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Victor O. Larsen   Date

An ϵ improvement to the asymptotic density of k-critical graphs

By

Victor O. Larsen
Doctor of Philosophy

Mathematics

Ronald Gould, Ph.D.
Advisor

Luke Postle, Ph.D.
Advisor

Dwight Duffus, Ph.D.
Committee Member

Vojtěch Rödl, Ph.D.
Committee Member

Accepted:

Lisa A. Tedesco, Ph.D.
Dean of the Graduate School

Date

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Victor O. Larsen
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An abstract of
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Abstract

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Given a graph G the *chromatic number*, denoted $\chi(G)$, is smallest number of colors necessary to color $V(G)$ such that no adjacent vertices receive the same color. A graph G is k-critical if $\chi(G) = k$ but every proper subgraph has chromatic number less than k. As k-critical graphs can be viewed as minimal examples of graphs with chromatic number k, it is natural to ask how small such a graph can be.

Let $f_k(n)$ denote the minimum number of edges in a k-critical graph on n vertices. The Ore construction, used to build larger k-critical graphs, implies that

$$f_k(n+k-1) \le f_k(n) + (k-1)\left(\frac{k}{2} - \frac{1}{k-1}\right).$$

A recent paper by Kostochka and Yancey provides a lower bound for $f_k(n)$ which implies that the asymptotic density $\phi_k := \lim_{n\to\infty} f_k(n)/n = \frac{k}{2} - \frac{1}{k-1}$.

In this work, we use the method of discharging to prove a lower bound on the number of edges which includes structural information about the graph. This lower bound shows that the asymptotic density of a k-critical graph can be increased by $\epsilon > 0$ by restricting to (K_{k-2})-free k-critical graphs.

We also prove that the graphs constructible from the Ore construction and K_k, called *k-Ore* graphs, are precisely the graphs which attain Kostochka and Yancey's bound. Moreover, we also provide results regarding subgraphs which must exist in k-Ore graphs. For the discharging argument, carried out in two stages, we also prove results regarding the density of nearly-bipartite subgraphs in k-critical graphs.

In the final chapter we examine the minimal set of subgraphs, called *k-critical structures*, which one needs to forbid to obtain an ϵ increase in asymptotic density. This lays the groundwork for future research into asymptotic density in k-critical graphs.

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to Rachel

Contents

Chapter 1

Introduction and history

1.1 Graph coloring

In this work, we assume that the reader has knowledge of basic concepts of graphs (formal definition, adjacency, connectivity, etc.). All non-standard terminology and notation will be specifically defined, but for any other definitions the reader can refer to a standard textbook such as [8]. This work was inspired by recent developments concerning Ore's Conjecture [16] on k-critical graphs. Let us briefly recall basic coloring definitions and then state Ore's Conjecture.

A *k-coloring* ϕ of a graph G is simply a labelling $\phi : V(G) \to [k]$. If the coloring ϕ has the additional property that for every $uv \in E(G)$ $\phi(u) \neq \phi(v)$, then we say that the k-coloring is *proper*. If there is a proper k-coloring of a graph G, then we say that G is *k-colorable*. Graph coloring has useful applications in scheduling, and in these applications we want to be as efficient as possible. Although we can easily prove that every graph G is $|V(G)|$-colorable, this is of no practical use. Rather, we are concerned with the *chromatic number* of a graph G, which is the smallest k for which G is k-colorable. We denote the chromatic number of a graph G by $\chi(G)$.

An early classic theorem regarding the chromatic number of a graph is Brooks' Theorem [3]. Recall that $\Delta(G)$ is the maximum degree of a graph.

Theorem 1.1. *If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.*

There is an obvious connection between the number of edges in a graph and the chromatic number of that graph. As we add more edges, there are more restrictions preventing an arbitrary k-coloring from being a proper k-coloring. Brooks' Theorem provides a bound on the chromatic number in terms of local density. It is trivial to show that $\chi(G) \leq 1 + \delta(G)$ by making a greedy coloring argument. Brooks' Theorem shows that we can be more efficient by 1 color class as long as G is not a complete graph or an odd cycle. For other classes of graphs, it is possible to be even more efficient.

The Borodin-Kostochka Conjecture [2] is as follows.

Conjecture 1.2. *If G is a connected graph with $\omega(G) \leq \Delta(G) - 1$ and $\Delta(G) \geq 9$, then $\chi(G) \leq \Delta(G) - 1$.*

Using the probabilistic method, this has been proven in [18] for very large $\Delta(G)$.

If we introduce further restrictions, we often obtain stronger results. A graph G is *planar* if it can be embedded in a plane such that no edges are crossing. A well-known coloring theorem for planar graphs is the celebrated Four Color Theorem [1].

Theorem 1.3. *Every planar graph is 4-colorable.*

This is of particular interest because the method of proof involves discharging, which is also the method used in this work. Discharging involves assigning charge to a graph in a particular way (we assign charge to each vertex), moving the charge around using discharging rules and reducible configurations to find a contradiction. One can contradict any sort of global hypothesis such as planarity, an assumption on maximum average degree, or—as we do in this work—number of edges.

The original proof of the Four Color Theorem is famously complex, involving nearly 2,000 configurations. That proof has been updated and streamlined to involve 'only' 633 configurations [19] but still remains highly detailed. The proof becomes much simpler, and the theorem stronger, if we introduce further restrictions on G. Namely, all planar graphs which are triangle-free are 3-colorable [9]. In our work as well, we obtain a strengthening of previous results if we forbid certain subgraphs. This is a common theme in the literature of coloring problems.

1.2 k-critical graphs

We return to the idea of density in a graph and its effect on the chromatic number. Removing an edge from a graph takes away one obstacle for an arbitrary k-coloring to be a proper k-coloring. Therefore, the chromatic number of $G - e$ is either $\chi(G)$ or $\chi(G) - 1$. The graphs we are concerned with are k-critical graphs. A graph G is *k-critical* if $\chi(G) = k$ and every proper subgraph is $(k-1)$-colorable. This is equivalent to requiring that $\chi(G - e) = \chi(G) - 1$ for each edge $e \in E(G)$.

A natural example of a k-critical graph is K_k. Also, using the Hajós construction (see [20], page 217) or the Ore composition (Definition 3.1) one can combine any two k-critical graphs to create other k-critical graphs. When studying coloring problems, the class of k-critical graphs is of particular interest as it is a natural starting point for general statements about k-chromatic graphs. For example, if we could prove the Borodin-Kostochka conjecture for every k-critical graph then it would follow that it holds for all graphs.

From this perspective, we say that k-critical graphs are minimal examples of k-chromatic graphs. It is natural to ask how small such graphs could be. This leads to the study of the parameter $f_k(n)$, the minimum number of edges in a k-critical graph on n vertices. Ore's Conjecture [16] is the following.

Conjecture 1.4. *If $k \geq 4$, then*

$$f_k(n+k-1) = f_k(n) + (k-1)\left(\frac{k}{2} - \frac{1}{k-1}\right).$$

Another measure of a k-critical graph is the *asymptotic density*, which we define to be $\phi_k := \lim_{n\to\infty} \frac{f_k(n)}{n}$. Ore's Conjecture [16], if proven, would imply that the limit ϕ_k exists and that $\phi_k = \frac{k}{2} - \frac{1}{k-1}$.

1.3 History of $f_k(n)$

We want to survey the history of bounds on $f_k(n)$. The first natural bound comes from a connectivity argument. Clearly $\delta(G) \geq k-1$ for any k-critical graph (or a greedy coloring will properly $(k-1)$-color G) so therefore

$$f_k(n) \geq \frac{k-1}{2}n.$$

In terms of asymptotic density, this shows that, if the limit ϕ_k exists then $\phi_k \geq \frac{k}{2} - \frac{1}{2}$ for k-critical graphs. Throughout the literature there have been many improvements on bounds for $f_k(n)$; however, we will focus our attention on ϕ_k.

The reason for examining ϕ_k comes from an observation about the Hajós construction (see [20], page 217). Given a k-critical graph G, if we look at the graph G' resulting from a Hajós construction of G and K_k, then we have added $k-1$ vertices and added one less than the number of edges in a K_k. That is,

$$|V(G')| = |V(G)| + k - 1 \text{ and } |E(G')| = |E(G)| + \frac{(k-2)(k+1)}{2}.$$

It follows that $f_k(n+k-1) \leq f_k(n) + \frac{(k-2)(k+1)}{2}$, which implies that ϕ_k exists and that

$$\phi_k \leq \frac{1}{k-1}\left(\frac{(k-2)(k+1)}{2}\right) = \frac{k}{2} - \frac{1}{k-1}.$$

This observation led Ore to pose Conjecture 1.4. Thus, it follows that $\frac{k}{2}-\frac{1}{2} \leq \phi_k \leq \frac{k}{2} - \frac{1}{k-1}$.

The first improvement to the lower bound on $f_k(n)$ came in 1957 when Dirac [6] showed that $f_k(n) \geq \frac{k-1}{2}n + \frac{k-3}{2}$ for $k \geq 4$ and $n \geq k+2$. When $n \neq 2k-1$, Kostochka and Stiebitz [12] obtain the improvement

$$f_k(n) \geq \frac{k-1}{2}n + k - 3.$$

Both of these results are significant in their own right, but in terms of asymptotic density of a k-critical graph these results offer no improvement over the first bound obtained by a connectivity argument.

Gallai [7] was able to improve the lower bound on asymptotic density, as well as calculate many exact values for $f_k(n)$. The bound he gives for $k \geq 4$ and $n \geq k+2$ is $f_k(n) \geq \frac{k-1}{2}n + \frac{k-3}{2(k^2-3)}n$. This was further improved to

$$f_k(n) \geq \frac{k-1}{2}n + \frac{k-3}{2(k^2-2k-1)}n$$

by Krivelevich [15]. Although these results bring the lower bound on ϕ_k closer to the upper bound, as k grows this improvement diminishes.

Ore's Conjecture was proven to be asymptotically true in a recent paper by Kostochka and Yancey [14]. They show the following result.

Theorem 1.5. *If $k \geq 4$ and G is k-critical then*

$$|E(G)| \geq \left\lceil \frac{(k+1)(k-2)|V(G)| - k(k-3)}{2(k-1)} \right\rceil .$$

Note that $\frac{(k+1)(k-2)}{2(k-1)} = \frac{k}{2} - \frac{1}{k-1}$, so Ore's Conjecture is asymptotically true. In a subsequent work [13], they verified that the graphs which attain this bound are exactly the k-Ore graphs (defined in Chapter 3). The method of proof of Theorem 1.5 is to define a *potential function* which is a measure of density in the graph. By bounding the potential of a k-critical graph using a discharging argument, they obtain a bound on the number of edges as a corollary. The proof in this work mirrors the proof of Theorem 1.5, with significant modifications to account for the extra information we give on the structure of k-critical graphs. Our aim is to increase asymptotic density past the upper bound of $\frac{k}{2} - \frac{1}{k-1}$ and to do this, we must forbid certain subgraphs. In Chapters 2–6, we lay out a proof which yields an improvement for (K_{k-3})-free k-critical graphs.

A similar attempt to push asymptotic density above the bound given by Ore's Conjecture was made by Postle [17] in the case where $k = 4$. For a 4-critical graph G, Theorem 1.5 implies that $|E(G)| \geq \frac{5}{3}|V(G)| - \frac{2}{3}$. However, in [17] it is shown that there exists an $\epsilon > 0$ such that

$$|E(G)| \geq \left(\frac{5}{3} + \epsilon\right)|V(G)| \tag{1.1}$$

when G is a 4-critical graph with girth at least 5. It can be shown that, in order to obtain any improvement in the asymptotic density, $\{K_3, C_4\}$ is, in fact, a minimal set of graphs one needs to forbid as subgraphs. For k-critical graphs with $k > 4$, a minimal set of subgraphs which we must forbid to obtain an improvement in asymptotic density is the set of *k-critical structures* (Definition 7.1). In Chapter 7, we examine a bound on these k-critical structures in k-Ore graphs. However, an analogous result to Equation 1.1 using k-critical structures for $k > 4$ cannot be obtained through the methods of this work.

1.4 Results

We now briefly summarize the results of this work and their implications. The most significant result of this paper is a strengthening of the conclusion of Theorem 1.5. We make a modification to the potential function which includes the function $T(G)$ (see Definition 3.3) which counts, in a particular way, the K_{k-2} and K_{k-1} subgraphs of a graph G.

Below is the precise formulation of the potential function used in this work to prove results on k-critical graphs for a fixed k. This is examined in depth in Chapter 4.

Definition 3.4 *Given a graph G, we define the* potential *of a graph to be*

$$\rho_\epsilon(G) := ((k-2)(k+1)+\epsilon)\,|V(G)| - 2(k-1)|E(G)| - \delta T(G)$$

for a fixed epsilon with $0 \leq \epsilon \leq \frac{4}{k^3-2k^2+3k}$ and $\delta = (k-1)\epsilon$. Because ϵ remains fixed throughout the proof, we omit this subscript. Using this new potential function, we are able to prove (for any $0 \leq \epsilon \leq \frac{4}{k^3-2k^2+3k}$) the following theorem via a discharging argument. The reader can find a definition of k-Ore graph in Chapter 3.

Theorem 3.5 *If G is a k-critical graph with $k \geq 4$ then*

1. $\rho(G) = k(k-3) + k\epsilon - 2\delta$ *if $G = K_k$,*
2. $\rho(G) \leq k(k-3) + |V(G)|\epsilon - \left(2 + \frac{|V(G)|-1}{k-1}\right)\delta$ *if G is k-Ore and $G \neq K_k$, and*
3. $\rho(G) \leq k(k-3) - 2(k-1)$ *if G is not k-Ore, for $k \geq 33$.*

On its own, this result is slightly opaque so we highlight important corollaries below.

Corollary 1.6. *If $k \geq 33$ and G is k-critical then*

$$|E(G)| \geq \left\lceil \frac{((k+1)(k-2)+\epsilon)\,|V(G)| - k(k-3) + 2\delta - k\epsilon - \delta T(G)}{2(k-1)} \right\rceil$$

where $\epsilon \leq \frac{4}{k^3-2k^2+3k}$ and $\delta = (k-1)\epsilon$.

Note that by setting $\epsilon = 0$, we get the same result as Theorem 1.5, although there is an extra restriction on k. Importantly, this gives us structural information about the graphs which realize or approach the lower bound on

ϕ_k given by Theorem 1.5. Graphs which have fewer K_{k-2} subgraphs and are close to the bound will have, in general, more edges than those with many K_{k-2} subgraphs.

In fact, if G is a (K_{k-2})-free k-critical graph, then $T(G) = 0$ and we have shown the following result. Let $\overline{f}_k(n)$ be the minimum number of edges in a (K_{k-2})-free k-critical graph on n vertices.

Corollary 1.7. *The asymptotic density* $\overline{\phi}_k := \lim_{n\to\infty} \frac{\overline{f}_k(n)}{n}$ *of* (K_{k-2})*-free* k*-critical graphs is bounded below by*

$$\overline{\phi}_k \geq \frac{(k-2)(k+1)+\epsilon}{2(k-1)}$$

where $\epsilon \leq \frac{4}{k^3-2k^2+3k}$.

If we restrict our attention to k-Ore graphs then Corollary 1.6 holds for $k \geq 4$ rather than just $k \geq 33$. Therefore, the structural implications of Corollary 1.6 hold for the family of k-Ore graphs even with smaller k. However, we are not able to obtain any increase in asymptotic density on this set of graphs because, by Lemma 3.11, there are no k-Ore graphs which are (K_{k-2})-free. In Chapter 7, we explore a second new definition of potential function. Instead of $T(G)$, we introduce the counting function $T_{cs}(G)$ which counts k-critical structures, which is a minimal set of subgraphs we must forbid to see an increase in asymptotic density. This yields the following result.

Theorem 7.2 *If* G *is* k*-Ore and* $k \geq 4$ *then for* $a = 7(k-1)$ *the following is true.*

1. *If* $T_{cs}(G) \geq 3$ *then* $T_{cs}(G) - 2 \geq \frac{|V(G)|-1}{2a}$ *and*
2. *if* $T_{cs}(G) \leq 2$ *then* $a \geq |V(G)|$.

The first statement of Theorem 7.2 is similar to the bound which gives the coefficient of δ in the second statement of Theorem 3.5. This observation gives hope that a theorem analogous to Theorem 7.2 which uses $T_{cs}(G)$ rather than $T(G)$ could be proven. However, such an endeavor is beyond the scope of this work.

When we restrict our attention to k-critical graphs which are not k-Ore, a stronger statement than Corollary 1.6 is possible. This is also a corollary to Theorem 3.5.

Corollary 1.8. *If $k \geq 33$ and G is a k-critical graph which is not k-Ore then*

$$|E(G)| \geq \left\lceil \frac{((k+1)(k-2)+\epsilon)\,|V(G)| - k(k-3) - \delta T(G)}{2(k-1)} \right\rceil + 1$$

where $\epsilon \leq \frac{4}{k^3-2k^2+3k}$ and $\delta = (k-1)\epsilon$.

If we set $\epsilon = 0$, this proves that k-critical graphs which are not k-Ore have at least one edge more than the bound given by Theorem 1.5. Therefore this work independently verifies the result of [13], that the graphs which attain the bound in Theorem 1.5 are exactly the k-Ore graphs.

Theorem 3.5 is the main result of this work. The first statement of the theorem is a straightforward calculation using the definition of potential and the second statement requires a lemma (Lemma 3.7) about how $T(G)$ behaves for k-Ore graphs. The rest of this work (excepting the final chapter) focuses on proving the third statement—which is the most complex of the three—via a discharging argument which is split up into two stages. In the method of discharging, each vertex is given initial charge relative to its degree. The assumption that G is a minimal counterexample to the third statement of Theorem 3.5 implies that the total charge on the graph is positive. Therefore, there must be vertices in $V(G)$ whose initial charge is positive (we say that these vertices are *unsatisfied*); this is exactly the vertices of degree $k-1$. To move towards a contradiction, we will create rules by which high degree vertices send charge to neighbors of degree $k-1$ while still retaining negative charge for themselves. Not all vertices can be satisfied in this way, so we finish with a second round of discharging.

In Chapter 2, we begin by looking at bounds on the number of edges from an independent set $I \subseteq V(G)$ of degree $k-1$ vertices in a k-critical graph G. This requires looking at kernel-perfect directed graphs and list-coloring arguments. We also state a result by Kierstead and Rabern [11] which is used to finish the discharging argument.

In Chapter 3, we define k-Ore graphs and the Ore composition operation. We then prove the first two statements of Theorem 3.5. We also introduce some structural lemmas about subgraphs that must exist in k-Ore graphs, which we use in the following chapters.

In Chapter 4, we begin our thorough investigation of the potential function ρ. Corollary 4.7 implies that every proper induced subgraph $G[R] \subsetneq G$ has potential that is higher than $\rho(G)$. Therefore $\rho(G) = \min\{\rho_G(R) \mid R \subseteq V(G)\}$, and we are able to use ρ as a global parameter for discharging. Because discharging arguments make a global assumption and then examine specific local structures, it is crucial for this global assumption to also hold on all induced subgraphs. The main goal of this chapter is to establish Lemma 4.19, which is used in determining the local structure near vertices of degree $k-1$ in a minimal counterexample to the third statement of Theorem 3.5. This will help inform the rules in the first stage of discharging.

In Chapter 5, we classify all vertices of degree $k-1$ into three different categories, depending on their local structure. The main lemma is Lemma 5.5 which shows that any adjacent vertices of degree $k-1$ must give rise to one of two structures, each of which has enough charge to compensate for the degree $k-1$ vertices near it.

In Chapter 6, we complete the proof by discharging in two stages. In the first stage, we ensure that two classes of degree $k-1$ vertices are satisfied. The third class cannot be directly satisfied this way so in the second stage, we examine the total charge on the graph and use it to get a bound on $|E(G)|$. From our assumption on potential and the result by Kierstead and Rabern, we have two other bounds on $|E(G)|$. These bounds cannot be simultaneously satisfied and we complete the proof of Theorem 3.5.

In Chapter 7, which is self-contained, we examine k-Ore graphs under a different definition of potential. We exclude different structures from k-critical graphs G, which gives more information about the structure of k-Ore graphs. These structures, called *k-critical structures*, are the minimal set of subgraphs which we have to forbid to obtain an ϵ improvement in the asymptotic density ϕ_k for k-critical graphs and thus the proof of Theorem 7.2 is far from straightforward. Without increasing the complexity far beyond the scope of this work, we cannot extend these results to general k-critical graphs.

Chapter 2

Graph definitions and preliminaries

2.1 Definitions

We begin with some graph notation that is frequently used in this work and then some preliminary lemmas which will be used in Chapter 6 during discharging. In this work, we focus on k-critical graphs, a special type of graph with chromatic number k. Recall the following definitions of the chromatic number and k-critical graphs. A *proper k-vertex-coloring*, or simply *proper k-coloring* is a labelling such that adjacent vertices of G receive different labels. The *chromatic number* of a graph G is the smallest k for which G is properly k-colorable. A *k-critical graph* is a graph G with chromatic number k but for any $e \in E(G)$, the graph $G - e$ is properly $(k-1)$-colorable.

There are a few other definitions for graphs that we use frequently throughout this work which we make note of below. Definitions not specifically stated in this work can be found in [8].

For a graph G we say that two subgraphs H_1, H_2 are *incident* if $V(H_1) \cap V(H_2) \neq \emptyset$, we say that an edge e is *incident* to a vertex x if x is one of the endpoints of the edge, and we say that an edge e is incident to an edge f if $x \in V(G)$ is an endpoint of both edges.

The *neighborhood*, $N_G(x)$, of a vertex $x \in V(G)$ is the set of all adjacent vertices. The *closed neighborhood*, $N_G[x]$, is the set $N_G(x) \cup \{x\}$.

For a graph G with two vertices x, y we define the graph $G/\underline{xy}$ to be the graph obtained from G by identifying the vertices x and y. That is, $V(G/\underline{xy}) := (V(G) \cup \{\underline{xy}\}) - \{x, y\}$ and the new vertex $\underline{xy}$ is adjacent to each vertex in $N_G(x) \cup N_G(y)$.

2.2 Almost-bipartite subgraphs in a k-critical graph

We begin this work on k-critical graphs by examining edge bounds of an almost-bipartite subgraph of a k-critical graph G. By *almost-bipartite* subgraph, we are referring to an induced subgraph on disjoint sets $A, B \subseteq V(G)$ such that A is an independent set. Because we only specify that A is independent, the induced subgraph $G[A \cup B]$ is not necessarily bipartite. In proving these results, we consider putting an orientation on edges of G to create a digraph D. On this digraph, we will make a list-coloring argument to achieve bounds on the number of edges in an almost-bipartite subgraph.

Definition 2.1. *For a graph G with disjoint vertex subsets A, B, we define $G(A, B)$ to be the bipartite graph obtained from $G[A \cup B]$ by removing all edges from $G[B]$ and $G[A]$. We define $e_G(A, B)$ to be the number of edges in $G(A, B)$.*

For a graph (or digraph) G, a *list* L is a mapping which assigns to each $v \in V(G)$ a set of available colors for v. Let $L(v)$ be a set of available colors for each $v \in V(G)$. We say that G is *L-colorable* if there is a proper coloring ϕ of G such that $\phi(v) \in L(v)$ for all v.

In [14], the authors prove (Lemmas 8 and 9) the following statement, where $\deg^+_D(v)$ is the out-degree of v in the digraph D.

Lemma 2.2. *Suppose that A is an independent set in a graph G and $B = V(G) - A$. Let D be the digraph obtained from G by replacing each edge in $G[B]$ by a pair of opposite arcs and replacing each edge connecting A with B by an arc with arbitrary orientation. If L is a list such that*

$$|L(v)| \geq 1 + \deg_D^+(v) \text{ for all } v \in V(D)$$

then D is L-colorable.

The following lemma is a specific case of another result (Corollary 11) in [14].

Lemma 2.3. *Let G be a k-critical graph with disjoint vertex subsets A, B such that*

1. *A is independent,*
2. $\deg_G(a) = k - 1$ *for each $a \in A$, and*
3. $\deg_G(b) = k$ *for each $b \in B$.*

Then $e_G(A, B) \leq 2|A| + 2|B|$.

We will need to prove similar lemmas where the vertices $b \in B$ all have degree $k + 1$, or have degree d with $k \leq d \leq k + 1$.

Lemma 2.4. *Let G be a k-critical graph with disjoint vertex subsets A, B such that*

1. *A is independent,*
2. $\deg_G(a) = k - 1$ *for each $a \in A$,*
3. *and* $\deg_G(b) = k + 1$ *for each $b \in B$.*

Then $e_G(A, B) \leq 3(|A| + |B|)$.

Proof. Suppose that the hypotheses of the lemma hold. If A or B are empty then $e_G(A, B) \leq 3(|A| + |B|)$ is trivial, so we may assume that they are not empty. Let $G' = G[A \cup B]$ and let $G'' = G(A, B)$. We claim that $\delta(G'') \leq 3$. Suppose, for sake of contradiction, that $\delta(G'') \geq 4$.

