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Kevin M. Wingfield   Date

Modular and Lexical Matchings in the Middle Levels Graph

By

Kevin M. Wingfield
Master of Science

Mathematics

_____
Dwight Duffus, Ph.D.
Advisor

_____
Ron Gould, Ph.D.
Committee Member

_____
Robert Roth, Ph.D.
Committee Member

Accepted:

_____
Lisa A. Tedesco, Ph.D.
Dean of the Graduate School

_____
Date

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By

Kevin M. Wingfield
B.S., Morehouse College, 2009

Advisor: Dwight Duffus, Ph.D.

An abstract of
A thesis submitted to the Faculty of the Graduate School
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Abstract

Modular and Lexical Matchings in the Middle Levels Graph
By Kevin M. Wingfield

Classes of explicitly defined matchings in the middle levels bipartite graph induced by the Boolean lattice are investigated. The original motivation for an investigation into the Hamiltonicity of the middle levels bipartite graph came from the conjecture of Havel. Here, we collect some known results and present some new observations that indicate that, when disjoint, the 2-factors obtained from taking the union of pairs of these matchings always contain short cycles.

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Chapter 1

Introduction

In the study of graphs, one widely investigated area is the search for Hamiltonian cycles in graphs. A Hamiltonian cycle is a simple cycle that contains each vertex of the graph. Beginning in the 1950's, progress was made on the problem of determining *sufficient conditions* for a graph to possess a Hamiltonian cycle. To this end, Dirac [7] and Ore [19] have two very well known results. These are archetypical *density* results, establishing that if a graph has enough edges it will be Hamiltonian. Dirac and Ore both ensure the existence of a sufficient number of edges by specifying a condition on the minimum degree on vertices and a condition on pairs of nonadjacent vertices, respectively.

The earliest investigations of Hamilton, the namesake of the Hamiltonian cycle, into the presence of such cycles was based upon a game. Given the solid

dodecahedron and considering each of its 20 vertices as a city of interest, the objective was to find a route that visited each city exactly once and ended at its starting point. A route with this property is an example of a Hamiltonian cycle in the graph whose vertices are given by the cities and edges, by those of the solid. Although this graph is quite sparse, it is highly symmetric, in the sense that its automorphism group is transitive on the vertices.

In 1970, Lovasz [18] asked if every connected vertex-transitive graph has a Hamiltonian path, noting that there are few examples of such graphs that are not Hamiltonian. In this thesis, we are concerned with a specific family of transitive graphs, defined as follows (see Section 1.1 for a more precise definition). Given a positive integer k, $\mathbf{B}_k$ denotes the graph whose vertex set is the collection of all k- and $(k+1)$-element subsets of a $(2k+1)$-element set, with edges defined by containment. This is the bipartite graph of the middle two levels of the Boolean lattice of all subsets of the $(2k+1)$-element set, ordered by containment. The question, now attributed to Havel [12], is whether $\mathbf{B}_k$ contains a Hamiltonian cycle for all k.

There has been progress on at least two fronts. First, computational work has established that $\mathbf{B}_k$ is Hamiltonian for all $k \leq 17$ [21]. Second, after results established that there are cycles with length a positive fraction

of the order of $\mathbf{B}_k$, Johnson [14] has shown that there is a cycle of length $(1 - o(1))|V(\mathbf{B}_k)|$.

Several researchers have approached the problem, with uniformly negative results, as follows. A Hamiltonian cycle in a graph is a special case of a 2-factor, that is, a spanning subgraph with the property that all of its vertices have degree 2. Since $\mathbf{B}_k$ is a graph with an even number of vertices, each 2-factor in $\mathbf{B}_k$ is the union of a pair of edge disjoint perfect matchings. Since any Hamiltonian cycle in $\mathbf{B}_k$ is the union of 2 perfect matchings, then one strategy for finding a Hamiltonian cycle in $\mathbf{B}_k$ was to assemble a large collection of explicitly defined perfect matchings and consider the union of two disjoint matchings. Beginning with Duffus, Sands and Woodrow [10] in 1988, the question of whether we can explicitly describe interesting classes of perfect matchings in $\mathbf{B}_k$, decide when they are disjoint, and determine if a Hamiltonian cycle could be built from the union of a pair of these was investigated. They began with the well-known class of lexicographic matchings and characterized those pairs which are disjoint. For each such pair, they showed that the 2-factor obtained from the union always contains "short" cycles.

We shall see two classes of matchings in $\mathbf{B}_k$, the modular matchings, con-

sidered in Chapter 2, and the lexical matchings, studied in Chapter 3. In each case, explicit descriptions are given and it is shown that, pairwise, the modular matchings can not give a Hamiltonian cycle. This holds for certain pairs of lexical matchings and we suspect all 2-factors include short cycles. In Chapter 4, a set of open problems regarding these matchings and their interactions are collected.

1.1   Definitions and notation

For this thesis, a graph is a finite, undirected, loopless graph without multiple edges. We shall use $[n]$ to denote the set $\{1, ..., n\}$ and $[i, j]$ to denote the set $\{i, i + 1, i + 2, ..., j\}$. The latter is to be understood modulo n, that is, for $n = 5, i = 4, j = 2$, $[4, 2] = \{4, 5, 1, 2\}$.

Let $\mathbf{P}(n)$ denote the set of all subsets of $[n]$, the power set, partially ordered by set containment. For $j = 0, 1, \ldots, n$, let $\mathbf{L}_j$ denote the collection of all j-element subsets of $[n]$. This collection of subsets makes up the jth level of the lattice $\mathbf{P}(n)$. For $n = 2k + 1$, let $\mathbf{B}_k$ denote the bipartite graph defined on the vertex set $\mathbf{L}_k \cup \mathbf{L}_{k+1}$ with A adjacent to B if $A \subset B$ or $B \subset A$. We adopted this notation from [9]. Given $A \subseteq [2k + 1]$ with $|A| = k$, we take

$A = \langle a_1, a_2, ..., a_k \rangle$ to mean $a_1 < a_2 < ... < a_k$ and denote its complement in

$[2k + 1]$ by $\bar{A}$. We let $\bar{A} = \langle \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{k+1} \rangle$ mean $\bar{a}_1 > \bar{a}_2 > ... > \bar{a}_{k+1}$. Let

$\sum A$ denote the sum of the elements of A.

