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Hasse Principle for Hermitian Spaces

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Doctor of Philosophy

Mathematics

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Abstract

Hasse Principle for Hermitian Spaces
By Zhengyao Wu

This dissertation proves new results on Hasse principle for Hermitian spaces. Let p be an odd prime. Let F be the function field of a curve over a p-adic field.

In a recent paper, Colliot-Thélène, Parimala and Suresh conjectured that a local-global principle holds for projective homogeneous spaces of connected linear algebraic groups over function fields of p-adic curves for $p \neq 2$. The first main result of this dissertation proves the following: Let A be a finite-dimensional simple F-algebra with an involution σ such that $F = Z(A)^\sigma$. Let $\varepsilon \in \{1, -1\}$ and $h\colon V \times V \to A$ an ε-hermitian space over (A, σ). Let X be a projective homogeneous space under

$$G = \begin{cases} \mathrm{SU}(A,\sigma,h) & \text{if } \sigma \text{ is of the first kind;} \\ \mathrm{U}(A,\sigma,h) & \text{if } \sigma \text{ is of the second kind.} \end{cases}$$

Let Ω be the set of all rank one discrete valuations on F. For each $v \in \Omega$, let F_v be the completion of F at v. Then

$$\prod_{v\in\Omega} X(F_v) \neq \emptyset \implies X(F) \neq \emptyset.$$

The proof implements patching techniques of Harbater, Hartmann and Krashen. As an application, we obtain a Springer-type theorem for isotropy of hermitian forms over odd degree extensions of function fields of p-adic curves.

Parihar and Suresh provided upper bounds for the u-invariant of hermitian spaces over division algebras over function fields of p-adic curves for $p \neq 2$. It was an open problem what their exact values are. The second main result of this dissertation proves the following: Let D be a central division algebra over F.

(1) If D is quaternion, then $u^+(D) = 6$ and $u^-(D) = 2$.

(2) Let L/F be a quadratic extension. If D is quaternion and $D \otimes_F L$ is division, then $u^0(D \otimes_F L) = 4$.

(3) If D is biquaternion, then $u^+(D) = 5$ and $u^-(D) = 3$.

The proof implements Larmour's theorem on Hermitian spaces over division algebras over complete discrete valued fields.

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Contents

CHAPTER 1

Generalities

1.1. Central simple algebras and Brauer groups

We refer readers to [GS06; Bou$_{A8}$; Gro68a; Gro68b; Gro68c] for details of central simple algebras and Brauer groups. Let K be a field. Let K_{alg} be the algebraic closure of K. Let K_{sep} be the separable closure of K in K_{alg}. The *absolute Galois group* of K is defined to be $\mathrm{Gal}(K_{\mathrm{sep}}/K) = \mathrm{Aut}(K_{\mathrm{alg}}/K)$.

Let A be a finite-dimensional associative unital algebra over K. Let $Z(A)$ be the center of A. We say that A is *central* if $Z(A) = K$. We say that A is *simple* if it has only 2 two-sided ideals $\{0\}$ and A. Every central division algebra D over K is a central simple algebra over K. Further, the matrix algebra $M_n(D)$ is a central simple algebra over K. By Wedderburn's theorem every central simple algebra A over K is of the form $A \simeq M_n(D)$ for a positive integer n and a central division K-algebra D. Here D is called the underlying division algebra of A. If $A \simeq M_n(K)$, we say that A *splits* over K.

Two central simple algebras are *Brauer equivalent* if they have isomorphic underlying division algebras. Let $[A]$ be the Brauer equivalence class of a central simple algebra A over K. The *Brauer group* $\mathrm{Br}(K)$ [GS06, Def. 2.4.9] is an abelian group with underlying set $\{[A] \mid A \text{ is a central simple algebra over } K\}$, the associative and commutative addition $[A]+[B] = [A \otimes_K B]$ for all pairs of central simple algebras A and B over K, the identity element $0 = [K] = [M_n(K)]$ and the inverse $-[A] = [A^{\mathrm{op}}]$ for all central simple algebra A over K, where A^{op} is the opposite algebra of A. We write ${}_n\mathrm{Br}(K)$ for the n-torsion subgroup of $\mathrm{Br}(K)$.