From G'' we construct a new graph H be splitting each vertex $b \in B$ into $\lceil d_{G''}(b)/4 \rceil$ vertices of degree at most 4. Let B'' be the partite set obtained from B in this manner. Then H is a bipartite graph, $\deg_H(a) \geq 4$ for each $a \in A$, and $\deg_H(b) \leq 4$ for each $b \in B''$. Hall's Theorem ([10]) gives a matching $M \subseteq E(H)$ that covers A. Because each edge in H corresponds to an edge in $G'' = G(A, B)$, we will also say that $M \subseteq E(G'')$

Now we construct a digraph D from G' by replacing each edge in $G[B]$ by a pair of opposite arcs, orienting all edges of $M \subseteq E(G'')$ toward A, and orienting all remaining edges in $E(G'')$ towards B. Because G is k-critical, we can properly $(k-1)$-color $G - \{A \cup B\}$ with ϕ using color set C. For each $v \in A \cup B$, we let

$$L(v) := C - \bigcup_{\substack{x \in V(G)-(A \cup B) \\ xv \in E(G)}} \phi(x).$$

Note that for any $v \in (A \cup B)$ we get $|L(v)| \geq (k-1) - \deg_G(v) + \deg_{G'}(v)$. Therefore for $a \in A$ and $b \in B$, we have $|L(a)| \geq \deg_{G'}(a)$ and $|L(b)| \geq \deg_{G'}(b) - 2$. It follows that $\deg^+_D(a) = \deg_{G'}(a) - 1 \leq |L(a)| - 1$. Further, because each $b \in B$ has at most $\lceil \frac{1}{4} \deg_{G''}(b) \rceil$ incident arcs which are oriented towards A and because each b has at least 4 neighbors in A it follows that

$$\deg^+_D(b) \leq \deg_{G'}(b) - \left\lfloor \frac{3}{4} \deg_{G''}(b) \right\rfloor \leq (|L(b)| + 2) - 3 = |L(b)| - 1.$$

By Lemma 2.2 D is L-colorable. But this implies that ϕ can be extended to properly $(k-1)$-color G, which is a contradiction. Therefore, we have shown that $\delta(G'') \leq 3$.

We now prove by induction on $(|A| + |B|)$ that $e_G(A, B) \leq 3(|A| + |B|)$. This is trivial for $|A| + |B| = 2$, which is the base case. Now suppose that

$|A|+|B| = \ell$ and that the lemma holds for $|A|+|B| < \ell$. Because $\delta(G'') \leq 3$, there is a vertex x with $\deg_{G''}(x) = d \leq 3$; without loss of generality, suppose that $x \in A$. By the inductive hypothesis $e_G(A - \{x\}, B) \leq 3(|A| + |B| - 1)$. Therefore $e_G(A, B) \leq 3(|A| + |B|) - 3 + d \leq 3(|A| + |B|)$. $\square$

Lemma 2.5. *Let G be a k-critical graph with disjoint vertex subsets A, B_0, B_1 and let $B := B_0 \cup B_1$. If*

1. *A is independent,*
2. *$\deg_G(a) = k - 1$ for each $a \in A$,*
3. *and $\deg_G(b) = k + i$ for each $b \in B_i$ with $0 \leq i \leq 1$.*

Then $e_G(A, B) \leq 2|A| + 2|B_0| + 4|B_1|$.

Proof. Suppose that the hypotheses of the lemma hold. If A or B are empty then $e_G(A, B) \leq 2|A|+2|B_0|+4|B_1|$ is trivial, so we may assume that they are not empty. Let $G' = G[A \cup B]$ and let $G'' = G(A, B)$. We claim that either there is some $a \in A$ with $\deg_{G''}(a) \leq 2$ or some $b_i \in B_i$ with $\deg_{G''}(b_i) \leq 2+2i$ for $0 \leq i \leq 1$. Suppose, for sake of contradiction, that $\deg_{G''}(a) \geq 3$ for all $a \in A$ and $\deg_{G''}(b_i) \geq 3 + 2i$ for all $b_i \in B_i$ with $0 \leq i \leq 1$.

From G'' we construct a new graph H be splitting each vertex $b \in B$ into $\lceil d_{G''}(b)/3 \rceil$ vertices of degree at most 3. Let B'' be the partite set obtained from B in this manner. Then H is a bipartite graph, $\deg_H(a) \geq 3$ for each $a \in A$, and $\deg_H(b) \leq 3$ for each $b \in B''$. Hall's Theorem ([10]) gives a matching $M \subseteq E(H)$ that covers A. Because each edge in H corresponds to an edge in $G'' = G(A, B)$, we will also say that $M \subseteq E(G'')$

Now we construct a digraph D from G' by replacing each edge in $G[B]$ by a pair of opposite arcs, orienting all edges of $M \subseteq E(G'')$ toward A, and orienting all remaining edges in $E(G'')$ towards B. Because G is k-critical, we can properly $(k-1)$-color $G - \{A \cup B\}$ with ϕ using color set C. For each $v \in A \cup B$, we let

$$L(v) := C - \bigcup_{\substack{x \in V(G)-(A\cup B)\\ xv \in E(G)}} \phi(x).$$

Note that for any $v \in (A \cup B)$ we get $|L(v)| \geq (k-1) - \deg_G(v) + \deg_{G'}(v)$. Therefore for $a \in A$ and $b_i \in B_i$ we have $|L(a)| \geq \deg_{G'}(a)$ and $|L(b_i)| \geq \deg_{G'}(b_i) - 1 - i$. It follows that $\deg^+_D(a) = \deg_{G'}(a) - 1 \leq |L(a)| - 1$. Further, because each $b_i \in B_i$ has at most $\lceil \frac{1}{3} \deg_{G''}(b_i) \rceil$ incident arcs which are oriented towards A and because each b_i has at least $3 + 2i$ neighbors in A it follows that

$$\deg^+_D(b_i) \leq \deg_{G'}(b_i) - \left\lfloor \frac{2}{3} \deg_{G''}(b_i) \right\rfloor \leq (|L(b_i)| + 1 + i) - \left\lfloor \frac{6+4i}{3} \right\rfloor = |L(b_i)| - 1.$$

By Lemma 2.2 D is L-colorable. But this implies that ϕ can be extended to properly $(k-1)$-color G, which is a contradiction. Therefore, we have shown that there is either some $a \in A$ with $\deg_{G''}(a) \leq 2$ or some $b_i \in B_i$ with $\deg_{G''}(b_i) \leq 2 + 2i$ for $0 \leq i \leq 1$.

We now prove by induction on $(|A| + |B|)$ that $e_G(A, B) \leq 2|A| + 2|B_0| + 4|B_1|$. This is trivial for $|A| + |B| = 2$, which is the base case. Now suppose that $|A| + |B| = \ell$ and that the lemma holds for $|A| + |B| < \ell$. Suppose that there is some $a \in A$ with $\deg_{G''}(a) = d \leq 2$. By the inductive hypothesis $e_G(A - \{a\}, B) \leq 2(|A| - 1) + 2|B_0| + 4|B_1|$. Therefore $e_G(A, B) \leq 2(|A| - 1) + 2|B_0| + 4|B_1| + d \leq 2|A| + 2|B_0| + 4|B_1|$.

Now suppose instead that there is some $b_i \in B_i$ with $\deg_{G''}(b_i) = d \leq 2+2i$. By the inductive hypothesis and assumption on degree of b_i it follows that

$$e_G(A, B) \leq (2|A| + 2|B_0| + 4|B_1| - 2 - 2i) + d \leq 2|A| + 2|B_0| + 4|B_1|.$$

$\square$

2.3 Further edge bound on k-critical graphs

The following lemma, used in the second stage of discharging in Chapter 6, is a restatement in terms of k-critical graphs of a theorem from Kierstead and Rabern (Theorem 4.5) [11]. Let $\text{mic}(G)$ be the maximum of $\sum_{v \in I} \deg_G(v)$ over all independent vertex subsets I of G.

Lemma 2.6. *Given a k-critical graph G with at least one vertex of degree $k-1$,*

$$2|E(G)| \geq (k-2)|V(G)| + mic(G) + 1.$$

In this work, we omit the $+1$ because the other two terms scale with $|V(G)|$.

Chapter 3

Ore compositions and k-Ore graphs

We can equivalently define a k-critical graph to be a graph G with chromatic number k such that every proper subgraph of G has chromatic number at most $k-1$. In this sense, k-critical graphs can be thought of as minimal graphs that are not $(k-1)$-colorable. It is natural to ask how small such graphs can be. Recall that $f_k(n)$ is the minimum number of edges of a k-critical graph on n vertices. In a recent paper, Kostochka and Yancey [14] proved that a k-critical graph must have edge density above a certain threshold. Namely, they proved that if $k \geq 4$, $n \geq k$ and $n \neq k+1$ then

$$f_k(n) \geq \left\lceil \left(\frac{k}{2} - \frac{1}{k-1}\right) n - \frac{k(k-3)}{2(k-1)} \right\rceil.$$

Further, this bound is tight since it is attained by an infinite class of k-critical graphs. In a subsequent paper, Kostochka and Yancey [13] proved that k-critical graphs which attain these bounds are precisely the *k-Ore graphs* described below. Therefore, in exploring new bounds on edge-density of k-critical graphs, it is important to begin our discussion with results on k-Ore graphs.

3.1 Ore composition and potential function

Definition 3.1. *An* Ore composition *of two graphs G_1 and G_2 is a graph obtained by the following procedure:*

1. *delete an edge xy from G_1,*
2. *split some vertex z of G_2 into two vertices z_1 and z_2 of positive degree,*
3. *identify x with z_1 and identify y with z_2.*

Note that an Ore composition of G_1 and G_2 does not necessarily obtain a unique graph. Further, the order in which we list the graphs is important; we say that G_1 (always listed first) is the *edge-side* and G_2 (always listed second) is the *split-side* of the composition. The identified vertices $\underline{xz_1}$ and $\underline{yz_2}$ are the *overlap* vertices of the composition. Further, we say that xy is the *replaced edge* of G_1 and that z is the *split vertex* of G_2. A graph G is a *k-Ore graph* if and only if it is in the smallestclass of graphs which is closed under the Ore composition operation and contains K_k. This means that any k-Ore graph can be obtained if we start with K_k and make repeated Ore compositions with copies of K_k and any intermediate graphs created along the way.

On occasion, we are interested in decomposing a k-Ore graph G into the graphs G_1 and G_2 that were used to obtain G. When we split a vertex z of the split-side, the resulting graph can be properly $(k-1)$-colored, but the new vertices z_1 and z_2 must receive different colors. Therefore, if we focus on colorings of G_1 then the restrictions caused by the replaced edge $xy \in E(G_1)$ are the same as the restrictions caused by the split-side of the composition. We may view the graph G as a graph G' isomorphic to G_1 where the edge $xy \in E(G')$ corresponds to a subgraph $D \subseteq G$ which is the split-side of the Ore composition of G after the vertex z has been split into two vertices. We call the edge $xy \in E(G')$ an *edge-replacement* to distinguish it from the edges which are in both G and G_1. Replacing an edge-replacement e in G' with its corresponding subgraph $D \subseteq G$ is equivalent to taking an Ore composition of G' and $D/\underline{xy}$.

Proposition 3.2. *Suppose that G is a k-Ore graph. Then there exists a graph $H = K_k$ with ℓ edge-replacements $\{e_1, \ldots, e_\ell\}$ corresponding to subgraphs $\{D_1, \ldots, D_\ell\}$ of G. The graph G can be obtained from H by replacing each e_i with D_i.*

Proof. Let G be a k-Ore graph. We will prove this by induction on $|V(G)|$. If G is K_k then the result is trivial, so we may assume that G is the Ore composition of two k-Ore graphs G_1 and G_2 with overlap vertices $\{x, y\}$. Let $D \subseteq G$ be the subgraph $G[(V(G) - V(G_1)) \cup \{x, y\}]$. This subgraph is isomorphic to the k-Ore graph G_2 after the split-vertex z is split into two vertices x, y of positive degree.

By induction G_1 can be reduced to $H_1 = K_k$ where the graph H_1 has r edge-replacements. Suppose that xy is an actual edge in H_1. Then G is reducible to $H = K_k$ where H has the same r edge-replacements as H_1 and also has the edge-replacement xy corresponding to D. Suppose instead that xy is not an edge in H_1. Then xy is an edge in a subgraph $D_1 \subseteq G_1$ which corresponds to an edge-replacement e_1 in H_1. Now G is reducible to $H = K_k$ where e_1 is an edge-replacement corresponding to a subgraph which is an Ore composition of D_1 and G_2. $\square$

Kostochka and Yancey's bound on $f_k(n)$ also gave an asymptotic confirmation of Ore's Conjecture; that is, Kostochka and Yancey's result proves that $\varphi_k := \lim_{n\to\infty} \frac{f_k(n)}{n} = \frac{k}{2} - \frac{1}{k-1}$. One of the goals of this work is to increase the asymptotic density when $k \geq 33$ for the class of k-critical graphs that do not contain K_{k-2} as a subgraph, which leads to a strengthening of Kostochka and Yancey's result.

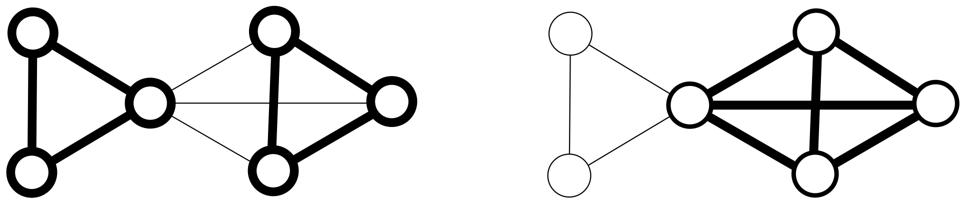

Figure 3.1: $H_1 \subseteq G$ is in bold on the left; $H_2 \subseteq G$ is in bold on the right.

In order to obtain this improvement, we define a graph function T which 'counts' the K_{k-1} and K_{k-2} subgraphs in a particular way.

Definition 3.3. *Suppose that the graph H is a disjoint union of r K_{k-1} and s K_{k-2} subgraphs. Then $T(H)$ is defined to be $2r + s$. More generally, we define $T(G)$ for an arbitrary graph G as follows:*

$$T(G) := \max_{H \subseteq G}\{T(H) \mid H \textit{ is a disjoint union of } K_{k-1} \textit{ and } K_{k-2} \textit{ components}\}.$$

We let $T(G)$ be the maximum over all choices of subgraphs $H \subseteq G$ which are disjoint collections of K_{k-1} and K_{k-2} subgraphs. For example, for $k = 5$, Figure 3.1 shows two subgraphs H_1 and H_2 of G. Both $T(H_1)$ and $T(H_2)$ are 2 and so it follows that $T(G) \geq 2$ (in fact, we can check that $T(G) = 2$ in this example). These two choices of subgraphs also highlight that there could be multiple subgraphs $H \subseteq G$ which witness $T(G)$. Throughout our proofs, we make no assumptions on how these subgraphs are chosen.

We can now define the potential of a graph.

Definition 3.4. *Given a graph G, we define the* potential *of a graph to be*

$$\rho_\epsilon(G) := ((k-2)(k+1) + \epsilon)\,|V(G)| - 2(k-1)|E(G)| - \delta T(G)$$

for a fixed ϵ with $0 \leq \epsilon \leq \frac{4}{k^3 - 2k^2 + 3k}$ and $\delta = (k-1)\epsilon$.

Because ϵ remains fixed throughout the proof, we omit this subscript. The theorem is then proven for any potential function ρ_ϵ with ϵ in the specified range. Different values of ϵ in this range will lead to different corollaries, as discussed in Chapter 1. The main goal of this work, and the aim of the remainder of this chapter and the subsequent three chapters, is to prove the following theorem about potential of k-critical graphs:

Theorem 3.5. *If G is a k-critical graph with $k \geq 4$ then*

1. *$\rho(G) = k(k-3) + k\epsilon - 2\delta$ if $G = K_k$,*
2. *$\rho(G) \leq k(k-3) + |V(G)|\epsilon - \left(2 + \frac{|V(G)|-1}{k-1}\right)\delta$ if G is k-Ore and $G \neq K_k$, and*
3. *$\rho(G) \leq k(k-3) - 2(k-1)$ if G is not k-Ore, for $k \geq 33$.*

When $k \geq 33$, this is a complete statement about potentials for all k-critical graphs. For $k < 33$ the first two statements hold but, due to constraints in the discharging argument that we use in Chapter 6, we are not able to prove the third statement for small k without significantly increasing the complexity of the discharging arguments. Note that $T(K_k) = 2$ and so the first assertion of Theorem 3.5 is immediate from the definition of $\rho(G)$. In the following Section, we will examine the function $T(G)$ where G is a k-Ore graph or an Ore composition and we will prove the second assertion of Theorem 3.5.

3.2 Results on k-Ore graphs

First, we prove the following lemma about Ore compositions:

Lemma 3.6. *If G is an Ore composition of G_1 and G_2, then $T(G) \geq T(G_1) + T(G_2) - 2$. Moreover, if $G_1 = K_k$ or $G_2 = K_k$ then $T(G) \geq T(G_1) + T(G_2) - 1$.*

Proof. Suppose that G is an Ore composition of G_1 and G_2 . Let e be the replaced edge of G_1 and z be the split vertex of G_2. From the definition of an Ore composition $T(G) \geq T(G_1 - e) + T(G_2 - \{z\})$. Note that $T(G_1 - e) \geq$

$T(G_1) - 1$ and $T(G_2 - \{z\}) \geq T(G_2) - 1$, because removing a single element can decrease $T(G)$ by at most 1. Thus, we get $T(G) \geq T(G_1) + T(G_2) - 2$ as desired. If $G_1 = K_k$ then $T(K_k - e) = 2$ for every edge $e \in E(G_1)$; also, if $G_2 = K_k$ then $T(K_k - \{v\}) = 2$ for every $v \in V(G_2)$. Removing an element from a K_k graph does not decrease $T(K_k)$. Therefore, it follows that $T(G) \geq T(G_1) + T(G_2) - 1$ if either G_1 or G_2 is K_k. Further, if both G_1 and G_2 are K_k then $T(G) = 4$. $\square$

Let G be a k-Ore graph. We can show inductively that there exists an $\ell \geq 0$ such that $|V(G)| = k + \ell(k-1)$ and $|E(G)| = \frac{(\ell+1)k(k-1)}{2} - \ell$. When G is a k-Ore graph that is not K_k, then it is an Ore composition and we can use Lemma 3.6 to get a bound on $T(G)$.

Lemma 3.7. *If G is a k-Ore graph and $G \neq K_k$ then $T(G) \geq 2 + \frac{|V(G)|-1}{k-1}$.*

Proof. We proceed by induction on $|V(G)|$. Since G is k-Ore and $G \neq K_k$, then G must be the Ore composition of two k-Ore graphs G_1 and G_2. Note that if G is an Ore composition of G_1 and G_2 then, by the definition of Ore composition, $|V(G)| = |V(G_1)| + |V(G_2)| - 1$ If G_1 and G_2 are both K_k graphs, then $|V(G)| = 2k - 1$ so we need to show that $T(G) \geq 4$. We know that $T(K_k - e) = T(K_k - \{z\}) = 2$ regardless of the choice of replaced edge e and split vertex z in the composition. Thus $T(G) \geq 4$ as desired.

Suppose that $G_1 = K_k$ and $G_2 \neq K_k$. Then by Lemma 3.6 and the inductive hypothesis, we have that

$$T(G) \geq T(G_2) + 1 \geq \left(2 + \frac{|V(G_2)| - 1}{k-1}\right) + 1 = 2 + \frac{|V(G)| - 1}{k-1}$$

as desired. A similar argument covers the case where $G_1 \neq K_k$ and $G_2 = K_k$.

Finally, suppose that neither G_1 nor G_2 is K_k. Then because $|V(G)| = |V(G_1)| + |V(G_2)| - 1$, it follows from Lemma 3.6 that

$$T(G) \geq \left(2 + \frac{|V(G_1)| - 1}{k-1}\right) + \left(2 + \frac{|V(G_2)| - 1}{k-1}\right) - 2 = \left(2 + \frac{|V(G)| - 1}{k-1}\right).$$

□

We obtain the second assertion of Theorem 3.5 as a corollary.

Corollary 3.8. *If G is a k-critical graph that is k-Ore and $G \neq K_k$ then $\rho(G) \leq k(k-3) + |V(G)|\epsilon - \left(2 + \frac{|V(G)|-1}{k-1}\right)\delta$.*

Proof. To prove this, we recall that

$$\rho(G) := ((k-2)(k+1) + \epsilon)\,|V(G)| - 2(k-1)|E(G)| - \delta T(G).$$

Also, a k-Ore graph which is not K_k has $k+\ell(k-1)$ vertices and $\frac{(\ell+1)k(k-1)}{2} - \ell$ edges for some $\ell \geq 1$. Therefore, using Lemma 3.7, it is a straightforward calculation to show that $\rho(G) \leq k(k-3) + |V(G)|\epsilon - \left(2 + \frac{|V(G)|-1}{k-1}\right)\delta$. □

3.3 Diamonds and emeralds

Definition 3.9. *A subgraph $D \subseteq G$ is a* diamond *of G if $D = K_k - uv$ and $\deg_G(x) = k-1$ for each $x \in V(D) - \{u, v\}$. The vertices u and v are called the* endpoints *of the diamond.*

Definition 3.10. *A subgraph $D \subseteq G$ is an* emerald *of G if $D = K_{k-1}$ and $\deg_G(x) = k-1$ for each $x \in V(D)$.*

Lemma 3.11. *If G is k-Ore and $v \in V(G)$, then there exists a diamond or emerald of G not containing v.*

Proof. We will prove this by induction on the order of G, $|V(G)|$. Let G be a k-Ore graph and let $v \in V(G)$ be an arbitrary vertex. If $G = K_k$ then $G - \{v\}$ is an emerald of G not containing v. Otherwise, we have that G is an Ore composition of two k-Ore graphs G_1 and G_2 with overlap vertices $\{a, b\}$. We choose this composition to minimize the number of vertices in the edge-side G_1. If $v \in V(G_1)$ then, inductively, there is a diamond or an emerald D in G_2 not containing the vertex $\underline{ab} \in V(G_2)$. Note that D is also a diamond or emerald of G not containing v.

Therefore, we may assume that $v \in V(G_2) - V(G_1)$. If $G_1 = K_k$, then G_1 is a diamond of G and we are done. Otherwise, G_1 is a composition of two k-Ore graphs H_1 and H_2 with overlap vertices $\{x, y\}$. If the edge $ab \in E(G_1)$ is in $E(H_2)$ then there is a decomposition of G with H_1 as the edge-sde, which contradicts the minimality of G_1. Thus, $ab \in E(H_1)$ and by induction there is a subgraph D of H_2 which is a diamond or emerald not containing $\underline{xy} \in V(H_2)$. Note that D is also disjoint from vertex v, and is also a diamond or emerald in G because $ab \in E(H_1)$. This completes the proof. $\square$

Note that one could prove as a corollary that there is a diamond or emerald in G disjoint from any edge $e \in E(G)$.

Lemma 3.12. *If G is k-Ore and $D = K_{k-1}$ is a subgraph of G, then either $G = K_k$ or there exists a diamond or emerald in G disjoint from D.*

Proof. We will prove this by induction on $|V(G)|$. Let G be a k-Ore graph and let $D = K_{k-1}$ be a subgraph of G. If $G = K_k$ then we are trivially done, so suppose that G is an Ore composition of two k-Ore graphs G_1 and G_2 with overlap vertices $\{a, b\}$. We choose this composition to minimize $|V(G_1)|$. Note that D lies entirely on the edge-side or the split-side of this composition. If $V(D) \subseteq V(G_1)$ then by Lemma 3.11 there is a diamond or an emerald in G_2 not containing the vertex $\underline{ab} \in V(G_2)$. This is also a diamond or an emerald of G and it is necessarily disjoint from D.

Therefore, we may assume that D lies on the split-side. We examine two cases based on if $V(D)$ contains one of the overlap vertices $\{a, b\}$ or not. First, suppose that $V(D)$ contains neither a nor b. Then if $G_1 = K_k$, then G_1 is a diamond that is disjoint from D. Otherwise, G_1 is a composition of two k-Ore graphs H_1 and H_2 with overlap vertices $\{x, y\}$. If the edge $ab \in E(G_1)$ is in $E(H_2)$ then there is a decomposition of G with H_1 as the edge-side, which contradicts the minimality of G_1. Thus, $ab \in E(H_1)$ and by Lemma 3.11 there is a diamond or an emerald in H_2 not containing $\underline{xy} \in V(H_2)$. Note that this does not contain a or b, so is also a diamond or emerald of G which is disjoint from D.

Now we suppose that $V(D)$ and $\{a, b\}$ are not disjoint. Because $ab \notin E(G)$, $|V(D) \cap \{a, b\}| = 1$ and without loss of generality, we assume that $a \in V(D)$. If $G_2 \neq K_k$ then by induction, there is a diamond or an emerald in G_2 disjoint from D. This is also disjoint from D in the graph G, so we may assume that $G_2 = K_k$. Because $\deg_{G_2}(\underline{ab}) = k - 1$, it follows that $b \in V(G)$ has exactly one neighbor on the split-side G_2. By Lemma 3.11, there is a diamond or emerald D' in G_1 disjoint from a. If D' is a diamond, then it is also a diamond of G that is disjoint from D because $b \in V(D')$ implies that b is an endpoint of D', since $ab \in E(G_1)$. If D' is an emerald that does not contain b, then it is an emerald of G that is disjoint from D. If D' is an emerald and $b \in V(D')$ then $\deg_{G_1}(b) = k - 1$. However, in G, b is no longer adjacent to a, and has exactly one neighbor on the split-side G_2. Therefore $\deg_G(b) = k - 1$ as well and D' is also an emerald of G that is disjoint from D. $\square$

Chapter 4

Potential and critical extensions

4.1 Potential

We now recall the definition of the potential of a graph.

Definition 4.1. *A graph G has* potential

$$\rho(G) = ((k-2)(k+1)+\epsilon)\,|V(G)| - 2(k-1)|E(G)| - \delta T(G).$$

We can also define this function on vertex subsets R of a graph G. If $R \subseteq V(G)$, then let

$$\rho_G(R) = ((k-2)(k+1)+\epsilon)\,|R| - 2(k-1)|E(G[R])| - \delta T(G[R])$$

where we reacall that $G[R]$ is the induced subgraph of G with vertex set R.

Below, we begin the list of complete subgraphs of a k-critical graph for a fixed k, ordered from smallest potential to largest potential. The complete subgraphs not listed all have larger potential than these four.

Fact 4.2.

1. $\rho(K_1) = k^2 - k - 2 + \epsilon$.

2. $\rho(K_{k-1}) = 2k^2 - 6k + 4 + (k-1)\epsilon - 2\delta$.

3. $\rho(K_2) = 2k^2 - 4k - 2 + 2\epsilon$.