A *Hamiltonian cycle* is a simple cycle that contains every vertex of a graph.

A *perfect matching* or a *1-factor* in $\mathbf{B}_k$ is a collection M of edges such that

each vertex of $\mathbf{B}_k$ is incident to exactly one edge of M. A 1-factorization of

$\mathbf{B}_k$ is a collection of $k+1$ disjoint perfect matchings of $\mathbf{B}_k$. For our purposes,

it will be convenient to consider a perfect matching to be an injection

$m : \mathbf{L}_k \to \mathbf{L}_{k+1}$ such that A is adjacent to $m(A)$, for all $A \in \mathbf{L}_k$. At times,

we will also regard the set $M = \{\{A, m(A)\} \mid A \in \mathbf{L}_k\}$ as the matching. An

r-factor in $\mathbf{B}_k$ is an r-regular spanning subgraph – thus, its edge set is the

union of r pairwise disjoint 1-factors.

Given a graph G and $u, v \in V(G)$, $d_G(u, v)$ is the distance from u to v in

G, that is, the minimum number of edges in a path with endpoints u and v.

We call d_G the *distance function* on G. For a subgraph H of G, d_H is the

distance function restricted to the edges of H.

1.2   Automorphisms in $\mathbf{B}_k$

For each n, let S_n denote the symmetric group on $[n]$. Then each $\phi \in S_{2k+1}$ induces a bijection on $\mathbf{L}_k$ and on $\mathbf{L}_{k+1}$ as follows. For any $A \in \mathbf{P}(2k+1)$, let $\widehat{\phi}(A) = \{\phi(a) | a \in A\}$. Note that for all $A \in \mathbf{L}_k$, $B \in \mathbf{L}_{k+1}$, $A \subset B$ if and only if $\widehat{\phi}(A) \subset \widehat{\phi}(B)$. Thus $\widehat{\phi}$ restricted to $\mathbf{L}_k \cup \mathbf{L}_{k+1}$ gives an automorphism of the bipartite graph $\mathbf{B}_k$. For each automorphism α of $\mathbf{B}_k$, there is some $\phi \in S_{2k+1}$ with $\alpha = \widehat{\phi}$. In [8], Duffus, Hanlon, and Roth provide a proof of this fact.

Note that for all 1-factors M of $\mathbf{B}_k$ and for all $\widehat{\phi} \in Aut(\mathbf{B}_k)$ as defined above, $\widehat{\phi}[M] = \{\{\widehat{\phi}(A), \widehat{\phi}(B)\} | \{A, B\} \in M\}$ is a 1-factor. Also, given disjoint matchings M and M' of $\mathbf{B}_k$, $M \cup M'$ and $\widehat{\phi}[M \cup M']$ are 2-factors in $\mathbf{B}_k$ that each have the same cycle structure. By cycle structure, we mean the sequence $(c_t | t \in \mathbb{N})$ where c_t is the number of cycles of length t.

Chapter 2

Modular Matchings

The first matchings we will consider are the modular matchings in $\mathbf{B}_k$. It is not clear when modular matchings were first defined. In [9], they are tied to lattice path combinatorics, and [9] is the first paper in which modular matchings are systematically studied in conjunction with the middle levels conjecture. In the sections that follow, we discuss some noteworthy results regarding modular matchings in $\mathbf{B}_k$, including the result that the union of any two modular matchings is not a Hamiltonian cycle.

2.1 Definitions and preliminaries

For $i \in [k+1]$, let $m_i : \mathbf{L}_k \to \mathbf{L}_{k+1}$ be defined as follows: for $A = \langle a_1, a_2, ..., a_k \rangle \in \mathbf{L}_k$, let

$$m_i(A) = A \cup \{\bar{a}_y\}, \text{ where } y \equiv (i + \sum A)(\text{mod } k+1). \qquad (2.1)$$

Thus $m_i(A)$ is the set obtained by adding the yth largest element of $\bar{A}$ to A, where $1 \leq y \leq k+1$. Let M_i be the set of edges of $\mathbf{B}_k$ of the form $\{A, m_i(A)\}$. The following lemma from [9] allows us to refer to m_i or M_i as the ith *modular matching*.

Lemma 2.1. *[9] For $i \in [k+1]$, M_i is a matching in $\mathbf{B}_k$ and $\{M_1, M_2, \ldots, M_{k+1}\}$ is a 1-factorization of $\mathbf{B}_k$.*

Proof. To see that m_i is a matching, we find a rule $b_i : \mathbf{L}_{k+1} \to \mathbf{L}_k$ such that $b_i \circ m_i$ is the identity map on $\mathbf{L}_k$. Define b_i by $b_i(B) = B - \{b_x\}$, where

$$x \equiv (i + \sum B)(\text{mod } k+1). \qquad (2.2)$$

We see that b_i removes the xth smallest element of B. Suppose that $A \in \mathbf{L}_k$ with the notation that precedes (2.1). Then $m_i(A) = A \cup \{\bar{a}_y\}$ and the following sequence of statements holds:

there are $(2k+1) - \bar{a}_y$ elements of $[2k+1]$ larger than $\bar{a}_y$;

$y-1$ of these elements are in $\bar{A}$;

$(2k+1) - \bar{a}_y - (y-1)$ are in A;

$$k - ((2k+2) - (\bar{a}_y + y)) \text{ elements of } A \text{ are less than } \bar{a}_y.$$

Computing modulo $k + 1$, the following holds:

$$k - ((2k+2) - (\bar{a}_y + y)) \equiv k + \bar{a}_y + y \pmod{k+1}$$

$$\equiv k + \bar{a}_y + (i + \sum A) \pmod{k+1}$$

$$\equiv -1 + i + \sum (A \cup \{\bar{a}_y\}) \pmod{k+1}.$$

But this means that $\bar{a}_y$ is the $i + \sum(A \cup \bar{a}_y) \pmod{k+1}$ smallest element of $(A \cup \{\bar{a}_y\})$. It follows that $b_i(m_i(A)) = b_i(A \cup \{\bar{a}_y\}) = A$.

