}{\mathcal{F}}$ is monochromatic.

□

4.1.2 The Partite Construction

Let $\mathcal{F}$ be a complete Steiner (r, t)-system and $\mathcal{G} \in \mathcal{S}_<(r, t)$ be given such that $V(\mathcal{F}) = \{1, 2, \ldots, k\}$ and $V(\mathcal{G}) = \{1, 2, \ldots, l\}$. Let $m = R_r(k, l; c)$ be the r-graph-Ramsey number of a k-uniform clique on l vertices $\mathcal{K}_l$ with c colors. Note that since the vertex set of $\mathcal{F}$ is linearly ordered, there is exactly one copy of $\mathcal{F}$ in each k-element set of vertices. Thus there is a 1-1 correspondence between edges of a k-uniform clique on m vertices $\mathcal{K}_m$ and copies of $\mathcal{F}$ in $\mathcal{K}_m$. Therefore, any c-coloring of $\binom{\mathcal{K}_m}{\mathcal{F}}$ yields a copy of $\mathcal{K}_l$ in which $\binom{\mathcal{K}_l}{\mathcal{F}}$ is monochromatic, and thus also a copy of $\mathcal{G}$ in which $\binom{\mathcal{G}}{\mathcal{F}}$ is monochromatic.

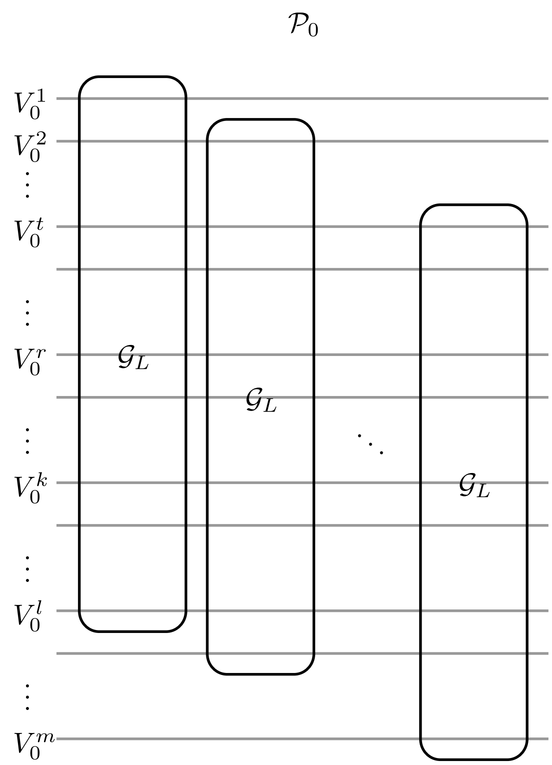


Figure 4.2: $\mathcal{P}_0$ of the partite construction

First we construct an auxiliary m-partite r-graph $\mathcal{P}_0$ as follows.

Let $V(\mathcal{P}_0) = V_0 = V_0^1 \cup V_0^2 \cup \cdots \cup V_0^m$ be ordered such that $V_0^1 < V_0^2 < \cdots < V_0^m$. For all $L \subset [m]$ such that $|L| = l$, set $L = \{i_1, i_2, \ldots, i_l\}$. Choose vertices $v^{i_1}, v^{i_2}, \ldots, v^{i_l}$ such that $v^{i_j} \in V_0^{i_j}$ for $1 \leq i \leq m, 1 \leq j \leq l$ such that $j \longmapsto v^{i_j}$ is an order preserving isomorphism between $\mathcal{G}$ and a copy of $\mathcal{G}$, which we denote by $\mathcal{G}_L$. We denote by $\mathcal{P}_0$ the disjoint union of $\binom{m}{l}$ copies of $\mathcal{G}_L$. Note that $|V(\mathcal{P}_0)| = \binom{m}{l}l$ and $|\mathcal{E}(\mathcal{P}_0)| = \binom{m}{l}|\mathcal{E}(\mathcal{G})|$. See Figure 4.2.

Consider an arbitrary, say lexicographic, ordering of $\binom{m}{k}$ k-sets such that $K_1 < K_2 < \cdots < K_{\binom{m}{k}}$ where $K_j = \{i_1(j), i_2(j), \ldots, i_k(j)\} \subset [m]$ for $1 \leq j \leq \binom{m}{k}$. We now construct an auxiliary m-partite r-graph $\mathcal{P}_1$ from $\mathcal{P}_0$ as follows.

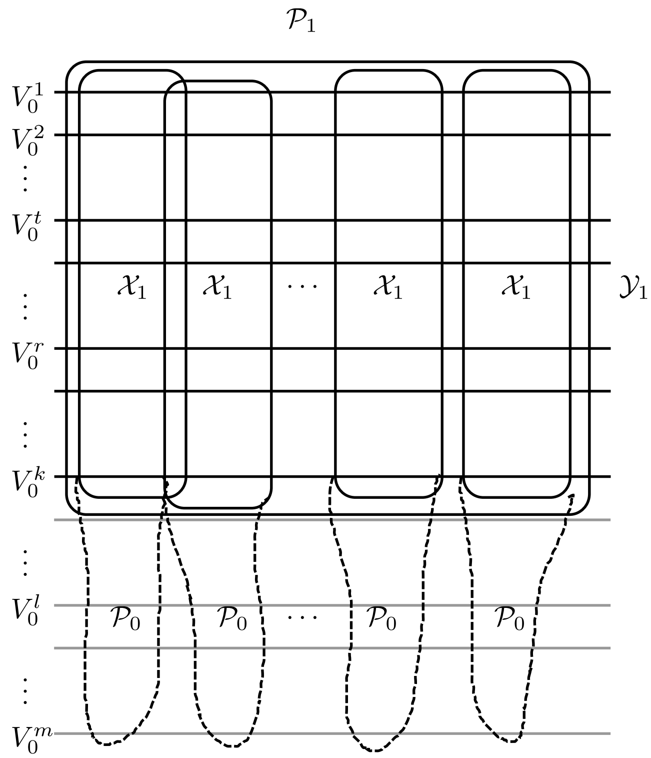


Figure 4.3: $\mathcal{P}_1$ of the partite construction

Take the vertex sets $V_0^{i_1(1)}, V_0^{i_2(1)}, \ldots, V_0^{i_k(1)}$ associated with the k-set K_1 (the first in the order of k-sets above), and let $\mathcal{X}_1 = \mathcal{P}_0[V_0^{i_1(1)} \cup V_0^{i_2(1)} \cup \cdots \cup V_0^{i_k(1)}]$ be the r-graph induced on $V_0^{i_1(1)}, V_0^{i_2(1)}, \ldots, V_0^{i_k(1)}$ in $\mathcal{P}_0$. Note that $\mathcal{X}_1 \in \mathrm{Part}(k, r, t)$ and $\mathcal{X}_1$ has the $\mathcal{F}$-union property (see Claim 4.1.7). By Lemma 4.1.1, there exists $\mathcal{Y}_1 \in \mathrm{Part}(k, r, t)$ with a system of copies $\mathcal{C}_1 \subset \binom{\mathcal{Y}_1}{\mathcal{X}_1}$ which has the property that any c-coloring of $\binom{\mathcal{Y}_1}{\mathcal{F}}$ yields a copy of $\mathcal{X}_1 \in \mathcal{C}_1 \subset \binom{\mathcal{Y}_1}{\mathcal{X}_1}$ in which $\binom{\mathcal{X}_1}{\mathcal{F}}$ is monochromatic. Now extend each copy of $\mathcal{X}_1 \in \mathcal{C}_1$ to a copy of $\mathcal{P}_0$ in such a way that for two distinct copies of $\mathcal{X}_1$ we have $\widetilde{\mathcal{X}_1} \subset \widetilde{\mathcal{P}_0}$ and $\widetilde{\widetilde{\mathcal{X}_1}} \subset \widetilde{\widetilde{\mathcal{P}_0}}$ such that $V(\widetilde{\mathcal{P}_0} \cap \widetilde{\widetilde{\mathcal{P}_0}}) = V(\widetilde{\mathcal{X}_1} \cap \widetilde{\widetilde{\mathcal{X}_1}})$. This completes the construction of $\mathcal{P}_1$. See Figure 4.3.