4. $\rho(K_{k-2}) = 3k^2 - 11k + 10 + (k-2)\epsilon - \delta$.

To prove Theorem 3.5, our strategy is to look at a minimal counterexample and use proof by contradiction. Therefore, we now need to precisely define what makes a graph minimal. Let G and H be two graphs.

Definition 4.3. *The graph H is* smaller *than G if either $|E(G)| > |E(H)|$, or $|E(G)| = |E(H)|$ and G has fewer pairs of vertices with the same closed neighborhood.*

Definition 4.4. *A graph G is* good *if it is k-critical and every smaller k-critical graph satisfies Theorem 3.5.*

Therefore, when we say that G is a minimal counterexample to the third statement of Theorem 3.5, we mean that it is a good graph which is not k-Ore and does not satisfy Theorem 3.5.

4.2 Critical extensions

If R is a proper subset of a k-critical graph G, then we can use the criticality of G to define an extension of R. First, note that if $R \subsetneq V(G)$, then we can $(k-1)$-color $G[R]$ with some proper coloring $\phi : R \to [k-1]$. We define the graph $G_{R,\phi}$ to be the graph obtained from G by identifying all vertices in $\phi^{-1}(i)$ to a single vertex x_i for $1 \leq i \leq k-1$, adding the edge $x_i x_j$ for $1 \leq i < j \leq k-1$, and then replacing any multiedges with single edges. Note that we don't remove adjacencies in G, so that if $u \in R$, $v \in V(G) - R$, and $uv \in E(G)$ then $vx_{\phi(u)} \in E(G_{R,\phi})$.

The following lemma is proved in [14], we include a proof for completeness.

Lemma 4.5. *If G is a k-critical graph with $R \subsetneq V(G)$. Then $G_{R,\phi}$ is not $(k-1)$-colorable.*

Proof. Suppose the result fails to hold. Let G be a k-critical graph with $R \subsetneq V(G)$ and let $\phi : R \to [k-1]$ be a proper $(k-1)$-coloring of $G[R]$. By assumption, there is a proper $(k-1)$-coloring ψ of $G_{R,\phi}$. Recall that $V(G_{R,\phi}) = (V(G) - R) \cup \{x_1, \ldots, x_{k-1}\}$ and that $G_{R,\phi}[\{x_1, \ldots, x_{k-1}\}]$ is a K_{k-1}. We change the labels of ψ so that $\psi(x_i) = i$ for each vertex $x_i \in V(G_{R,\phi})$. However, $\psi|_{V(G)-R} \cup \phi|_R$ is now a proper $(k-1)$-coloring of G, which is a contradiction. Therefore, the result must hold. $\square$

By Lemma 4.5, we know that there is a k-critical subgraph $W \subseteq G_{R,\phi}$. Because G is k-critical, W must contain at least one vertex in $\{x_1, \ldots, x_n\}$. We let $X := V(W) \cap \{x_1, \ldots, x_n\}$ and $R' := (R \cup V(W)) - X$. Note that $R \subsetneq R' \subseteq V(G)$. We say that R' is a W*-critical extension* of R and that X is the *core* of the extension.

Lemma 4.6. *If R' is a W-critical extension of $R \subsetneq V(G)$ with core X, then*

$$\rho_G(R') \leq \rho_G(R) + \rho(W) - \rho(K_{|X|}) - \delta T(K_{|X|}) + \delta|X|. \tag{4.1}$$

Proof. The three elements of a graph that contribute to potential are vertices, edges, and the number of K_{k-1} and K_{k-2} subgraphs. Because $R' = R \cup V(W) - X$, each side of Equation 4.1 counts the same number of vertices. For the edges, each side of Equation 4.1 counts some edges that the other does not. Note that $\rho_G(R')$ counts edges in G from R to $V(W) - X$. The right-hand side counts edges in $G_{R,\phi}$ from X to $V(W) - X$. All other edges (those inside $G[R]$ and those inside $W - \{X\}$) are accounted for by both sides of Equation 4.1. However each edge from R to $V(W) - X$ in G maps to an edge from X to $V(W) - X$. Therefore we have the desired inequality with respect to the edge-contribution to potential.

Finally we look at the K_{k-1} and K_{k-2} subgraphs. Now $T(G[R'])$ must be at least the sum of $T(G[R])$ and $T(W - \{X\})$. Also, $T(W - \{X\})$ must be at least $T(W) - |X|$ because each $x_i \in X$ could be in, at worst, one subgraph counted by $T(W)$. Therefore, we get the desired inequality in Equation 4.1. $\square$

Corollary 4.7. *For good graphs G with $R \subsetneq V(G)$, $\rho_G(R') \leq \rho_G(R) - 2(k - 1) - \delta$.*

Proof. By Fact 4.2, we can see that $\rho(K_{|X|}) + \delta T(K_{|X|}) - \delta|X|$ is minimized when $|X| = 1$. Therefore, Lemma 4.6 becomes

$$\rho_G(R') \leq \rho_G(R) + \rho(W) - (k^2 - k - 2 + \epsilon - \delta). \tag{4.2}$$

If W is K_k, then because $T(W) = T(W - \{x\})$ for $x \in X$ we can ignore the $+\delta$ in Equation 4.2. Therefore for $W = K_k$ it follows that

$$\rho_G(R') \leq \rho_G(R) + (k^2 - 3k + k\epsilon - 2\delta) - (k^2 - k - 2 + \epsilon)$$

$$= \rho_G(R) - 2(k - 1) + (k - 1)\epsilon - 2\delta.$$

If W is k-Ore and $W \neq K_k$ then it follows from Theorem 3.5 that

$$\rho_G(R') \leq \rho(G) + (k(k-3) + |V(G)|\epsilon - \left(2 + \frac{|V(G)| - 1}{k - 1}\right)\delta) - (k^2 - k - 2 + \epsilon - \delta).$$

Because $\delta \geq (k - 1)\epsilon$, this implies that $\rho_G(R') \leq \rho_G(R) - 2(k - 1) - \delta$.

Finally, if W is not k-Ore, then $\rho(W) \leq k(k - 3) - 2(k - 1)$ because G is a good graph. Therefore we get $\rho_G(R') \leq \rho_G(R) - 4(k - 1) - \epsilon$ and because $-2(k - 1) - \epsilon \leq \delta$ we have proven the corollary. $\square$

Corollary 4.7 implies that in a minimal counterexample G every proper subgraph $G[R] \subsetneq G$ has potential that is higher than $\rho(G)$ because we can continue making critical extensions, lowering the potential and growing the subset until we reach an extension which is all of $V(G)$. Therefore $\rho(G) =$

$\min\{\rho_G(R) \mid R \subseteq V(G)\}$, and we are able to use ρ as a global parameter for discharging.

Suppose that R' is a W-critical extension of $R \subsetneq V(G)$ with core X. We define $|X|$ to be the *core size* of the W-critical extension. If $R' = V(G)$ then we say that R' is a *spanning* critical extension. Note that if $R' \neq V(G)$ then $G[R']$ is $(k-1)$-colorable; however, the coloring ϕ used on R cannot be extended to a proper $(k-1)$-coloring of R' by construction. The extension R' is, in fact, a minimal subset of $V(G)$ to which ϕ cannot be extended.

Finally, we define a *complete* critical extension. The extension R' is complete if both $G[R'-R] = W[R'-R]$ and there is a one-to-one correspondence between edges in W from $R'-R$ to X, to edges in G from $R'-R$ to R. If an extension is complete, then the edge contribution in Lemma 4.6 is the same on each side of Equation 4.1. That is,

$$|E(G[R'])| = |E(G[R])| + |E(W)| - |E(K_{|X|})|.$$

For a general W-critical extension R', it is possible that the left hand side of the above equation is larger. If we have $|E(G[R'])| = |E(G[R])| + |E(W)| - |E(K_{|X|})| + i$ then we say that the extension is *i-incomplete*.

4.3 Collapsible sets and edge-additions

Given a graph G and a subset of its vertices $R \subseteq V(G)$, the *boundary* of R, denoted by $\partial_G R$, is defined to be the set $\partial R := \{v \in R \mid uv \in E(G)$ for some $u \in V(G) - R\}$. If R is a subset of the vertices of two graphs G and H, we will specify $\partial_G R$ and $\partial_H R$ for the boundary of R in each respective graph.

Definition 4.8. *A proper vertex subset R is* collapsible in G *if for every proper $(k-1)$-coloring of R, the vertices of ∂R receive the same color.*

We call such a set collapsible, because the subset R behaves like a fat vertex with respect to coloring.

Proposition 4.9. *A subset $R \subsetneq V(G)$ is collapsible in G if and only if every critical extension has a core of size 1, is spanning, and is complete.*

Proof. Suppose, first, that R is collapsible in G. Let ϕ be an arbitrary proper $(k-1)$-coloring of $G[R]$. Permute the colors, if necessary, so that ∂R receives color 1, and consider the graph $G_{R,\phi}$ and a W-critical extension R'. The vertex $x_1 \in V(G_{R,\phi})$ is a cut-vertex. Let $A = (V(G) - R) \cup \{x_1\}$. Because a k-critical graph cannot contain a cut-vertex and must have at least k vertices, we know that W is a subgraph of $G_{R,\phi}[A]$ and that the core X contains only x_1.

Now suppose, for sake of contradiction, that $R' \subsetneq V(G)$ and let $y \in V(G) - R'$. Then we can properly $(k-1)$-color $G - \{y\}$ with ψ; this induces a proper $(k-1)$-coloring in R and, without loss of generality, we assume that $\psi(\partial R) = 1$. It follows that using ψ to color $W - \{x_1\}$ and coloring x_1 with 1 gives a proper $(k-1)$-coloring of W. Therefore, we have shown that R' has core size 1 and is spanning, and we now show that it is complete.

If $G[R' - R]$ has an edge f that is not in $W[R' - R]$, then we can properly $(k-1)$-color $G - f$ with ψ and the same argument as above gives a proper $(k-1)$-coloring of W. Suppose that an edge $ux_1 \in E(W)$ corresponds to two edges $uv_1, uv_2 \in E(G)$ with $v_1, v_2 \in R$. Then $G - uv_1$ can be $(k-1)$-colored by ψ. Because R is collapsible, $\psi(v_1) = \psi(v_2)$ is distinct from $\psi(u)$ and thus ψ is actually a proper $(k-1)$-coloring of G which is a contradiction. Therefore, R' must also be a complete extension. Because R' and ϕ were chosen arbitrarily, this shows the forward direction of the proposition.

We now prove the reverse implication. Suppose, for sake of contradiction, that every W-critical extension of R has a core of size 1, is spanning, and is complete and that ϕ is a proper $(k-1)$-coloring of R where $\phi(u) \neq \phi(v)$ for $u, v \in \partial R$. Without loss of generality, we may assume that $\phi(u) = 1$ and $\phi(v) = 2$. We pick a W-critical extension R' with core X. Because $|X| = 1$ we may assume that $x_2 \notin X$. Because $v \in \partial R$ and R' is spanning, there is an edge zv in G from $R' - R$ to R that has no corresponding edge in W as $x_2 \notin V(W)$. Therefore R' is at least 1-incomplete, which is a contradiction. $\square$

Lemma 4.10. *There is no 2-cut in a minimal counterexample to the third statement of Theorem 3.5.*

Proof. Let G be a minimal counterexample to the third statement of Theorem 3.5. Suppose that there is a 2-cut $\{x, y\}$. Then by Dirac [5] deleting $\{x, y\}$ leaves us with two components H_1 and H_2, $\tilde{G_1} = G - H_2$ is $(k-1)$-colorable by ϕ where $\phi(x) = \phi(y)$, and $\tilde{G_2} = G - H_1$ is $(k-1)$-colorable by ψ where $\psi(x) \neq \psi(y)$. We could claim that G is an Ore composition of $\tilde{G_1} + xy$ and $\tilde{G_2}/\underline{xy}$ if we are able to show that x and y have no common neighbors in $\tilde{G_2}$.

Suppose that z is such a vertex. Then $G - xz$ is $(k-1)$-colorable by ϕ and because $\tilde{G_1} \subseteq G - xz$, $\phi(x) = \phi(y)$. However, $\phi(z) \neq \phi(y)$ because they are adjacent and so ϕ is also a proper $(k-1)$-coloring of G. This is a contradiction, so no such z exists. Thus we have shown that G is an Ore composition of $\tilde{G_1} + xy$ and $\tilde{G_2}/\underline{xy}$, which we rename to G_1 and G_2 (respectively) for the remainder of the proof.

Because G is not k-Ore, at most one of G_1 and G_2 is k-Ore. First, suppose that G_2 is not k-Ore. Any time we have an Ore composition of graphs we know that

$$\rho(G) = \rho(G_1) + \rho(G_2) - k^2 + 3k - \epsilon + \delta\left(T(G_1) + T(G_2) - T(G)\right).$$

By Lemma 3.6 and because G_2 is not k-Ore, it follows that $\rho(G) \leq \rho(G_1) - 2(k-1) - \epsilon + 2\delta \leq \rho(G_1)$. Therefore G_1 must be k-Ore because it is smaller than G. If $G_1 \neq K_k$ and $|V(G_1)| = n$, then the second assertion of Theorem 3.5 yields

$$\rho(G) \leq \left(k(k-3) + n\epsilon - (n-1)\frac{\delta}{k-1} - 2\delta\right) - 2(k-1) - \epsilon + 2\delta$$

$$= k(k-3) - 2(k-1) + (n-1)\left(\epsilon - \frac{\delta}{k-1}\right) \leq k(k-3) - 2(k-1),$$

which is a contradiction. So it follows that G_1 must be K_k. But in this case Lemma 3.6 gives $T(G_1) + T(G_2) - T(G) \leq 1$, so we get $\rho(G) \leq (k(k-3) + k\epsilon - 2\delta) - 2(k-1) - \epsilon + \delta$. Because $\delta \geq (k-1)\epsilon$, this is a contradiction. The argument works exactly the same when we assume that G_1 is not k-Ore. Therefore we have shown that a minimal counterexample G must be at least 3-connected. □

Definition 4.11. *Let G be a k-critical graph. An* edge-addition *in G is a non-edge xy such that for some k-critical graph H with $xy \in E(H)$, $H - xy \subseteq G$ and $V(H) \subsetneq V(G)$.*

Note that if a k-critical graph G has a collapsible subset $R \subsetneq V(G)$ then G admits an edge-addition. The connectivity of a k-critical graph means $|\partial_G R| \geq 2$ and R collapsible implies that xy is an edge addition in G for $x, y \in \partial_G R$.

Lemma 4.12. *A minimal counterexample G to the third statement of Theorem 3.5 admits no edge-addition.*

Proof. Let G be a k-critical graph such that G is not k-Ore, $\rho(G) > k(k-3) - 2(k-1)$, and G is the smallest k-critical graph that satisfies (1) and (2). Suppose, for sake of contradiction, that there is a non-edge xy such that $G + xy$ contains a k-critical subgraph H where $V(H) \subsetneq V(G)$, .

Of all the edge-additions that exist in G, pick xy so that we minimize the number of vertices in the k-critical subgraph H. Let $R = V(H)$. Then $\rho_G(R) \leq \rho(H) + 2(k-1) + \delta$ where equality implies that xy is in a K_{k-1} or K_{k-2} subgraph counted by $T(H)$. Because R is a proper subset, we can make a W-critical extension R' and by Corollary 4.7 we have

$$\rho_G(R') \leq \rho_G(R) - 2(k-1) - \delta \leq \rho(H). \tag{4.3}$$

Since $\rho(G) \leq \rho_G(R') \leq \rho(H)$ and H is smaller than G, it follows that H must be k-Ore and $\rho(H) \leq k(k-3)$.

We show now that R must also be collapsible in G. If R is not collapsible, then we may assume that we chose the W-critical extension R' so that it is either not spanning, not complete, or has core of size larger than 1. If R' is not spanning then we can make another extension and by Corollary 4.7 and Equation 4.3 we have $\rho(G) \leq \rho_G(R'') \leq \rho(H) - 2(k-1) - \delta \leq k(k-3) - 2(k-1)$, which is a contradiction. If R' is i-incomplete for $i \geq 1$ then, because each uncounted edge lowers the potential $\rho_G(R')$ by $2(k-1)$, we get $\rho(G) \leq \rho_G(R') \leq \rho(H) - i2(k-1)$. This is also a contradiction. Finally, if the W-critical extension R' has a core of size larger than 1 then by Fact 4.2 $\rho(K_{|X|})$ is minimized when $|X| = k-1$ rather than when $|X| = 1$. One can check that using $|X| = k-1$ instead of $|X| = 1$ in Corollary 4.7 implies that

$$\rho_G(R') \leq [\rho_G(R) - 2(k-1) - \delta] - (k^2 - 5k + 6 + (k-2)\epsilon - (k-2)\delta).$$

Using Equation 4.3, and because $2(k-1) \leq k^2 - 5k + 6 + (k-2)\epsilon - (k-2)\delta$ when $k \geq 6$, it follows that $\rho_G(R') \leq \rho(H) - 2(k-1) \leq k(k-3) - 2(k-1)$, which is a contradiction. Therefore, we have shown by Proposition 4.9 that R is collapsible in G.

By Lemma 4.10 there is no 2-cut in G, so $|\partial R| \geq 3$ and we can say $w \in \partial R - \{x, y\}$. Suppose $H = K_k$. Then $G[R] = K_k - e$ and because $|\partial R| \geq 3$ it is not possible for R to be collapsible since the largest color class of any proper $(k-1)$-coloring of $K_k - e$ is 2. Therefore, we may assume that H is an Ore composition of k-Ore graphs H_1 and H_2 with overlap vertices $\{a, b\}$. By the minimality of H, $xy \in E(H_1)$.

Because there is no 2-cut in G, then there must be $u, v \in \partial R$ such that $u \in V(H_1) - V(H_2)$ and $v \in V(H_2) - V(H_1)$. Now consider a proper $(k-1)$-coloring ϕ of $G[R]$. We know that $\phi(u) = \phi(v)$ because R is collapsible, and that $\phi(a) \neq \phi(b)$. Without loss of generality, let $\phi(a) = 1$ and $\phi(b) = 2$. If $\phi(u) \notin \{1, 2\}$, then we may assume that $\phi(u) = 3$. In this case, switching the colors 3 and 4 on H_1 only will yield a proper $(k-1)$-coloring of $G[R]$ in which u and v receive different colors, which is a contradiction. Otherwise, we may assume that $\phi(u) = 1$. Now either $\phi(v) = \phi(a) = 1$ in every proper $(k-1)$-coloring of R, or we can switch $\phi(v)$ with another color to produce a proper $(k-1)$-coloring of R where $\phi(v) \neq \phi(u)$. In the latter case we contradict the fact that R is collapsible, and in the former case av is a non-edge of G such that $G + av$ has a k-critical subgraph of size at most $|V(H_2)| + 1$, which contradicts the minimality of H. Therefore no edge-addition can exist in a minimal counterexample to the third statement of Theorem 3.5. □

Corollary 4.13. *If G is a minimal counterexample to the third statement of Theorem 3.5, then G has no collapsible subsets R with $|R| \geq 2$.*

Proof. Suppose that G has a collapsible subset $R \subsetneq G$ and that $|R| \geq 2$. Then the connectivity of a k-critical graph implies that $|\partial R| \geq 2$. Thus because R is collapsible, for distinct vertices $x, y \in \partial R$ the edge xy is an edge-addition, which is a contradiction. □

Using the definitions of diamond and emerald, we have the following corollary.

Corollary 4.14. *A minimal counterexample to the third statement of Theorem 3.5 cannot have a diamond or emerald.*

Proof. If G has a diamond with endpoints u, v, then $\{u, v\}$ is a 2-cut, which contradicts Lemma 4.10. If G has an emerald D, then let $R = V(G) - V(D)$. Either ∂R receives only one color in each proper $(k-1)$-coloring of $G[R]$ or there exists some proper $(k-1)$-coloring ϕ where $|\phi(\partial R)| > 1$. If such a ϕ exists then we can extend ϕ to $V(D)$ and properly $(k-1)$ color all of G, which is a contradiction. But if ∂R receives one color in every proper $(k-1)$-coloring of $G[R]$ then for $x, y \in \partial R$ we have that xy is an edge-addition, which contradicts Lemma 4.12. □

Note that for a graph G with a subgraph $G' \subseteq G$, it is possible for $D \subseteq G'$ to be a diamond (or emerald) of G' but not be a diamond (or emerald) of G. For D to be a diamond (or emerald) of both G and G', we need the edges from $V(D)$ to $V(G) - V(D)$ to be the same in G as in G' (although extra edges in G to the endpoints of a diamond are allowable).

4.4 Generalization to i-collapsible and $(i+1)$-edge-additions

Now we generalize our previous definitions of collapsible subsets and edge-additions to further examine the structure of a minimal counterexample to the third statement of Theorem 3.5.

Definition 4.15. *A proper vertex subset $R \subsetneq V(G)$ is* i-collapsible *in G if for all proper $(k-1)$-colorings ϕ of $G[R]$ using color set C*

$$\min_{c \in C} \left| \left\{ uv \in E(G) \mid u \in \phi^{-1}(C - c) \cap R \text{ and } v \in V(G) - R \right\} \right| \leq i. \tag{4.4}$$

Thus, a proper vertex subset R is i-collapsible if there is a majority color class in $\phi(\partial_G R)$ which covers all but at most i edges from R into $V(G) - R$. Note that 0-collapsible sets correspond to our previous definition of collapsible sets. We also generalize the definition of edge-addition such that $i = 0$ corresponds to the previous definition of edge-addition.

Definition 4.16. *Let G be a k-critical graph. An* (i+1)-edge-addition *in G is a set S of at most $(i+1)$ non-edges such that there exists a k-critical graph H with $S \subseteq E(H)$, $H - S \subseteq G$, and $V(H) \subsetneq V(G)$.*

Note that because no proper subgraph of G is k-critical, a 1-edge-addition cannot come from a set of 0 edges. Thus, this definition for $i = 0$ is identical to the previous definition of edge-addition.

Proposition 4.17. *If $R \subsetneq V(G)$ is a vertex subset where all W-critical extensions of R are spanning, have core size 1, and are at most i-incomplete then R is i-collapsible in G.*

Proof. To show that R is i-collapsible in G we need to show that Equation 4.4 holds for all proper $(k-1)$-colorings of R. Let ϕ be an arbitrary proper $(k-1)$-coloring using color set C and let R' be a W-critical extension using that coloring. By hypothesis, $R' = V(G)$ and we may permute colors of ϕ so that the core of the extension is the vertex x_1. Thus each edge from $\phi^{-1}(C - \{1\}) \cap R$ to $V(G) - R$ is counted by $|E(G[R'])|$ but not by $|E(G[R])| + |E(W)| - |E(K_{|X|})|$. Because R' is at most i-incomplete, there can be at most i such edges. Therefore

$$\left|\left\{uv \mid u \in \phi^{-1}(C - \{1\}) \cap R \text{ and } v \in V(G) - R\right\}\right| \leq i.$$

and by definition R is i-collapsible. □

The following is an extension of a result (Lemma 16) in [14].

Proposition 4.18. *If G is a k-critical graph with a i-collapsible subset $R \subsetneq V(G)$ for $i \leq (k-3)/2$ then G admits an $(i+1)$-edge-addition.*

Proof. Let G be a k-critical graph and let $R \subsetneq V(G)$ be an i-collapsible subset for $i \leq (k-3)/2$. Suppose, for sake of contradiction, that G admits no $(i+1)$-edge-addition. For each $u \in \partial R$, let $w(u) = |\{uv \in E(G) \mid v \in V(G) - R\}|$. Because G is a k-critical graph, it is $(k-1)$-edge-connected and thus $\sum_{u \in \partial R} w(u) \geq k - 1$.

Now $w : \partial R \to \{1, 2, \ldots\}$ is an integral positive weight function that satisfies the hypotheses of Lemma 16 from [14]. Therefore, for each $0 \leq i \leq$

$(k-3)/2$ there exists a graph H with $V(H) = \partial R$ and $|E(H)| = i+1$ such that for every independent set M in H with $|M| \geq 2$

$$\sum_{u \in \partial R - M} w(u) \geq i+1.$$

That means that because G admits no $(i+1)$-edge addition, $G[R] + E(H)$ is $(k-1)$-colorable using color set C by ϕ. But ϕ is also a proper $(k-1)$-coloring of $G[R]$ and, further, every color class $c \in C$ of size at least 2 has

$$|\{uv \in E(G) \mid u \in \phi^{-1}(C-c) \cap R \text{ and } v \in V(G) - R\}| \geq i+1.$$

This does not immediately contradict that R is i-collapsible, because it gives no information about color classes of size 1. To complete the proof, we now examine the cases in [14] more closely.

Let $\partial R = \{u_1, \ldots, u_s\}$ and, without loss of generality, assume that $w(u_1) \geq w(u_j)$ for all $2 \leq j \leq s$.

Case 1. *Suppose* $w(u_2) + \cdots + w(u_s) \geq i+2$.

For this case, we may further assume that $w(u_1) \geq w(u_2) \geq \cdots \geq w(u_s)$. Choose the largest j such that $w(u_j) + \cdots w(u_s) \geq i+1$. Let $\alpha = (i+1) - (w(u_{j+1}) + \cdots + w(u_s))$. Therefore $\alpha \leq w(u_j)$. Since $i \leq (k-3)/2$ and $w(u_1) + \cdots + w(u_s) \geq k-1$, we also have $w(u_1) + \cdots + w(u_j) \geq i+1+\alpha$. By the choice of j and the ordering of the vertices, $0 < \alpha \leq w(u_j) \leq w(u_1)$. We draw α edges each connecting u_1 with u_j and $i+1-\alpha$ edges connecting $\{u_{j+1}, \ldots, u_s\}$ with $\{u_1, \ldots, u_j\}$ so that for each ℓ, the degree of u_ℓ in the obtained multigraph H is at most $w(u_\ell)$. Because G admits no $(i+1)$-edge-addition this gives a proper $(k-1)$-coloring ϕ of $G[R] + E(H)$ using color set C, which is also a proper $(k-1)$-coloring of $G[R]$. Let $c \in C$ be any color class.

