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Log-Canonical Rings of Graph Curves

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An abstract of
a thesis submitted to the Faculty of James T. Laney School of Graduate Studies
of Emory University in partial fulfillment
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in Mathematics

2016

Abstract

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By William Baker

I generalize David Zureick-Brown and John Voight's work on log-canonical rings to graph curves. I use a paper of Noot as a starting point. I outline some of the difficulties in developing Max Noether-like and Petri-like theorems. I work out theorems for the generators of most well behaved graph curves. I also find a useful construction for hyperelliptic graph curves.

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1 Introduction

In this thesis we discuss various theorems about the structure of the log-canonical rings associated to a graph curve.

Definition A graph curve is a curve whose irreducible components are each isomorphic to $mathbbP^1$ and such that each intersection is transverse (i.e. locally looks like $xy = 0$). In particular, a graph curve admits no triple intersections. The genus of a graph curve is the maximal number of generators for the group of loops of the dual graph of the graph curve, see [N] for details. We say that a graph curve is hyperelliptic when it has a $2-1$ function to a genus 0 graph curve. Noot described the canonical rings of these graph curves, including showing that Max Noether's theorem applies.

Definition Given a curve, X, and a divisor D, we say that $H^0(X, D)$ is a vector space of rational functions, $f : X \to P^1$, such that $D - Div(f)$ is effective. We continue by defining a canonical ring to be a graded ring of the form $\bigoplus H^0(X, nK)$, where the multiplication is function multiplication. Also a log-canonical ring is a graded ring of the form $\bigoplus H^0(X, n(K+E)$, where E is an effective divisor. Max Noether's theorem states that canonical rings associated to non-hyperelliptic curves are generated in degree 1. I will use ideas from [V,Z] to further generalise these results to the log-canonical rings. These rings are the graded rings $\bigoplus H^0(X, n(K + E))$, where K is a canonical divisor of the graph curve X, and E is an effective divisor.

Remark Multiplication is just multiplication of the elements as functions, similar to canonical rings. When the curve is clear I will use the notation R_{K+E} to refer the log-canonical ring. The largest complications with graph curves is that even for curves of genus greater than or equal to than 1, the canonical divisor is not necessarily effective, and that $mathbbP^1$ is not the only genus 0 curve. The most frequently used fact is that Riemann-Roch Theorem applies to our graph curves. Note that K will always refer to a canonical divisor. We will always choose our log divisor to be away from K.

First we prove a theorem that will allow us to only consider convenient divisors up to equivalence.

Theorem 1. If two divisors D_1 and D_2 of a curve X are equivalent, then the rings $\bigoplus H^0(X, nD_1)$, and $\bigoplus H^0(X, nD_2)$, are graded ring-isomorphic.

Proof. Suppose we have D and E as two equivalent divisors, so that $f : X \to P^1$, and $D = E + Div(f)$. We can define a map, $\phi : R_D \to R_E$ where $\phi(g) = f^n g$ with g being a homogeneous element, and n is the graded piece of g, we can extend this map to nonhomogeneous elements by using the addition property of ring homomorphisms. We have a few things to check. First the image of the map actually lands in R_E compatibly with the grading.That is if g is in the nth graded piece of R_D then $g \in H^0(X, nD)$ so $nD - div(g) < 0$, and therefore$nE - div(gf^n) = nE - ndiv(f) - div(g) = nE - n(D - E) - div(g) > 0$. Also,taking$gandh$to be in the nth graded piece inR_D then $\phi(g+h) = f^n(g+h) = \phi(g)+\phi(h)$. To check multiplication, suppose g is homogenous in the nth piece and h is homogeneous in the mth graded piece. We see that $\phi(gh) = f^{n+m}gh = f^n g f^m h = \phi(g)\phi(h)$. We also see that ϕ is an isomorphism since we can similarly define the inverse function by multiplication by f^{-n}. $\square$

We wish to do away with curves that do not have effective canonical divisors, many of the Riemann-Roch style arguments break down in these cases. Noot demonstrates that any edge or vertices on the dual graph of our curve where all the holomorphic differentials vanish is the base locus. These are the parts of the graph that no loops go through, since holomorphic differentials correspond to loops. The canonical ring associated with a curve is isomorphic to the canonical ring associated to the curve where the components corresponding to base locus removed. However, when we add log points, the result might not be as degenerate as we can making the divisors effective.