Let A be a central simple algebra over a field K. The *dimension* of A is a square. The *degree* of A is defined to be $\deg(A) = \sqrt{\dim_K A}$. The *index* of A is defined to be $\operatorname{ind}(A) = \deg(D)$, where D is the underlying division algebra of A over K. The *period* (or *exponent*) of A is defind to be the order of $[A]$ in $\operatorname{Br}(K)$ and is denoted by $\operatorname{per}(A)$. A theorem of Brauer [GS06, Prop. 4.5.13] says that $\operatorname{per}(A) | \operatorname{ind}(A)$ and they have the same prime factors.

Example 1.1.1. Let K be a field of characteristic not 2. Suppose $a, b \in K^*$. Let $(a, b)_K$ denote the *quaternion algebra* over F generated by $\{1, i, j, ij\}$ with relations $i^2 = a$, $j^2 = b$, $ij = -ji$. Every quaternion algebra is a central simple algebra of degree 2, period 1 or 2 and index 1 or 2.

Cyclic algebras and cross product algebras are other important examples of central simple algebras.

Example 1.1.2. A field K is *quasi-finite* if it is perfect and there exists $s \in \operatorname{Gal}(K_{\text{sep}}/K)$ and an isomorphism $\widehat{\mathbb{Z}} \to \operatorname{Gal}(K_{\text{sep}}/K)$ given by $1 \mapsto s$. By [Ser79, XIII, §2, Prop. 5], if K is a quasi-finite field, then $\operatorname{Br}(K)$ is trivial. By [Ser79, XIII, §2, Prop. 3], if L is a fintie field extension of K, then L is a quasi-finite field and hence $\operatorname{Br}(L)$ is trivial.

For example, $\mathbb{F}_q$ and $\mathbb{C}((t))$ are quasi-finite fields.

Let R be a commutative ring. Let A be an R-algebra. We say that A is an *Azumaya algebra* over R if $Z(A) = R$ and A is a projective left module over $A \otimes A^{\text{op}}$. By [AG60a, Th. 2.1], an algebra A over a field K is a central simple algebra if and only if A is an Azumaya algebra over K. Two Azumaya algebras A_1 and A_2 are *Brauer equivalent* if there exists finitely generated faithful projective modules P_1 and P_2 over R such that $A_1 \otimes_R \operatorname{End}_R(P_1) \simeq A_2 \otimes_R \operatorname{End}_R(P_2)$. Let $[A]$ be the Brauer equivalence class of an Azumaya A over R. The *Brauer group* $\operatorname{Br}(R)$ [AG60a, p. 368] is an abelian group with underlying set $\{[A] \mid A \text{ is an Azumaya algebra over } R\}$, the associative and commutative multiplication $[A] + [B] = [A \otimes_R B]$ for all pairs of Azumaya algebras A and B over K, the identity element $0 = [R] = [\operatorname{End}_R(P)]$ where

P is a finitely generated faithful projective module over R, the inverse $-[A] = [A^{\mathrm{op}}]$ for all Azumaya algebra A over R, where A^{op} is the opposite algebra of A.

The following result will be used in the proof of our main result theorem 2.3.6.

Proposition 1.1.3. [AG60a, Cor. 6.2]. Let R be a complete local ring with residue field k. Then the canonical quotient map induces an isomorphism $\mathrm{Br}(R) \simeq \mathrm{Br}(k)$.

Let A be a ring. A map $\sigma\colon A \to A$ is called an *involution* if $\sigma(x+y) = \sigma(x)+\sigma(y)$, $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in A$.

Let A be a central simple algebra over a field K. Let $K^\sigma = \{x \in K \mid \sigma(x) = x\}$. An involution σ on A is *of the first kind* if $[K : K^\sigma] = 1$; it is *of the second kind* if $[K : K^\sigma] = 2$. Let $A^\sigma = \{x \in A \mid \sigma(x) = x\}$ and let $d = \deg(A)$. An involution σ on A is *orthogonal* if it is of the first kind and $\dim_K(A^\sigma) = \dfrac{d(d+1)}{2}$; it is *symplectic* if it is of the first kind and $\dim_K(A^\sigma) = \dfrac{d(d-1)}{2}$; it is *unitary* if it is of the second kind (i.e. $\dim_K(A^\sigma) = d^2$).

Remark 1.1.4. If A is a central simple algebra over a field K with an involution σ of the first kind, then $\mathrm{per}(A) = 2$ and hence $[A] \in {}_2\mathrm{Br}(K)$. The reason is that σ defines an isomorphism $A \simeq A^{\mathrm{op}}$.