Since $\phi^{-1}(c)$ is independent, it follows that

$$\sum_{u \in \partial R - \phi^{-1}(c)} w(u) \geq \sum_{u \in \partial R - \phi^{-1}(c)} d_H(u) \geq \frac{1}{2} \sum_{u \in \partial R} d_H(u) = i+1.$$

This contradicts the fact that R is i-collapsible in G.

Case 2. *Suppose* $w(u_2) + \cdots + w(u_s) \leq i + 1$.

We let $E(H) = \{u_1u_j \mid 2 \leq j \leq s\}$. Because G admits no $(i+1)$-edge-addition this gives a proper $(k-1)$-coloring of $G[R] + E(H)$ using color set C, which is also a proper $(k-1)$-coloring of $G[R]$. If c is any color class not containing u_1 then $w(u_1) \geq i + 1$, and that color class does not satisfy the definition of i-collapsible. Suppose that $\phi(u_1) = 1$. Note that u_1 is the only vertex in ∂R that receives color 1, so it follows that $w(u_2) + \cdots + w(u_s) \leq i$ because R is i-collapsible.

Let ψ be a proper $(k-1)$-coloring of $G[(V(G) - R) \cup \{u_1\}]$ such that $\psi(u_1) = 1$. Because G is k-critical, $\overline{\psi} = \psi|_{V(G)-R} \cup \phi|_R$ is not a proper coloring. We pick ψ so that the number of edges from $\partial_G R$ to $V(G) - R$ that have endpoints colored the same by $\overline{\psi}$ is minimized. Without loss of generality, $\phi(u_2) = 2$ and one of its neighbors in $V(G) - R$ also receives color 2.

We will switch color class 2 in ϕ with another color class so that $\overline{\psi}$ colors u_2 differently than all of its neighbors without increasing the number of edges from ∂R to $V(G) - R$ with endpoints colored the same. We cannot switch 2 with 1 in ϕ, so there remains $(k-2)$ choices. For each of the at most i edges u_jv from $\partial R - \{u_1\}$ to $V(G) - R$, we remove from the set of choices $\phi(u_j)$ if $\phi(u_j) \neq 2$ or $\phi(v)$ if $\phi(u_j) = 2$. This leaves at least $k - 2 - i \geq (k-1)/2 \geq 1$ choice. By switching color class 2 in ϕ, we contradict our choice of ψ. Thus, either $\overline{\psi}$ is a proper $(k-1)$-coloring of G or R is not i-collapsible, and we have completed the proof. □

Lemma 4.19. *In a minimal counterexample G to the third statement of Theorem 3.5, there is no proper subset $R \subsetneq V(G)$ with $|R| \geq 2$ and $\rho_G(R) < \rho(G) + 2(i+1)(k-1) + \delta$ for $1 \leq i \leq \frac{k-4}{2}$. Further, G does not admit an i-edge-addition for $1 \leq i \leq \frac{k-4}{2}$.*

Proof. We prove this by induction on i. First we examine the case where $i = 1$. Lemma 4.12 shows that there is no 1-edge addition.

Now suppose, for sake of contradiction, that $R \subsetneq V(G)$ is a proper vertex subset where $\rho_G(R) < \rho(G) + 4(k-1) + \delta$. Taking an arbitrary W-critical extension R', Corollary 4.7 gives us $\rho_G(R') \leq \rho_G(R) - 2(k-1) - \delta < \rho(G) + 2(k-1)$. If the extension is not spanning then we can extend again and get $\rho_G(R'') < \rho(G)$, which is not possible. If the extension is not complete, then we lose at least an extra $2(k-1)$ when extending from R to R'. This gives $\rho_G(R') < \rho(G)$ which is not possible. Lastly, if the core of the extension R' has size larger than 1, then the potential equation for an extension (Equation 4.2) in Corollary 4.7 uses $|X| = k-1$ rather than $|X| = 1$. Using Fact 4.2, one can check that, for $k \geq 6$, the potential of the extension goes down at least an additional $2(k-1)$. Again, this implies that $\rho_G(R') < \rho(G)$ which is not possible. Therefore, we have shown that R is a collapsible set in G which contradicts Corollary 4.13.

We now prove this for general i. The inductive hypothesis is that there is no proper subset R of size at least 2 and $\rho_G(R) < \rho(G) + 2i(k-1) + \delta$ and, further, there are no j-edge-additions for $j < i$. For sake of contradiction, we assume that R is a proper subset of size at least 2 and that $\rho_G(R) < \rho(G) + 2(i+1)(k-1) + \delta$. By Corollary 4.7, a W-critical extension R' has potential $\rho_G(R') \leq \rho_G(R) - 2(k-1) - \delta < \rho(G) + 2i(k-1)$. The inductive hypothesis implies that $R' = V(G)$, so all extensions are spanning. Also, because $\rho(G) = \rho_G(R')$, the right hand side of the inequality cannot drop by $2i(k-1)$, and so the extension can be at most $(i-1)$-incomplete. If the core of R' has size larger than 1, then Equation 4.2 becomes

$$\rho_G(R') \leq \rho_G(R) + \rho(W) - (2k^2 - 6k + 4 + (k-1)\epsilon - 2\delta).$$

Because $\rho(G) = \rho_G(R')$ we can use the assumption on $\rho_G(R)$ and the bound

$i \leq \frac{k-4}{2}$ to obtain the following inequality:

$$\rho(G) < \rho(G) + 2(i+1)(k-1) + \delta + \rho(W) - (2k^2 - 6k + 4 + (k-1)\epsilon - 2\delta)$$

which simplifies to

$$0 < -k^2 + 3k - 2 + 3\delta - (k-1)\epsilon + \rho(W).$$

Whether W is k-Ore or not, it is a smaller k-critical graph than G so by Theorem 3.5 the right hand side of the inequality is negative, which is a contradiction. Therefore, the core of the extension R' has size 1.

By Proposition 4.17, R is $(i-1)$-collapsible. Since every proper subset of size at least 2 with $\rho_G(R) < \rho(G) + 2(i+1)(k-1) + \delta$ is $(i-1)$-collapsible by the above argument, we may assume that we chose an $(i-1)$-collapsible set R of minimal size. By Proposition 4.18, G admits an i-edge-addition and $H \subseteq G[R] + S$is a k-critical subgraph. Because G does not admit an $(i-1)$-edge-addition, $|S| = i$.

Pick a set S of exactly i edges that minimizes the order of k-critical graph $H \subseteq G[R] + S$. Let $R_0 = V(H)$; then $\rho_G(R_0) \leq \rho(H) + 2i(k-1) + i\delta$.

First, we suppose that H is not k-Ore. Then because H is smaller than G, $\rho(H) \leq k(k-3) - 2(k-1) < \rho(G)$. Thus, like R, R_0 is a proper subset of size at least 2 with $\rho_G(R_0) < \rho(G) + 2(i+1)(k-1) + \delta$. Also $R_0 \subseteq R$, so by the minimality of R it follows that $R = R_0$.

As before, every extension of R is spanning, has core size 1, and is at most $(i-1)$-incomplete. Further, there must be some extension R' that is $(i-1)$-incomplete. Otherwise, all extensions are at most $(i-2)$-incomplete and by Proposition 4.17 there is a $(i-1)$-edge-addition, which would contradict the inductive hypothesis. Choose such an $(i-1)$-incomplete extension R' and examine the potential $\rho_G(R')$. Because R' is a spanning and $(i-1)$-incomplete extension, Lemma 4.6 and the i-edge addition in $G[R]$ give that

$$\rho(G) = \rho_G(R') \leq \rho_G(R) + \rho(W) - 2(i-1)(k-1) - (k^2 - k - 2 + \epsilon - \delta)$$

$$\leq \rho(H) + 2i(k-1) + i\delta + \rho(W) - 2(i-1)(k-1) - (k^2 - k - 2 + \epsilon - \delta)$$
$$< \rho(G) + \rho(W) + (i+1)\delta - \epsilon + 2(k-1) - k^2 + k + 2.$$

This calculation shows that $0 < \rho(W) - k^2 + 3k + (i+1)\delta - \epsilon$. If W is not k-Ore, then this is a contradiction because $2(k-1) > (i+1)\delta - \epsilon$. Therefore W is k-Ore. Because $|X| = 1$, Lemma 3.11 implies that there is a diamond or an emerald D in $W - \{X\}$. If it is a diamond, then this is a 1-edge addition in G, which contradicts Lemma 4.12. If it is an emerald, then for each $u_i \in V(D)$ we have $\deg_{W-\{X\}}(u) = k-1$. However, in G, there could be other edges incident to u_i namely, those that cause the $(i-1)$-incompleteness of R'. So at least $(k-1) - (i-1) = k - i$ vertices $u_i \in V(D)$ have degree $k-1$ in G.

Suppose that $u_1, \ldots, u_\ell$ are the vertices with $\deg_G(u_i) = k-1$. For each i in $1 \leq i \leq \ell$, label the unique vertex in $N_G(u_i) - V(D)$ as v_i. If any $v_i \neq v_j$ for $1 \leq i < j \leq \ell$, then $v_i v_j$ is an edge-addition in G. To see this, we properly $(k-1)$-color $G - \{u_i, u_j\}$. There are two colors not used on the partial coloring of D, suppose they are 1 and 2. If $\phi(v_i) \in \{1, 2\}$ then we color u_j with $\phi(v_i)$ and color u_i with the other. Otherwise, we can greedily color u_j, u_i in that order. Either way, we have obtained a proper $(k-1)$-coloring of G. Therefore, $v_1 = v_2 = \cdots = v_\ell$. Adding an edge from v_1 to each of $u_{\ell+1}, \ldots, u_{k-1}$ yields a K_k. This $(i-1)$-edge-addition is a contradiction.

Therefore, the graph H obtained by adding i edges to $G[R]$ must be k-Ore. Recall that we chose the set $|S|$ of i edges to minimize the order of H. We first examine the case where H is not K_k and then the case where $H = K_k$.

If H is not K_k, then H is an Ore composition of k-Ore graphs H_1 and H_2 with overlap vertices $\{a, b\}$. Let $\overline{S} = S \cap E(H_1)$ be the new edges on the edge-side of the composition. If $|\overline{S}| \leq i - 2$ then $\overline{S} \cup \{ab\}$ is a $(i-1)$-edge-addition, which is a contradiction. If $|\overline{S}| = i - 1$ then $\overline{S} \cup \{ab\}$ is a i-edge-addition that contradicts the minimal order of H. Therefore $S \subseteq E(H_1)$ and $H_2 - \{\underline{ab}\} \subseteq G$. By Lemma 3.11 there is a diamond or an emerald D in

$H_2 - \{\underline{ab}\}$ away from the split-vertex of the composition. If D is a diamond, then it admits a 1-edge-addition which is also a 1-edge-addition in G. This contradicts Lemma 4.12.

Therefore D is an emerald of $H_2 - \{\underline{ab}\}$; however, D is not necessarily an emerald in G because a vertex in $V(D)$ could lie in $\partial_G R$. In any proper $(k-1)$-coloring of $G[R]$, the vertices $V(D) = \{u_1, u_2 \ldots, u_{k-1}\}$ all receive different colors. Recall that R' has core size 1, is spanning, and is at most $(i-1)$ incomplete; thus, there are at most i vertices in $V(D) \cap \partial_G R$, since only one can share a color with X. Let $u_1, \ldots, u_\ell$ be the vertices in $V(D) - \partial_G R$. These vertices have a common neighbor v_1 outside of $V(D)$, otherwise we have a 1-edge-addition. Adding an edge from v_1 to each u_i with $\ell < i \leq k-1$ is an i-edge-addition which gives a K_k and thus contradicts the minimality of H.

Now we turn to the case where $H = K_k$. Let $V(H) = \{u_1, \ldots, u_k\}$ and note that $G[R] + S = H$ by the minimality of R. We label the vertices so that $u_1 u_k \in S$. Thus we can properly $(k-1)$-color $G[R]$ with ϕ using color set $C = [k-1]$ such that $\phi(u_i) = i$ for $1 \leq i \leq k-1$ and $\phi(u_k) = 1$. Because R is $(i-1)$-collapsible in G, we have

$$\min_{c \in C} \left|\left\{uv \in E(G) \mid u \in \phi^{-1}(C-c) \cap R \text{ and } v \in V(G) - R\right\}\right| \leq i - 1. \quad (4.5)$$

Note that $\deg_H(u) = k-1$ for any $u \in R$, and because G is a k-critical graph we have $\deg_G(u) \geq k-1$ as well. Thus, for each $u \in R$ there are at least as many edges of G from u to $V(G) - R$ as the number of edges of S incident with u. This means that

$$\left|\left\{uv \in E(G) \mid u \in \phi^{-1}(C-c) \cap R \text{ and } v \in V(G) - R\right\}\right| \geq i$$

for each $c \neq 1$ and so that color class does not witness Equation 4.5. It follows that the color class $c = 1$ must witness Equation 4.5 and, hence, must cover all but at most $i-1$ endpoints of edges in S.

This implies that every edge in $S - \{u_1u_k\}$ is incident to either u_1 or u_k. If u_1u_2 and u_ku_3 are both in S, then we can switch the labels u_2 and u_k. Then no color class covers all but at most $i-1$ endpoints of edges in S and we obtain a contradiction. Therefore S either forms a star subgraph, or $|S| = 3$ and S forms a triangle subgraph. We examine these cases separately.

Suppose that S is a star and that u_1 is the center of the star. In this case we show that $G[R - \{u_1\}]$ is an emerald in G, which is a contradiction. In order to satisfy Equation 4.5 each leaf of the star has exactly one edge to $V(G) - R$ and every $u \in R$ not incident with S has no edges to $V(G) - R$. Therefore $\deg_G(u_i) = k - 1$ for $2 \leq i \leq k$ and, by definition, $G[R - \{u_1\}]$ is an emerald in G.

Suppose that S is a triangle and that $S = \{u_1u_2, u_2u_3, u_1u_3\}$. In order to satisfy Equation 4.5 for any proper $(k-1)$-coloring of $G[R]$ each u_i for $1 \leq i \leq 3$ has exactly two edges to $V(G) - R$ and every $u \in R$ not incident with S has no edges to $V(G) - R$. However, by assumption of the lemma, $i = 3$ implies that $k \geq 10$. This is a contradiction because k-critical graphs are $(k-1)$-edge-connected, and there are only 6 edges from R to $V(G) - R$. This contradiction completes the proof. $\square$

Corollary 4.20. *In a minimal counterexample to the third statement of Theorem 3.5 G, any proper vertex subset $R \subsetneq V(G)$ with $|R| \geq 2$ has potential $\rho_G(R) \geq \rho(G) + k^2 - 3k + 2 + \delta$.*

Proof. By Lemma 4.19, $\rho_G(R) \geq \rho(G) + 2(i+1)(k-1) + \delta$ for all $i \leq \frac{k-4}{2}$. One can check that this implies that $\rho_G(R) \geq \rho(G) + k^2 - 3k + 2 + \delta$ as claimed. $\square$

We are also able to get degree conditions on neighbors of vertices of low degree, which will be frequently used when discharging.

Lemma 4.21. *In a minimal counterexample G to the third statement of Theorem 3.5, let x and y be adjacent vertices such that* $\deg_G(x) = k-1$ *and* $N_G[x]$ *is not a subset of* $N_G[y]$*. Then* $\deg_G(y) \geq |N_G(x) \cap N_G(y)| + 1 + \frac{k-3}{2}$.

Proof. Suppose that G is a minimal counterexample to the third statement of Theorem 3.5 and let x and y be adjacent vertices such that $\deg_G(x) = k-1$ and let $w \in N_G[x] - N_G[y]$.

For any proper $(k-1)$-coloring of $G - \{x\}$, the vertices of $N_G(x)$ must all receive distinct colors. Further, it is not possible to switch the color of y to match w. Therefore, for any proper $(k-1)$-coloring ϕ of $G - \{x\}$, there must be some vertex in $N_G[y] - N_G[x]$ which is colored with $\phi(w)$.

If we add edges from w to every vertex in $N_G[y] - N_G[x]$, this creates a k-critical subgraph of $G - \{x\}$. Therefore $|N_G[y] - N_G[x]| \geq \frac{k-3}{2}$, because otherwise we contradict Lemma 4.19. This gives the desired bound on $\deg_G(y)$. $\square$

Chapter 5

Cloning

In this chapter, we define a second reduction operation called cloning. Cloning will help us determine what types of vertices can be adjacent to vertices of degree $k-1$ in a minimal counterexample to the third statement of Theorem 3.5. It is crucial to understand the structure near these vertices if we are to succeed with a discharging argument, because vertices of degree $k-1$ are the vertices which need to be sent charge in order to complete the full proof of Theorem 3.5. Many results in this section are about structures that exist around vertices of *low* degree, meaning vertices of degree $k-1$. By examining graphs resulting from this cloning operation, we are able to identify the configurations which need to be accounted for by our discharging rules.

Definition 5.1. *Let G be a k-critical graph with $xy \in E(G)$ such that* $\deg_G(x) = k-1$. *We define the operation of* cloning x with y *to mean constructing a new graph $G_{y\to x}$ such that $V(G_{y\to x}) = (V(G) - \{y\}) \cup \{\tilde{x}\}$ and $E(G_{y\to x}) = E(G - \{y\}) \cup \{\tilde{x}v \mid v \in N_G(x)\} \cup \{\tilde{x}x\}$.*

5.1 Clusters

We also use the notion of a cluster defined in [14]. Recall that for a graph G and vertex x, the closed neighborhood of x is the set $N_G[x] = N_G(x) \cup \{x\}$.

Definition 5.2. *A* cluster *is a maximal set $R \subseteq V(G)$ such that for every $x \in R$, $\deg_G(x) = k-1$ and for every pair $x, y \in R$, $N_G[x] = N_G[y]$.*

A cluster, then, is a set of pairwise adjacent vertices of low degree. Further, if $x \in V(G)$ is in a cluster C and $xy \in E(G)$ then when we clone x with y, the new vertex $\tilde{x}$ is added to the cluster C. Because large clusters will create areas in the graph that need quite a lot of charge, we want to bound the size of clusters in a minimal counterexample to the third statement of Theorem 3.5.

Proposition 5.3. *Let C be a cluster in a minimal counterexample G to the third statement of Theorem 3.5. Then $|C| \leq k-3$. Furthermore, if $C \subseteq S$ such that $G[S] = K_{k-1}$, then $|C| \leq \frac{k+1}{2}$.*

Proof. Suppose that G is a minimal counterexample to the third statement of Theorem 3.5 and that C is a cluster of size s in G. Clearly $s < k$ because otherwise $G = K_k$, which cannot be a counterexample. If $s = k-1$ then C is an emerald of G. If $s = k-2$, then we label the neighbors of x outside of C with u and v. Note that the graph $G[C \cup \{u, v\}]$ is a diamond of G. Therefore, by Corollary 4.14 we have $s \leq k-3$.

Suppose that C is in a K_{k-1} subgraph. Label the vertices of the K_{k-1} with $\{x_1, \ldots, x_{k-1}\}$, so that $x_i \in C$ for $1 \leq i \leq s$. For $i \leq s$, all x_i share a common neighbor y outside of the K_{k-1}. If $s \geq \frac{k+2}{2}$ then this corresponds to at most $\frac{k-4}{2}$ edges that we can add between y and x_i for $i > s$ to obtain a K_k. By Lemma 4.19, this $\frac{k+2}{2}$-edge-addition is a contradiction. □

Notice that if we clone a degree $k-1$ vertex x using a vertex y with $\deg_G(y) > k-1$ then $G_{y \to x}$ is a smaller graph than G. Further, if both x and y have degree $k-1$ in G, x is in a cluster of size s and y is in a different cluster of size t with $s \geq t$ then $G_{y \to x}$ is smaller than G. The following lemma is true for any k-critical graph G, not just a minimal counterexample to Theorem 3.5.

Lemma 5.4. *Given a k-critical graph G if $xy \in E(G)$, x is in a cluster of size s, $\deg_G(y) \leq k-2+s$, and x and y are not in a common cluster then $G_{y\to x}$ is not properly $(k-1)$-colorable.*

Proof. Suppose that G is a k-critical graph with $xy \in E(G)$ such that x is in a cluster C of size s and $\deg_G(y) = k-2+s$. Let ϕ be a proper $(k-1)$-coloring of $G_{y\to x}$. Consider $\phi|_{V(G)-\{y\}}$ as a partial proper coloring of G. We extend ϕ by coloring y using a color distinct from its $k-2$ neighbors not in C. Because G is k-critical this is not a proper coloring; we may assume that $\phi(x) = \phi(y)$. This is the only impediment to ϕ being a proper $(k-1)$-coloring of G. However, we can now color x with $\phi(\tilde{x})$ instead and we have properly $(k-1)$-colored G. This contradiction completes the proof. $\square$

We return to a minimal counterexample G to the third statement of Theorem 3.5. Suppose that $xy \in E(G)$ and that $\deg_G(x) = k-1$, $\deg_G(y) \leq k-2+s$, and x and y are not in a common cluster. Now consider the graph $G_{y\to x}$. By Lemma 5.4, $G_{y\to x}$ is not $(k-1)$-colorable. Therefore, we let $H \subseteq G_{y\to x}$ be a k-critical subgraph. The new vertex $\tilde{x}$ has degree $k-1$ in H which, for $R = V(H) - \{\tilde{x}\}$, implies that

$$\rho_G(R) \leq \rho(H) - (k-2)(k+1) - \epsilon + 2(k-1)(k-1) + \delta,$$

which we write as

$$\rho_G(R) \leq \rho(H) + k^2 - 3k + 4 - \epsilon + \delta. \tag{5.1}$$

Lemma 5.5. *Suppose that G is a minimal counterexample to the third statement of Theorem 3.5 and $xy \in E(G)$ such that*

1. *x is in a cluster C_x of size s,*

2. *$\deg_G(y) \leq k-2+s$, and*

3. *if y is in a cluster C_y then $C_y \neq C_x$ and $|C_y| = t \leq s$.*

Then for a k-critical subgraph $H \subseteq G_{y\to x}$ either H is k-Ore or $H = G_{y\to x}$. Moreover, $H = G_{y\to x}$ is only possible if $\deg_G(y) = k-1$.

Proof. Let G be a minimal counterexample to the third statement of Theorem 3.5 and suppose that there is an $xy \in E(G)$ such that x is in a cluster C_x of size s and $\deg_G(y) \leq k-2+s$. Suppose further that if y is in a cluster C_y then $C_y \neq C_x$ and $|C_y| = t \leq s$. We clone x using y to obtain the graph $G_{y\to x}$. By Lemma 5.4, there exists a k-critical subgraph $H \subseteq G_{y\to x}$. The conditions on cluster size ensure that $H \subseteq G_{y\to x}$ is smaller than G. Suppose, for sake of contradiction, that H is not k-Ore and $H \neq G_{y\to x}$.

Let $R = V(H) - \tilde{x}$. Equation 5.1 gives a bound on $\rho_G(R)$ which we need to use. Let R' be a W-critical extension of R in G. By Lemma 4.6 we get $\rho(G) \leq \rho_G(R') \leq \rho_G(R) + \rho(W) - \big(\rho(K_{|X|}) + \delta T(K_{|X|}) - \delta|X|\big)$. We will examine this potential more closely in cases based on the size of X.

Suppose first that $1 < |X| < k-1$. Then from Fact 4.2 and Equation 5.1 it follows that

$$\rho(G) \leq \rho(H) + \rho(W) - k^2 + k + 6 - 3\epsilon + 3\delta.$$

Because $H \subseteq G_{y\to x}$ is smaller than G, is not a k-Ore graph, and G is a minimal counterexample, this implies that $k^2 - k - 6 + 3\epsilon - 3\delta < \rho(W)$. This is a contradiction to Theorem 3.5, since W is smaller than G and thus $\rho(W) \leq k^2 - 3k$.

Suppose instead that $|X| = k-1$. Then Fact 4.2 implies that

$$\rho(G) \leq \rho(H) + \rho(W) - k^2 + 3k - k\epsilon + k\delta. \tag{5.2}$$

Again, $H \subseteq G_{y\to x}$ is smaller than G, is not a k-Ore graph, and G is a minimal counterexample. Therefore this implies that $k^2 - 3k + k\epsilon - k\delta < \rho(W)$. This is a contradiction unless W is k-Ore, so we may assume that W is k-Ore. Under this assumption, the inequality in Equation 5.2 is tight in the sense that lowering the right hand side by $2(k-1)$ leads to a contradiction. Thus, if the W-critical extension R' is not spanning or is not complete, then the right hand side of Equation 5.2 is lowered by at least $2(k-1)$ and we have a contradiction.

Suppose that W is K_k, then because R' is spanning, there is only one vertex in $V(G)-R$. Therefore, $V(G) = R \cup \{y\}$ and $V(H) = R \cup \{\tilde{x}\}$ and it follows that $|V(G)| = |V(H)|$. However, H is smaller than G and thus $\rho(H) \leq \rho(G)$. Also, because their vertex sets overlap except at one vertex, $T(H)$ and $T(G)$ differ by at most 1. Therefore, it is not possible for G to have more edges than H, and we also have $|E(G)| = |E(H)|$. If $\deg_G(y) > k-1$ then this contradicts the fact that H is smaller than G. If $\deg_G(y) = k-1$ then it follows that $H = G_{y \to x}$, which contradicts our assumption on H.

If W is not K_k, then $X \subseteq V(W)$ is a K_{k-1} subgraph and by Lemma 3.12 there is an emerald or a diamond D in W disjoint from X. If D is a diamond then because $V(D) \subseteq V(W) - X \subseteq V(G)$ there is an edge-addition in G, which contradicts Lemma 4.12. If D is an emerald, then D is also a K_{k-1} subgraph of G. However, because the extension R' is complete, each $x \in V(D)$ has at most one adjacency with each color class in X and thus $\deg_G(x) = k-1$. Therefore D is an emerald of G, which contradicts Corollary 4.14.