In any case, I will assume that we get noneffective canonical divisors precisely in the case where we have edges of the dual graph where there are no loops, we will call such edges bridges, as they connect two subgraphs together. From Noot we have a clear picture of the canonical case, that is that the canonical ring is a direct product of the canonical rings of the connected parts after removing the bridges. I will refer to these parts as loop connected subgraphs,or loop connected subcurve. So we need to understand how we can write the log canonical rings hopefully in terms of the loop connected subcurves.

A purely algebraic approach is to describe the log canonical ring as a fibered product of the log canonical rings of the loop connected subcurves. This is because the bridge is an intersection point, which causes an agreement condition between evaluating functions on each of the loop connected subcurves at the intersection point.

As an example suppose a graph curve X, the canonical divisor K has exactly 2 negative points (counting multiplicity), let us call these points P and Q. We see that, $H^0(X, K + P + Q) = H^0(X, K) \oplus < 1 >$, where 1 is the constant function along every component of X. This is clear from Rieman-Roch. It is possible for a graph curve to have arbitrarily high amounts of negative points though, simply consider attaching arbitrary amounts of P_1 to a graph curve, such that each of these "leaves" intersect the graph curve exactly once. Such a graph curve has a negative point on each leaf.

Another important consideration is that these bridged graph curves have many zero divisors in their log canonical rings, in fact take the previous example, with the two negative points, and let us say that there are two loop connected subcurves. I say that for $H^0(X, K)$, we can choose a basis of functions that are each zero along one of the loop connected subcurves. The reason for this goes back to our graph curve, the holomorphic differentials which correspond to our functions correspond to the loops. We can choose a generating set of our loops, where each of these generators lies in only one of our loop connected components, these generators then correspond to our basis of functions. Now suppose we introduce a log divisor $K + P + Q + E_1 + E_2$, such that E_1 and E_2 are effective defined away from K. Also let us say that E_1 consists purely of points on our first loop connected subcurve X_1 and E_2 on our second, X_2.

For now assume that P is on X_1 and Q is on X_2. We have that $H^0(X, K + P + E_1)$, and $H^0(X, K + Q + E_2)$, are subspaces of $H^0(X, K + P + Q + E_1 + E_2)$. We see that we can write a basis of each of these subspaces again in functions that are zero along either X_1 or X_2, since every function in $H^0(X, K)$ is zero at our intersection point, and adding log points to only on side of that intersection does not give us the freedom to have a non zero function on the other side of the intersection point. Now, these two subspaces account for all but one dimension of $H^0(X, K + P + Q + E_1 + E_2)$, but the last function is simply the constant function as previously described. Now also notice that if we allow multiplication, all of our basis functions are zero divisors except our constant function. In fact, since higher graded pieces of our log canonical ring can be written in the same log divisor form, this gives us a description of our log canonical ring, it is just a direct product of the log canonical rings of each of our loop connected subcurves, with the equivalence relation that the constant function in both rings are the same. I believe that it is possible to use this procedure inductively to understand graph curves with any amount of loop connected components, and any amount of negative points. The major complication is that if there are many bridges, adding two log points will not be enough to rectify the non effectiveness.

Unless said otherwise, for the remainder of this paper I will assume that our curve has only one loop connected component, and thus all canonical divisors will be effective. I will now give some Max Noether like-theorems.

2 Main Theorems

Theorem 2. Suppose we have the divisors $K+E+P$ and $K+E$ on a curve X, such that K is the canonical divisor, E is an effective divisor with degree greater or equal to 1, and P is an effective divisor of degree 1. If R_{K+E} is generated by elements of degree n or less, than so is the R_{K+E+P}.