Example 1.1.5. Let $A = (a, b)_K$ be a quaternion algebra as in example 1.1.1. Let σ be a K-linear map on A given by $\sigma(i) = -i$ and $\sigma(j) = -j$. Then σ is a symplectic involution and it is called the *canonical* involution on A. Let τ be a unitary involution on A. Suppose $k = K^\tau$ and $K = k(\sqrt{\lambda})$ for some $\lambda \in k^* \setminus k^{*2}$. Let ι be the nontrivial automorphism of K over k such that $\iota(\sqrt{\lambda}) = -\sqrt{\lambda}$. By a theorem of Albert [KMRT98, Prop. 2.22], $A \simeq A_0 \otimes_k K$ for some quaternion algebra A_0 over k and $\tau \simeq \sigma_0 \otimes \iota$ where σ_0 is the canonical involution on A_0.

1.2. Hermitian spaces and Witt groups

We refer readers to [Sch85; Knu91; Bou$_{\mathrm{A9}}$] for details of Hermitian forms and Witt groups. Let K be a field of characteristic not 2. Let A be a central simple algebra over K. Let V be finitely generated right A-module. Suppose $A \simeq M_m(D)$ for a central division algebra D over K. Then $V \simeq (D^m)^s$ for an integer $s \geq 0$. Then $\dim_K(V) = sm \dim_K(D) = s \deg(A) \operatorname{ind}(A)$. The *reduced dimension* [KMRT98, Def. 1.9] of V over A is defined to be $\operatorname{rdim}_A(V) = \dim_K(V)/\deg(A) = s \operatorname{ind}(A)$.

Let σ be an involution on A such that $K^\sigma = k$ Suppose $\varepsilon \in \{1, -1\}$. A map $h\colon V \times V \to A$ is an ε*-hermitian form* over (A, σ) if $h(x_1+x_2, y) = h(x_1, y)+h(x_2, y)$, $h(x, y_1+y_2) = h(x, y_1)+h(x, y_2)$ for all $x, x_1, x_2, y, y_1, y_2 \in V$; $h(xa, yb) = \sigma(a)h(x, y)b$ for all $a, b \in A$, $x, y \in V$; $h(y, x) = \varepsilon\sigma(h(x, y))$ for all $x, y \in V$. If $\varepsilon = 1$, h is called a *hermitian* form; if $\varepsilon = -1$, h is called a *skew-hermitian* form.

Let $V^* = \operatorname{Hom}_A(V, A)$. Then V^* has a right A-module structrue given by

$$(f * a)(x) = \sigma(a)f(x) \text{ for all } f \in V^*,\ a \in A \text{ and } x \in V.$$

Then h gives a right A-module homomorphism $\widetilde{h}\colon V \to V^*$ such that $\widetilde{h}(x)(y) = h(x, y)$ for all $x, y \in V$. We say that h is an ε-hermitian *space* if $\widetilde{h}$ is an isomorphism. Let $E = \operatorname{End}_A(V)$ and let $\tau = \operatorname{ad}_h$ be the *adjoint involution* of h, i.e. $h(x, f(y)) = h(\tau(f)(x), y)$ for all $f \in E$ and $x, y \in V$.

The *rank* of h is defined to be

$$\operatorname{Rank}(h) = \frac{\dim_K(V)}{\deg(A)\operatorname{ind}(A)} = \frac{\operatorname{rdim}_A(V)}{\operatorname{ind}(A)} = s.$$

Let K be a field of characteristic not 2. Let D is be a division algebra over K with an involution σ. Let V be a finite dimensional right vector space over D. Then $V \simeq D^n$. Let h be an ε-hermitian space over (D, σ). There exists $a_1, \ldots, a_n \in D^*$ such that $\sigma(a_i) = \varepsilon a_i$ and $h(x, y) = \sigma(x_1)a_1y_1+\cdots+\sigma(x_n)a_ny_n$ for all $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in V$. We simply write $h \simeq \langle a_1, \ldots, a_n\rangle$ and hence $\operatorname{Rank}(h) = \dim_D(V) = n$.

Suppose $\operatorname{rdim}(V) = 2r$ and ad_h is orthogonal. The *determinant* of h is $\det(h) = \operatorname{Nrd}_{\operatorname{End}_A(V)/K}(f) \in K^*/K^{*2}$ for $f \in \operatorname{End}_A(V)$ such that $\operatorname{ad}_h(f) = -f$. By [KMRT98, Prop. 7.1], the definition is independent of the choice of f. The *discriminant* of h is defined to be $\operatorname{disc}(h) = (-1)^r \det(h)$. In particular, if $A = D$ is division, ad_h is orthogonal and $h \simeq \langle a_1, \ldots, a_{2m}\rangle$, then $r = m \deg(D)$, $\det(h) = \mathrm{Nrd}_{D/K}(a_1 a_2 \cdots a_{2m}) \in K^*/K^{*2}$ and $\operatorname{disc}(h) = (-1)^{m\deg(D)} \mathrm{Nrd}_{D/K}(a_1 a_2 \cdots a_{2m}) \in K^*/K^{*2}$. The proof is similar to [KMRT98, Prop. 7.3(c)].