Now we examine the final case; suppose that every W critical extension has core size $|X| = 1$. With this assumption and Equation 5.1, it follows that

$$\rho_G(R') \leq \rho(H) + \rho(W) - 2k + 6 - 2\epsilon + 2\delta. \tag{5.3}$$

If R' is not a spanning W-critical extension, then R' must satisfy Corollary 4.20. Therefore it follows that $\rho(G) + k^2 - 3k + 2 + \delta \leq \rho(H) + \rho(W) - 2k + 6 - 2\epsilon + 2\delta$. Because H is smaller than G and is not k-Ore, this implies that $k^2 - k - 4 + 2\epsilon - \delta < \rho(W)$, which is a contradiction. Therefore R' is spanning.

We now check how incomplete this extension can be. If R' is i-incomplete then Equation 5.3 becomes

$$\rho_G(R') \leq \rho(H) + \rho(W) - 2k + 6 - 2i(k-1) - 2\epsilon + 2\delta.$$

The left hand side is $\rho(G)$, since R' is spanning. Thus, since H is not k-Ore and is smaller than G, this yields

$$0 < \rho(W) - 2k + 6 - 2i(k-1) - 2\epsilon + 2\delta. \tag{5.4}$$

We know that $\rho(W) \leq \rho(K_k)$ since it is also smaller than G, so if we assume that $i \geq \frac{k-3}{2}$ then we get $0 < -k + 3 + (k-2)\epsilon$. This is a contradiction, so we know that R' is at most $\frac{k-4}{2}$-incomplete.

If every W-critical extension of R is at most $\frac{k-6}{2}$-incomplete, then by Proposition 4.17 R is $\frac{k-6}{2}$-collapsible. Then Proposition 4.18 implies that G admits a $\frac{k-4}{2}$-edge-addition, which contradicts Lemma 4.19. Therefore, we can pick a W-critical extension R' so that the extension is either exactly $\frac{k-5}{2}$-incomplete or $\frac{k-4}{2}$-incomplete, depending on parity of k.

Suppose that W is not k-Ore. Then using the fact that $\rho(W) \leq k^2 - 3k - 2(k-1)$ and $i \geq \frac{k-5}{2}$, Equation 5.4 becomes $0 < k + 1 - 2\epsilon + 2\delta - 2(k-1)$, which is a contradiction. Therefore we may assume that W is k-Ore.

By Lemma 3.11, there exists a subgraph $D \subseteq W - \{X\} \subseteq G$ which is a diamond or emerald of W. If D is a diamond, then it admits a 1-edge-addition. This is also a 1-edge-addition in G, which contradicts Lemma 4.12. Therefore, D is an emerald of W. Because of the incompleteness of the extension, it is possible that D is not an emerald in G. However, R' is at most $\frac{k-4}{2}$ incomplete, so there are at most $\frac{k-4}{2}$ vertices of D that do not have degree $k-1$ in G. If $u_1, u_2 \in V(D)$ have degree $k-1$ in G and are adjacent to $w_1, w_2 \in V(G) - V(D)$ (respectively) with $w_1 \neq w_2$, then w_1w_2 is a 1-edge-addition in G. Thus, there exists a vertex $w \in V(G) - V(D)$ such that all $u \in V(D)$ with $\deg_G(u) = k-1$ are adjacent to w. But now $G[V(D) \cup \{w\}]$ is a K_k missing at most $\frac{k-4}{2}$ edges. This $\frac{k-4}{2}$-edge-addition in G contradicts Lemma 4.19.

Therefore, all cases lead to contradiction, and it must be that H is k-Ore or $H = G_{y \to x}$. Moreover, the only case (when $W = K_k$ and $|X| = k-1$) that contradicts the assumption that $H \neq G_{y \to x}$ required $\deg_G(y) = k-1$. Further, if H is k-Ore and $H = G_{y \to x}$ then $W = K_k$ and $|X| = k-1$. Now if $\deg_G(y) > k-1$ then the right hand side of Equation 5.2 is at least $2(k-1)$ smaller, which is a contradiction. So it follows that $H = G_{y \to x}$ is only possible if $\deg_G(y) = k-1$. $\square$

5.2 Gadgets and K_{k-3} subgraphs

We now explore the structure of subgraphs of a minimal counterexample to the third statement of Theorem 3.5.

Definition 5.6. *A* gadget*, H°, is a graph obtained from a k-Ore graph H by deleting a vertex x of degree $k-1$ in a cluster of size at least 2. For a graph G, a* gadget of G *is a subgraph in G that is a gadget.*

Definition 5.7. *A* key vertex *of a k-Ore graph H is a vertex which is on the edge-side of every possible Ore composition that yields H. We also define a key vertex of a gadget to be a vertex which is a key vertex of the corresponding k-Ore graph.*

Using this new language, we obtain the following corollary to Lemma 5.5 regarding adjacent vertices of degree $k-1$ which belong to different clusters.

Corollary 5.8. *Suppose that G is a minimal counterexample to the third statement of Theorem 3.5 and $xy \in E(G)$ such that*

1. *x is in a cluster C_x of size s,*
2. $\deg_G(y) \leq k-2+s$, *and*
3. *if y is in a cluster C_y then $C_y \neq C_x$ and $|C_y| = t \leq s$.*

Then x is a key vertex of a gadget of G or x is in a K_{k-3} subgraph of G. Moreover, if the latter is true then $\deg_G(y) = k - 1$*.*

Proof. Let G be a minimal counterexample to the third statement of Theorem 3.5 and suppose that there is an $xy \in E(G)$ such that x is in a cluster C_x of size s and $\deg_G(y) \leq k - 2 + s$. Suppose further that if y is in a cluster C_y then $C_y \neq C_x$ and $|C_y| = t \leq s$. Then by Lemma 5.5, there is a k-critical graph H which is a subgraph of $G_{y \to x}$ and either H is k-Ore or $H = G_{y \to x}$. Note that if $H = G_{y \to x}$ then $\deg_G(y) = k - 1$.

If H is k-Ore, then $H^\circ = H - \{\tilde{x}\}$ is a gadget of G which contains the vertex x. Suppose that $H = K_k$. Then trivially every vertex of H° is a key vertex. Suppose instead that H is an Ore composition of two k-Ore graphs H_1 and H_2 with overlap vertices $\{a, b\}$. If $\tilde{x} \in V(H_2)$ then ab is a 1-edge-addition in G, which contradicts Lemma 4.12. Therefore $\tilde{x}$ must be on the edge-side of every Ore composition that yields H. Because $\tilde{x}$ and x have the same closed neighborhoods in H, this is also true of x.

If H is not k-Ore, then $H = G_{y \to x}$. Because $\deg_G(x) = \deg_G(y)$, we know that there are the same number of edges in H and G. But the hypothesis $s \geq t$ implies that H is smaller than G and so H follows Theorem 3.5. Therefore H must have lower potential than G. Because they have the same number of edges, the potential can only have dropped because $T(H) > T(G)$. This implies that $\tilde{x}$, and hence x, is in a K_{k-2} subgraph of H or is in a K_{k-1} subgraph of H. In either case, x is in a K_{k-3} subgraph of G. $\square$

Using Corollary 5.8, we will define the various types of vertices of degree $k - 1$ which can exist in G. A vertex $u_1 \in V(G)$ of degree $k - 1$ could be in a K_{k-3} subgraph or could be a key vertex of a gadget. It is possible that such a vertex is not found via Corollary 5.8 (meaning there is no adjacent cluster to u_1). We call these vertices *structure-vertices*. A vertex u_2 could belong to a cluster C_{u_2} which is adjacent to no other clusters. If such a vertex

u_2 is a key vertex of a gadget or is in a K_{k-3} subgraph, then we will group with with the other structure-vertices. If such a vertex u_2 is not a structure-vertex, then we call it a *lone-vertex*. If a vertex u_3 of degree $k-1$ has an adjacent cluster C_v but is not a structure-vertex, then by Corollary 5.8 it follows that $|C_{u_3}| < |C_v|$ (equality implies that u_3 is a structure-vertex) and that C_v contains key vertices of a gadget or is in a K_{k-3} subgraph. We say that such a vertex u_3 is a *near-vertex*. In this way, we partition all vertices of degree $k-1$ in G into these three disjoint categories: structure-vertices, lone-vertices, and near-vertices.

Proposition 5.9. *If x is a structure-vertex in a minimal counterexample to the third statement of Theorem 3.5, then x cannot be adjacent to two near-vertices y, z, each belonging to a different cluster.*

Proof. Let G be a minimal counterexample to the third statement of Theorem 3.5 and suppose that x is a structure vertex with two near-vertex neighbors y, z such that y and z belong to different clusters.

If $yz \in E(G)$ then, without loss of generality, we may assume that $G_{y\to z}$ is smaller than G. By Corollary 5.8, z is a structure-vertex of G and cannot be a near-vertex. This is a contradiction.

If $yz \notin E(G)$ then, in the graph $G_{x\to z}$ the degree of y is $k-2$. Therefore, by Lemma 5.4, there is a k-critical subgraph $H \subseteq G_{x\to z}$. Because $y \notin V(H)$, we know that H is smaller than G and therefore, by the proof of Corollary 5.8, z is a key vertex of a gadget. This contradicts the fact that z is a near-vertex. $\square$

5.3 Bounds on degrees of neighbors of structure-vertices

By Proposition 3.2, every k-Ore graph H can be reduced to a K_k where some number of edges of the K_k are replaced by split-side k-Ore graphs in a composition operation when constructing H. We call such an edge an *edge-replacement*. Recall that each edge-replacement uv corresponds to a subgraph $D \subseteq G$ where, for any proper $(k-1)$-coloring of D, u and v must receive different colors. If x is a key vertex in H, then x must be a vertex of the K_k. Similarly, a gadget H° can be reduced to a K_{k-1} with edge-replacements. A key vertex x of H° will always be a vertex of this K_{k-1}, and if $\deg_{H^\circ}(x) = k-2$ then the neighbors of x in H° are also key vertices of H. This may not be immediately clear, but is shown in the proof of the following lemma.

Lemma 5.10. *Let x be a key vertex of a gadget H° of a minimal counterexample G to the third statement of Theorem 3.5, and let* $\deg_G(x) = k-1$. *If y is a different key vertex of H° then either $N_G[x] \subseteq N_G[y]$ or* $\deg_G(y) \geq 3(k-3)/2 + 1$.

Proof. Let G be a minimal counterexample to the third statement of Theorem 3.5 and suppose that H° is a gadget of G with key vertices x, y where $\deg_G(x) = k-1$. If $N_G[x] \subseteq N_G[y]$, then we are done so we may assume that $N_G[x]$ is not a subset of $N_G[y]$. We define H' to be a reduction of H° to a K_{k-1} graph where some edges are edge-replacements; by definition, x and y are in $V(H')$.

Suppose that there is an edge $xu \in E(H')$ which is an edge-replacement that corresponds to $D \subseteq G$. The graph D is $(k-1)$-colorable but x and u must receive different colors. Thus, in any proper $(k-1)$-coloring ϕ of D, the color $\phi(u)$ must be given to a vertex in $N_D(x)$. But then we can add to D a set of edges S from u to each vertex in $N_D(x)$ and get a k-critical subgraph in $D+S$. By Lemma 4.19, it follows that $|N_D(x)| \geq \frac{k-3}{2}$. Therefore, because $\deg_G(x) = k-1$, the vertex x can not be incident with any edge-replacements in H'.

From this, it follows that $xy \in E(G)$. Also, $\deg_{H^\circ}(x) = k-2$ and x must have one neighbor w in $V(G) - V(H^\circ)$. By Lemma 4.21 it follows that

$$\deg_G(y) \geq |N_G(x) \cap N_G(y)| + 1 + \frac{k-3}{2} = 3(k-1)/2 + 1.$$

Suppose that y is not incident with any edge-replacement in H'. Then $V(H') - \{x, y\} = N_G(x) \cap N_G(y)$ and we get $\deg_G(y) \geq 3(k-1)/2 + 1$. Suppose that y is incident with an edge-replacement yu in H' that corresponds to a subgraph $C \subseteq G$. As above, we can show that $|N_C(y)| \geq \frac{k-3}{2}$. Therefore, for each $u \in V(H') - \{x, y\}$ that is not in $N_G(x) \cap N_G(y)$, we get a contribution of at least $\frac{k-3}{2}$ to the degree of y in G. We have, in fact, proven a slightly stronger statement than Lemma 5.10. Namely, that if $N_G[x]$ is not a subset of $N_G[y]$ then $\deg_G(y) \geq ((k-3-i) + i\frac{k-3}{2}) + 1 + \frac{k-3}{2}$, where i is the number of edge-replacements of H' incident to y. $\square$

Lemma 5.11. *Let x be a key vertex in a gadget of G, where G is a minimal counterexample to the third statement of Theorem 3.5 and* $\deg_G(x) = k-1$. *Then x has at least $\frac{k-3}{2}$ neighbors of degree at least* $3(k-3)/2 + 1$.

Proof. Let G be a minimal counterexample to the third statement of Theorem 3.5 such that H° is a gadget of G. Suppose that x is a key vertex of H° and that $\deg_G(x) = k-1$. We define H' to be a reduction of H° to a K_{k-1} graph where some edges could be edge-replacements. By definition of a key vertex, $x \in V(H')$ and the same argument as in the proof of Lemma 5.10 shows that x is not adjacent to any edge-replacements and x has exactly one neighbor w in $V(G) - V(H^\circ)$. We partition $V(H')$ into sets A, B where $A := \{u \in V(H') \mid uw \in E(G)\}$ and $B := \{u \in V(H') \mid uw \notin E(G)\}$. By Lemma 5.10, each vertex $b \in B$ has $\deg_G(b) \geq 3(k-1)/2 + 1$.

We can add to G a set S of $|B|$ edges from w to each vertex $b \in B$. Because each vertex in $A \cup B \cup \{w\}$ needs a unique color, either this is a $|B|$-edge-addition or $V(G) = V(H^\circ) \cup \{w\}$. In the first case, by Lemma 4.19, it follows that $|B| > \frac{k-4}{2}$. We show now that the second case is impossible.

Suppose that $V(G) = V(H^\circ) \cup \{w\}$. Then H is a k-Ore graph with vertex set $V(H^\circ) \cup \{\tilde{x}\}$ and $\deg_H(\tilde{x}) = k - 1$. The potential of H is at most $\rho(H) \leq k(k-3) + k\epsilon - 2\delta$. Deleting the vertex $\tilde{x}$ and adding w (with its adjacencies in G) gives us the graph G. This operation affects only one vertex so $|T(H) - T(G)| \leq 1$. If $\deg_G(w) > k-1$ then $\rho(G) \leq \rho(H) - 2(k-1) + \delta \leq k(k-3) - 2(k-1) + k\epsilon - \delta$ which is a contradiction.

Therefore it follows that $\deg_G(w) = k - 1$. But now in any proper $(k-1)$-coloring ϕ of $G - \{w\}$, the subset $N_G(w)$ must have exactly one vertex of each color. There exists at least one edge-replacement $uv \in E(H')$, otherwise $G = K_k$ which is a contradiction. Let $D \subseteq G$ be the corresponding subgraph. By Lemma 4.10, there cannot be a 2-cut in G so there must be an edge from w to some vertex $z \in V(D) - \{u, v\}$. We may assume that ϕ colors D such that $\phi(z)$ is different from $\phi(u)$ and $\phi(v)$, because otherwise $\{zu, zv\}$ is a 2-edge-addition which is a contradiction for $k \geq 8$. However, because $k - 1 \geq 4$ we can switch the color class $\phi(z)$ so that $|\phi(N_G(w))| = k - 2$. This allows us to properly $(k-1)$-color G, which is a contradiction. □

Lemma 5.12. *If x is in a K_{k-3} subgraph of G, where G is a minimal counterexample to the third statement of Theorem 3.5, and* $\deg_G(x) = k - 1$ *then x has at least $(k-9)/6$ neighbors of degree at least* $3(k-3)/2 - 1$.

Proof. Let G be a minimal counterexample to the third statement of Theorem 3.5 such that $x \in V(G)$ is in a K_{k-3} subgraph $D \subseteq G$ and $\deg_G(x) = k - 1$. The vertex x has three neighbors $w_1, w_2, w_3 \in V(G) - V(D)$. We partition $V(D)$ into sets A, B where $A := \{u \in V(D) \mid uw_i \in E(G) \text{ for } 1 \leq i \leq 3\}$ and $B := \{u \in V(D) \mid uw_i \notin E(G) \text{ for some } 1 \leq i \leq 3\}$.

We can add to G the edges $\{w_1w_2, w_1w_3, w_2w_3\}$ and w_ib for $1 \leq i \leq 3$ and $b \in B$. Adding this set S of $3 + 3|B|$ edges gives a K_k subgraph so is a $(3 + 3|B|)$-edge-addition. By Lemma 4.19 it follows that $|B| \geq \frac{k-9}{2}$.

For $b \in B$, we know that $N_G[x]$ is not a subset of $N_G[b]$. Also, x and b are each adjacent to the $k-5$ other vertices in $V(D)$. Therefore by Lemma 4.21, b has degree at least $(k-5) + 1 + \frac{k-3}{2} = 3(k-3)/2 - 1$. □

Lemma 5.13. *If x is in a K_{k-3} subgraph $D \subseteq G$, where G is a minimal counterexample to the third statement of Theorem 3.5,* $\deg_G(x) = k-1$, *and x has a neighbor $y \in V(G) - V(D)$ which is in a different cluster, then x has at least $(k-7)/2$ neighbors of degree at least* $3(k-3)/2 - 1$.

Proof. Let G be a minimal counterexample to the third statement of Theorem 3.5 such that $x \in V(G)$ is in a K_{k-3} subgraph $D \subseteq G$, x has a neighbor $y \in V(G) - V(D)$, and both x and y have degree $k-1$ in G. Further, x and y are in different clusters. We partition $V(D) - \{x\}$ into sets A, B where $A := \{u \in V(D) - \{x\} \mid uy \in E(G)\}$ and $B := \{u \in V(D) - \{x\} \mid uy \notin E(G)\}$.

By Lemma 4.21, x and y cannot have too many neighbors in common because $\deg_G(y) = k-1$. We get that $k-1 = \deg_G(y) \geq |A| + 1 + \frac{k-3}{2}$, from which it follows that $|A| \leq \frac{k-1}{2}$. But $V(D) = A \cup B \cup \{x\}$ so this implies that $|B| \geq \frac{k-7}{2}$.

Similar to Lemma 5.12, one can show using Lemma 4.21 that $b \in B$ has degree at least $3(k-3)/2 - 1$. □

Chapter 6

Discharging

In this chapter, we complete the proof of the third statement of Theorem 3.5 using a discharging argument. The method of discharging, described very elucidatingly in [4], is a global versus local argument. We use potential as a global measure on the graph, and have already shown how certain local structures (2-vertex cuts, i-edge-additions, i-collapsible sets, etc.) are not allowed in a minimal k-critical graph which is not k-Ore and has potential greater than $k(k-3)-2(k-1)$. There still remain areas of low potential such as gadgets or clusters of size $k-3$, and the goal of this chapter is to show how the local structure at these areas contradicts the global hypothesis on potential. We will give each vertex charge relative to its degree, and then spread that charge throughout the graph using bounds obtained in previous chapters. The goal of a general discharging argument is to give each vertex charge which is not positive. This will lead to a contradiction, whereby the total charge on the minimal counterexample G gives that $\rho(G) > k(k-3)-2(k-1)$ is not possible.

However, we will go through discharging in two stages where only the first stage is done in the traditional style. In the first stage, we send packets of charge along edges, pushing from areas of high negative charge to areas which have positive initial charge. This gives a bound on each vertex which does not fall into four specific sets (vertex subsets L, M, P, and Q defined in Definition 6.2), and also gives us a bound on the global charge of the graph.

However, these bounds only hold for $k \geq 33$, which gives the restriction on the third statement of Theorem 3.5. It is possible that the third statement is true for other k as well; however, it would take a much more complex set of discharging rules to handle these cases.

In the second stage of discharging looks at averaging charge across the graph, letting subsets with negative charge compensate for subsets with positive charge rather than pushing charge along a specific edge. In the end, these arguments combine with the results of Chapter 2 to give a bound on $2|E(G)|$. Using other bounds on this parameter, such as the bound established by Kierstead and Rabern [11], we will arrive at a contradiction for $\epsilon \leq \frac{4}{k^3-2k^2+3k}$.

6.1 Discharging: set-up and first stage

Let G be a minimal counterexample to the third statement of Theorem 3.5. We define a charge function $w : V(G) \to \mathbb{R}$ so that

$$w(v) := (k-2)(k+1) + \epsilon - \deg_G(v)(k-1).$$

Note that $\sum_{v \in V(G)} w(v) = \rho(G) + \delta T(G) \geq \rho(G)$. Below, we show the initial charge for vertices of varying degrees in G.

Fact 6.1.

1. *If* $\deg_G(v) = k-1$ *then* $w(v) = k-3+\epsilon$.
2. *If* $\deg_G(v) = k$ *then* $w(v) = -2+\epsilon$.
3. *If* $\deg_G(v) = k+1$ *then* $w(v) = -k-1+\epsilon$.
4. *If* $\deg_G(v) = k+\ell$ *for* $\ell \geq 2$ *then* $w(v) = -2-\ell(k-1)+\epsilon$.

We now define four disjoint subsets of vertices which will need to be dealt with in the second stage of discharging.

Definition 6.2. *We define the following* special *vertex subsets.*

$$L := \{v \in V(G) \mid \deg_G(v) = k-1 \textit{ and } v \textit{ has no neighbors of degree } k-1\},$$

$$M := \{v \in V(G) \mid v \textit{ is in a cluster of size 2 and is a lone-vertex}\},$$

$$P := \{v \in V(G) \mid \deg_G(v) = k\},$$

$$Q := \{v \in V(G) \mid \deg_G(v) = k+1\}.$$

Further, we remove a vertex x from $L \cup M$ if x is a structure-vertex. Therefore, $L \cup M$ contains only lone-vertices.

Let R be the set $V(G) - (L \cup M \cup P \cup Q)$ which contains the rest of the vertices of G. After two stages of discharging we will arrive at a contradiction to our assumption that $\rho(G) > k(k-3) - 2(k-1)$. In the first stage we have two rules to follow.

Discharging Rule #1: Every vertex of degree at least $k+2$ reserves charge $-2+\epsilon$ and sends the remaining charge equally to all neighbors. Note that a vertex of degree d which follows this rule has charge $k^2 - k - d(k-1)$ after reserving $-2+\epsilon$. Therefore, it sends out charge $\left(\frac{k}{d} - 1\right)(k-1)$ to each of its neighbors.

Discharging Rule #2: Every vertex of degree $k-1$ which is a key vertex of a gadget or in a K_{k-3} sends charge $-(k-1)/t$ to each vertex of an adjacent cluster of size $t < k-3$ which is not a key vertex of a gadget or in a K_{k-3} subgraph.

Note that any vertex which follows Rule #2 is in a cluster of size $s \geq t$ and therefore, by Proposition 5.9, sends charge at most $-(k-1)$.

Now let $w'(v)$ be the charge function after applying Rules #1 and #2 to G.

Lemma 6.3. *For every vertex $v \in R = V(G) - (L \cup M \cup P \cup Q)$ applying Rules #1 and #2 give $w'(v) \leq -2+\epsilon$.*

Proof. We need to check this for vertices of degree $k+2$ or higher, and also for lone-vertices in a cluster of size r (where $3 \leq r \leq k-4$), structure-vertices, and near-vertices in a cluster of size $r \leq k-4$. We will do this on a case by case basis.

Case 1. *Suppose* $\deg_G(v) \geq k+2$.

Let v be such a vertex. By Rule #1, v has reserved charge $-2+\epsilon$ and so $w'(v)$ follows the desired inequality.

Case 2. *Suppose* $\deg_G(v) = k-1$ *and* v *is in a cluster of size* r *(where* $3 \leq r \leq k-4$*) which is not adjacent to another cluster.*

Let C be a cluster of size r where $3 \leq r \leq k-4$, and let $v \in C$. Because $\deg_G(v) = k-1$, v has $k-r$ neighbors in $V(G) - C$. Label these vertices $y_1, y_2, \ldots, y_{k-r}$. By Corollary 5.8, if any y_i has degree less than $k-1+r$ then we can move to the case where v is in a K_{k-3} subgraph or is a key vertex of a gadget. Therefore, each y_i follows Rule #1 and sends charge $\left(\frac{k}{k-1+r} - 1\right)(k-1)$ to v. Therefore the new charge of v is $w(v) + \left(\frac{k}{k-1+r} - 1\right)(k-1)(k-r)$. One can check that the second derivative of this with respect to r is positive for all $k > 1$.

$$\frac{d^2}{dr^2}[w'(v)] = \frac{2k(1-3k+2k^2)}{(r+k-1)^3}.$$

Therefore, we only need to check that $w'(v) \leq -2+\epsilon$ for the endpoints $r = 3$ and $r = k-4$.

When $r = 3$, v starts with charge $k-3+\epsilon$ and receives a total of $\frac{-2(k-1)(k-3)}{k+2}$ charge. Thus $w'(v) = -k+9-\frac{30}{k+2}+\epsilon$, and for $k \geq 8$ we get $w'(v) \leq -2+\epsilon$.

When $r = k-4$, v starts with charge $k-3+\epsilon$ and receives a total of $\frac{(5-k)(k-1)4}{2k-5}$ charge. Thus $w'(v) = -k+4+\frac{15}{2k-5}+\epsilon$, and for $k \geq 8$ we get $w'(v) \leq -2+\epsilon$.