In general, we repeat this procedure to construct an auxiliary m-partite r-graph $\mathcal{P}_{j+1}$ from $\mathcal{P}_j$ for $1 \le j \le \binom{m}{k} - 1$ as follows.

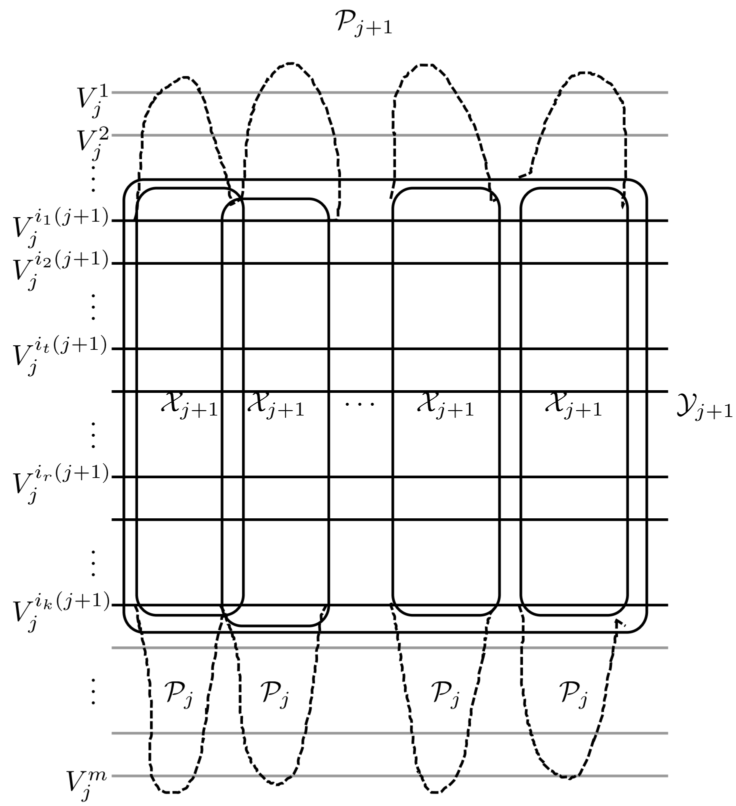


Figure 4.4: $\mathcal{P}_{j+1}$ of the partite construction

Take the vertex sets $V_j^{i_1(j+1)}, V_j^{i_2(j+1)}, \ldots, V_j^{i_k(j+1)}$ associated with the k-set K_{j+1} (the $(j+1)^{\text{st}}$ in the order of k-sets), and let $\mathcal{X}_{j+1} = \mathcal{P}_j[V_j^{i_1(j+1)} \cup V_j^{i_2(j+1)} \cup \ldots \cup V_j^{i_k(j+1)}]$ be the r-graph induced on $V_j^{i_1(j+1)}, V_j^{i_2(j+1)}, \ldots, V_j^{i_k(j+1)}$ in $\mathcal{P}_j$. Note that $\mathcal{X}_{j+1} \in \operatorname{Part}(k, r, t)$ and $\mathcal{X}_{j+1}$ has the $\mathcal{F}$-union property (see Claim

4.1.7). By Lemma 4.1.1, there exists $\mathcal{Y}_{j+1} \in \mathrm{Part}(k, r, t)$ with a system of copies $\mathcal{C}_{j+1} \subset \binom{\mathcal{Y}_{j+1}}{\mathcal{X}_{j+1}}$ which has the property that any c-coloring of the copies of $\binom{\mathcal{Y}_{j+1}}{\mathcal{F}}$ yields a copy of $\mathcal{X}_{j+1} \in \mathcal{C}_{j+1} \subset \binom{\mathcal{Y}_{j+1}}{\mathcal{X}_{j+1}}$ in which $\binom{\mathcal{X}_{j+1}}{\mathcal{F}}$ is monochromatic. Now extend each copy of $\mathcal{X}_{j+1} \in \mathcal{C}_{j+1}$ to a copy of $\mathcal{P}_j$ in such a way that for two distinct copies of $\mathcal{X}_{j+1}$ we have $\widetilde{\mathcal{X}_{j+1}} \subset \widetilde{\mathcal{P}_j}$ and $\widetilde{\widetilde{\mathcal{X}_{j+1}}} \subset \widetilde{\widetilde{\mathcal{P}_j}}$ such that $V(\widetilde{\mathcal{P}_j} \cap \widetilde{\widetilde{\mathcal{P}_j}}) = V(\widetilde{\mathcal{X}_{j+1}} \cap \widetilde{\widetilde{\mathcal{X}_{j+1}}})$. This completes the construction of $\mathcal{P}_{j+1}$. See Figure 4.4.

With $\binom{m}{k}$ applications of Lemma 4.1.1, we repeat the procedure above to construct auxiliary m-partite r-graphs $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{\binom{m}{k}}$. We will observe that $\mathcal{P}_j, 0 \leq j \leq \binom{m}{k}$, is a partial Steiner (r, t)-system in the following claim.

Claim 4.1.6. *For* $0 \leq j \leq \binom{m}{k}$, $\mathcal{P}_j \in \mathcal{S}_<(r, t)$.

Proof. Proceed by induction on j. For the base case $j = 0$, we observe that $\mathcal{P}_0$ is the disjoint union of partial Steiner (r, t)-systems, and is, therefore, a partial Steiner (r, t)-system itself. Assume that $\mathcal{P}_j$ is a partial Steiner (r, t)-system by the induction hypothesis. We need to show that $\mathcal{P}_{j+1}$ is a partial Steiner (r, t)-system.

Assume that $\mathcal{P}_{j+1} \notin \mathcal{S}_<(r, t)$ for sake of contradiction. Then there exists $E_1, E_2 \in \mathcal{E}(\mathcal{P}_{j+1})$ such that $|E_1 \cap E_2| \geq t$ for $E_1 \neq E_2$. Clearly both E_1 and E_2 cannot belong to $\mathcal{E}(\widetilde{\mathcal{P}_j})$ for some $\widetilde{\mathcal{P}_j} \in \binom{\mathcal{P}_{j+1}}{\mathcal{P}_j}$ by induction assumption. Therefore, $E_1 \in \mathcal{E}(\widetilde{\mathcal{P}_j})$ and $E_2 \in \mathcal{E}(\widetilde{\widetilde{\mathcal{P}_j}})$ for $\widetilde{\mathcal{P}_j} \neq \widetilde{\widetilde{\mathcal{P}_j}}$. Consequently, $E_1 \cap E_2 \subset V(\widetilde{\mathcal{X}_{j+1}}) \cap V(\widetilde{\widetilde{\mathcal{X}_{j+1}}})$ contains t distinct vertices $v^{i_1}, v^{i_2}, \ldots, v^{i_t}$.