It is clear from (2.1) that for distinct indices i, j in $[k + 1]$ and for all $A \in \mathbf{L}_k$, $m_i(A) \neq m_j(A)$. Since $\mathbf{B}_k$ is a $(k + 1)$-regular graph, it follows that $\{M_1, M_2, \ldots, M_{k+1}\}$ is a 1-factorization of $\mathbf{B}_k$. $\qquad\square$

For instance, with $k = 7$, $i = 3$ and $A = \{1, 2, 4, 5, 8, 9, 11\}$, $m_i(A) = A \cup \{13\}$. Since $y \equiv (3 + \sum A) \equiv 3 \bmod 8$ and 13 is the 3rd largest element of $\bar{A}$, then m_i adds 13 to the set A. And with $B = A \cup \{13\}$, $b_i(B) = B - \{13\}$. Since $x \equiv (3 + \sum B) \equiv 8 \bmod 8$ and 13 is the 8th smallest element of B, then b_i removes 13 from the set B.

Let us see that the modular matchings are invariant under rotation or shift. We let $\sigma = (1 \ 2 \ldots 2k + 1) \in S_n$ and refer to σ as a rotation or shift and say that σ^j is a rotation or shift by j, $j = 1, 2, \ldots, 2k + 1$. Also, for any $\tau \in S_{2n+1}$, and for any $A \subseteq [2k + 1]$, let $\tau[A] = \{\tau(a) \mid a \in A\}$.

Lemma 2.2. *[9] For* $i = 1, 2, \ldots, k+1$ *and for all* $A \in \mathbf{L}_k$, $m_i(\sigma[A]) = \sigma[m_i(A)]$.

Proof. Let $A = \langle a_1, a_2, \ldots, a_k \rangle$, set $y \equiv i + \sum A \pmod{k+1}$, so $m_i(A) = A \cup \{\bar{a}_y\}$. First suppose that $2k+1 \notin A$. Then $\sum \sigma[A] \equiv \sum A - 1 \pmod{k+1}$, so m_i adds the $y - 1^{\text{st}}$ largest element of $\overline{\sigma[A]}$. Moreover, $\bar{a}_y + 1 = \sigma(\bar{a}_y)$ is the $y - 1^{\text{st}}$ largest element of $\sigma[A]$. Thus $\sigma[m_i(A)] = m_i(\sigma[A])$.

Now suppose that $2k + 1 \in A$. Then $\sum \sigma[A] \equiv \sum A \pmod{k+1}$. In this case m_i adds the yth largest element of their complements to both A and $\sigma[A]$, respectively. These are the elements $\bar{a}_y$ and $\bar{a}_y + 1 = \sigma(\bar{a}_y)$, thus, $\sigma[m_i(A)] = m_i(\sigma[A])$. $\square$

Lemma 2.3. *Let C be the cycle in the 2-factor $m_i \cup m_j$ of $\mathbf{B}_k$ containing A with $|A| = k$. Then $d_C(A, \sigma[A]) = d_C(\sigma^t[A], \sigma^{t+1}[A])$, where $t \in [2k]$ and d_C denotes the distance on C.*

The proof of Lemma 2.3 follows immediately from Lemma 2.2.

2.2   Pairwise unions of modular matchings

In [13], Horak *et al.* observed that the 2-factor in $\mathbf{B}_k$ defined as the union of modular matchings M_i and M_{i+1} must contain short cycles, and, so, can-

not give a Hamilton cycle in $\mathbf{B}_k$. Their approach is generalized to give the following result.

Theorem 2.4. *For $k > 2$, the union of any two modular matchings in $\mathbf{B}_k$ is not a Hamiltonian cycle.*

Before we proceed to the proof of the theorem, let us investigate the relationship between b_i, b_j and m_i, m_j. For the remainder of this section, we write $u \equiv v$ to mean $u \equiv v \pmod{k+1}$. Suppose $|B| = k + 1$, $B = \langle b_1, b_2, ..., b_{k+1} \rangle$ and $b_i(B) = B - \{b_x\}$. Then b_x is the xth smallest element of B, where $x \equiv i + \sum B$. Suppose $b_j(B) = B - \{b_z\}$. Then b_z is the zth smallest element of B, where $z \equiv j + \sum B$. Now the question of interest becomes: what is the relationship between b_x and b_z? Notice $z \equiv j + \sum B \equiv (j - i) + i + \sum B$. Thus b_z is $(x + (j - i))^{\text{th}}$ smallest element of B. Therefore $b_z = b_{x+j-i}$. Thus the difference between the element that b_i removes from B and the element that b_j removes from B is a move of $(j - i)$ positions to the right in B with the possibility for a wraparound.

There is a similar relationship between the element that m_i adds to A and the element that m_j adds to A. Suppose $|A| = k$, $A = \langle a_1, a_2, ..., a_k \rangle$ and $m_i(A) = A \cup \{\bar{a}_y\}$. Then $\bar{a}_y$ is the yth largest element of $\bar{A}$, where $y \equiv i + \sum B$. Suppose $m_j(A) = A \cup \{\bar{a}_w\}$. Then $\bar{a}_w$ is the wth largest

element of A, where $w \equiv j + \sum A$. Then $w \equiv j + \sum A \equiv ((j-i)+i) + \sum A$.

Thus $\bar{a}_w$ is $(y + (j-i))^{\text{th}}$ largest element of A. Therefore $\bar{a}_w = \bar{a}_{y+j-i}$. Thus

the difference between the element that m_i adds to A and the element that

m_j adds to A is a move of $(j-i)$ positions to the right in $\bar{A}$ from $\bar{a}_y$ to $\bar{a}_w$

with the possibility for a wraparound. We now proceed with the proof of

Theorem 3.4.

Proof. In order to prove the theorem, it suffices to show that the cycle in the

2-factor $M_i \cup M_j$ containing a particular set A is never a Hamiltonian cycle.

Let M_i and M_j be any two distinct modular matchings in $\mathbf{B}_k$. Assume $k > 1$.

Without loss of generality, suppose $i < j$. Let C be the cycle in the 2- factor

which contains $A = [k]$. Let $d = j - i$. Suppose that $m_i(A) = A \cup \{\bar{a}_y\}$ and

consider the following two cases.