Case 3. *Suppose* $\deg_G(v) = k-1$ *and* v *is a key vertex of a gadget.*

Let v be a key vertex of a gadget. By Lemma 5.11, v has at least $\frac{k-3}{2}$ neighbors of degree at least $\frac{3}{2}(k-3)+1$; we will call these high-degree neighbors. For $k \geq 21$, high-degree neighbors have degree at least $\frac{4}{3}k$. Recall that each high-degree neighbor sends charge $\left(\frac{k}{d}-1\right)(k-1)$, where d is the degree of the vertex, by Rule #1. Therefore v gets charge $\frac{-1}{4}(k-1)$ of less from each high-degree neighbor. Also, by Rule #2, v may give away, at worst, charge $-(k-1)$ to adjacent near-vertices. This would raise $w'(v)$ by at most $(k-1)$. Therefore it follows that

$$w'(v) \leq \frac{-(k-1)}{4} \cdot \frac{k-3}{2} + k - 3 + \epsilon + (k-1) = \frac{-1}{8}\left(k^2 - 20k + 19\right) - 2 + \epsilon.$$

For $k \geq 19$, we have $w'(v) \leq -2 + \epsilon$.

Case 4. *Suppose* $\deg_G(v) = k-1$ *and* v *is in a* K_{k-3} *subgraph.*

Let v be a vertex of degree $k-1$ that is in a K_{k-3} subgraph D. By Lemma 5.12, v has at least $\frac{k-9}{6}$ neighbors of degree at least $\frac{3}{2}(k-3)-1$; we will call these high-degree neighbors. For $k \geq 33$, high-degree neighbors have degree at least $\frac{4}{3}k$. Recall that each high-degree neighbor sends charge $\left(\frac{k}{d}-1\right)(k-1)$ by Rule #1. Therefore v gets charge $\frac{-1}{4}(k-1)$ of less from each high-degree neighbor. As long as v does not give away charge by Rule #2, we have

$$w'(v) \leq \frac{-(k-1)}{4} \cdot \frac{k-9}{6} + k - 3 + \epsilon = \frac{-1}{24}\left(k^2 - 64k + 81\right) + \epsilon.$$

For $k \geq 33$ it follows that $w'(v) \leq -2 + \epsilon$.

If v also gives away charge by Rule #2, then we may assume that v has a neighbor $u \in V(G) - V(D)$ which is in a different cluster. At worst, v has given away $-(k-1)$ charge, which will increase $w'(v)$ by at most that amount. Lemma 5.13 implies, however, that there are more high-degree neighbors so v has more charge to give. In this case we have

$$w'(v) \leq \frac{-(k-1)}{4} \cdot \frac{k-7}{2} + k - 3 + \epsilon + (k-1) = \frac{-1}{8}\left(k^2 - 24k + 39\right) + \epsilon.$$

One can check that for $k \geq 23$, this gives $w'(v) \leq -2 + \epsilon$.

Case 5. *Suppose* $\deg_G(v) = k - 1$ *and* v *is in a cluster of size* $r < k - 3$ *which is adjacent to a vertex of degree* $k - 1$ *which is a key vertex of a gadget or in a* K_{k-3} *subgraph.*

Let v be in a cluster C_v of size $r < k - 3$, and let u be an adjacent vertex of degree $k - 1$ which is a key vertex of a gadget or in a K_{k-3} subgraph. Let u be in a different cluster, C_u, of size s. If $s \leq r$ then by Corollary 5.8, v is a key vertex of a gadget or is in a K_{k-3} subgraph and is covered by Case 3 or Case 4. Thus we may assume that $s > r$. By Rule #2, v and each other vertex in C_v receives $-(k - 1)/r$ charge from each vertex of C_u. Because $s > r$, the final charge on v is

$$w'(v) \leq k - 3 + \epsilon - \frac{k-1}{r} \cdot s < -2 + \epsilon.$$

□

Note that there was an approximation in Cases 3 and 4, where we used $\frac{4}{3}k$ as the degree of a high-degree vertex rather than the actual degree. If we instead use the precise degree, the equations are more complicated to solve. Using a CAS one can check that, even without approximating, Lemma 6.3 only holds for $k \geq 33$. Therefore, we paid no penalty in strength of argument when we used numbers for which the calculations were easier.

6.2 Second stage: averaging charge over the graph

Now we are done with the first stage of discharging. The second stage involves global arguments about total charge on G. By Lemma 6.3, we know that the charge on each vertex in $R = V(G) - (L \cup M \cup P \cup Q)$ is at most $-2 + \epsilon$.

However, Rule #1 also had an effect on the sets L and M because a vertex in one of these special sets could have had neighbors of degree $k+2$ or higher. A charge of $-\frac{2(k-1)}{k+2}$ (the charge sent by a vertex of degree $k+2$) or less is sent along each edge from a vertex in R to a vertex in $L \cup M$. We count the number of such edges with the function $e(L \cup M, R)$. Because vertices of L and M are not adjacent to vertices of degree $k-1$ that are not in the same cluster, Rule #2 had no effect on any of the four special sets. Therefore, the total charge on the graph is

$$\sum_{v \in V(G)} w(v) = \sum_{v \in V(G)} w'(v) \leq \epsilon|V(G)| - 2|R| + (k-3)|L| + (k-3)|M| - 2|P|$$
$$-(k+1)|Q| - \frac{2(k-1)}{k+2} e(L \cup M, R). \tag{6.1}$$

To further investigate this, we need a bound on the number of edges counted by $e(L \cup M, R)$. For a vertex $v \in L$ each edge incident with v goes to an edge in either $P \cup Q$ or R. For a vertex $v \in M$ each edge (other than the edge inside the cluster) goes to an edge in either Q or R. It is not possible for there to be an edge from $v \in M$ to P because, by Corollary 5.8, v would be a structure-vertex and therefore would not belong to M. Using this, it follows that

$$e(L \cup M, R) = (k-1)|L| - e(L, P \cup Q) + (k-2)|M| - e(M, Q). \tag{6.2}$$

We now combine two bounds on $e(L, P \cup Q)$. First, we make the simple observation that $e(L, P \cup Q) \leq (k-1)|L|$. Using Lemma 2.5 with the vertex sets L, P, and Q it follows that, for any real numbers $a, b \geq 0$ such that $a + b = 1$,

$$e(L, P \cup Q) \leq a\left(|L|(k-1)\right) + b\left(2|L| + 2|P| + 4|Q|\right).$$

By letting $a = \frac{k-4}{2k-2}$ we get

$$e(L, P \cup Q) \leq \left(\frac{k-4}{2}|L|\right) + \left(\frac{k+2}{k-1}|L| + \frac{k+2}{k-1}|P| + \frac{2(k+2)}{k-1}|Q|\right). \tag{6.3}$$

Also, by Lemma 2.4 we have that $e(M, Q) \leq 3|M| + 6|Q|$. Substituting this and Equation 6.3 into Equation 6.2, we obtain the following bound on $e(L \cup M, R)$:

$$e(L \cup M, R) \geq \frac{(k+2)(k-3)}{2(k-1)}|L| - \frac{k+2}{k-1}|P| + (k-5)|M| - \frac{8k-2}{k-1}|Q|.$$

Using this we can rewrite Equation 6.1 as

$$\sum_{v \in V(G)} w(v) \leq \epsilon|V(G)| - 2|R| + (k-3)(|L| + |M|) - 2|P| - (k+1)|Q|$$

$$-(k-3)|L| + 2|P| - \frac{2(k^2-6k+5)}{k+2}|M| + \frac{16k-4}{k+2}|Q|.$$

This equation simplifies to

$$\sum_{v \in V(G)} w(v) \leq \epsilon|V(G)| - 2|R| - \frac{k^2-5k+11}{k+2}|M| - \frac{k^2-15k+6}{k+2}|Q|$$

$$\leq \epsilon|V(G)| - 2(|V(G)| - |L| - |P|)$$

because the fractional coefficients on $|M|$ and $|Q|$ are both at least 2.

Recall our assumption that G is a minimal counterexample. This implies that $\rho(G) > k(k-3) - 2(k-1) > 0$ and thus we have $0 < \sum_{v \in V(G)} w(v) \leq \epsilon|V(G)| - 2|V(G)| + 2(|L| + |P|)$. It follows that

$$|L| + |P| > |V(G)| \left(1 - \frac{\epsilon}{2}\right).$$

This tells us that all but fewer than $n\epsilon/2$ of the vertices in G are in L or P. We will use this information and the potential of G to get lower and upper bounds for $2|E(G)|$. Note that $2|E(G)| \geq k|P| + (k-1)|L|$, so we get a lower bound

$$2|E(G)| > k|V(G)| \left(1 - \frac{\epsilon}{2}\right) - |L|. \tag{6.4}$$

Also, $\rho(G) > k(k-3) - 2(k-1) > 0$ implies that

$$2|E(G)| < \left(k - \frac{2}{k-1} + \frac{\epsilon}{k-1}\right)|V(G)|. \tag{6.5}$$

It follows from combining Equations 6.4 and 6.5 that

$$|L| > \frac{|V(G)|}{k-1}\left(2 - \epsilon - \frac{\epsilon(k^2-k)}{2}\right). \tag{6.6}$$

This bound on L now allows us to build a new lower bound on $2|E(G)|$ which will lead to a contradiction and complete the proof of Theorem 3.5. Recall that $\mathrm{mic}(G)$ is the maximum of $\sum_{v\in I}\deg_G(v)$ over all independent vertex subsets I of G. The special set L is an independent set of G by construction, and so for a minimal counterexample G it follows that

$$\mathrm{mic}(G) \geq |L|(k-1) > |V(G)|\left(2 - \epsilon - \frac{\epsilon(k^2-k)}{2}\right).$$

Using Lemma 2.6, we update the lower bound on $2|E(G)|$ and, on subsequent lines, make a closer comparison of the upper and lower bound.

$$\left(k + \frac{\epsilon-2}{k-1}\right)|V(G)| > 2|E(G)| > (k-2)|V(G)| + \mathrm{mic}(G)$$

$$\left(k + \frac{\epsilon-2}{k-1}\right)|V(G)| > |V(G)|\left((k-2) + 2 - \epsilon - \frac{\epsilon(k^2-k)}{2}\right)$$

$$\frac{\epsilon-2}{k-1} > -\epsilon - \frac{\epsilon(k^2-k)}{2}$$

$$\frac{4}{k^3-2k^2+3k} < \epsilon.$$

Therefore, because $\epsilon \leq \frac{4}{k^3-2k^2+3k}$, these two bounds lead to a contradiction. This contradiction implies that there is no minimal counterexample G to Theorem 3.5 which is k-critical, not k-Ore, and has $\rho(G) > k(k-3)-2(k-1)$. Therefore, the third statement of Theorem 3.5 has been proven. As stated in Section 1, this result gives us a variety of corollaries. First, we have shown a bound on potential for all k-Ore graphs which includes information about subgraphs in the k-Ore graphs. As the size of k-Ore graphs grows, the potential goes down by at least $\delta - (k-1)\epsilon \geq 0$ for every increase in $k-1$ vertices (which corresponds to an Ore composition with K_k).

Second, the graphs which are not k-Ore are at least $2(k-1)$ less than $k(k-3)$. This corresponds to being one edge worth of potential away from the upper bound on potential shown by Kostochka and Yancey [14]. For $\epsilon = \delta = 0$ this proves independently (for the case where $k \geq 33$) the result in [13] that the k-critical graphs which attain the bound in [14] are exactly the k-Ore graphs.

We have also shown that for k-critical graphs without K_{k-2} or K_{k-1} subgraphs (for these graphs, $T(G) = 0$) that the asymptotic density is above that given by Ore's Conjecture. That is, where $\overline{f}_k(n)$ is the minimum number of edges in a (K_{k-2})-free k-critical graph on n vertices, we have shown that

$$\overline{\phi}_k := \lim_{n\to\infty} \frac{\overline{f}_k(n)}{n} \geq \frac{k}{2} - \frac{1}{k-1} + \frac{\epsilon}{2(k-1)}.$$

Chapter 7

$T_{cs}(G)$, a modification of $T(G)$

In this chapter, we take a second approach to building a potential function. This second potential function encodes different information, but still yields a similar overall result for k-Ore graphs. In the new potential function, the ϵ increase on the coefficient of $|V(G)|$ is paid for with $T_{cs}(G)$ (defined below), which counts structures called *k-critical structures*. Each k-critical structure contains either a K_{k-1} or a K_{k-2}, so therefore the ϵ increase can be paid with a much lower cost than the function $T(G)$ used in the previous chapters.

The k-Ore graphs are precisely the class of graphs which attain the bound in [14]. This is proved in [13] and also is independently proved in this work as a corollary to Theorem 3.5. Therefore, k-Ore graphs are of particular importance when examining density of edges in a k-critical graph and it is worthwhile to obtain a second bound on edge density in k-Ore graphs. Further, this result gives us reason to believe that the ϵ increase in asymptotic edge density that (K_{k-2})-free k-critical graphs have could be extended to a broader class of k-critical graphs.

Before stating the main result of this chapter, we need to establish some terminology. Let K_{k-1}^{-} be the complete graph on $k-1$ vertices minus one edge. We define a family of graphs $\mathcal{T}$, where $G \in \mathcal{T}$ if and only if

1. for special vertices $u, v \in V(G)$ the graph $G - \{u, v\}$ is K_{k-1}^{-},

2. $\deg_G(u) + \deg_G(v) = k - 1$,

3. $N_G(u) \cap N_G(v) = \emptyset$,

4. and $\deg_G(v)$ and $\deg_G(u)$ are both at least 2.

Note that if G is a graph in the family $\mathcal{T}$, then when we identify the special vertices $u, v \in V(G)$ we obtain a diamond (Definition 3.9).

Definition 7.1. *Let G be any graph. A subgraph $H \subseteq G$ is a k-*critical structure *if either $H = K_{k-1}$, $H = K_k - 2K_2$, or $H \in \mathcal{T}$. We then define $T_{cs}(G)$ to be the maximum number of vertex-disjoint k-critical structures in the graph G.*

In this chapter, our aim is to prove that there is a linear bound on the number of k-critical structures in a k-Ore graph. To do this we will prove the following theorem.

Theorem 7.2. *If G is k-Ore and $k \geq 4$ then for $a = 7(k-1)$ the following is true.*

1. *If $T_{cs}(G) \geq 3$ then $T_{cs}(G) - 2 \geq \frac{|V(G)|-1}{2a}$ and*

2. *if $T_{cs}(G) \leq 2$ then $a \geq |V(G)|$.*

7.1 k-critical structures

Recall that a subgraph $D \subseteq G$ is a diamond of G if $D = K_k - uv$ and $\deg_G(x) = k - 1$ for each $x \in V(D) - \{u, v\}$. The vertices $u, v \in V(D)$ are called *endpoints* and we now define the vertices in $V(D) - \{u, v\}$ to be *interior vertices.* Note that if D is a diamond of G with endpoints u, v, then G is an Ore composition of K_k and G^* for some graph G^*. Further, in the graph G, if we delete the interior vertices of D and identify u and v to a single vertex then we obtain a graph that is isomorphic to G^*. We call this process a *vertex-collapse of* D, or simply a collapse of D. When we collapse a diamond $D \subseteq G$ with endpoints u, v, we will always label the resulting graph G^* and call the single vertex in $V(G^*) - V(G)$ a *fat vertex* and label it $\underline{uv}$. Also note that for any edge $e \in E(D)$ either $D - e$ contains a K_{k-1}, or $D - e = K_k - 2K_2$. Therefore $D - e$ always contains a k-critical structure.

We recall what it means to split a vertex, which leads to a definition of another important subgraph called a *k-split*. Given a graph G, and a vertex $x \in V(G)$, we can create a new graph H by splitting the vertex x. We split x by deleting it, adding two new vertices x_1, x_2, and connecting them to vertices in $V(G) - \{x\}$ so that $N_H(x_1) \cup N_H(x_2) = N_G(x)$, $N_H(x_1) \cap N_H(x_2) = \emptyset$, and neither $N_H(x_1)$ nor $N_H(x_2)$ is empty. If we identify the vertices x_1, x_2 in H the resulting graph is denoted $H/\underline{x_1x_2}$. Note that $H/\underline{x_1x_2}$ is isomorphic to G if H is constructed as above. Let $\mathcal{H}$ be the family of graphs obtained from K_k by splitting a vertex $x \in V(K_k)$ such that x_1 and x_2 both have degree at least 2. Note that every graph in $\mathcal{H}$ has $k-1$ vertices of degree $k-1$ and two special vertices of degree at most $k-3$.

Definition 7.3. *A subgraph $D \subseteq G$ is called a* k-split *if D is isomorphic to a graph in $\mathcal{H}$ and $\deg_G(x) = k-1$ for each vertex $x \in V(D)$ with $\deg_D(x) = k-1$. These vertices are called* interior vertices *of the k-split and the two special vertices of degree at most $k-3$ in D we call the* endpoints *of the k-split.*

Note that if D is a k-split with endpoints u, v in G, then G is an Ore composition of G' and K_k for some graph G'. Further, in the graph G, if we delete the interior vertices of D and add the edge uv then we obtain a graph that is isomorphic to G'. We call this process an *edge-reduction* of D. The new edge e is called an *edge-replacement* to indicate that it came from an edge-reduction of a k-split. Let $f \in E(D)$ be any edge in a k-split. Then $D - f$ either contains a K_{k-1} or $D - f$ is a graph in the family $\mathcal{T}$. Therefore $D - f$ always contains a k-critical structure.

In the proof of the main result of this chapter, we will make use of graph functions θ^* and θ' which we define now. If G contains $r \geq 1$ diamonds, then we label the diamonds $D_1, D_2 \ldots, D_r$. We let G_1 be the graph obtained from G by vertex-collapsing D_1 with endpoints u_1, v_1 to the fat vertex $\underline{u_1 v_1}$. Suppose that some D_k with $k > 1$ was incident to D_1 in G. As the interior vertices of a diamond are only adjacent to other vertices of that diamond, these two diamonds must intersect in G at an endpoint. We may assume that D_k has endpoints u_1, v_k in G for $v_k \neq v_1$. But in G_1 one of the endpoints is now the fat vertex $\underline{u_1 v_1}$. We say that D_k remains intact despite the relabelling of its endpoints. Therefore, every D_k with $k > 1$ is a diamond in G_1 and we can vertex-collapse D_2 to obtain the graph G_2. At each step, we collapse the diamond with smallest index D_k creating the graph G_k, until we reach the end and have obtained the graph G_r. We say that $\theta^*(G) = G_r$; this can be equivalently thought of as collapsing all diamonds in G simultaneously.

If G contains $r \geq 1$ k-splits, then we label the k-splits $D_1, D_2, \ldots, D_r$. We let G_1 be the graph obtained from G by edge-reducing D_1. Because the interior vertices of D_1 are only adjacent in G to vertices of D_1, every k-split D_k with $k > 1$ remains intact in G_1. Therefore we can edge-reduce, one at a time, the k-split with smallest index D_k creating the graph G_k. At the end of this process, we have obtained the graph G_r. We say that $\theta'(G) = G_r$; this can be equivalently thought of as edge-reducing all k-splits in G simultaneously.

7.2 Preliminaries

The following result is the analog of Lemma 3.6 and will be needed for the proof of the Theorem 7.2

Lemma 7.4. *If G is an Ore composition of G_1 and G_2 then $T_{cs}(G) \geq T_{cs}(G_1) + T_{cs}(G_2) - 2$. Moreover, if $G_1 = K_k$ or $G_2 = K_k$ then $T_{cs}(G) \geq$*

$T_{cs}(G_1) + T_{cs}(G_2) - 1$. *Lastly, if both* G_1 *and* G_2 *are* K_k, *then* $T_{cs}(G) = 2$.

Proof. Suppose that G is an Ore composition of G_1 and G_2. Let e be the replaced edge of G_1 and z be the split vertex of G_2. From the definition of an Ore composition $T_{cs}(G) \geq T_{cs}(G_1 - e) + T_{cs}(G_2 - \{z\})$. Note that $T_{cs}(G_1 - e) \geq T_{cs}(G_1) - 1$ and $T_{cs}(G_2 - \{z\}) \geq T_{cs}(G_2) - 1$, because removing a single element can remove at most one k-critical structure. Thus, we get $T_{cs}(G) \geq T_{cs}(G_1) + T_{cs}(G_2) - 2$ as desired. If $G_1 = K_k$ then $T_{cs}(K_k - e) = 1$ for every edge $e \in E(G_1)$; also, if $G_2 = K_k$ then $T_{cs}(K_k - \{v\}) = 1$ for every $v \in V(G_2)$. Thus removing an element from a K_k graph does not remove a k-critical structure. Therefore, it follows that $T_{cs}(G) \geq T_{cs}(G_1) + T_{cs}(G_2) - 1$ if either G_1 or G_2 is K_k. Further, if both G_1 and G_2 are K_k then $T_{cs}(G) = 2$. $\square$

Proposition 7.5. *Let* G *be a* k*-Ore graph. Then* $T_{cs}(G) \leq 1$ *if and only if* $G = K_k$.

Proof. Note that $T_{cs}(K_k) = 1$. Now let G be a vertex-minimal k-Ore graph where $T_{cs}(G) \leq 1$ and $G \neq K_k$. Because G is k-Ore but not K_k, G is an Ore composition of G_1 and G_2 where G_1 and G_2 are both k-Ore. By Lemma 7.4 it follows that

$$1 \geq T_{cs}(G) \geq T_{cs}(G_1) + T_{cs}(G_2) - 2$$

so $3 \geq T_{cs}(G_1) + T_{cs}(G_2)$. Hence $T_{cs}(G_i) \leq 1$ for $i = 1$ or $i = 2$, and by the minimality of G we conclude that $G_i = K_k$. In this case Lemma 7.4 instead implies that $T_{cs}(G_1) + T_{cs}(G_2) \leq 2$ so, in fact, $G_1 = G_2 = K_k$. However, in this case $T_{cs}(G) = 2$ by Lemma 7.4. This contradiction proves the proposition. $\square$

The following lemma is similar to Lemma 3.11, however this lemma covers only k-Ore graphs G with $T_{cs}(G) = 2$

Lemma 7.6. *If* G *is* k*-Ore and* $T_{cs}(G) = 2$ *then*

1. *for all $e \in E(G)$ either $T_{cs}(G - e) \geq 2$ or there exists a diamond or a k-split in $G - e$.*

2. *for all $x \in V(G)$ either $T_{cs}(G - \{x\}) \geq 2$ or there exists a diamond or a k-split in $G - \{x\}$.*

Proof. We will prove the lemma by induction on $|V(G)|$. Let G be a k-Ore graph with $T_{cs}(G) = 2$ such that all smaller k-Ore graphs satisfy the conclusions of the lemma. Note that $G \neq K_k$ by Proposition 7.5 because $T_{cs}(G) = 2$. Thus, G is an Ore composition of two k-Ore graphs G_1 and G_2. Let $f \in E(G_1)$ and $y \in V(G_2)$ be the replaced edge of G_1 and split vertex of G_2, respectively. We claim that $T_{cs}(G_i) \leq 2$ for $i = 1, 2$, and prove this claim by contradiction. Suppose $T_{cs}(G_1) \geq 3$. If $T_{cs}(G_2) \geq 2$ then we have by Lemma 7.4 that $2 = T_{cs}(G) \geq T_{cs}(G_1) + T_{cs}(G_2) - 2 \geq 3$, which is a contradiction. If $T_{cs}(G_2) = 1$ then $G_2 = K_k$ and Lemma 7.4 implies that $2 = T_{cs}(G) \geq T_{cs}(G_1) + T_{cs}(G_2) - 1 \geq 3$, which is also a contradiction. Thus, we conclude that $T_{cs}(G_1) \leq 2$. A similar argument proves that $T_{cs}(G_2) \leq 2$ as well.

Now let $e \in E(G)$ from the hypotheses of the lemma be given. There are two cases, either $e \in E(G_1)$ or $e \in E(G_2)$. Suppose $e \in E(G_1)$. For $T_{cs}(G_2) = 2$, then by induction $T_{cs}(G_2 - \{y\}) \geq 2$ or there exists a diamond or k-split in $G_2 - \{y\}$. Because $G_2 - \{y\} \subseteq G - e$, any k-critical structures, diamonds, or k-splits in $G_2 - \{y\}$ also exist in $G - e$ and the lemma holds. If instead $T_{cs}(G_2) = 1$, then the graph $G_2' \subseteq G$ obtained by splitting G_2 at the vertex y is a k-split. Removing e from G does not destroy this k-split, so there exists a k-split in $G - e$.

Now we examine the case where $e \in E(G_2)$. If $T_{cs}(G_1) = 2$ then by induction $T_{cs}(G_1 - f) \geq 2$ or there exists a diamond or k-split in $G_1 - f$. But $G_1 - f \subseteq G - e$ because $e \in E(G_2)$, so any k-critical structures, diamonds, or k-splits in $G_1 - f$ also exist in $G - e$ and the lemma holds. If $T_{cs}(G_1) = 1$

then $G_1 - f \subseteq G - e$ is a diamond. Therefore, we have shown that the first statement of the lemma holds in all cases.

We now turn to the second statement. Let $x \in V(G)$ be given. Suppose $x \in V(G_2)$. If $T_{cs}(G_1) = 2$ then by induction $T_{cs}(G_1 - f) \geq 2$ or there exists a diamond or k-split in $G_1 - f$. But, as before $G_1 - f \subseteq G - \{x\}$, so any k-critical structures, diamonds, or k-splits in $G_1 - f$ also exist in $G - \{x\}$ and the lemma holds. If $T_{cs}(G_1) = 1$, then $G_1 - f$ is a diamond in $G - \{x\}$. This case includes the situation where x is an overlap vertex of G_1 and G_2.

Suppose instead that $x \in V(G_1) - V(G_2)$. In this case, if $T_{cs}(G_2) = 2$ then inductively $T_{cs}(G_2 - \{y\}) \geq 2$ or there exists a diamond or k-split in $G_2 - \{y\}$, and these structures also exist in $G - \{x\}$ because $G_2 - \{y\} \subseteq G - \{x\}$. If $T_{cs}(G_2) = 1$, then the graph G_2' obtained by splitting G_2 at the vertex y is a k-split. Because $x \notin V(G_2)$, this k-split G_2' will be intact after removing the vertex x from G. This completes the proof of the lemma. □

For the following proposition we show that that to obtain a smaller k-Ore graph, a diamond can be vertex-collapsed and a k-split can be edge-reduced.