By Lemma 4.1.5, there exists a copy $\mathcal{F}_1$ of $\mathcal{F}$ in the intersection of $\mathcal{E}(\widetilde{\mathcal{X}_{j+1}}) \cap \mathcal{E}(\widetilde{\widetilde{\mathcal{X}_{j+1}}})$ such that $\{v^{i_1}, v^{i_2}, \ldots, v^{i_t}\} \subset E$ for some $E \in \mathcal{E}(\mathcal{F}_1)$. In particular, there exists $E \in \mathcal{E}(\widetilde{\mathcal{X}_{j+1}}) \cap \mathcal{E}(\widetilde{\widetilde{\mathcal{X}_{j+1}}})$ such that $\{v^{i_1}, v^{i_2}, \ldots, v^{i_t}\} \subset E \in$

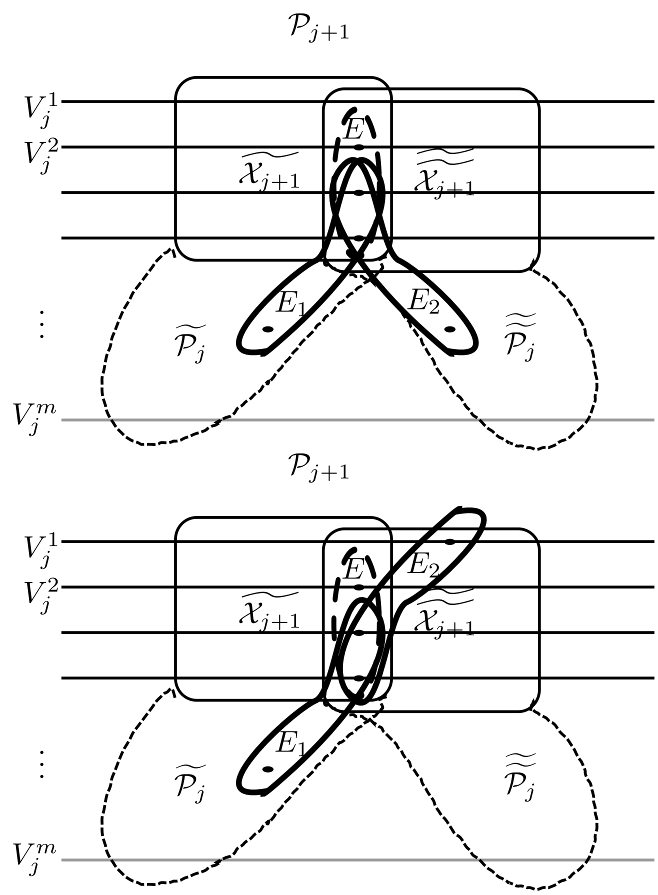


Figure 4.5: Edges of type E_1 and E_2 are *not* in $\mathcal{P}_{j+1}$ (illustrated for $t = 2, r = 3, k = 4$)

$\mathcal{E}(\mathcal{Y}_{j+1})$. But then either $E, E_1 \in \widetilde{\mathcal{P}_j} \in \mathcal{S}_<(r,t)$ or $E, E_2 \in \widetilde{\mathcal{P}_j} \in \mathcal{S}_<(r,t)$, and hence $|E \cap E_1| < t$ or $|E \cap E_2| < t$, a contradiction. See Figure 4.5.

Therefore, $|E_1 \cap E_2| < t$ for all $E_1, E_2 \in \mathcal{E}(\mathcal{P}_{j+1})$, and hence $\mathcal{P}_{j+1} \in \mathcal{S}_<(r,t)$ as desired.

$\square$

In the partite construction, we noted that $\mathcal{X}_j$ had the $\mathcal{F}$-union property.

Next we prove a stronger statement that $\mathcal{P}_j$, $0 \le j \le \binom{m}{k}$, has the $\mathcal{F}$-union property, by induction:

Claim 4.1.7. *If $\mathcal{G} \in S_<(r,t)$ has the $\mathcal{F}$-union property, then $\mathcal{P}_j$ has the $\mathcal{F}$-union property for all j, $0 \le j \le \binom{m}{k}$.*

Proof. Proceed by induction on j. For the base case, $j = 0$, we note that $\mathcal{P}_0$ has the $\mathcal{F}$-union property because and $\mathcal{P}_0$ is the disjoint union of copies of $\mathcal{G} \in S_<(r,t)$ with the $\mathcal{F}$-union property. Assume $\mathcal{P}_j$ has the $\mathcal{F}$-union property by the induction hypothesis. We need to show that $\mathcal{P}_{j+1}$ has the $\mathcal{F}$-union property.

First we observe that each $\widetilde{\mathcal{F}} \in \binom{\mathcal{P}_{j+1}}{\mathcal{F}}$ must belong to some $\widetilde{\mathcal{P}_j} \in \binom{\mathcal{P}_{j+1}}{\mathcal{P}_j}$ by distinguishing two cases. Indeed, if $\widetilde{\mathcal{F}}$ contains a vertex $v \in V(\mathcal{X}_{j+1})$, where $\mathcal{X}_{j+1} = \mathcal{P}_j[V_j^{i_1(j+1)} \cup V_j^{i_2(j+1)} \cup \ldots \cup V_j^{i_k(j+1)}]$, then v belongs to a unique copy $\widetilde{\mathcal{P}_j} \in \binom{\mathcal{P}_{j+1}}{\mathcal{P}_j}$. By construction, however, all vertices of $\widetilde{\mathcal{F}}$ must belong to $\widetilde{\mathcal{P}_j}$. This is because every t-element set of vertices of $\widetilde{\mathcal{F}}$ is in some edge which by construction of $\mathcal{P}_{j+1}$ must belong to $\mathcal{P}_j$. On the other hand, if $\widetilde{\mathcal{F}} \subset \mathcal{X}_{j+1}$, then by construction $\widetilde{\mathcal{F}}$ corresponds to a sequence of elements of $\binom{\mathcal{X}_{j+1}}{\mathcal{F}}$ which is an element of $HJC(n, \mathcal{Q})$. Every line containing this sequence corresponds to a copy of $\mathcal{P}_j$ in $\mathcal{P}_{j+1}$.

Now we are ready to proceed with the proof of Claim 4.1.7. Assume that $\mathcal{P}_{j+1}$ does not have the $\mathcal{F}$-union property for sake of contradiction. Then there exists $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{E}(\mathcal{P}_{j+1})$ sharing an edge $E \in \mathcal{E}(\mathcal{F}_1) \cap \mathcal{E}(\mathcal{F}_2)$. Clearly both $\mathcal{F}_1$ and $\mathcal{F}_2$ cannot belong to $\mathcal{E}(\widetilde{\mathcal{P}_j})$ by the induction assumption. Thus, there exist $\widetilde{\mathcal{P}_j}$ and $\widetilde{\widetilde{\mathcal{P}_j}} \in \binom{\mathcal{P}_{j+1}}{\mathcal{P}_j}$ with $\mathcal{F}_1 \subset \widetilde{\mathcal{P}_j}$ and $\mathcal{F}_2 \subset \widetilde{\widetilde{\mathcal{P}_j}}$. Consequently, $E \in \mathcal{E}(\mathcal{X}_{j+1})$ because $\mathcal{E}(\widetilde{\mathcal{P}_j}) \cap \mathcal{E}(\widetilde{\widetilde{\mathcal{P}_j}}) \subset \mathcal{E}(\mathcal{X}_{j+1})$. In view of Lemma 4.1.5, this however means that there is a copy $\mathcal{F}_3 \in \binom{\mathcal{X}_{j+1}}{\mathcal{F}}$ which belongs to both

$\widetilde{\mathcal{P}_j}$ and $\widetilde{\widetilde{\mathcal{P}_j}}$ containing this edge. But then either $\mathcal{F}_1, \mathcal{F}_3$ or $\mathcal{F}_2, \mathcal{F}_3$ are two distinct copies of $\mathcal{F}$ belonging to either $\widetilde{\mathcal{P}_j}$ or $\widetilde{\widetilde{\mathcal{P}_j}}$, contradicting the induction assumption.

$\square$

Recall that $\mathcal{X}_{j+1}$ is induced on a subset of vertices of $\mathcal{P}_j$, i.e. $\mathcal{X}_{j+1} = \mathcal{P}_j[V_j^{i_1(j+1)} \cup V_j^{i_2(j+1)} \cup \ldots \cup V_j^{i_k(j+1)}]$, $\mathcal{X}$. Since $\mathcal{P}_j$ has the $\mathcal{F}$-union property as shown in Claim 4.1.7, $\mathcal{X}_{j+1}$ has the $\mathcal{F}$-union property as well.