   Case 1: Suppose $\bar{a}_y > k + d$. As defined in Section 1.1, let d_C be the

distance function, restricted to the cycle C. We will show that the cycle

containing A is not Hamiltonian. Below we follow the path beginning with

A and following the edges of matching M_i then M_j.

$$A = [k],$$

$$m_i(A) = [k] \cup \{\bar{a}_y\},$$

$$b_j(B) = A_1 = [1, d-1] \cup [d+1, k] \cup \{\bar{a}_y\},$$

$$m_i(A_1) = B_1 = [1, d-1] \cup [d+1, k] \cup \{k+d, \bar{a}_y\},$$

$$b_j(B_1) = A_2 = [1, d-2] \cup [d+1, k] \cup \{k+d, \bar{a}_y\},$$

$$m_i(A_2) = B_2 = [1, d-2] \cup [d+1, k] \cup \{k+d-1, k+d, \bar{a}_y\},$$

$$\vdots$$

$$b_j(B_{d-2}) = A_{d-1} = \{1\} \cup [d+1, k] \cup [k+d-(d-1)+2, k+d] \cup \{\bar{a}_y\},$$

$$m_i(A_{d-1}) = B_{d-1} = \{1\} \cup [d+1, k] \cup [k+2, k+d] \cup \{\bar{a}_y\},$$

$$b_j(B_{d-1}) = A_d = [d+1, k] \cup [k+2, k+d] \cup \{\bar{a}_y\},$$

$$m_i(A_d) = B_d = [d+1, k+d] \cup \{\bar{a}_y\},$$

$$b_j(B_d) = A_{d+1} = [d+1, k+d].$$

To see that b_j removes $d = j - i$ from B, observe that $b_i(B) = B - \{\bar{a}_y\}$, where $\bar{a}_y = b_{k+1}$. Move d elements to the right in B, wrapping around, to select $b_{k+1+d} = d$, which b_j removes. To see that m_i adds $k+d$ to A_1, observe

that $m_j(A_i) = A_1 \cup \{d\}$, where $d = \bar{a}_1$. Now move d elements to the right

in $\bar{A}$ to select $\bar{a}_{1+d} = k + d$, which m_i adds. Now, the subsequent additions

and deletions follow the same pattern, producing the sequence of sets given

above.

Notice that $A_{d+1} = \sigma^d[A]$ and $d_C(A, \sigma^d[A]) = 2(d+1)$. We know that

$d_C(\sigma^d[A], \sigma^{2d}[A]) = d_C(\sigma^{kd}[A], \sigma^{(k+1)d}[A]) = 2(d+1)$ by Lemma 2.3. If each

of the $2k + 1$ shifts of A are on C, $|C| = 2(2k+1)(d+1)$. Since it is

not necessarily the case that each of the $2k + 1$ shifts of A are on C, then

$|C| \leq 2(2k+1)(d+1)$.

Case 2: Suppose $k + 1 \leq \bar{a}_y \leq k + d$. Then the path beginning at A and

following the edges of M_i then M_j is given below. The sequence of insertions

and deletions of elements follow the same pattern as in Case 1.

$$A = [k],$$

$$m_i(A) = B = [k] \cup \{\bar{a}_y\},$$

$$b_j(B) = A_1 = [1, d-1] \cup [d+1, k] \cup \{\bar{a}_y\},$$

$$m_i(A_1) = B_1 = [1, d-1] \cup [d+1, k] \cup \{\bar{a}_y, k+d+1\},$$

$$b_j(B_1) = A_2 = [1, d-1] \cup [d+2, k] \cup \{\bar{a}_y, k+d+1\},$$

$$m_i(A_2) = B_2 = [1, d-1] \cup [d+2, k] \cup \{\bar{a}_y, k+d+1, k+d+2\},$$

$$\vdots$$

$$b_j(B_{d-2}) = A_{k-d} = [1, d-1] \cup \{k\} \cup \{\bar{a}_y\} \cup [k+d+1, 2k-1],$$

$$m_i(A_{d-1}) = B_{k-d} = [1, d-1] \cup \{k\} \cup \{\bar{a}_y\} \cup [k+d+1, 2k],$$

$$b_j(B_{d-1}) = A_{k-d+1} = [1, d-1] \cup \{\bar{a}_y\} \cup [k+d+1, 2k]$$

$$m_i(A_d) = B_{k-d+1} = [1, d-1] \cup \{\bar{a}_y\} \cup [k+d+1, 2k+1],$$

$$b_j(B_d) = A_{k-d+2} = [1, d-1] \cup [k+d+1, 2k+1],$$

Since $A_{k-d+2} = \sigma^{k+d}[A]$, $d_C(A, \sigma^{k+d}[A]) = 2(k-d+2)$. Since all of the $2k+1$ shifts of A could lie on C, $|C| \leq 2(2k+1)(k-d+2)$. In both cases, we see that $|C| \leq 2(2k+1)(k+1) < 2\binom{2k+1}{k}$. Therefore for all $1 \leq i < j \leq k+1$, $M_i \cup M_j$ is not a Hamiltonian cycle in $\mathbf{B}_k$. $\square$

2.3 3-factors of modular matchings

In contrast to the fact that the 2-factor $M_i \cup M_{i+1}$ is disconnected and contains short cycles, Horak *et al.* [13] showed that the union of three consecutive modular matchings is connected.

Lemma 2.5. *[13] For any $A \in \mathbf{L}_k$ with $A \neq [k]$ and $i \in [k+1]$, the spanning subgraph of $\mathbf{B}_k$ formed by the edges in $M_i \cup M_{i+1} \cup M_{i+2}$ contains a path P that starts at A and ends in a set $B \in \mathbf{L}_k$ of smaller weight.*

Proof. Let $A = \langle a_1, a_2, ..., a_k \rangle$. Suppose $m_{i+1}(A) = B = A \cup \{\bar{a}_y\}$ and assume that there is some element of A that is larger than $\bar{a}_y$. Thus $B = \langle a_1, a_2, \ldots, \bar{a}_y, \ldots, a_k \rangle$. Now relabel $B = \langle x_1, x_2, \ldots, x_{k+1} \rangle$. Notice that $\bar{a}_y \neq x_{k+1} = a_k$. Thus $\bar{a}_y = x_l$ for some $l \in [k]$. Then by definition $b_{i+1}(B) = B - \{\bar{a}_y\} = B - \{x_l\}$. So, b_{i+1} removes the lth smallest element of B. Since b_{i+1} removes the lth smallest element of B, then b_{i+2} must remove the $(l+1)^{\text{st}}$ smallest element of B. Now let $C = b_{i+2}(m_{i+1}(A))$. Then $C = b_{i+2}(B) = B - \{x_{l+1}\}$. Then

$$\sum C = \sum B - \{x_{l+1}\} < \sum B - \{x_l\}$$
$$= \sum B - \{\bar{a}_y\}$$
$$= \sum A.$$

Thus the weight of C is less than that of A. Therefore the path that starts at A and follows first the edge of M_{i+1} and then the edge of M_{i+2} has the required property.