Proposition 7.7. *Suppose that G is k-Ore. If G contains a diamond D then the graph G^* obtained by vertex-collapsing D is also k-Ore. If G contains a k-split D then the graph G' obtained by edge-reducing D is k-Ore.*

Proof. Suppose that neither statement of the proposition holds. Then let G be a counterexample with a minimum number of vertices; note that $G \neq K_k$ as this has no diamonds or k-splits. Thus, we have that G is an Ore composition of two k-Ore graphs G_1 and G_2.

Suppose that D is a diamond of G. Then we consider the graph H where G is an Ore composition of K_k and H with overlap vertices which are the endpoints of $D \subseteq G$. Note that H is the graph G^* in the statement of the proposition, so it remains to prove that H is k-Ore. Note that D must be a diamond of G_1 or G_2; assume for now that D is a diamond of G_1. By the minimality of G it follows that G_1 is an Ore composition of K_k and G_1' where G_1' is a k-Ore graph. But now we can realize H as an Ore composition of G_1' and G_2, both of which are k-Ore; therefore, H is k-Ore. The method of argument when D is a diamond of G_2 is identical.

Now suppose that $D \subseteq G$ is a k-split. The k-split D must be a subgraph of G_1 or G_2; assume $D \subseteq G_1$. Further, G is also an Ore composition of H and K_k with overlap vertices which are the endpoints of D. By the minimality of G it follows that G_1 is an Ore composition of G_1^* and K_k where G_1^* is k-Ore. Now H is a Ore composition of G_1^* and G_2, both of which are k-Ore; therefore, H is k-Ore. The method of argument when $D \subseteq G_2$ is identical. $\square$

7.3 Main result on k-Ore graphs

Here we restate and then prove the main result of this chapter. This proof is broken up into 10 claims.

Theorem 7.2. *If G is k-Ore and $k \geq 4$ then for $a = 7(k-1)$ the following is true.*

- *If $T_{cs}(G) \geq 3$ then $T_{cs}(G) - 2 \geq \frac{|V(G)|-1}{2a}$ and*
- *if $T_{cs}(G) \leq 2$ then $a \geq |V(G)|$.*

Proof. Suppose the theorem does not hold. Then let G be a counterexample with a minimum number of vertices. That is, let G be a k-Ore graph where either $T_{cs}(G) \geq 3$ and $T_{cs}(G) - 2 < \frac{|V(G)|-1}{2a}$ or $T_{cs}(G) \leq 2$ and $a < |V(G)|$, and let any k-Ore graph with fewer vertices than G follow the inequalities of Theorem 7.2. By Proposition 7.5, we may assume that $G \neq K_k$ and $T_{cs}(G) \geq 2$. Therefore G is an Ore composition of two k-Ore graphs G_1 and G_2. Note that G_1 and G_2 follow the inequalities of 7.2.

Claim 7.8. *No two diamonds are incident in G.*

Proof. Suppose two diamonds D, D' are incident in G. Note that two distinct diamonds could only intersect at their endpoint vertices; thus, we say that D has endpoints u, v and that D' has endpoints v, w. Collapsing D into a fat vertex $\underline{uv}$ still leaves D' intact, although the endpoint $v \in V(D')$ is now the fat vertex $\underline{uv}$. We can collapse this diamond as well. Then both diamonds have collapsed into a single fat vertex $\underline{uvw}$, and we have a new graph G' which has $2(k-1)$ fewer vertices than G.

By Proposition 7.7, G' is k-Ore, so by the minimality of G, Theorem 7.2 applies to G'. We can recover G from G' by expanding the fat vertex back into two diamonds, one at a time. By doing this we have possibly removed a k-critical structure in G' which was using the vertex $\underline{uvw}$; however, we have also introduced two K_{k-1} subgraphs, namely $D - \{v\}$ and $D' - \{v\}$. Therefore, if $t = T_{cs}(G')$ it follows that

$$T_{cs}(G) \geq t + 1. \tag{7.1}$$

If $G' = K_k$ then, in fact, we can ensure that the fat vertex is not in a k-critical structure and expanding the fat vertex into two adjacent diamonds adds two k-critical structures while removing none. Therefore we have $T_{cs}(G) = 3$ and $|V(G)| = 3k - 2$. By supposition we have that $T_{cs}(G) - 2 < \frac{|V(G)|-1}{2a}$, but because this requires $2a < 3(k-1)$, we have a contradiction, so $G' \neq K_k$.

Therefore, Proposition 7.5 implies that $T_{cs}(G') \geq 2$, so $T_{cs}(G) \geq 3$ by Equation 7.1. If $T_{cs}(G') = 2$, then by the minimality of G we have that $|V(G')| \leq a$. But then $1 \leq (T_{cs}(G') + 1) - 2 \leq T_{cs}(G) - 2$ and because G is a counterexample, we have

$$1 \leq T_{cs}(G) - 2 < \frac{|V(G)| - 1}{2a} = \frac{|V(G')| + 2k - 3}{2a} \leq \frac{a + 2k - 3}{2a}.$$

This implies that $a < 2k - 3$, which is a contradiction.

Now we assume that $T_{cs}(G') \geq 3$. Then by the minimality of G we have $T_{cs}(G') - 2 \geq \frac{|V(G')|-1}{2a}$. Because $2a \geq 2(k-1)$ and G has $2(k-1)$ more vertices than G' it follows that

$$T_{cs}(G) - 2 \geq T_{cs}(G') - 1 \geq \frac{|V(G')| - 1}{2a} + 1 \geq \frac{|V(G')| + 2(k-1) - 1}{2a}$$
$$= \frac{|V(G)| - 1}{2a}.$$

Thus G is not a counterexample at all. Therefore we have proven that two diamonds are not incident in G. □

We now consider $\theta^*(G)$, which is obtained by taking all diamonds in G and simultaneously collapsing them. By Claim 7.8, we know that any vertex in $\theta^*(G)$ corresponds either to a single vertex in G, or a single diamond in G. By repeated applications of Proposition 7.7 we can show that $\theta^*(G)$ is k-Ore.

Claim 7.9. $\theta^*(G) \neq K_k$.

Proof. Suppose, for sake of contradiction, that $\theta^*(G) = K_k$. If $T_{cs}(G) = 2$ then G can have at most two diamonds. In this case $|V(G)| \leq 3k - 2$ so G is not a counterexample at all. Therefore we may assume that $T_{cs}(G) \geq 3$.

Let ℓ be the number of vertices in $\theta^*(G)$ which are fat vertices. If $\ell = 1$, then G is an Ore composition of K_k and K_k and $T_{cs}(G) = 2$. So we may assume that $\ell \geq 2$. Suppose $\ell = 2$. Then we have that $1 \leq T_{cs}(G) - 2 < \frac{|V(G)|-1}{2a}$. But $|V(G)| = 3k - 2$ and $2a \geq 3(k-1)$, so this is a contradiction.

Therefore, we can assume that $\ell \geq 3$. Then $T_{cs}(G) \geq \ell$ and $|V(G)| = k - \ell + k\ell = (\ell+1)k - \ell$. Because $a \geq 3k - 3$ we have

$$\ell - 2 \leq T_{cs}(G) - 2 < \frac{|V(G)| - 1}{2a} = \frac{(\ell+1)(k-1)}{2a} \leq \frac{(\ell+1)(k-1)}{2 \cdot 3(k-1)} = \frac{\ell+1}{6}.$$

For $\ell \geq 3$ this inequality is a contradiction. Therefore, we have proven that $\theta^*(G) \neq K_k$. □

Claim 7.10. *If $D \subseteq \theta^*(G)$ is a diamond then at most one vertex of D is a diamond in G. That is, at most one vertex of D is a fat vertex. Furthermore, this vertex can only be an interior vertex.*

Proof. Suppose that there is a diamond $D \subseteq \theta^*(G)$ where $\ell \geq 2$ vertices of D are fat vertices, meaning they were diamonds in G. We replace each fat vertex of D with a diamond to recover the subgraph $\widehat{D} \subseteq G$ which can be vertex-collapsed to D; that is, $\theta^*(\widehat{D}) = D$. Further, let $\theta^*(D) = \underline{x}$. Let G' denote the graph obtained from G by collapsing $\widehat{D}$ to D, and let G'' denote the graph obtained from G' by collapsing D to $\underline{x}$.

Suppose $G'' = K_k$. Then $T_{cs}(G'') = 1$ and $T_{cs}(G) \geq \ell + 1 \geq 3$, because when we replace $\underline{x}$ with $\widehat{D}$ we leave a $K_{k-1} \subseteq G - \widehat{D}$ intact, and we add ℓ new K_{k-1} subgraphs from the diamonds in $\widehat{D}$. We also have $|V(G)| = k + (k-1) + \ell(k-1) = (\ell+2)(k-1)+1$. Because G is a counterexample with $T_{cs}(G) \geq 3$ we have $T_{cs}(G) - 2 < \frac{|V(G)|-1}{2a}$. This gives us the string of inequalities

$$\ell - 1 \leq T_{cs}(G) - 2 < \frac{|V(G)|-1}{2a} = \frac{(\ell+2)(k-1)}{2a}.$$

This implies that $a < \frac{\ell+2}{2\ell-2}(k-1)$, which has a maximum value (when $\ell = 2$) of 2. But $a \geq 2(k-1)$, so this is a contradiction and G'' cannot be K_k.

Thus, we may assume that $T_{cs}(G'') \geq 2$. If $T_{cs}(G'') = 2$ then by the minimality of G it follows that $|V(G'')| \leq a$, so $|V(G)| \leq a + 3(k-1)$. Then we have

$$1 \leq T_{cs}(G) - 2 < \frac{|V(G)|-1}{2a} \leq \frac{a+3k-4}{2a}. \tag{7.2}$$

This implies that $a < 3k-4$, which is a contradiction. If $T_{cs}(G'') \geq 3$ then by the minimality of G it follows that $T_{cs}(G'') - 2 \geq \frac{|V(G'')|-1}{2a}$. Also, $T_{cs}(G'')+1 \leq T_{cs}(G)$, because if we replace $\underline{x}$ with $\widehat{D}$, we possibly destroy one k-critical structure that contained $\underline{x}$, but we add at least two K_{k-1} subgraphs from the diamonds of $\widehat{D}$. Now we have

$$\frac{|V(G)|-1}{2a} > T_{cs}(G) - 2 \geq T_{cs}(G'') - 1 \geq \frac{|V(G'')|-1}{2a} + 1, \tag{7.3}$$

which implies that $|V(G)| > |V(G'')| + 2a$. But $|V(G'')| + 3(k-1) \geq |V(G)|$ and it follows that $3(k-1) > 2a$, which is a contradiction. Therefore, there is no diamond $D \subseteq \theta^*(G)$ with more than one fat vertex.

Now we suppose that one the endpoints of the diamond $D \subseteq \theta^*(G)$ is a fat vertex. Then $T_{cs}(\widehat{D}) = 2$, and $T_{cs}(G'') + 1 \leq T_{cs}(G)$ because replacing $\underline{x}$ with $\widehat{D}$ may remove one k-critical structure, but also will add the two k-critical structures in $\widehat{D}$. If $G'' = K_k$, then $T_{cs}(G) = T_{cs}(\widehat{D}) + 1 = 3$ because the K_{k-1} of $K_k - \{\underline{x}\}$ remains intact. Also $|V(G)| = 3k - 2$. But this implies that $1 < \frac{3(k-1)}{2a}$, which is a contradiction. So we may assume that $G'' \neq K_k$. As $T_{cs}(G) \geq T_{cs}(G'') + 1$, it follows from Proposition 7.5 that $T_{cs}(G) \geq 3$. Because of this and the fact that $|V(G)| = |V(G'')| + 2(k-1) < |V(G'')| + 3(k-1)$, we have the same hypotheses that led to the contradictory Equations 7.2 and 7.3. □

In fact, each diamond D in $\theta^*(G)$ has exactly one fat vertex. If any diamond in $D \subseteq \theta^*(G)$ had no fat vertices, then $D \subseteq G$ so would have been collapsed to a vertex in $\theta^*(G)$. When we expand the fat vertex of $D \subseteq \theta^*(G)$ and recover $\widehat{D} \subseteq G$, we define this structure to be a *super-diamond*. Using this language, we say that each diamond in $\theta^*(G)$ comes from a super-diamond in G.

Claim 7.11. *No two diamonds intersect in $\theta^*(G)$.*

Proof. Let D_1, D_2 be two diamonds in $\theta^*(G)$ whose vertex sets intersect. Note that the diamonds can only share endpoints, so we may assume that D_1 has endpoints u, v and that D_2 has endpoints v, w. We collapse D_1 to a fat vertex $\underline{uv}$; by Proposition 7.7, the resulting graph H is k-Ore, and either D_2 is a diamond in G^* with endpoints $\underline{uv}, w$, or H is K_k.

First suppose that H is K_k. Then $T_{cs}(\theta^*(G)) = 2$, and $|V(\theta^*(G))| = 2k-1$. By Claim 7.10 each diamond in $\theta^*(G)$ has exactly one fat vertex, and they do not overlap. Thus, $|V(G)| = 4k-3$. If $T_{cs}(G) = 2$, then $|V(G)| = 4k-3 \leq a$ is a contradiction. If $T_{cs}(G) \geq 3$ then $1 \leq T_{cs}(G) - 2 < \frac{4k-4}{2a}$ implies that $a < 2(k-1)$, which is also a contradiction. Therefore, we can assume that

H is not K_k. Then by Proposition 7.7, when we collapse $D_2 \subseteq H$ so that we get the single fat vertex $\underline{uvw}$, the resulting graph will be k-Ore.

Now let $C = D_1 \cup D_2$ be the subgraph of $\theta^*(G)$ that consists of both diamonds; $T_{cs}(C) = 2$. We can replace each fat vertex in C with a diamond in order to recover $\widehat{C} \subseteq G$. Note that $\theta^*(\widehat{C}) = C$ so $T_{cs}(\widehat{C}) \geq 2$, and that C can collapse further to $\underline{uvw}$ as above. Let G' be the graph obtained from G by collapsing $\widehat{C} \subseteq G$ to C, and let G'' be the graph obtained from G' by collapsing C to $\underline{uvw}$. Then $|V(G)| = |V(G'')| + 4(k-1)$. If G'' is K_k, then $T_{cs}(G) \geq 3$, because $G'' - \{\underline{uvw}\}$ is a K_{k-1} that remains intact when we substitute $\widehat{C}$ for the fat vertex. So $1 \leq T_{cs}(G) - 2 < \frac{|V(G)|-1}{2a}$ implies that $2a < |V(G)| - 1 = 5(k-1)$, which is a contradiction. Therefore we may assume that G'' is not K_k. Then $T_{cs}(G) \geq T_{cs}(G'') + 1$ because replacing $\underline{uvw}$ with $\widehat{C}$ possibly removes a k-critical structure in G'' but adds at least 2 from $\widehat{C}$, and by Proposition 7.5 $T_{cs}(G'') \geq 2$ which gives $T_{cs}(G) \geq 3$. But then

$$\frac{|V(G)|-1}{2a} > |T_{cs}(G)| - 2 \geq T_{cs}(G'') - 1 \geq \frac{|V(G'')|-1}{2a} + 1$$

implies that $|V(G)| > |V(G'')| + 2a$, meaning that $4(k-1) > 2a$, and this is a contradiction.

Thus, we have shown that no such diamonds D_1, D_2 can exist in $\theta^*(G)$. □

We now consider $\theta^*(\theta^*(G))$, which is obtained by taking all diamonds in $\theta^*(G)$ and simultaneously collapsing them. By Claims 7.8 and 7.11, we know that every vertex in $\theta^*(\theta^*(G))$ corresponds to either a single vertex in G, a single diamond in G, or a single super-diamond in G.

Claim 7.12. *$\theta^*(\theta^*(G))$ is not K_k.*

Proof. Suppose, for sake of contradiction, that $\theta^*(\theta^*(G)) = K_k$. First suppose that $T_{cs}(G) = 2$. Since $T_{cs}(G) \geq T_{cs}(\theta^*(G))$ it follows that $\theta^*(G)$ can have at most two diamonds. Thus $|V(\theta^*(G))| \leq 3k-2$ and $|V(G)| \leq 5k-4$, which is a contradiction as $a > 5k-4$.

Therefore, we may assume that $T_{cs}(G) \geq 3$. Let ℓ be the number of diamonds in $\theta^*(G)$. Then $|V(G)| \geq k + 2\ell(k-1) = (2\ell+1)(k-1)+1$ because each diamond in $\theta^*(G)$ is a super-diamond in G by Claim 7.10. If $\ell \leq 2$ then we have $1 \leq T_{cs}(G) - 2 < \frac{|V(G)|-1}{2a} \leq \frac{5(k-1)}{2a}$ which is a contradiction as $a > 5(k-1)$. If $\ell \geq 3$, then because each diamond in $\theta^*(G)$ corresponds to a super-diamond in G and each super-diamond contains a k-critical structure, we have $T_{cs}(G) \geq \ell$. But $a > 4(k-1)$ so it follows that

$$\ell - 2 \leq T_{cs}(G) - 2 < \frac{|V(G)|-1}{2a} \leq \frac{(2\ell+1)(k-1)}{2a} \leq \frac{(2\ell+1)(k-1)}{2 \cdot 4(k-1)} = \frac{2\ell+1}{8}.$$

For $\ell \geq 3$, this inequality is a contradiction. Therefore, we have shown that $\theta^*(\theta^*(G)) \neq K_k$. $\square$

Claim 7.13. *There are no diamonds in $\theta^*(\theta^*(G))$.*

Proof. Let $D \subseteq \theta^*(\theta^*(G))$ be a diamond with endpoints u, v. Replace each fat vertex with either a super-diamond or diamond so that we recover $\widehat{D} \subseteq G$ and $\theta^*(\theta^*(\widehat{D})) = D$. Note that D must have at least one fat vertex that corresponds to a super-diamond in G, otherwise $\theta^*(\theta^*(\widehat{D}))$ would be a single fat vertex instead of a diamond. Let G' be the graph obtained from G by collapsing $\widehat{D}$ to D, and let G'' be the graph obtained from G' by further collapsing D to a fat vertex $\underline{uv}$. Let ℓ be the number of fat vertices in D; as noted above, $\ell \geq 1$. Also, $|V(G)| \leq |V(G'')| + (2\ell+1)(k-1)$.

Suppose first that $\ell \geq 2$. Then $T_{cs}(D) \geq \ell$. When we replace $\underline{uv}$ with $\widehat{D}$, we lose at most one k-critical structure in G'', but gain at least ℓ structures from $\widehat{D}$. Thus $T_{cs}(G) \geq T_{cs}(G'') + \ell - 1$. As long as $T_{cs}(G'') \geq 2$, this implies that $T_{cs}(G) \geq 3$. Otherwise, if $G'' = K_k$, then we lose no structures when expanding $\underline{uv}$, as $G''-\{\underline{uv}\} = K_{k-1}$. So in that case $T_{cs}(G) \geq T_{cs}(G'')+\ell \geq 3$. In all cases $T_{cs}(G) \geq 3$ and thus it follows that $T_{cs}(G) - 2 < \frac{|V(G)|-1}{2a}$.

If $T_{cs}(G'') \leq 2$ then $T_{cs}(G) - 2 \geq \ell - 1$ and $|V(G'')| \leq a$. From these observations, it follows that

$$\begin{aligned} \ell - 1 \leq T_{cs}(G) - 2 < \frac{|V(G)|-1}{2a} < \frac{|V(G)|}{2a} \leq \frac{a + (2\ell+1)(k-1)}{2a} \\ = \frac{1}{2} + \frac{(2\ell+1)(k-1)}{2a}. \end{aligned}$$

But this implies that $a < \frac{2\ell+1}{2\ell-3}(k-1)$. When $\ell \geq 2$ it follows that $a < 5(k-1)$, but this is a contradiction.

Thus we may assume that $T_{cs}(G'') \geq 3$. In this case,

$$\frac{|V(G)|-1}{2a} > T_{cs}(G) - 2 \geq T_{cs}(G'') + \ell - 3 \geq \frac{|V(G'')|-1}{2a} + \ell - 1$$

which implies that $|V(G)| > |V(G'')| + 2a(\ell - 1)$. But we know that with ℓ fat vertices in D, $|V(G'')| + (2\ell + 1)(k - 1) \geq |V(G)|$. Combining these two inequalities, we get $(2\ell + 1)(k - 1) > 2a(\ell - 1)$, which can be written as $\frac{2\ell+1}{\ell-1}(k - 1) > 2a$. Because $\frac{2\ell+1}{\ell-1}$ is a decreasing function for $\ell \geq 2$, we have $5(k - 1) > 2a$, which is a contradiction.

We now suppose that $\ell = 1$. In this case, the fat vertex $\underline{x} \in V(D)$ must correspond to a super-diamond $C \subseteq G$. Note that $C \subseteq \widehat{D}$. Also note that a super-diamond contains a K_{k-1} that does not intersect either endpoint of the super-diamond. Thus the vertices $V(D) - \{\underline{x}\}$ either form a K_{k-1} or form a graph in $\mathcal{T}$ that uses the endpoints of C as well. Either way $T_{cs}(\widehat{D}) = 2$ and thus $T_{cs}(G) \geq 3$.

If $T_{cs}(G'') \leq 2$ then $|V(G'')| \leq a$ by the minimality of G. By hypothesis, $1 \leq T_{cs}(G) - 2 < \frac{|V(G)|-1}{2a}$ and so $2a < |V(G)| - 1$. But $|V(G)| \leq |V(G'')| + 3(k-1) \leq a+3(k-1)$ so this implies that $a < 3k-4$, which is a contradiction.

If $T_{cs}(G'') \geq 3$ then by the minimality of G, $T_{cs}(G'') - 2 \geq \frac{|V(G'')|-1}{2a}$. So then

$$\frac{|V(G)|-1}{2a} > T_{cs}(G) - 2 \geq T_{cs}(G'') - 1 \geq \frac{|V(G'')|-1}{2a} + 1.$$

This implies that $|V(G)| > |V(G'')+2a$, but we also have $|V(G'')|+3(k-1) \geq |V(G)|$ so then $3(k-1) > 2a$, which is a contradiction.

Therefore we have shown that $\theta^*(\theta^*(G))$ has no diamonds. $\square$

Claim 7.14. *If $D \subseteq \theta^*(\theta^*(G))$ is a k-split then no more than one vertex of D comes from a diamond or super-diamond.*

Proof. Suppose that D is a k-split with endpoints u, v where $\ell \geq 2$ vertices of D are fat vertices. We replace each fat vertex of D with a diamond or super-diamond to recover $\widehat{D} \subseteq G$, where $\theta^*(\theta^*(\widehat{D})) = D$. We can also edge-reduce D to an edge-replacement $e = uv$ by adding the edge $e = uv$ and deleting all interior vertices. Let G' be the graph obtained from G by collapsing $\widehat{D}$ to D and let G'' be the graph obtained from G' by edge-reducing D to e. Replacing an edge-replacement with a k-split adds $k-1$ vertices to a graph, and replacing a fat vertex with a diamond or super-diamond adds at most $(2k-2)$ vertices to a graph. Then $|V(G)| \leq |V(G'')| + (2\ell+1)(k-1)$. We claim that $T_{cs}(G) \geq 3$, and to show this we inspect three cases when $T(G'') \geq 2$, and then show how their conclusions are stronger for $G'' = K_k$.

Case 1. *Suppose $\ell \geq 3$*

When we replace the edge-replacement $e = uv$ with $\widehat{D}$ we lose at most two k-critical structures in G'' because u and v may be fat vertices that are also in k-critical structures in G''. However, we gain at least $\ell \geq 3$ k-critical structures in $\widehat{D}$. Therefore $T_{cs}(G) \geq T_{cs}(G'') - 2 + \ell$.

Case 2. *Suppose $\ell = 2$ and both fat vertices are endpoints of D.*

When we replace the edge-replacement e with $\widehat{D}$, we lose at most two structures. However, as $D - \{u, v\} = K_{k-1}$, this k-critical structure remains when u, v are replaced with diamonds and it follows that $T_{cs}(\widehat{D}) \geq 3$. Therefore $T_{cs}(G) \geq T_{cs}(G'') - 2 + 3 = T_{cs}(G'') + 1$.

Case 3. *Suppose $\ell = 2$ and at least one fat vertex in D is an interior vertex.*

We replace the edge-replacement $e \in E(G'')$ with $\widehat{D}$. If e is in a k-critical structure in G'' then we lose that k-critical structure, but we gain at least ℓ k-critical structures in $\widehat{D}$. Hence, $T_{cs}(G) \geq T_{cs}(G'') - 1 + \ell$. Otherwise, $e \in E(G'')$ is not in a k-critical structure, and we lose at most one k-critical structure, as one of the endpoints of e may be a fat vertex that is split in $\widehat{D}$. However, we gain at least ℓ k-critical structures in $\widehat{D}$, so here as well $T_{cs}(G) \geq T_{cs}(G'') - 1 + \ell$. Therefore we always have $T_{cs}(G) \geq T_{cs}(G'') + 1$ when $T_{cs}(G'') \geq 2$.

Now consider when $G'' = K_k$. We can lose at most one structure in G'' since only one exists. Therefore, in Case 1 we have $T_{cs}(G) \geq T_{cs}(G'') - 1 + \ell \geq 3$. In Case 2, we have $T_{cs}(G) \geq T_{cs}(G'') - 1 + 3 \geq 3$. And in Case 3 only one endpoint of D can be a fat vertex. Without loss of generality let it be v. Then $G'' - \{v\} = K_{k-1}$ remains intact in G so $T_{cs}(G) \geq T_{cs}(G'') + 2 = 3$.