To prove Theorem 3.1.5 for a complete Steiner (r, t)-system $\mathcal{F}$, it remains to show that $\mathcal{P}_{\binom{m}{k}}$ is the desired Ramsey r-graph.

Claim 4.1.8. *$\mathcal{H} = \mathcal{P}_{\binom{m}{k}}$ is the desired Ramsey r-graph of Theorem 3.1.5 for a complete Steiner (r, t)-system $\mathcal{F}$.*

Proof. Given a c-coloring of the copies of $\mathcal{F}$ in $\mathcal{P}_{\binom{m}{k}}$. First, we consider only the copies of $\mathcal{F}$ induced on $V^{i_1(\binom{m}{k})} \cup V^{i_2(\binom{m}{k})} \cup \cdots \cup V^{i_k(\binom{m}{k})}$ of $\mathcal{P}_{\binom{m}{k}}$. Since $\mathcal{Y}_{\binom{m}{k}}$ is Ramsey for $\mathcal{X}_{\binom{m}{k}}$, there exists $\widetilde{\mathcal{X}_{\binom{m}{k}}}$, a copy of $\mathcal{X}_{\binom{m}{k}}$, contained in $\mathcal{Y}_{\binom{m}{k}}$ in which $\binom{\widetilde{\mathcal{X}_{\binom{m}{k}}}}{\mathcal{F}}$ is monochromatic. Let $\widetilde{\mathcal{P}_{\binom{m}{k}-1}} \supset \widetilde{\mathcal{X}_{\binom{m}{k}}}$ be a copy of $\mathcal{P}_{\binom{m}{k}-1}$ with $\binom{\widetilde{\mathcal{X}_{\binom{m}{k}}}}{\mathcal{F}}$ monochromatic. We repeat this process to obtain copies

$$\widetilde{\mathcal{P}_j} \text{ of } \mathcal{P}_j, j = \binom{m}{k} - 1, \ldots, 0$$

each associated with a copy $\widetilde{\mathcal{X}_{j+1}}$ of $\mathcal{X}_{j+1}$ satisfying $\widetilde{\mathcal{X}_{j+1}} \subset \widetilde{\mathcal{P}_j}$ and with $\binom{\widetilde{\mathcal{X}_{j+1}}}{\mathcal{F}}$ monochromatic.

Since $\widetilde{\mathcal{P}_{\binom{m}{k}-1}} \supset \widetilde{\mathcal{P}_{\binom{m}{k}-2}} \supset \cdots \supset \widetilde{\mathcal{P}_0}$, the resulting $\widetilde{\mathcal{P}_0}$ has the property that the color of each copy $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ in $\widetilde{\mathcal{P}_0}$ depends only on a k-tuple of sets

$V^{i_1}(\widetilde{\mathcal{P}_0}), V^{i_2}(\widetilde{\mathcal{P}_0}), \ldots, V^{i_k}(\widetilde{\mathcal{P}_0})$ with the property that $V(\widetilde{\mathcal{F}}) \cap V^{i_j}(\widetilde{\mathcal{P}_0}) \neq \emptyset$ for all $j = 1, 2, \ldots, k$. In other words, the color of each $\mathcal{F}$ in $\widetilde{\mathcal{P}_0}$ depends only on $K_j \subset [m], 1 \leq j \leq \binom{m}{k}$ with the property that

$$\widetilde{\mathcal{F}} \subset \bigcup_{i \in K_j} V^i(\widetilde{\mathcal{P}_0}). \tag{5}$$

Let $\gamma\colon \binom{m}{k} \longrightarrow [c]$ be a c-coloring assigning to each $K_j \in \binom{[m]}{k}$ a color common to all copies of $\mathcal{F}$ satisiying (5). Since $m = R_r(k, l; c)$, there exists $V^{i_1}(\widetilde{\mathcal{P}_0}), V^{i_2}(\widetilde{\mathcal{P}_0}), \ldots, V^{i_l}(\widetilde{\mathcal{P}_0})$ with all crossing copies of $\mathcal{F}$ monochromatic. Due to the construction of $\mathcal{P}_0$, we obtain a copy of $\mathcal{G}$ in which $\binom{\mathcal{G}}{\mathcal{F}}$ is monochromatic.

□

4.1.3 Proof of Theorem 3.1.5 for other positive cases: $\mathcal{F}$ is an edge and $|V(\mathcal{F})| < t$

We have just proved Theorem 3.1.5 for the case $\mathcal{F}$ is a complete Steiner (r,t)-system. It remains to discuss the cases when $\mathcal{F}$ is an edge or $|V(\mathcal{F})| < t$. In either of these cases the proof of the partite lemma can be repeated almost verbatim (in fact, in somewhat simplified form). In the partite construction, we needed to make sure that $\mathcal{P}_{j+1}$ is a partial Steiner (r,t)-system provided $\mathcal{P}_j$ is. The proof of this fact is based on Claim 4.1.6. When $\mathcal{F}$ is an edge, the proof is the same as that of Claim 4.1.6. In the case $|V(\mathcal{F})| < t$, the reason is much simpler since any two copies of $\mathcal{P}_j$ intersect in at most a $t-1$-partite r-graph, any two edges of $\mathcal{P}_{j+1}$ intersect in less than t vertices, provided $\mathcal{P}_j$ has the same property.

Chapter 5

Upper Density of Quasi-random Hypergraphs

5.1 Introduction

Recall that for fixed $l \geq 2$, an l-*graph* $G = (V, E)$ is an l-uniform hypergraph with vertex set V and edge set $E \subseteq \binom{V}{l}$, or a subset of the l-tuples of V. For $K \subseteq V$ and $|K| = k$, we denote the l-subgraph of G *induced* by K as $G[K] = (K, E \cap \binom{K}{l})$. The *density* of such an l-graph G is defined by $d(G) = |E|/\binom{|V|}{l}$.

Let $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be a sequence of l-graphs with $G_n = (V_n, E_n)$ such that $|V_n| \to \infty$ as $n \to \infty$. We define the density $d(\mathcal{G})$ of a sequence $\mathcal{G}$ as $d(\mathcal{G}) = \lim_{n\to\infty} d(G_n)$ (if the limit exists). We will consider only graph sequences for which the limit $d(G_n)$ exists as $n \to \infty$.

Setting

$$\sigma_k(\mathcal{G}) = \max_n \max_{K \in \binom{V_n}{k}} d(G_n[K]),$$

an averaging argument yields that $\{\sigma_k(\mathcal{G})\}_{k=2}^{\infty}$ is a non-increasing non-negative sequence and so the limit $\overline{d}(\mathcal{G}) = \lim_{k\to\infty} \sigma_k(\mathcal{G})$ exists. We call this limit $\overline{d}(\mathcal{G})$

the *upper density* of $\mathcal{G}$.

The result we present in this chapter are motivated by a theorem of Erdős [8].

Theorem 5.1.1. *For every $\epsilon > 0, l \geq 2$ and $t > 0$, there exists n such that any l-graph with n vertices and ϵn^l edges contains a complete l-partite l-graph $K^{(l)}_{t,t,\ldots,t}$. Consequently, for any sequence $\mathcal{G}$ of l-graphs with $d(\mathcal{G}) > 0$, $\overline{d}(\mathcal{G}) \geq l!/l^l$.*

In this note we are interested in a similar problem if we restrict to quasi-random l-graphs.

Definition 5.1.2. *Given $\epsilon > 0$ and $\alpha > 0$, we define an (α, ϵ)-*quasi-random *hypergraph to be an l-graph $Q = (V, E)$ with the property that for all $W \subseteq V$, $d(Q[W]) = \alpha(1 \pm \epsilon)$ for $|W| \geq \epsilon n$ where $|V| = n$. A sequence $\mathcal{Q} = \{Q_n\}_{n=1}^{\infty}$ of (α, ϵ_n)-quasi-random l-graphs is* quasi-random *if ϵ_n is decreasing and $\epsilon_n \to 0$ as $n \to \infty$.*

Note that for $l = 2$ quasi-random graphs must contain arbitrarily large cliques as $\epsilon_n \to 0$ and thus any quasi-random sequence of 2-graphs with $d(\mathcal{Q}) > 0$ necessarily satisfies $\overline{d}(\mathcal{Q}) = 1$. In this note we prove a related result for $l \geq 3$.