We may assume that $\bar{a}_y$ is the largest element of $B = A \cup \{\bar{a}_y\}$. Let s and z be the first and the last element of the last interval of A preceding $\bar{a}_y$. Clearly, $s < \bar{a}_y$. Furthermore, $s > 1$ since $A \neq [k]$. We proceed by considering two cases based on the length of the last interval of A preceding $\bar{a}_y$.

First suppose the length of the last interval is 1 (i.e., $s = z$ and $s + 1 \notin A$). Thus $m_{i+1}(A) = B = A \cup \{\bar{a}_y\} = \langle a_1, a_2, ..., s, \bar{a}_y \rangle$. Since m_{i+1} adds element $\bar{a}_y$ to A, then b_{i+1} must remove $\bar{a}_y$ from B. So, b_{i+1} removes the $k + 1^{\text{st}}$ smallest element of B. Therefore b_i must remove the kth smallest element of B, namely s. So, effectively $b_i \circ m_{i+1}$ trades s for $\bar{a}_y$.

Set $D = b_i(B) = B - \{s\} = \langle a_1, a_2, ..., a_{k-2}, \bar{a}_y \rangle$. Consider $b_{i+2} \circ m_{i+1}(D)$. From above, we know that m_i adds s to D. Therefore

$$m_{i+1}(D) = D \cup \{s - 1\} = \langle a_1, a_2, ..., s - 1, \bar{a}_y \rangle.$$

Since m_{i+1} adds $s-1$ to D, then b_{i+2} must remove $\bar{a}_y$. Let $E = b_{i+2} \circ m_{i+1}(D)$. Thus, E differs from A in that it has $s - 1$ in place of s. Therefore $\sum E < \sum A$. So, the path starting at A and following the edges of the matchings

M_{i+1}, M_i, M_{i+1}, M_{i+2} in order gives the desired path.

Now suppose that the length of the last interval before $\bar{a}_y$ is more than 1. Then $A = \langle a_1, a_2, ..., s, ..., z \rangle$ and the definition of the matchings give:

$$m_{i+1}(A) = B = \langle a_1, a_2, ..., s, s+1, ..., z, \bar{a}_y \rangle,$$

$$D = b_i(B) = \langle a_1, a_2, ..., s, s+1, ..., z-1, \bar{a}_y \rangle,$$

$$m_{i+1}(D) = \langle a_1, a_2, ..., s-1, s, s+1, ..., z-1, \bar{a}_y \rangle,$$

$$E = b_i \circ m_{i+1}(D) = \langle a_1, a_2, ..., s-1, s+1, ..., z-1, \bar{a}_y \rangle,$$

$$m_{i+1}(E) = \langle a_1, a_2, ..., s-1, s+1, ..., z-1, z, \bar{a}_y \rangle,$$

$$F = b_{i+2} \circ m_{i+1}(E) = \langle a_1, a_2, ..., s-1, s+1, ..., z \rangle.$$

From the list of sets above, we see that F differs from A in that it has $s-1$ in place of s. Therefore $\sum F < \sum A$. So, the path starting at A and following the edges of the matchings M_{i+1}, M_i, M_{i+1}, M_{i+2}, M_{i+1}, M_{i+2} in order gives the desired path.

$\square$

The above lemma shows that every set other than $[k]$ is connected to a set of smaller weight. In other words, it shows that every set is connected to $[k]$ and so, the subgraph of $\mathbf{B}_k$ induced by M_i, M_{i+1}, and M_{i+2} is connected.

Theorem 2.6. *[13] For $i \in [k+1]$, the union of the matchings M_i, M_{i+1}, and M_{i+2} is a connected spanning cubic subgraph of $\mathbf{B}_k$.*

In [13], it is shown that the union of three consecutive modular matchings induces a 3-connected spanning cubic subgraph of $\mathbf{B}_k$. In order to show that the union of three consecutive modular matchings is 3-connected, they proved by contradiction that there is no edge-cut of size 1 or 2. They offer two consequences of the fact that the spanning subgraph of $\mathbf{B}_k$ with edge set $M_i \cup M_{i+1} \cup M_{i+2}$ is cubic and 3-connected. Firstly, if it turns out that $\mathbf{B}_k$ is not Hamiltonian, this shows an example of a non-Hamiltonian, cubic, 3-connected graph. Secondly, Paulraja [20] proved that every 3-connected cubic graph G has a Hamiltonian prism. For a graph G, the prism of G is the graph consisting of two copies of G with a 1-factor joining corresponding vertices. The property that G has a Hamiltonian prism is closely related to traceability properties of G, described in [13].

Chapter 3

Lexical Matchings

One of the most well-known combinatorial facts about the full Boolean lattice $\mathbf{P}(n)$ of subsets of $[n]$, ordered by containment, is that $\mathbf{P}(n)$ has a symmetric chain decomposition. A lattice has a symmetric chain decomposition if it can be partitioned into chains of subsets, consecutive with respect to containment, such that each chain begins with a set of size r if and only if it ends with one of size $n - r$. In the case $n = 2k + 1$, the set of "middle edges" of these chains provides a matching in $\mathbf{B}_k$. One particular symmetric chain decomposition of the Boolean lattice has several different descriptions, ranging from the original induction argument of DeBruijn, Tengbergen and Kruyswijk [5] to the explicit bracketing rule provided by Greene and Kleitman [8]. We are interested in the matching in $\mathbf{B}_k$ given by the middle edges of the symmetric chains. In the next section we give a brief outline of these

symmetric chain decompositions and their relation to the *lexical matchings* introduced by Kierstead and Trotter [17].