Now we have shown the claim that $T_{cs}(G) \geq 3$ and will show contradictions regardless of whether $T_{cs}(G'') \leq 2$ or $T_{cs}(G'') \geq 3$. Suppose first that $T_{cs}(G'') \leq 2$. By the minimality of G it follows that $|V(G'')| \leq a$. Now if $\ell = 2$ then $|V(G)| \leq a + 5(k-1)$. But then it follows that

$$1 \leq T_{cs}(G) - 2 < \frac{|V(G)| - 1}{2a} \leq \frac{a + 5k - 6}{2a}$$

and this implies that $a < 5k - 6$ which is a contradiction. Therefore we may assume that $\ell \geq 3$. Then the argument of Case 1 implies that $T_{cs}(G) \geq \ell$ so

$$\begin{aligned} \ell - 2 \leq T_{cs}(G) - 2 < \frac{|V(G)| - 1}{2a} &\leq \frac{a + (2\ell+1)(k-1) - 1}{2a} \\ &< \frac{1}{2} + \frac{(2\ell+1)(k-1)}{2a}. \end{aligned}$$

It follows that $\ell - \frac{5}{2} < \frac{(2\ell+1)(k-1)}{2a}$ and thus $a < \frac{2\ell+1}{2\ell-5}(k-1)$. But $\frac{2\ell+1}{2\ell-5}$ is a decreasing function on $\ell \geq 3$ with a maximum at $\ell = 3$. Therefore we have shown that $a < 7(k-1)$, which is a contradiction.

Now suppose instead that $T_{cs}(G'') \geq 3$. Then by the minimality of G it follows that $T_{cs}(G'') - 2 \geq \frac{|V(G'')|-1}{2a}$. If $\ell = 2$ then

$$\frac{|V(G)|-1}{2a} > T_{cs}(G) - 2 \geq T_{cs}(G'') - 1 \geq \frac{|V(G'')|-1}{2a} + 1$$

which implies that $|V(G)| > |V(G'')| + 2a$. However, $|V(G'')| + 5(k-1) \geq V(G'')$ so we have shown that $5(k-1) > 2a$, which is a contradiction. Therefore we may assume that $\ell \geq 3$. Then from Case 1 it follows that

$$\frac{|V(G)|-1}{2a} > T_{cs}(G) - 2 \geq T_{cs}(G'') - 4 + \ell \geq \frac{|V(G'')|-1}{2a} + \ell - 2.$$

This implies that $|V(G)| > |V(G'')| + 2a(\ell-2)$. But $|V(G'')| + (2\ell+1)(k-1) \geq |V(G)|$ so together these inequalities show that $\frac{2\ell+1}{\ell-2}(k-1) > 2a$. When $\ell \geq 3$ this implies that $7(k-1) > 2a$, which is a contradiction.

Therefore, we have shown that any k-split $D \in \theta^*(\theta^*(G))$ has at most one fat vertex.

□

Claim 7.15. *No two k-splits are incident in $\theta^*(\theta^*(G))$.*

Proof. Suppose that D_1, D_2 are k-splits in $\theta^*(\theta^*(G))$ with endpoints u, v and v, w respectively. By Claim 7.14 it follows that each of D_1 and D_2 has at most one fat vertex. Also, these fat vertices are disjoint, as they must be interior vertices. We can edge-reduce D_1 to a edge-replacement $e = uv$ and by Proposition 7.7 the resulting graph H is k-Ore. Also, $H \neq K_k$ as $D_2 \subseteq H$ has $k+1$ vertices. Then we edge-reduce D_2 to a edge-replacement $f = vw$.

Let $C = D_1 \cup D_2$ be the subgraph of $\theta^*(\theta^*(G))$ that consists of both k-splits. $T_{cs}(C) = 2$. If there are fat vertices in C, then we replace them with a diamond or super-diamond to recover $\widehat{C} \subseteq G$, and it follows that $T_{cs}(\widehat{C}) \geq 2$. Note that $\widehat{C} = C$ is possible. Let G' be the graph obtained from G by collapsing $\widehat{C}$ to C and let G'' be the graph obtained from G' by edge-reducing $D_1 \subseteq C$ to $e = uv$ and $D_2 \subseteq C$ to $f = vw$.

When we replace e with D_1 we gain a k-critical structure overall unless the edge e is used in a k-critical structure of G''. Similarly, replacing f with D_2 gains a k-critical structure overall unless the edge f is used in a k-critical structure of G''. But if e, f are both used in a k-critical structure, then they are part of the same k-critical structure because they are incident at the vertex v and k-critical structures are disjoint. So when we expand the edge-replacements e, f we lose at most one structure, and gain $T_{cs}(C) = 2$. Thus $T_{cs}(G) \geq T_{cs}(G') \geq T_{cs}(G'') + 1$ because $T_{cs}(\widehat{C}) \geq T_{cs}(C)$.

Suppose first that $G'' = K_k$. Then $|V(G')| = k + 2(k-1) = 3k - 2$ and $|V(G)| \leq 3k - 2 + 2(2k-2) = 7k - 6$ because each k-split in G' may have one fat vertex that was a super-diamond in G. But $G'' - \{v\} = K_{k-1}$ and this is not lost by replacing edge-replacements adjacent to v with k-splits. So $T_{cs}(G) \geq T_{cs}(\widehat{C}) + 1 \geq 3$. Then it follows that $1 \leq T_{cs}(G) - 2 < \frac{|V(G)|-1}{2a} \leq \frac{7(k-1)}{2a}$, which is a contradiction.

Suppose now that $T_{cs}(G'') = 2$. Then by the minimality of G, $|V(G'')| \leq a$. Because $T_{cs}(G) \geq T_{cs}(G'') + 1 = 3$ it follows that

$$1 \leq T_{cs}(G) - 2 < \frac{|V(G)| - 1}{2a} \leq \frac{|V(G'')| + 6(k-1) - 1}{2a} \leq \frac{a + 6(k-1) - 1}{2a}.$$

From this it follows that $a < 6k - 7$, which is a contradiction.

Finally, we suppose that $T_{cs}(G'') \geq 3$. Then $\frac{|V(G)|-1}{2a} > T_{cs}(G) - 2 \geq T_{cs}(G'') - 1 \geq \frac{|V(G'')|-1}{2a} + 1$, so it follows that $|V(G)| > |V(G'')| + 2a$. But $|V(G'')| + 6(k-1) \geq |V(G)|$ and these two inequalities show that $6(k-1) > 2a$, which is a contradiction.

Therefore we have shown that no two k-splits are incident in $\theta^*(\theta^*(G))$. $\square$

We now consider $\theta'(\theta^*(\theta^*(G)))$ which is obtained by taking all k-splits in $\theta^*(\theta^*(G))$ and simultaneously edge-reducing them. By Claim 7.14 no edge-replacement is incident to a fat vertex and by Claim 7.15 no edge-replacement shares an endpoint with another edge-replacement.

Claim 7.16. *$\theta'(\theta^*(\theta^*(G)))$ is not K_k.*

Proof. To reduce notation, let $G' := \theta^*(\theta^*(G))$ and $G'' := \theta'(\theta^*(\theta^*(G)))$ for this proof. Suppose, for the sake of contradiction, that $G'' = K_k$. By Claim 7.12, $\theta^*(\theta^*(G)) \neq K_k$ so there must be at least one edge-replacement in G''. Let m be the number of edge-replacements in G'' and let ℓ be the number of fat vertices in G''. Note that $m \geq 1$; also, $2m + \ell \leq k$ because each edge-replacement is adjacent to two distinct vertices which are not fat vertices.

It follows that $T_{cs}(G') \geq m$ as each edge-replacement in G'' is a k-split in G' with a K_{k-1} subgraph, and $T_{cs}(G) \geq m + \ell$ as each fat vertex in G'' corresponds to a diamond or super-diamond in G. Additionally, we know that $|V(G'')| = k$, and $|V(G')| = k + m(k-1)$. As each k-split may have a fat vertex not counted by ℓ, and all fat vertices in G' could be collapsed super-diamonds, it follows that $|V(G)| \leq k + m(k-1) + m(2k-2) + \ell(2k-2) = (3m + 2\ell + 1)(k-1) + 1$.

First, suppose that $m \geq 3$. Then $T_{cs}(G) \geq m + \ell \geq 3$ and because G is a counterexample it follows that

$$m + \ell - 2 \leq T_{cs}(G) - 2 < \frac{|V(G)| - 1}{2a} \leq \frac{(3m + 2\ell + 1)(k-1)}{2a}.$$

So then $2a < \frac{3m+2\ell+1}{m+\ell-2}(k-1)$. The derivative of $\frac{3m+2\ell+1}{m+\ell-2}$ with respect to ℓ is $\frac{-(m+5)}{(m+\ell-2)^2}$ so is negative for all $m \geq 3$. Thus this function is maximized when $\ell = 0$ and we have $2a < \frac{3m+1}{m-2}(k-1)$. Because $m \geq 3$ this implies that $2a < 10(k-1)$, which is a contradiction.

Second, suppose that $m = 2$. For $\ell \geq 1$ we get $T_{cs}(G) \geq m + \ell \geq 3$. So it follows that

$$\ell = m + \ell - 2 \leq T_{cs}(G) - 2 < \frac{|V(G)| - 1}{2a} \leq \frac{(3m + 2\ell + 1)(k-1)}{2a}$$
$$= \frac{(2\ell + 7)(k-1)}{2a}.$$

This gives $2a < \frac{2\ell+7}{\ell}(k-1)$ and for $\ell \geq 1$ implies that $2a < 9(k-1)$, which is a contradiction.

Therefore we may assume that $\ell = 0$ when $m = 2$. But in this case $T_{cs}(G) \geq 3$ as well because each edge-replacement e, f in G'' contributes to a K_{k-1} in G' and $G'' - \{e, f\}$ is a $K_k - 2K_2$, which is a k-critical structure that remains intact in G' and G. So then $1 \leq T_{cs}(G) - 2 < \frac{|V(G)|-1}{2a} \leq \frac{7(k-1)}{2a}$, which is a contradiction because $2a \geq 7(k-1)$.

Finally, suppose that $m = 1$. For $\ell \geq 2$ we get $T_{cs}(G) \geq m + \ell \geq 3$ so

$$\ell - 1 \leq m + \ell - 2 \leq T_{cs}(G) - 2 < \frac{|V(G)| - 1}{2a} \leq \frac{(2\ell + 4)(k-1)}{2a}.$$

This implies that $2a < \frac{2\ell+4}{\ell-1}(k-1)$ and for $\ell \geq 2$ it follows that $2a < 8(k-1)$, which is a contradiction.

So we may assume that $\ell \leq 1$ when $m = 1$. Then $|V(G)| \leq (3m+2\ell+1)(k-1)+1 \leq 6k-5$. If $T_{cs}(G) = 2$, then we have $a < |V(G)| \leq 6k-5$ which is a contradiction. If $T_{cs}(G) \geq 3$ then we have $1 \leq T_{cs}(G) - 2 < \frac{|V(G)|-1}{2a} \leq \frac{6(k-1)}{2a}$. This implies that $2a < 6(k-1)$ which is also a contradiction.

We have shown that all values of m lead to contradiction, so therefore $G'' = \theta'(\theta^*(\theta^*(G))) \neq K_k$. □

Claim 7.17. *$\theta'(\theta^*(\theta^*(G)))$ contains no diamonds or k-splits.*

Proof. Suppose that $\theta'(\theta^*(\theta^*(G)))$ has a diamond D. It follows from Claim 7.13 that $\theta^*(\theta^*(G))$ has no diamonds so it follows that D must have at least one edge-replacement. Let $\overline{D}$ be the structure in $\theta^*(\theta^*(G))$ obtained from G by replacing each edge-replacement of D with a k-split. Then we can replace each fat vertex in $\overline{D}$ with either a super-diamond or diamond to recover $\widehat{D} \subseteq G$. Let G' be the graph obtained from G' by collapsing $\widehat{D}$ to $\overline{D}$ and let G'' be the graph obtained from G'' by edge-reducing any k-splits in $\overline{D}$. Thus $D \subseteq G''$. Further, let G''' be the graph obtained by collapsing D to a fat vertex $\underline{x}$. Each of these graphs are k-Ore by repeated applications of Proposition 7.7.

Let m be the number of edge-replacements in D and let ℓ be the number of fat vertices in D. Note that $m \geq 1$. Then $T_{cs}(\overline{D}) \geq m$ and $T_{cs}(\widehat{D}) \geq m + \ell$. We claim that $T_{cs}(\widehat{D}) \geq 2$. If $m \geq 2$ this is clear, and if $m = 1$, then let e be the edge-replacement in D. $D - e$ is either a $K_k - 2K_2$ if e is incident to neither endpoint of D, or contains a K_{k-1} if e is incident to an endpoint of D. Thus $T_{cs}(\widehat{D}) \geq T_{cs}(\overline{D}) = 2$.

When replacing $\underline{x}$ with $\widehat{D}$ we possibly lose one k-critical structure in G''' but gain at least $\max\{2, m+\ell\}$ from $\widehat{D}$. We claim now that $T_{cs}(G) \geq 3$. For $T_{cs}(G''') \geq 2$ there are two cases depending on $m + \ell$. When $(m + \ell) \geq 2$ then $T_{cs}(G) \geq T_{cs}(G''') - 1 + T_{cs}(\widehat{D}) \geq 1 + m + \ell$ and when $(m + \ell) = 1$ then $T_{cs}(G) \geq T_{cs}(G''') - 1 + 2 \geq 3$. The remaining case is when $T_{cs}(G''') = 1$ and thus $G''' = K_k$ by Proposition 7.5. Here, $G''' - \{\underline{x}\}$ is a K_{k-1} that remains intact in G. So in fact, no k-critical structures from G''' are lost when expanding to $\widehat{D}$ and it follows that $T_{cs}(G) \geq 1 + m + \ell$ when $(m + \ell) \geq 2$ and $T_{cs}(G) \geq 1 + T_{cs}(\widehat{D}) \geq 3$ when $(m + \ell) = 1$. We have shown that $T_{cs}(G) \geq 3$ in all cases and, in particular we have

$$T_{cs}(G) \geq \max\{1, T_{cs}(G''') - 1\} + 2 \text{ when } m + \ell = 1 \text{ and}$$

$$T_{cs}(G) \geq \max\{1, T_{cs}(G''') - 1\} + m + \ell \text{ when } m + \ell \geq 2. \tag{7.4}$$

We also want a bound on $|V(G)|$. By construction, $|V(G'')| = |V(G''')| + (k - 1)$ and $|V(G')| = |V(G''')| + (m + 1)(k - 1)$. Because each k-split in G' may have contained a fat vertex not counted by ℓ, it follows that

$$\begin{aligned} |V(G)| &\leq |V(G''')| + (m + 1)(k - 1) + 2(m + \ell)(k - 1) \\ &= |V(G''')| + (3m + 2\ell + 1)(k - 1). \end{aligned} \tag{7.5}$$

First, suppose that $T_{cs}(G''') \leq 2$ so that $|V(G''')| \leq a$. When $m+\ell \geq 2$ it follows that

$$m+\ell-1 \leq T_{cs}(G)-2 < \frac{|V(G)|-1}{2a} \leq \frac{a+(3m+2\ell+1)(k-1)-1}{2a}$$
$$< \frac{1}{2} + \frac{3(m+\ell)+1}{2a}(k-1).$$

This implies that $a < \frac{3(m+\ell)+1}{2(m+\ell)-3}(k-1)$ and for $(m+\ell) \geq 2$ we get $a < 7(k-1)$, which is a contradiction. When $(m+\ell)=1$ then it must be that $m=1$ and $\ell=0$. But now because $1 \leq T_{cs}(G)-2 < \frac{|V(G)|-1}{2a} \leq \frac{a+4(k-1)-1}{2a}$, it follows that $a < 4k-5$, which is also a contradiction.

Now suppose that $T_{cs}(G''') \geq 3$. When $m+\ell \geq 2$ then we have, because G''' follows Theorem 7.2, $\frac{|V(G)|-1}{2a} > T_{cs}(G)-2 \geq (T_{cs}(G''')-1+m+\ell)-2 \geq \frac{|V(G''')|-1}{2a}-1+m+\ell$. This implies that $|V(G)| > |V(G''')|+2a(m+\ell-1)$. But we also have that $|V(G)| \leq |V(G''')|+(3m+2\ell+1)(k-1) \leq |V(G''')|+(3(m+\ell)+1)(k-1)$. Together, these inequalities show that $2a < \frac{3(m+\ell)+1}{(m+\ell)-1}(k-1)$. For $(m+\ell) \geq 2$ it follows that $2a < 7(k-1)$, which is a contradiction.

When $m+\ell=1$ then $\ell=0$ and we have

$$\frac{|V(G)|-1}{2a} > T_{cs}(G)-2 \geq (T_{cs}(G''')-1+2)-2 \geq \frac{|V(G''')|-1}{2a}+1.$$

This implies that $|V(G)| > |V(G''')|+2a$ but we also have that $|V(G''')|+4(k-1) \geq |V(G)|$. Together these inequalities show that $4(k-1) > 2a$, which is a contradiction. Therefore there can be no diamonds D in $\theta'(\theta^*(\theta^*(G)))$.

Now suppose that $\theta'(\theta^*(\theta^*(G)))$ contains a k-split C. Because all k-splits in $\theta^*(\theta^*(G))$ are now edge-replacements in $\theta'(\theta^*(\theta^*(G)))$, at least one edge of C must be an edge-replacement. Let $\overline{C}$ be the subgraph of $\theta^*(\theta^*(G))$ obtained from C by replacing each edge-replacement of C with a k-split. Then we can replace each fat vertex in $\overline{C}$ with either a super-diamond or diamond to recover $\widehat{C} \subseteq G$. Let G' be the graph obtained from G by collapsing $\widehat{C}$ to $\overline{C}$ and let G'' be the graph obtained from G' by edge-reducing any k-splits in $\overline{C}$. Thus $C \subseteq G''$. Further, let G''' be the graph obtained from G'' by edge-reducing C to the edge-replacement $f = uv$. Each of these graphs are k-Ore by repeated applications of Proposition 7.7.

Let m be the number of edge-replacements in C and let ℓ be the number of fat vertices in C. Note that $m \geq 1$. Also $T_{cs}(\overline{C}) \geq m$ and $T_{cs}(\widehat{C}) \geq m + \ell$. We claim that $T_{cs}(\widehat{C}) \geq 2$. If $m \geq 2$ this is clear, and if $m = 1$, then let e be the edge-replacement in C. $C - e$ is either in $\mathbb{H}$ if e is incident to neither endpoint of C, or contains a K_{k-1} if e is incident to an endpoint of C. Thus $T_{cs}(\widehat{C}) \geq T_{cs}(\overline{C}) = 2$.

When replacing $f = uv$ with $\widehat{C}$ we possibly lose one k-critical structure in G''' but gain at least $\max\{2, m+\ell\}$ from $\widehat{C}$. Exactly as in the diamond case above, if $T_{cs}(G''') \geq 2$ then $T_{cs}(G) \geq T_{cs}(G''') - 1 + T_{cs}(\widehat{C})$. Also, if $T_{cs}(G''') = 1$ then $G''' = K_k$ by Proposition 7.5 and $G''' - f$ contains a K_{k-1} that remains intact in G. So in fact, no k-critical structures from G''' were lost and it follows that $T_{cs}(G) \geq 1 + m + \ell$ when $(m + \ell) \geq 2$ and $T_{cs}(G) \geq 1 + T_{cs}(\widehat{C}) \geq 3$ when $(m + \ell) = 1$.

Therefore when G''' has a k-split, Equations 7.4 and 7.5 also hold, just as they did when G''' had a diamond. Thus, the same arguments will also lead to contradictions for k-splits $C \subseteq \theta'(\theta^*(\theta^*(G)))$. Therefore, we have shown that $\theta'(\theta^*(\theta^*(G)))$ contains no diamonds or k-splits. $\square$

End of proof of Theorem 7.2. Now we have a graph $H = \theta'(\theta^*(\theta^*(G)))$ which is not K_k, and contains no diamonds or k-splits. By repeated applications of Proposition 7.7 we know that H is k-Ore. Because H is not K_k it follows that H is an Ore composition of two k-Ore graphs H_1 and H_2 with overlap vertices $\{u, v\}$. Because H contains no diamonds or k-splits, $T_{cs}(H_i) \geq 2$ for $i = 1, 2$.

We replace all edge-replacements in $E(H)$ and all fat vertices in $V(H) - \{u, v\}$ with the structures that were originally subgraphs of G. This gives a graph G' where $G' = G$ if neither u nor v is a fat vertex in H, or where we could obtain G' from G by collapsing the necessary diamonds or super-diamonds into the fat vertices u or v. Thus it follows that $T_{cs}(G) \geq T_{cs}(G')$. Also G' is k-Ore and we can realize G' as an Ore composition of two k-Ore graphs G'_1 and G'_2 with overlap vertices $\{u, v\}$ (the same as the overlap vertices of H). Because $T_{cs}(G'_i) \geq T_{cs}(H_i)$ for $i = 1, 2$, we know that $T_{cs}(G'_i) \geq 2$ for $i = 1, 2$. We break the proof of the theorem into three cases.

First, we consider the case where $T_{cs}(G'_1) = T_{cs}(G'_2) = 2$. If $T_{cs}(G'_1) = 2$ then it follows that $T_{cs}(H_1) = 2$. But $H_1 - uv$ has no diamonds, so Lemma 7.6 implies that $T_{cs}(H_1 - uv) = 2$. Thus, $T_{cs}(G'_1 - uv) = 2$ as well. A similar argument shows that $T_{cs}(G'_2 - \{\underline{uv}\}) = 2$. But this gives 4 disjoint critical structures in G' so we know that $4 \leq T_{cs}(G') \leq T_{cs}(G)$. Also, by the minimality of G, we have that $|V(G'_i)| \leq a$ for $i = 1, 2$. Thus $|V(G')| \leq 2a - 1$ and because the vertices u, v could be at worst super-diamonds, it follows that $|V(G)| \leq 2a - 1 + 4(k - 1)$. Putting these inequalities together we have that

$$2 \leq T_{cs}(G) - 2 < \frac{|V(G)| - 1}{2a} \leq 1 + \frac{4k - 5}{2a}.$$

This implies that $2a < 4k - 5$, which is a contradiction.

Second, we consider the case where one of $T_{cs}(G'_1)$ and $T_{cs}(G'_2)$ is 2, and the other is at least 3. Suppose that $T_{cs}(G'_1) = 2$. Then, by Lemma 7.6, it follows that $T_{cs}(G'_1 - e) = 2$. Thus $T_{cs}(G) \geq T_{cs}(G') \geq 2 + T_{cs}(G'_2) - 1 = T_{cs}(G'_2) + 1$ by Lemma 7.4. Also, we have that $|V(G'_1)| \leq a$ which implies that $|V(G)| \leq |V(G'_2)| + a + 4k - 5$. However we then have the string of inequalities

$$T_{cs}(G'_2) - 1 \leq T_{cs}(G) - 2 < \frac{|V(G)| - 1}{2a} \leq \frac{|V(G'_2)| - 1}{2a} + \frac{a + 4k - 5}{2a}.$$

Because Theorem 7.2 holds for G'_2, this implies that $1 < \frac{a+4k-5}{2a}$, or that $a < 4k-5$ which is a contradiction. A similar argument yields a contradiction when $T_{cs}(G'_2) = 2$ and $T_{cs}(G'_1) \geq 3$.

Third, we consider the case where $T_{cs}(G_1')$ and $T_{cs}(G_2')$ are both at least 3. If $u, v \in V(G')$ are not fat vertices, then $G' = G$ and we have that $|V(G)| = |V(G_1')| + |V(G_2')| - 1$. Then it follows from Lemma 7.4 that

$$T_{cs}(G) - 2 \geq T_{cs}(G_1') + T_{cs}(G_2') - 4 \geq \frac{|V(G_1')| - 1}{2a} + \frac{|V(G_2')| - 1}{2a} = \frac{|V(G)| - 1}{2a}.$$

This contradicts the fact that G is a counterexample.

Therefore we may assume that at least one of u or v is a fat vertex in G'. We claim that $T_{cs}(G) \geq T_{cs}(G_1') + T_{cs}(G_2') - 1$; if this can be shown, then it follows that

$$\begin{aligned} T_{cs}(G) - 2 &\geq T_{cs}(G_1') + T_{cs}(G_2') - 4 + 1 \\ &\geq \frac{|V(G_1')| - 1}{2a} + \frac{|V(G_2')| - 1}{2a} + \frac{2a}{2a} \geq \frac{|V(G)| - 1}{2a} \end{aligned}$$

where the last inequality is because $|V(G)| \leq |V(G_1')| + |V(G_2')| - 1 + 4(k-1) \leq |V(G_1')| + |V(G_2')| - 1 + 2a$. This would contradict the fact that G is a counterexample.

To show the claim we will first assume that exactly one of u, v is a fat vertex in G' and, without loss of generality, suppose that it is u. The graphs $G_2' - \{\underline{uv}\}$ and $G_1' - \{u\}$ are disjoint subgraphs in G, so it follows using an argument similar to the proof of Lemma 7.4 that $T_{cs}(G) \geq T_{cs}(G_1') - 1 + T_{cs}(G_2') - 1$. No k-critical structure in this lower bound uses the vertex $u \in V(G')$ and so when we expand that fat vertex, we gain an additional k-critical structure. Therefore $T_{cs}(G) \geq T_{cs}(G_1') + T_{cs}(G_2') - 1$.

Now assume that both u and v are fat vertices in G'. The graphs $G_2' - \{\underline{uv}\}$ and $G_1' - \{u, v\}$ are disjoint subgraphs in G, so it follows using an argument similar to the proof of Lemma 7.4 that $T_{cs}(G) \geq T_{cs}(G_1') - 2 + T_{cs}(G_2') - 1$. No k-critical structure in this lower bound use the vertices $u, v \in V(G')$ and so when we expand the two fat vertices, we gain two additional k-critical structures. Therefore $T_{cs}(G) \geq T_{cs}(G_1') + T_{cs}(G_2') - 1$.

We have shown that $T_{cs}(G) \geq T_{cs}(G_1') + T_{cs}(G_2') - 1$ whenever at least one of u or v is a fat vertex in G', and thus the case where $T_{cs}(G_1')$ and $T_{cs}(G_2')$

are both at least 3 will also lead to a contradiction. This completes the proof of Theorem 7.2. $\square$

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