Theorem 5.1.3. *For a sequence $\mathcal{Q}$ of quasi-random l-graphs with $d(\mathcal{Q}) > 0$,*

(i) $\overline{d}(\mathcal{Q}) \geq \frac{(l-1)!}{l^{l-1}-1}$ and

(ii) when $l = 3$ there exists a quasi-random sequence of 3-graphs with $\overline{d}(\mathcal{Q}) = \frac{1}{4}$.

For $l > 3$, however, we do not know if $\overline{d}(\mathcal{Q}) \geq \frac{(l-1)!}{l^{l-1}-1}$ could not be replaced by a larger number. Our results for $l = 3$ are shown in the Section 2.1 and a similar construction may be applied to generalize the result for all l-graphs, proving Theorem 5.1.3(i).

A number α is a *jump* if there exists a constant $c = c(\alpha)$ such that given any sequence of l-graphs $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ if $\overline{d}(\mathcal{G}) > \alpha$, then $\overline{d}(\mathcal{G}) \geq \alpha + c$. It follows from the Erdős-Stone Theorem that all non-negative numbers less than 1 are jumps for graphs and it follows from Theorem 5.1.1 that all non-negative numbers less than $l!/l^l$ are jumps for l-graphs. Erdős conjectured that, analagous to graphs, all numbers less than 1 are jumps for l-graphs as well. This conjecture was disproved by Frankl and Rödl in [12] who showed that there are an infinite number of non-jumps for all $l \geq 3$. However, these non-jumps were found to occur at relatively large densities. While the smallest case of determining whether $l!/l^l$ is a jump is still open and likely a difficult problem, our result shows that under the further assumption of quasi-randomness that $l!/l^l$ is indeed a jump for all $l \geq 3$.

We extend the concept of jumps to sequences of quasi-random l-graphs.

Definition 5.1.4. *A number α is a* jump for quasi-random l-graphs *if there exists a constant $c = c(\alpha)$ such that given any sequence of quasi-random l-graphs $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ if $\overline{d}(\mathcal{G}) > \alpha$, then $\overline{d}(\mathcal{G}) \geq \alpha + c$.*

Theorem 5.1.3(i) implies that every number between 0 and $\frac{(l-1)!}{l^{l-1}-1}$ is a jump for quasi-random l-graphs. Further we will show that for $l = 3$ this interval can be improved from $[0, \frac{1}{4})$ to [0,0.3192) given the following question of Erdős [9] is answered positively.

Question 5.1.5. *Let $c > 0$ and $\mathcal{Q} = \{Q_n\}_{n=1}^{\infty}$ be a quasi-random sequence of 3-graphs. If $d(\mathcal{Q}) = \frac{1}{4} + c$, then does each Q_n contain $K_4^{(3)} - e$ as $n \to \infty$?*

More formally, we prove the following in Section 5.3.

Theorem 5.1.6. *A positive answer to Question 5.1.5 implies that any quasi-random sequence $\mathcal{Q}$ with $d(\mathcal{Q}) > \frac{1}{4}$ satisfies $\overline{d}(\mathcal{Q}) > 0.3192$.*

Very recently, Glebov, Kráľ' and Volec in [13] answered Question 5.1.5 in the positive using Razborov's flag-algebra method [29]. This result confirms our assertion in Theorem 5.1.6.

In Section 5.4, we include remarks and questions for future study of quasi-random l-graphs with $l > 3$ and other possibilities for jumps for quasi-random 3-graphs.

5.2 Proof of Theorem 5.1.3

5.2.1 The lower bound

Our proof is based on the following lemma proved in [2] and [21].

Lemma 5.2.1. *For all $\alpha > 0$ and $\epsilon > 0$, there exists $\delta > 0$, $m > 0$ and $n_0 > 0$ such that if $Q = (V, E)$ is an (α, δ)-quasi-random l-graph with $|V| = n \geq n_0$ vertices then $Q[M]$ is (α, ϵ)-quasi-random for at least $\frac{1}{2}\binom{n}{m}$ m-sets $M \in \binom{n}{m}$.*

Going forward in this subsection, we restrict to $l = 3$ for simplicity. Essentially the same statements may be applied to general l-graphs.

Given a 3-graph F, $\alpha > 0$ and $\epsilon > 0$, we write $(\alpha, \epsilon) \to F$ to denote the fact that every (α, ϵ)-quasi-random 3-graph R contains F. Let F and H be 3-graphs. For F, H, and $v \in V(F)$, we define F_H^v to be the 3-graph as follows.

(i) $V(F_H^v) = V(F) \cup V(H) - \{v\}$ and

(ii) $E(F_H^v) = E(F - \{v\}) \cup E(H) \cup \bigcup_{u \in V(H)} \{\{a,b,u\} \colon \{a,b,v\} \in E(F)\}$

In other words, to obtain F_H^v from F, replace v with $V(H)$ and add all the edges in H as well as the edges of type $\{a,b,u\}$ where $u \in V(H)$ and $\{a,b,v\} \in E(F)$. In this construction we will assume that F and H are vertex disjoint and thus $|V(F_H^v)| = |V(F)| + |V(H)| - 1$ and $|E(F_H^v)| = |E(F)| + |E(H)| + |V(H) - 1||\{e \in E(F) \colon v \in e\}|$.

Using the notation stated above, we observe the following.

Lemma 5.2.2. *For all $\alpha > 0$, $\epsilon > 0$, $\gamma > 0$ and 3-graphs F and H, there exists $\delta = \delta(\alpha, \epsilon, \gamma) > 0$ such that if $(\alpha, \epsilon) \to F$ and $(\alpha, \gamma) \to H$, then $(\alpha, \delta) \to F_H^v$.*

Proof. Let $|V(F)| = f$ and let $v \in V(F)$. Given $\alpha > 0$ and $\epsilon > 0$ such that $(\alpha, \epsilon) \to F$, let $\delta_{L(2.1)}$ and $m = m(\alpha, \epsilon)$ be the constants ensured by Lemma 5.2.1. Consider an (α, δ)-quasi-random hypergraph Q on n vertices. Set $\delta \leq \min(\delta_{L(2.1)}, \gamma/2m^f)$. We want to show that Q must contain F_H^v. By Lemma 5.2.1, $R = Q[M]$ is (α, ϵ)-quasi-random for at least $\frac{1}{2}\binom{n}{m}$ M's. By assumption ($(\alpha, \epsilon) \to F$) each such (α, ϵ)-quasi-random $Q[M]$ contains a copy of F. Consequently, the number of $Q[M]$'s with each containing a copy of F is at least $\frac{1}{2}\binom{n}{m}$. On the other hand, each copy of F is in at most $\binom{n-f}{m-f}$ different $Q[M]$'s. Thus, we have at least

$$\frac{1}{2}\binom{n}{m} / \binom{n-f}{m-f} = \frac{\binom{n}{f}}{2\binom{m}{f}} > \frac{1}{2}(\frac{n}{m})^f = cn^f$$

distinct copies of F in Q, where $c = c(m(\alpha, \epsilon), f) = 1/2m^f$. Set $V(F) = \{u_1, u_2, \ldots, u_{f-1}, v\}$ and let $F^{copy} = F^c$ be a copy of F in Q with $V(F^c) = \{u_1^c, u_2^c, \ldots, u_{f-1}^c, v^c\}$ so that $u_i \to u_i^c$ for $i = 1, 2, \ldots, f-1$ and $v \to v^c$ is an isomorphism.