3.1 Definitions and background results

It was observed in [5] that a product of two chains $C \times D$ has a symmetric chain decomposition. Suppose we are given a product of symmetric chain orders $P \times Q$. Then any product of chains consisting of one chain from the symmetric chain decomposition of P and one chain from Q is a symmetric interval in $P \times Q$. The set of all these symmetric intervals partitions $P \times Q$. Induction gives a symmetric chain decomposition of a product of $C_1 \times C_2 \times \ldots \times C_n$ of chains [5]. Aigner [1] showed that this symmetric chain partition on chain products has other descriptions in terms of a lexicographic order and a bracketing scheme. Let's specialize to the product of 2-element chains and regard $\mathbf{P}(n)$ as both the set of subsets of $[n]$ ordered by containment and as the product of n 2-element chains, that is, the set of $0, 1$-sequences with the coordinate-wise order.

Given the usual ordering on $[n]$, we order the a level $\mathbf{L}_k$ as follows: we say that A_i precedes A_j in the lexicographical ordering if and only if, $A_i =$

$\{m_1, m_2, \ldots, m_k\}$, with $m_1 < \ldots < m_k$ and $A_j = \{n_1, n_2, \ldots, n_k\}$, with $n_1 < \ldots < n_k$, we have that if t is the minimum subscript such that $m_t \neq n_t$ then $m_t < n_t$. Let A_1, A_2, ..., A_n, $n = \binom{2k+1}{k}$, be the subsets in $\mathbf{L}_k$, ordered lexicographically. Let B_1, B_2, ... B_n be the subsets in $\mathbf{L}_{k+1}$, ordered lexicographically. Now define $M : \mathbf{L}_k \to \mathbf{L}_{k+1}$ inductively by: (i) $M(A_1) = B_1$; (ii) $M(A_2) = B_2$, ..., $M(A_{l-1}) = B_{l-1}$ having been defined, $M(A_l)$ is the smallest B_h in the lexicographic order on the B's which contains A_l and which is not equal to any of $M(A_1)$, $M(A_2)$, ..., $M(A_{l-1})$. Then Aigner [1] observed M is a bijection from $\mathbf{L}_k$ to $\mathbf{L}_{k+1}$ and M agrees with the inductive matching in [5]. M is called the *lexicographic matching*.

In [2], there is a starred exercise that outlines a proof that the lexicographic matching M can be described in terms of a bracketing procedure. But before we proceed to this bracketing version of the lexicographic matching, we need an alternate view of a set S. Let $S \subset [2k+1]$. We can represent S with respect to the usual ordering on $[2k+1]$ as the binary $(2k+1)$-tuple $(a_1, \ldots, a_{2k+1})$, where

$$a_t = \begin{cases} 1 & \text{if } t \in S \\ 0, & \text{otherwise.} \end{cases}$$

We now describe the afore mentioned bracketing procedure which can be found in [16]. Consider the binary representation of a set S. Now starting

from the left, move right. When a zero is encountered, it becomes unmatched, possibly only temporarily. When a one is encountered, it is matched to the rightmost unmatched zero, and this zero is no matched as well. If there are currently no unmatched zeros, then this one is unmatched. We continue in this manner until we reach the end of the sequence. The left most unmatched zero is the position of the element that the lexicographic matching will add to the set S. Duffus, Hanlon, and Roth [8] provide a detailed proof of Aigner's result which makes it not too difficult to see that the bracketing procedure corresponds to the 0-lexical matching of Kierstead and Trotter [17] that is verified in Theorem 3.4.

There are at least two ways to generalize the lexicographic matching to get a family of matchings. One such generalization is to consider the $(2k+1)!$ orderings of $[2k+1]$ and define the lexicographic matching associated with each ordering, then study how these matchings interact with each other. Work in this area was done by Duffus, Sands, and Woodrow in [10]. In [10], they give necessary and sufficient conditions for two lexicographic matchings to be disjoint. Also in [10], they show that for $k > 1$, the union of two lexicographic matchings in $\mathbf{B}_k$ is never a Hamiltonian cycle.

Additionally, the lexicographic matching can be generalized to construct

new matchings called lexical matchings. In order to discuss the lexical matchings, we import the notation used by Kierstead and Trotter in [17]. For subsets R and S of $[n]$, define the *S-split* of R, denoted by R/S, to be $R/S = |R \cap S| - |R \cap \bar{S}|$. For each $x \in \bar{S}$, let $D_S(x)$ denote $\{y \in \bar{S} - \{x\} : [y, x)/S < 0\}$ and $d_S(x)$ denote $|D_S(x)|$. Think of $D_S(x)$ as the set of $y \in \bar{S}$ for which $[y, x)$ is deficient in elements of S.

Lemma 3.1. *[17] Let $S \subset [2k + 1]$ with $|S| = k$. If x and w are distinct elements of $\bar{S}$ then $D_S(w) \subset D_S(x)$ or $D_S(x) \subset D_S(w)$.*

Proof. Notice that

$$[w, x)/S + [x, w)/S = |[w, x) \cap S| - |[w, x) \cap \bar{S}| + |[x, w) \cap S| - |[x, w) \cap \bar{S}|$$

$$= |[2k + 1] \cap S| - |[2k + 1] \cap \bar{S}|$$

$$= [2k + 1]/S = -1.$$

Since $[w, x)/S$ and $[x, w)/S$ are integers, then exactly one of them is negative.

Without loss of generality suppose $[w, x)/S$ is negative. We will show that $D_S(w) \subset D_S(x)$. Since $[w, x)/S$ is negative, by the definition of $D_S(x)$ $w \in D_S(x)$. But note that $w, x \notin D_S(w)$. Now suppose $y \in D_S(w)$. If $y \in (x, w)$ then $[y, x)/S = [y, w)/S + [x, w)/S < 0$ and so $y \in D_S(x)$. Otherwise, $y \in (w, x)$ and $[y, x)/S = [y, w)/S - [x, w)/S < 0$ and $y \in D_S(x)$. ☐

One immediate consequence of Lemma 3.1 is that if x and w are distinct

elements of $\bar{S}$ then $d_S(x)$ and $d_S(w)$ are distinct integers.

Corollary 3.2. *[17] Let $S \subset [2k+1]$ with $|S| = k$. Then d_S is a well-defined*

bijection from $\bar{S}$ to $\{0, k\}$.

Proof. By definition, d_S is bounded between 0 and k, and by Lemma 3.1 the

values are distinct. □

For each $S \subset [2k + 1]$ with $|S| = k$, let e_S be the inverse of d_S. So,

$d_S(e_S(i)) = i$. We can think of $e_S(i)$ as the element of $\bar{S}$ such that there are

i y's in $\bar{S}$ with $[y, e_S(i))/S < 0$.