Proof. The theorem sets us up for an inductive argument. Without loss of generality we can insist on P being away from the support of K. First notice that R_{K+E} can be viewed as a subring of R_{K+E+P}, where we just included each function into the larger graded pieces. Now suppose R_{K+E+P} has a generator that is of degree greater than n, say n_0, call such a function with minimal pole order at P, y. This function clearly cannot be in R_{K+E} viewed as a subring, and thus y must have have a pole at the point P, say it is a pole of order m. We see that $0 < m \leq n_0$ since y is an element of $H^0(n_0(K+E+P))$ However,since $H^0(K+E)$ lies in $H^0(K+E+P)$ as a codimension 1 subspace by Riemann-Roch, we can pick an element in our first graded piece of R_{K+E+P}, x such that x has a pole of order 1 at P.

Finally we can introduce an element in R_{K+E} with no pole or zero at the point P. This is clear if both $K+E$ is effective, we can choose the constant function 1. Let us call this element r. Now consider the quantity $y - x^m r^{n_0-m}$, firstly this subtraction is entirely in the n_0 graded piece, secondly the order of the pole at P is strictly less than m. Now suppose it has a pole of order 0 at P. Then it is not a generator of the ring. If this element is a generator of the ring, we have a contradiction on the pole minimality of y. Therefore $y - x^m r^{n_0-m}$ is not a generator. However, then y can be written as the sum of two non generating elements which is a contradiction. Therefore there can be no such generator y in the n_0 graded piece. $\square$

Theorem 3. Any log-canonical ring associated to a non-hyperelliptic graph curve X, of the form R_{K+P} where P is a positive point, is generated in degrees up to but no greater than 3.

Proof. In fact we will see that rings of this form are generated in degree 3. Let us assume that $H^0(X, K)$ has a function r that has no pole or zero at P (the constant function works). Consider $H^0(2K+P)$ as a codimension 1 subspace of $H^0(2K+2P)$, we can then extend a basis to include some element y in $H^0(2K+2P)$ with a double pole at P. This element is in the second graded part of R_{K+P}, and it has a double pole at P, but from Rieman-Roch, there is no element in $H^0(K+P)$ with a pole at P, and thus y is a generator of our ring. Also, consider $H^0(3K+2P)$ as a codimension 1 subspace of $H^0(3K+3P)$, we can then extend a basis to include some element z in $H^0(2K+2P)$ with a triple pole at P. z must also be a generator, for suppose it is not, then there must be some product of the first and second graded pieces that yields a pole of order three function at the point P, but this cannot be, since the first graded piece has no functions with poles as P. Now suppose we have a generator, w, in degree $n > 3$. If w has no pole at P, we have a contradiction since R_K is generated in degree 1. Now suppose w has a pole of order 1. First, this means that our curve has a genus greater than 1, since K is the identity divisor if effective by degree considerations. If K is the identity divisor than so is nk, and so by Riemann-Roch no function could have a pole of order 1 at P. So if genus is greater than one, then by Riemann-Roch, $H^0(X, 2K)$ is a codimension one subspace of $H^0(X, 2K+P)$, so we can find a degree 2 function, y_1 with a single pole at P. Consider that quantity $aw - by_1r^{n-2}$, where a, b are scalars with appropriate scaling, so as to cancel the poles at P. This is a homogeneous element in the nth graded part of the ring, and has no pole at P. Thus this element cannot be a generator, so w cannot be a generator either.

We wish to consider what happens when w has a pole of greater order than 1. Let us add a hypothesis that w has the lowest ordered pole at P for possible generators of degree n. Also notice that w cannot have a pole at P greater than n.Now suppose w has a pole of order $2m$ at P. Consider the element $aw - by^m r^{n-2m}$ with a, b being the necessary scalars to cancel the pole. Thus this homogeneous element of degree n has a pole strictly smaller than $2m$, thus is not a generator so w is not either. Alternatively suppose the pole at P is of the form $2m+1$. Consider the element $aw - bzy^{m-1}r^{n-2m-1}$, with a, b being appropriate scalars. By the same argument we have a contradiction. □

Theorem 4. Suppose we have a nonhyperelliptic graph curve X, with a divisor $K+2P$, where K is a canonical divisor, and P is a point. R_{K+2P} is generated in degree 2.