For each of the cn^f copies F^c of F, consider an ordered $(f-1)$-tuple $(u_1^c, u_2^c, \ldots, u_{f-1}^c)$. Since the total number of $(f-1)$-tuples of vertices of Q is bounded by $n(n-1)\ldots(n-(f-1)) \leq n^{f-1}$ we infer that there exists an $(f-1)$-tuple of vertices $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_{f-1}$ of Q contained in $cn^f/n^{f-1} \sim cn$ copies F^c of F. Consider a set S, $|S| = cn = n/cm^f$, of vertices $\overline{v}$ each of which together with $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_{f-1}$ induces a copy F^c of F. Due to the (α, δ)-quasi-randomness of Q and the fact that $\delta \leq \gamma/2m^f = c\gamma$, $Q[S]$ is (α, γ)-quasi-random and, therefore, due to the assumption of Lemma 5.2.2, contains a copy of H with vertex set $V(H) = \{v_1, \ldots, v_{|V(H)|}\}$. Since each v_i $(1 \leq i \leq |V(H)|)$ together with $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_{f-1}$ span a copy F^c of F, we infer that $\{\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_{f-1}, v_1, \ldots, v_{|V(H)|}\}$ spans a copy of F_H^v. Thus, $(\alpha, \delta) \to F_H^v$.

$\square$

Before we prove Theorem 5.1.3(i) for $l = 3$, we construct an auxilliary sequence of 3-graphs $\mathcal{G} = \{G_i\}_{i=1}^{\infty}$ with density tending to $\frac{1}{4}$. We will then show that G_i is in Q_n for n large enough. Let G_1 be a 3-graph with three vertices and one edge. For $i > 1$, let G_i be the 3-graph obtained by taking 3 vertex disjoint copies of G_{i-1}, and adding all edges with exactly one vertex in each copy. For instance, G_2 has 9 vertices and $3 + 3^3 = 30$ edges.

Since $|V(G_i)| = 3|V(G_{i-1})| = 3^i$ and

$$\begin{aligned}
|E(G_i)| &= |V(G_{i-1})|^3 + 3|E(G_{i-1})| \\
&= 3^{3(i-1)}(1 + \frac{1}{9} + \ldots \frac{1}{9^{i-1}}) \\
&= 3^{i-1}\frac{(3^i-1)(3^i+1)}{8},
\end{aligned}$$

the density of G_i as $i \to \infty$ is

$$\lim_{i\to\infty} d(G_i) = \lim_{i\to\infty} \frac{3^{i-1}\frac{(3^i-1)(3^i+1)}{8}}{\binom{3^i}{3}} = \lim_{i\to\infty} \frac{1}{4}\left(\frac{3^i+1}{3^i-2}\right) = \frac{1}{4}.$$

Consider an arbitrary sequence of (α, δ_n)-quasi-random 3-graphs $\mathcal{Q} = \{Q_n\}_{n=1}^{\infty}$ with $d(Q_n) = \alpha(1 \pm \delta_n) > 0$ where $\delta_n \in (0,1)$, δ_n is decreasing and $\delta_n \to 0$ as $n \to \infty$. We will show that there exists $n_1 < n_2 < n_3 < \dots$ such that for $n \geq n_i$, Q_n contains G_i. Based on our density calculation of G_i above, $\overline{d}(\mathcal{Q}) \geq \frac{1}{4}$.

Since Q_n contains G_1 whenever $\delta_n < \alpha$, it remains to show the following claim by induction on i.

Claim 5.2.3. *Assuming $(\alpha, \delta_{n_i}) \to G_i$, there exists n_{i+1} such that $(\alpha, \delta_{n_{i+1}}) \to G_{i+1}$*

Proof. Our goal is to find n_{i+1} so that $(\alpha, \delta_n) \to G_{i+1}$ for all $n \geq n_{i+1}$. This will be achieved in three applications of Lemma 5.2.2 as shown in Figure 5.1. We will construct hypergraphs F', F'', F''' with $G_i \subseteq F' \subseteq F'' \subseteq F''' = G_{i+1}$ and n', n'', n''' with $n_i < n' < n'' < n''' = n_{i+1}$ such that

$$(\alpha, \delta_n) \to F^{(i)} \text{ for all } n \geq n^{(i)} \tag{*}$$

Set $V(G_1) = \{a, b, c\}$, $H = G_i$, and $\gamma = \delta_{n_i}$. Below we will describe appropriate choices of F, ϵ and v to obtain graphs $F^{(i)}$, $i = 1, 2, 3$ satisfying (*).

a) Set $F = G_1$, $\epsilon = \delta_1$ and $v = a$. Since $(\alpha, \delta_1) \to G_1$ and $(\alpha, \delta_{n_i}) \to G_i$, by Lemma 5.2.2 there exists $\delta' = \delta(\alpha, \delta_1, \delta_{n_i})$ such that $(\alpha, \delta') \to F^a_{G_i}$.

b) Set $F' = F^a_{G_i}$, $\epsilon = \delta'$ and $v = b$. Since $(\alpha, \delta') \to F'$ and $(\alpha, \delta_{n_i}) \to G_i$, by Lemma 5.2.2 there exists $\delta'' = \delta(\alpha, \delta', \delta_{n_i})$ such that $(\alpha, \delta'') \to F'^b_{G_i}$.

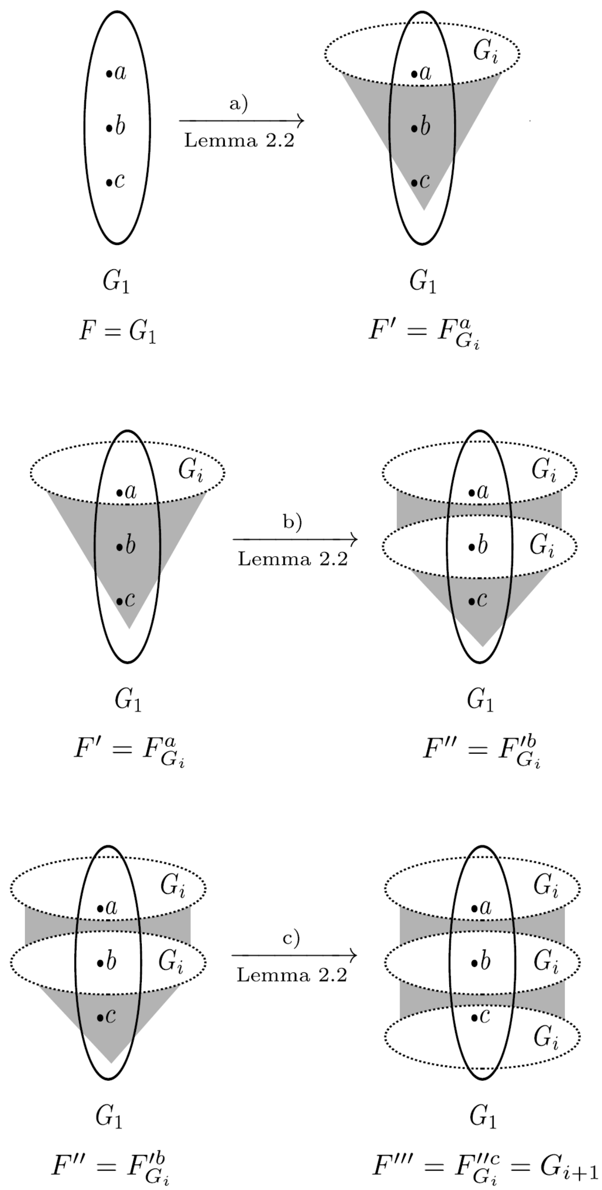


Figure 5.1: Three applications of Lemma 2.2 prove Claim 2.3

c) Set $F'' = F'^{b}_{G_i}$, $\epsilon = \delta''$ and $v = c$. Since $(\alpha, \delta'') \to F''$ and $(\alpha, \delta_{n_i}) \to G_i$, by Lemma 5.2.2 there exists $\delta''' = \delta(\alpha, \delta'', \delta_{n_i})$ such that $(\alpha, \delta''') \to F''^{c}_{G_i}$.