Now we consider another description of which element of $[2k + 1]$ the i-

lexical matching adds to a set S that will be more convenient for our purposes

due to Duffus, Hanlon and Roth [8]. Let $(a_1, \ldots, a_{2k+1})$ be a binary $(2k+1)$-

cycle containing k 1's and $(k + 1)$ 0's. In order to locate the element x

that the i-lexical matching adds to S, we only need to find position x in the

$(2k + 1)$-cycle representation of S such that $a_x = 0$ and there are exactly i

intervals of the form $[y, x)$, $y \notin S$, for which the number of 0's is greater than

the number of 1's. The i-lexical matching, M_i, with respect to the standard

order on $[2k + 1]$, is defined by $M_i(S) = S \cup \{e_S(i)\}$. In order to see that the

i-lexical matching defines a matching in $\mathbf{B}_k$, let us first consider the following lemma due to Kierstead and Trotter [17].

Lemma 3.3. [17] *If S and T are distinct elements of $\mathbf{L}_k$ such that $S \cup \{x\} = T \cup \{y\}$ for some $x, y \in [2k+1]$, then $D_S(x) \subset D_T(y)$ or $D_T(y) \subset D_S(x)$.*

Proof. Note that $x \in T - S$, $y \in S - T$, and $R/S = R/T$ if $x, y \in \bar{R}$. Thus $(x, y)/T + (y, x)/S = -1$ and so exactly one of $(x, y)/T$ and $(y, x)/S$ is negative. Say $(y, x)/S$ is negative. We show that $D_T(y) \subset D_S(x)$. First choose $z \in (y, x) \cap \bar{S}$ such that $[z, x)/S = -1$. Since $x \in T - S$, we have that

$$[z, x)/T = [z, x)/S + 1 + (x, y)/T \geq 0$$

and $z \in D_S(x) - D_T(y)$. Now suppose that $w \in D_T(y)$. Then $w \neq x$ and $w \neq y$. If $w \in (x, y)$, then

$$[w, x)/S = [w, y)/T + 1 + (y, x)/S < 0$$

and so $w \in D_S(x)$. Otherwise $w \in (y, x)$ and

$$[w, x)/S = [w, y)/T - 1 - (x, y)/T < 0$$

and again $w \in D_S(x)$.

$\square$

Theorem 3.4. *For $i = 0, 1, \ldots, k$, M_i is a matching in $\mathbf{B}_k$.*

Proof. Clearly $S \subset M_i(S)$. Thus it suffices to show that M_i is one-to-one. The fact that M_i is indeed one-to-one follows immediately from Lemma 3.3.

$\square$

In Chapter 2, we saw that the modular matchings are σ-invariant. We will later make use of the fact that the lexical matchings are also σ-invariant.

Lemma 3.5. *For $i = 0, 1, \ldots, k$ and for all $S \in \mathbf{L}_k$, $m_i(\sigma[S]) = \sigma[m_i(S)]$.*

This result follows immediately from the fact that each $S \subset [2k + 1]$ has a $(2k + 1)$-cycle representation.

3.2 Partial results

In this section, we will use the method explored in Section 2.3 to show that for certain choices of i and j, the union of the i-lexical and j-lexical matchings yield a 2-factor which is not a Hamiltonian cycle.

Theorem 3.6. *For $k > 1$. Let $i \in \{0, 1, \ldots, k\}$ and $A = [k]$. Then the cycle containing A in the 2-factor obtained from the union of the i-lexical and $(k - i)$-lexical matchings, $M_i \cup M_{k-i}$, is not a Hamiltonian cycle.*

Proof. Let $i \in \{0, 1, \ldots, k\}$ and $A = [k]$. Let C be the cycle of $M_i \cup M_{k-i}$ which contains A. Without loss of generality assume $i < k - i$.

In order to see that m_i adds $(k + i + 1)$ to A, consider the $(2k + 1)$-cycle representation of A. So, $A = (1, 1, \ldots, 1, 0, 0, \ldots, 0)$ where A has a sequence of k 1's followed by a sequence of $(k + 1)$ 0's. Now consider the 0 in position $(k + i + 1)$. We want to show $d_A(k + i + 1) = i$. In the $(2k + 1)$-cycle representation of A, there is a sequence of i 0's between the kth and $(k + i + 1)^{\text{st}}$ positions in the cycle. Thus, for each $y \in [k + 1, k + i]$, $[y, k+i+1)/A < 0$. Therefore each of the i 0's here contribute to $d_A(k+i+1)$. For each $y \in [k+i+2, 2k+1]$, $[y, k+i+1)$ contains all of the k 1's in A. Since there are only k 0's not including the 0 in position $(k + i + 1)$ in the cycle representation of A, then for each $y \in [k+i+2, 2k+1]$, $[y, k+i+1)/A \geq 0$. Therefore none of these y's contribute to the number of intervals for which the 0's outnumber the 1's. And indeed $m_i(A) = A \cup \{k + i + 1\}$.

In order to see that m_{k-i} removes $(i + 1)$ from B. We want to locate the 1 in the xth position of the $(2k+1)$-cycle representation of B such that there are $k - i$ intervals of the form $[y, x)$, $y \notin B$, for which the number of 0's outnumber the number of 1's. Consider the element $(i + 1)$. By counting the number of intervals of the form $[y, i + 1)$ with $y \in \bar{B}$ such that the 0's

outnumber the 1's, we see that the $(k-i)$-lexical matching removes $(i+1)$ from B.

$$A = [k],$$

$$m_i(A) = B = [k] \cup \{k+i+1\},$$

$$b_{k-i}(B) = A_1 = [i] \cup [i+2, k] \cup \{k+i+1\}.$$

Continuing with this kind of analysis, we see that the path beginning with A and following the edges of matching M_i then M_{k-i} is given below.