Example 1.2.1. If $A = K$, $\sigma = \mathrm{Id}_K$ and $\varepsilon = 1$, then a hermitian form h is a symmertric bilinear form and $q_h(x) = h(x, x)$ for all $x \in V$ is a quadratic form, i.e. a homogeneous map $V \to K$ of degree 2.

Conversely, let $q\colon V \to K$ be any quadratic form. Its has an associated symmetric bilinear form $b_q(x, y) = \frac{1}{2}(q(x+y)-q(x)-q(y))$ for all $x, y \in V$. Then b_q is a hermitian form over (K, Id_K).

An ε-hermitian space h over (A, σ) is called *isotropic* if there exists $x \neq 0$, $x \in V$ such that $h(x, x) = 0$; otherwise h is called *anisotropic*. A right sub-F-module W of V is called a *totally isotropic subspace* if $h(x, y) = 0$ for all $x \in W$. Let E be a central simple algebra over K with an involution τ. We say that τ is *isotropic* if there exists $f \neq 0$, $f \in E$ such that $\tau(f)f = 0$; otherwise τ is called *anisotropic*. A right ideal I of E is called a *totally isotropic ideal* if $\tau(f)g = 0$ for all $f, g \in E$. Let $E = \mathrm{End}_D(V)$ and let $\tau = \mathrm{ad}_h$ be the adjoint involution of h. Then h is isotropic if and only if ad_h is isotropic. When $A = D$ is division, W is a totally isotropic subspace of V if and only if $I = \mathrm{Hom}_D(V, W)$ is a totally isotropic ideal of E [see KMRT98, Prop. 6.2]. Here

$$\mathrm{rdim}_D(W) = \frac{\dim_K(W)}{\deg(D)} = \frac{\dim_K(W)\cdot\dim_K(V)}{\deg(D)\cdot\dim_K(V)} = \frac{\dim_K(I)}{\deg(E)} = \mathrm{rdim}_E(I).$$

Example 1.2.2. [Knu91, Ch. 1, 3.5]. Let A be a central simple algebra over a field K. Let σ be an involution on A. Let V be a finitely generated right A-module. Let $(V \oplus V^*, \mathbb{H})$ be an ε-hermitian space over (D, σ) defined by

$$\mathbb{H}((x, f), (y, g)) = f(y) + \varepsilon\sigma(g(y))$$

for all $x, y \in V$ and $f, g \in V^*$. Then $\mathbb{H}$ has totally isotropic subspaces $V \oplus 0$ and $0 \oplus V^*$. The space $(V \oplus V^*, \mathbb{H})$ is called the *hyperbolic plane* of V.

Let $\mathrm{Herm}^{\varepsilon}(A, \sigma)$ denote the category of ε-hermitian spaces over (A, σ). The Hermitian u-invariant [Mah05, Def. 2.1] of (A, σ, ε) is defined to be:

$$u(A, \sigma, \varepsilon) = \sup\{n | \text{there exists an anisotropic } h \in \mathrm{Herm}^{\varepsilon}(A, \sigma), \mathrm{Rank}(h) = n.\}$$

Suppose that σ and τ are involutions on A. Mahmoudi has proved that [Mah05, Prop. 2.2] if σ and τ are of the same type, then $u(A, \sigma, \varepsilon) = u(A, \tau, \varepsilon)$; if σ is orthogonal and τ is symplectic, then $u(A, \sigma, \varepsilon) = u(A, \tau, -\varepsilon)$; if σ is unitary, then $u(A, \sigma, 1) = u(A, \sigma, -1)$. Thus we have only three types of Hermitian u-invariants [Mah05, Rem. 2.3], we denote:

$$u(A, \sigma, \varepsilon) = \begin{cases} u^+(A), & \text{if } \varepsilon = 1 \text{ and } \sigma \text{ is orthogonal,} \\ & \text{or, } \varepsilon = -1 \text{ and } \sigma \text{ is symplectic;} \\ u^-(A), & \text{if } \varepsilon = -1 \text{ and } \sigma \text{ is orthogonal,} \\ & \text{or, } \varepsilon = 1 \text{ and } \sigma \text{ is symplectic;} \\ u^0(A), & \text{if } \sigma \text{ is unitary .} \end{cases}$$

where u^+ is called the *orthogonal* Hermitian u-invariant, u^- is called the *symplectic* Hermitian u-invariant and u^0 is called the *unitary* Hermitian u-invariant.