Proof. Like usual we want a first degree function with neither a zero nor a pole at P, call this function r. We have a function in degree one with a double pole at P by Riemann-Roch, call this function x. For our second graded piece, notice that $H^0(X, 2K+2P)$ is a codimension one subspace of $H^0(X, 2K+3P)$, so we have a function, call it y. This function has a pole of order 3 at P, and it cannot be written as a product of of degree 1 elements, as none of these elements have an odd pole at P. Now we know that the ring is generated by no more than degree 3 elements, so we need only check that there are no possible generators in degree 3. Let us suppose we have such a generator z. It must have a pole of order $1, 2, 3, 4, 5, 6$ at P. It cannot have a pole of order 1 by the argument in the previous theorem. For the other cases consider the elements r^2x, ry, rx^2, xy, x^3, by scaling and subtraction we can arrive at a contradiction. □

Theorem 5. Suppose we have a nonhyperelliptic graph curve X, with a divisor $K+P+Q$, where K is a canonical divisor, and P and Q are points. R_{K+P+Q} is generated in degree 2.

Proof. Like usual we want a first degree function with neither a zero nor a pole at P, call this function r. We have a function in degree one with a single pole at P and Q by Riemann-Roch, call this function x. For our second graded piece, notice that $H^0(X, 2K+P)$ is a codimension one subspace of $H^0(X, 2K+2P)$, so we have a function, call it y. This function has a pole of order 2 at P, and it cannot be written as a product of of degree 1 elements, as none of these elements have a pole at P, and no pole at Q. If X is not genus 1, we also have a function that has a pole of order 1 at P, call this function y_1. Now we know that the ring is generated by no more than degree 3 elements, so we need only check that there are no possible generators in degree 3. I will show that multiplication from $H^0(X, K+P+Q) \oplus H^0(X, 2K+2P+2Q)$ surjects onto $H^0(X, 3K+3P+3Q)$. We already have all the functions in $H^0(X, 3K)$ by Max Noether, and consider the image generated by, $ry_1, r^2x, ry, rx^2, xy, x^3$. These elements are clearly all linearly independent, and by Riemann-roch, they generate $H^0(X, 3K+3P+3Q)$. together with the canonical elements. For the genus 1 case just exclude ry_1. □

The situation becomes more complicated for the question of when log canonical rings become generated in degree 1. In order for a graph curve with genus greater than 1 to be generated in degree 1, we must have a function in $H^0(X, K)$ that has a simple zero at one of our log points without being uniformly zero on that component. If we have such a function, the argument goes similarly to the previous ones. I hypothesis that such a function is guaranteed to exist if the our point is on a component that intersects the rest of the graph curve at least 3 times.

Theorem 6. *Suppose we have a nonhyperelliptic graph curve X, with a divisor $K + 3P$, where K is a canonical divisor, and P is a point. R_{K+3P} is generated in degree 1, if we have a degree 1 function, z that has a simple zero at P, or if X has a genus of 0, or 1.*

Proof. All we must prove is that $H^0(X, K+3P) \oplus H^0(X, K+3P)$ surjects onto $H^0(X, 2K+6P)$. All we need to do is produce elements with poles of orders 1 through 6 in the second degree. By Riemann-Roch we have degree 1 elements x_2 and x_3, with poles of order 2 and 3 at P respectively. Now suppose X is genus 0, and that $K+3P$ is effective, then by Riemann Roch, $H^0(X, K+3P)$ is dimension 2, and in fact we can write out basis functions as 1, and x. By Riemann-Roch, each graded piece is one dimension higher, but since there can be no relations between 1 and x, the ring is visibly graded in degree 1. Note that the log divisor could be any divisor of the form $K + E$ where E is effective and degree 3. Now suppose X is genus 1, then there are no degree 2 elements with a single pole at P. Thus together with the canonical functions, we have the functions $rx_2, rx_3, (x_2)^2, x_2x_3, (x_3)^2$.

If X has genus higher than 1 we need only consider one additional product zx_2. This function has a simple pole at P as required. □

Similar proofs exist for divisors of the form $K + E$ where E is effective and degree 3.