$$m_i(A_1) = B_1 = [i] \cup [i+2, k] \cup \{k+i, k+i+1\},$$

$$b_{k-i}(B_1) = A_2 = [i-1] \cup [i+2, k] \cup \{k+i, k+i+1\},$$

$$m_i(A_2) = B_2 = [i-1] \cup [i+2, k] \cup \{k+i-1, k+i, k+i+1\}$$

$$\vdots$$

$$b_{k-i}(B_{i-1}) = A_i = \{1, 2\} \cup [i+2, k] \cup [k+3, , k+i+1],$$

$$m_i(A_{i-1}) = B_{i-1} = \{1,2\} \cup [i+2,k] \cup [k+2, k+i+1],$$

$$b_{k-i}(B_{i-1}) = A_i = \{1\} \cup [i+2,k] \cup [k+2, k+i+1],$$

$$m_i(A_i) = B_i = \{1\} \cup [i+2, k+i+1],$$

$$b_{k-i}(B_i) = A_{i+1} = [i+2, k+i+1],$$

From the above sequence of sets, we see that the i-lexical matching fills in the set $[k+1, k+i+1]$ and the $(k-i)$-lexical matching removes elements from $[i+1]$. Thus, the i-lexical matching must add $(i+1)$ elements before merging the last two intervals and the $(k-i)$-lexical matching must remove $(i+1)$ elements before removing the entire interval $[i+1]$.

Since $A_{i+1} = \sigma^{i+1}[A]$, $d_C(A, \sigma^{i+1}[A]) = 2(i+1)$. Since there are $2k+1$ shifts of A and at most all of them lie on C, $|C| \leq 2(2k+1)(i+1) < 2\binom{2k+1}{k}$. Therefore for $i \in \{0, 1, \ldots, k\}$, $M_i \cup M_{k-i}$ is not a Hamiltonian cycle in $\mathbf{B}_k$. $\square$

Suppose now that M and M' are disjoint matchings in $\mathbf{B}_k$ and that $M \cup M'$ is a 2-factor with c_t cycles of length t ($t \in \mathbb{N}$). Then for all $\phi \in S_{2k+1}$, let $\widehat{\phi}$ be the induced automorphism of $\mathbf{B}_k$. As noted in Section 1.2, $\widehat{\phi}[M \cup M'] =$

$\widehat{\phi}[M] \cup \widehat{\phi}[M']$ is a 2-factor in $\mathbf{B}_k$ with c_t cycles of length t. In particular $M \cup M'$ is a Hamiltonian cycle in $\mathbf{B}_k$ if and only if $\widehat{\phi}[M] \cup \widehat{\phi}[M']$ is a Hamiltonian cycle. Let us use this easy observation with lexical matchings to obtain an easy proof of a result in [17].

Lemma 3.7. *Let* $\rho = (1 \ 2k+1)(2 \ 2k)\ldots(k \ k+2)(k+1) \in S_{2k+1}$. *Then* $\widehat{\rho}(M_i) = M_{k-i}$ *for* $i = 0, 1, \ldots, k$.

Proof. We must show that if $m_i(A) = A \cup \{x\}$ then $m_{k-i}(\widehat{\rho}(A)) = \widehat{\rho}(A) \cup \{\rho(x)\}$. Suppose $m_i(A) = A \cup \{x\}$. Then $d_A(x) = i$. Thus, there are exactly i intervals in the $(2k+1)$- cycle representation of A such that $[y, x)/A < 0$ with $y \notin A$. That is, there are exactly i y's such that on the interval $[y, x)$ the number of 0's is greater than the number of 1's in A. Since there are a total of k y's in A and i of them have the property that the interval $[y, x)$ has more 0's than 1's, then there must be $k - i$ y's in the cycle representation of A such that $[y, x)/A > 0$. For each of these $k - i$ y's with $[y, x)/A > 0$, the interval $[y, x)$ contains more 1's than 0's. For each of these $k - i$ y's, the interval $(x, y]$ contains more 0's than 1's in the cycle representation of A. Notice, that ρ is a reflection across the $(k+1)$ position. So, $\widehat{\rho}((x, y]) = [\rho(y), \rho(x))$. Since ρ does not add or remove elements, the number of 0's in $[\rho(y), \rho(x))$ is still greater than the number of 1's. Thus for each $y \notin A$ such that $(x, y]/A < 0$,

$[\rho(y), \rho(x))/\widehat{\rho}(A) < 0$. By definition $m_{k-i}(\widehat{\rho}(A)) = \widehat{\rho}(A) \cup \{\rho(x)\}$. $\square$

Chapter 4

Future Work

The search for a Hamiltonian cycle in $\mathbf{B}_k$ has proven to be a challenge even for small values of k. The size of $\mathbf{B}_k$ is exponential in k, so calculations for quite small values of k require a considerable amount of computing (see [21]. Although it is true that a Hamiltonian cycle in $\mathbf{B}_k$ must be the union of two disjoint 1-factors, determining when two perfect matchings are disjoint is not always easy and it is certainly not easy to determine the length of the cycles obtained from their union. In this thesis, we saw that the union of two disjoint modular matchings always yields a short cycle and that the union of the i- and $(k - i)$-lexical matchings always contains a short cycle. Below we give a list of other avenues yet to be fully explored in the quest to make progress on the middle levels problem.

1. For k sufficiently large, show that the union of the i- and j-lexical

matchings always contain a short cycle.

2. Investigate the cubic graphs, the 3-factors, obtained from the edge set of 3 distinct lexical matchings.

3. In [17], Kierstead and Trotter created the i-lexical $\mathcal{M}_i$ family by acting on the i-lexical matching with the automorphism group of $\mathbf{B}_k$. They showed that the families of $\mathcal{M}_i$ and $\mathcal{M}_j$ matchings do not intersect for $0 \leq i < j \leq \frac{k}{2}$ and that the size of the family of i-lexical matchings is $(2k)!$, with the only permutations that map an i-lex to an i-lex being one of the $(2k + 1)$ shifts. They pose the following problem: show that no two i-lex matchings can form a Hamiltonian cycle.

4. Regarding modular matchings, investigate the 3-factor obtained from $M_i \cup M_j \cup M_r$ for any i, j, r. Also concerning modular matchings, in [9] the action of S_{2k+1} on the modular matchings is analyzed. They show that the shift invariance of modular matchings and that $\rho(M_i) = M_j$ where $i + j \equiv 1 \pmod{k + 1}$. This is analogous to Lemma 3.7. They ask if it is always true that for any matchings M in any 1-factorization of $\mathbf{B}_k$ into shift-invariant matchings, is $\widehat{\rho}[M]$ also in the factorization?

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