Observe that $F''' = F''^{c}_{G_i} = G_{i+1}$. Consequently $(\alpha, \delta_n) \to G_{i+1}$ for all n with $\delta_n \leq \delta'''$.

□

In a similar way to Claim 5.2.3 one can show a slightly more general fact stated below as Proposition 5.2.5. First we define the lexicographic product of two 3-graphs.

Definition 5.2.4. *The* lexicographic product *of two 3-graphs F and H with vertex sets U and W respectively is a 3-graph $F \cdot H$ with vertex set $U \times W$ and with $\{(u_1, w_1), (u_2, w_2), (u_3, w_3)\} \in E(F \cdot H)$ if $\{u_1, u_2, u_3\} \in E(F)$ or if $u_1 = u_2 = u_3$ and $\{w_1, w_2, w_3\} \in E(H)$.*

Proposition 5.2.5. *For all $\alpha > 0$, $\epsilon > 0$, $\gamma > 0$ and 3-graphs F and H there exists $\delta = \delta(\alpha, \epsilon, \gamma) > 0$ such that $(\alpha, \epsilon) \to F$ and $(\alpha, \gamma) \to F$ implies $(\alpha, \delta) \to F \cdot H$.*

5.2.2 The upper bound for l=3

It remains to show there exists a sequence of quasi-random 3-graphs with upper density $\frac{1}{4}$.

Proof. Consider a random tournament T_n on n vertices in which pairs are assigned arc direction with probability $\frac{1}{2}$. Let R_n be a 3-graph with $V(R_n) = V(T_n)$ and $E(R_n)$ consisting of vertex sets of all directed 3-cycles in T_n. This 3-graph was first considered by Erdős and Hajnal in [10] in the context of Ramsey theory.

It is well-known (see [9]) that R_n is $(\frac{1}{4}, \delta_n)$-quasi-random with $\delta_n \to 0$ as $n \to \infty$. On the other hand it follows from the well known result of

Kendall and Babington Smith [19] that any tournament on n vertices has at most $\frac{1}{24}(n^3 - n)$ directed 3-cycles (cf. [20]) and so no subgraph of any R_n has density larger than $\frac{1}{4} + o(1)$. Thus the upper density of the sequence $\mathcal{R} = \{R_n\}_{n=1}^{\infty}$ is at most $\frac{1}{4} + o(1)$ establishing (ii) of Theorem 5.1.3.

□

5.3 Proof of Theorem 5.1.6

For $l = 3$, Theorem 5.1.3(i) implies that every number in $[0, \frac{1}{4})$ is a jump for quasi-random 3-graphs. In this section, we prove that $\frac{1}{4}$ is a jump as well and, more precisely, any number in $[\frac{1}{4}, 0.3192)$ is a jump for quasi-random 3-graphs given Question 5.1.5 is answered positively. To this end, we use a recent result of Glebov, Král' and Volec who in [13] answered Question 5.1.5 using a computer aided proof based on Razborov's flag-algebra method [29].

Proof. Given a sequence of quasi-random 3-graphs $\mathcal{Q} = \{Q_n\}_{n=1}^{\infty}$ with $\overline{d}(\mathcal{Q}) > \frac{1}{4}$, any Q_n with $n \geq n_0$ contains $K_4^{(3)} - e$ by [13]. In a way similar to the proof of Theorem 5.1.3(i) we will first construct a sequence of 3-graphs $\mathcal{F} = \{F_n\}_{n=1}^{\infty}$ such that $F_n \subseteq Q_n$ and $\lim_{n\to\infty} d(F_n) = \frac{3}{10}$. Subsequently we will alter it to a sequence of 3-graphs $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ in which $\lim_{n\to\infty} d(G_n) \approx 0.3192$.

Let $F_1 = K_4^{(3)} - e$ with $V(F_1) = \{a_1, a_2, a_3, b\}$ and $E(F_1) = \{\{a_1, a_2, b\}, \{a_1, a_3, b\}, \{a_2, a_3, b\}\}$. Let A_i $(1 \leq i \leq 3)$ and B be copies of $K_4^{(3)} - e$. We obtain F_2 by taking four vertex disjoint copies of F_1, with vertex set A_i, $1 \leq i \leq 3$, and B and adding edges of type $\{a_i, a_j, b\}$ where $a_i \in A_i, a_j \in A_j, b \in B, 1 \leq i < j \leq 3$. Note that $|V(F_2)| = 4^2 = 16$ and $|E(F_2)| = 3(4) + 4^3(3)$. In other words $F_2 = F_1 \cdot F_1$ is the lexicographic product of two copies of F_1. We continue in this fashion to construct the sequence $\mathcal{F}$. For $i > 1$, let $F_i = F_1 \cdot F_{i-1}$ be the 3-graph obtained by taking four vertex disjoint copies of F_{i-1}, and adding edges in a similar way as described above. Since $|V(F_i)| = 4|V(F_{i-1}| = 4^i$ and

$$|E(F_i)| = 3|V(F_{i-1})| + 4^3|E(F_{i-1})| = 3 \cdot 4^{i-1}(1 + 4^2 + \ldots + 4^{2(i-1)}) = \frac{4^{i-1}}{5}(16^i - 1),$$

the density of F_i as $i \to \infty$ is

$$\lim_{i \to \infty} d(F_i) = \lim_{i \to \infty} \frac{\frac{4^{i-1}}{5}(16^i - 1)}{\binom{4^i}{3}} = \frac{3}{10}.$$

In a similar way as in the proof of Theorem 5.1.3(i), one can show that for all i there exists n such that F_i is contained in Q_n. Thus, every number between 0 and $\frac{3}{10}$ is a jump for quasi-random 3-graphs.

One can improve $\frac{3}{10}$ to 0.3192 by considering conveniently chosen "blow ups" of F_i. We will describe this in more detail now. Setting $V(F_i) = \{1, 2, \ldots, \nu_i\}$, we first observe (similarly as in Lemma 5.2.2) that for each i, there exists an n_i so that 3-graphs Q_n, $n \geq n_i$, contain $c_i|V(Q_n)|^{\nu_i}$ copies of F_i. Hence by Theorem 5.1.1, Q_n contains a t-blowup $F_i * t$ of F_i, more precisely, a graph with vertex set $\bigcup_{j=1}^{\nu_i} W_j$, $|W_1| = \cdots = |W_{\nu_i}| = t$ and $\{\widetilde{a}, \widetilde{b}, \widetilde{c}\} \in E(F_i * t)$ if $\{a, b, c\} \in E(F_i)$. In order to maximize the density, we consider graphs F_i with different vertices "blown up" to sets of different cardinalities.

More precisely, set $\alpha = \frac{2}{5}(4\sqrt{6} - 9) \approx 0.2154$ and to each vertex $\overline{x} = (x_1, \ldots, x_i) \in V(F_i)$ assign a weight $w(\overline{x}) = (1 - 3\alpha)^j \alpha^{i-j}$ where j represents the number of b's among entries of $\overline{x}$ and for t large consider a blow-up G_i of F_i with each vertex $\overline{x}$ "blown-up" by $w(\overline{x}) * t$ vertices. Using this iterated construction, one can calculate that every number between 0 and $\frac{1}{19}(9 - 2\sqrt{6}) \approx 0.3192$, where $\frac{1}{19}(9 - 2\sqrt{6}) = \lim_{i \to \infty} d(G_i)$, is a jump for quasi-random 3-graphs.