Theorem 7. *Given a hyperelliptic graph curve X, a canonical divisor, K, can be constructed from the $2-1$ map, f, to a genus 0 curve Y, so that K has the property of being invariant under involution, and $f(K)$ is a log canonical divisor on Y.*

Proof. Consider the preimage of intersection points of Y, by continuity, these must be intersection points of X. Also by continuity, the preimage of any point of X must be either entirely intersection points, or entirely non intersection points. From this we can conclude that intersections of X either map in pairs to intersections of Y, or a single intersections maps with itself to a nonintersecting point of Y. This is clear because intersection points are double points. Similarly, we can track the components, and find that they must map $2-1$ onto themselves, or map with another component to a component of Y. Now given a component of Y, Y_i we wish to give it a divisor D_i. Let n_i be the number of points of Y_i whose preimage contains an intersection point of X, and let d_i be 1 is the preimage of Y_i is a single component. $D_i = (n_i + d_i - 2)P$, where P is a point away from the images of intersection points on the component Y_i. It is clear that $D = \sum(D_i)$ is a log canonical divisor of Y. Now I say that the pull back of D is a canonical divisor of X. First I will check that the degree of D is appropriately $g-1$, where g is the genus of X. Let us define A to be the total number of intersections of X, B the number of intersections that do not map to intersections of Y, C the number of components of X, and F the number of components that map $2-1$ onto themselves. We see that $A - C + 1 = g$, by analyzing the dual graph. By counting we get that $\frac{A-B}{2} - \frac{C+F}{2} + 1 = 0$, and therefor $B + F = g + 1$. Now this implies, by counting, that $deg(D) = g - 1$. Thus the pullback of D, which we will not refer to as K is the right degree. Also notice that $dim(H^0(Y, D)) = g$ by Riemann-Roch, so we can pull back these functions as well to get that $dim(H^0(X, K) = g$. Finally we will check to make sure K is indeed a canonical divisor by checking degree component wise. Suppose a component X_i of X does not map $2-1$ onto itself, then $deg(K_i) = deg(D_i) = n_i - 2$ and n_i is precisely the number of intersections on X_i. Alternatively suppose X_j maps $2-1$ onto itself, then $deg(K_i) = 2deg(D_i) = 2(n_i - 2 + 1) = 2n_i - 2$. I say that $2n_i$ is precisely the number of intersections of X_j since f is 2_1, and it is not possible for a non intersection point on Y to map to a single intersection point on X_j. Since K is the pull back of D, we have proved it is invariant under involution. □

Theorem 8. Let X be a hyperelliptic graph curve with canonical divisor K and E, an effective divisor of degree 2, such that E is not invariant under our hyperelliptic involution. The associated log canonical ring, R_{K+E+P}, where P is a point, is generated in degree 1 if like in the nonhyperelliptic case there exists a function in R_K that has a simple zero at P.

Proof. First I need to consider the log canonical ring associated to Y and the divisor D (from last theorem). As previously noted R_D must be generated in degree 1. Thus let $x'_1, \dots x'_g$, be a set of degree 1 functions that generate the ring, and then let $y'_1, ..., y'_(2g-1)$ be a basis for our second graded piece (so all of these functions are sums and products of the x'_i. Now let us pull these functions back to X, to get $x_1, \dots x_g$, and $y_1, \dots, y_{2g-1}$ respectively. Notice that these y_i are still obtained from the x_i. Now I say that $H^0(X, K+E) \oplus H^0(X, K)$ surjects onto $H^0(X, 2K+E)$ by the multiplication map. My argument will be extremely similar to [V,Z]. $H^0(X, K)$ is a codimension 1 subspace of $H^0(X, K+E)$, so let us choose a function z with poles at the points of E. By assumption on E, z is not invariant under involution. Notice that there can be no relations in degree 2 containing just z and the x_i. Let $a(x)$ and $b(x)$ be polynomials of variables $x_1, \dots, x_g$, a relation containing z would look like $a(x)z = b(x)$, which is a contradiction when we apply our involution. Thus, $y_1, \dots, y_{2g-1}, zx_1, \dots zx_g$ are all linearly independent functions, and by Riemann-Roch they span the the target space.

With this we have amended the difficulties of the hyperellipticity, and the argument continues as in the not hyperelliptic case, noting that the image of multiplication is onto the canonical part. The proof then follows the same. I should note that assuming D is effective, if the image of P is on a component where D restricted to that component is of positive degree, than it is a simple matter to find a function constant on the other components, and with a simple zero at P. $\square$

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