$\square$

5.4 Concluding remarks

In Section 2.2 we considered $\mathcal{R} = \{R_n\}_{n=1}^{\infty}$, a sequence of quasi-random 3-graphs formed by random tournaments T_n, and observed that $d(\mathcal{R}) = \overline{d}(\mathcal{R}) = \frac{1}{4}$. There are other quasi-random sequences of 3-graphs with density equal to upper density. Consider the quasi-random sequences $\mathcal{Q} = \{Q_n\}_{n=1}^{\infty}$ described in [32]: Let χ be a random $(k-1)$-coloring of pairs of $\{1, \ldots, n\}$ and define the edges of Q_n to be all triples $\{i, u, v\}$ such that $\chi(\{i, u\}) \neq \chi(\{i, v\})$. It can be shown that $d(\mathcal{Q}) = \overline{d}(\mathcal{Q}) = 1 - \frac{1}{k-1}$. In summary, if $\alpha \in \{\frac{1}{4}, \frac{1}{2}, \frac{2}{3}, \ldots\}$, then there is a sequence of quasi-random 3-graphs with $d(\mathcal{Q}) = \overline{d}(\mathcal{Q})$. Are there any others?

We proved that a sequence of quasi-random l-graphs $\mathcal{Q}$ with $d(\mathcal{Q}) > 0$ has $\overline{d}(\mathcal{Q}) \geq \frac{(l-1)!}{l^{l-1}-1}$. In particular, we showed that this bound is the best possible when $l = 3$. For $l = 4$, it is not clear to the authors if there exists a quasi-random sequence of 4-graphs with upper density equal to $\frac{3!}{4^3-1} = \frac{2}{21}$.

Theorem 5.1.3(i) implies that every quasi-random sequence of l-graphs with positive density has upper density at least $\frac{(l-1)!}{l^{l-1}-1}$. For $l = 3$ this is the best possible, but we were unable to show an analagous fact for $l > 3$. One can observe that $\frac{(l-1)!}{l^{l-1}-1}$ cannot be replaced by a number larger than $\frac{(l-1)!}{(l-1)^{l-1}}$. In order to see this, consider the quasi-random sequence $\mathcal{Q} = \{Q_n\}_{n=1}^{\infty}$ with vertex set $V(Q_n) = \{1, \ldots, n\} = [n]$. Let χ be a random $(l-1)$-coloring of pairs of $[n]$. Define the edge set $\{i, v_1, \ldots, v_{l-1}\} \in E(Q_n)$ if and only if all pairs $\{i, v_1\}$, $\ldots$, $\{i, v_{l-1}\}$ have different color. One can observe that $d(\mathcal{Q}) = \overline{d}(\mathcal{Q}) = \frac{(l-1)!}{(l-1)^{l-1}}$.

Bibliography

[1] F. Abramson and L.A. Harrington. Models without indiscernibles. *The Journal of Symbolic Logic*, 43(3):572–600, 1978.

[2] N. Alon, W. de la Vega, R. Kannan, and M. Karpinski. Random sampling and approximation of MAX-CSP problems. In *Proceedings of the 34th annual ACM symposium on Theory of computing*, pages 232–239. ACM Press, 2002.

[3] Noga Alon and Joel H. Spencer. *The Probabilistic Method*. Wiley, New York, 1992.

[4] W. Deuber. *A generalization of Ramsey's theorem, in: Infinite and Finite Sets, Vol. 1: 323-332, Colloquia Mathmematics Societatis János Bolyai, Vol. 10*. North-Holland, Amsterdam/London, 1975.

[5] Walter Deuber. Partitionstheoreme für graphen. *Commentarii Mathematici Helvetici*, 50(1):311–320, 1975.

[6] P. Erdős and A. Hajnal. On chromatic number of graphs and set systems. *Acta Mathematica Hungarica*, 17(1-2):61–99, 1966.

[7] P. Erdős, A. Hajnal, and L. Posa. *Strong embeddings of graphs into colored graphs, in: Infinite and Finite Sets, Vol. 1: 585-595, Colloquia Mathmematics Societatis János Bolyai, Vol. 10*. North-Holland, Amsterdam/London, 1975.

[8] P. Erdős. On extremal problems of graphs and generalized graphs. *Israel Journal of Mathematics*, 2:183–190, 1964.

[9] P. Erdős. Problems and results on graphs and hypergraphs: Similarities and differences. In J. Nešetřil and V. Rödl, editors, *Mathematics of Ramsey Theory*, pages 223–233. Springer-Verlag, 1990.

[10] P. Erdős and A. Hajnal. On Ramsey like theorems. *Proceedings: Conference in Combinatorial Mathematics, Mathematics Institute, Oxford University*, pages 123–140, 1972.

[11] P. Erdős and A.H. Stone. On the structure of linear graphs. *The Bulletin of the American Mathematical Society*, 52:1087–1091, 1946.

[12] P. Frankl and V. Rödl. Hypergraphs do not jump. *Combinatorica*, 4(2):149–159, 1984.

[13] R. Glebov, D. Král', and J. Volec. A problem of Erdős and Sós on 3-graphs. *Submitted*.

[14] R. Gould. *Graph Theory*. Dover Books on Mathematics, 2013.

[15] Ronald L. Graham, Bruce Rothchild, and Joel Spencer. *Ramsey Theory*. Wiley-Interscience, New York, 1990.

[16] A. W. Hales and R.I. Jewett. Regularity and positional games. In Ira Gessel and Gian-Carlo Rota, editors, *Classic Papers in Combinatorics*, Modern Birkhuser Classics, pages 320–327. Birkhuser Boston, 1987.

[17] Alexander S. Kechris, Vladimir G. Pestov, and Stevo Todorcevic. Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups. *Geometric and Functional Analysis*, 15(1):106–189, 2005.

[18] P. Keevash. The existence of designs. *Submitted*.

[19] M.G. Kendall and B. Babington Smith. On the method of paired comparisons. *Biometrika*, 31:324–345, 1940.

[20] J. W. Moon. *Topics on Tournaments.* Holt, Rhinehart and Winston, 1968.

[21] D. Mubayi and V. Rödl. Uniform edge distribution in hypergraphs is hereditary. *Electronic Journal of Combinatorics*, 11:1–32, 2004.

[22] J. Nešetril. Ramsey Theory, Handbook of Combinatorics, R. Graham et al. (eds.), 1331–1403, 1995.

[23] J. Nešetřil and V. Rödl. Partitions of finite relational and set systems. *Journal of Combinatorial Theory, Series A*, 22(3):289–312, 1977.

[24] J. Nešetřil and V. Rödl. On a probabilistic graph-theoretic method. *Proceedings of the American Mathematical Society*, 72(2):417–421, 1978.

[25] J. Nešetřil and V. Rödl. Sparse ramsey graphs. *Combinatorica*, 4(1):71–78, 1984.

[26] J. Nešetřil and V. Rödl. Strong ramsey theorems for steiner systems. *Transactions of the American Mathematical Society*, 303(1):183–192, 1987.

[27] Jaroslav Nešetřil and Vojtěch Rödl. Type theory of partition properties of graphs. In *Recent advances in graph theory (Proc. Second Czechoslovak Sympos., Prague, 1974)*, pages 405–412, 1975.

[28] F. P. Ramsey. On a problem of formal logic. *Proceedings of London Mathematical Society*, 30(3):264–286, 1930.

[29] A. Razborov. Flag algebras. *J. Symbolic Logic*, 72(4):1239–1282, 2007.

[30] F. Roberts. Applications of ramsey theory. *Discrete Applied Mathematics*, 9:251–261, 1984.

[31] V. Rödl. The dimension of a graph and generalized Ramsey theorems. *Master's thesis, Charles University*, 1973.

[32] V. Rödl. On universality of graphs with uniformly distributed edges. *Discrete Mathematics*, 59(1-2):125–134, 1986.

[33] J. Spencer. Restricted Ramsey configurations. *Journal of Combinatorial Theory, Series A*, 19:278–286, 1975.

[34] P. Turán. On an extremal problem in graph theory. *Mat. Lapok*, 48:436–452, 1941.