Let A be a central simple algebra over a field K. Let σ be an involution on A. Let $\varepsilon \in \{1, -1\}$. Suppose (V_1, h_1) and (V_2, h_2) are two ε-hermitian spaces over (A, σ), their *orthogonal sum* $(V_1 \oplus V_2, h_1 \perp h_2)$ is defined to be

$$(h_1 \perp h_2)((x_1, x_2), (y_1, y_2)) = h_1(x_1, y_1) + h_2(x_2, y_2)$$

for all $x_1, x_2, y_1, y_2 \in V$. Isomorphism classes of ε-hermitian spaces over (A, σ) with respect to $\perp$ form an abelian monoid. The Grothendieck group $\mathrm{KU}^{\varepsilon}(A, \sigma)$ of this abelian monoid is an abelian group. Orthogonal sums of hyperbolic planes are called *hyperbolic spaces.* Then *Witt group* $W^{\varepsilon}(A,\sigma)$ is the quotient of $\mathrm{KU}^{\varepsilon}(A,\sigma)$ by its subgroup of classes of hyperbolic spaces [see Knu91, Ch. 1, 10].

In particular, if $A = D$ is a central division algebra, by Witt's decomposition [Knu91, Ch. 1, 6.1.1], an ε-hermitian space h over (D,σ) can be written uniquely as

$$h \simeq h_{\mathrm{an}} \perp h_{\mathrm{hyp}},$$

where h_{an} is anisotropic and h_{hyp} is hyperbolic. Two ε-hermitian spaces h_1 and h_2 over (D,σ) are Witt equivalent if $(h_1)_{\mathrm{an}} \simeq (h_2)_{\mathrm{an}}$. Let $[h]$ denote the Witt equivalence class of h. The $W^{\varepsilon}(D,\sigma)$ is an abelian group with underlying set

$$\{[h] \mid h \text{ is an } \varepsilon \text{ hermitian space over } (D,\sigma).\}$$

the associative and commutative addition $[h_1]+[h_2] = [h_1 \perp h_2]$ for all ε-hermitian spaces h_1 and h_2 over (D,σ), the identity element $0 = [\mathbb{H}]$, the inverse $-[h] = [-h]$ for all ε-hermitian space h over (D,σ).

Let (K,v) be a discrete valued field with valuation ring R_v, maximal ideal m_v and residue field $k(v) = R_v/m_v$, $\mathrm{char}(k(v)) \neq 2$. Let $(\widehat{R_v}, \widehat{m_v})$ be the completion of (R_v, m_v) and $K_v = \mathrm{Frac}(\widehat{R_v})$. Let $\widehat{v}$ be the extension of v to K_v. We have $k(\widehat{v}) = \widehat{R_v}/\widehat{m_v} = k(v)$. Let D be a finite-dimensional division algebra over K with an involution σ such that $Z(D)^{\sigma} = K$. *Suppose that $D \otimes_K K_v$ is a division algebra over K_v.* By [CF67, ch. II, 10.1], $\widehat{v}$ extends to a valuation v' on $Z(D \otimes_K K_v)$ such that

$$v'(x) = \frac{1}{[Z(D\otimes_K K_v) : K_v]} v(N_{Z(D\otimes_K K_v)/K_v}(x))$$

for all $x \in (D\otimes_K K_v)^*$. By [Wad86], v' extends to a valuation w on $D \otimes_K K_v$ such that

$$w(x) = \frac{1}{\mathrm{ind}(D\otimes_K K_v)} v'(\mathrm{Nrd}_{D\otimes_K K_v/Z(D\otimes_K K_v)}(x))$$

for all $x \in (D\otimes_K K_v)^*$. The restriction of w to D is a valuation on D and $w(x) = \dfrac{1}{\mathrm{ind}(D)} v(\mathrm{Nrd}_{D/K}(x))$ for all $x \in D^*$. Since $\mathrm{Nrd}_{D/K}(x) = \mathrm{Nrd}_{D/K}(\sigma(x))$, we have

$w(\sigma(x)) = w(x)$ for all $x \in D$. Since $\mathrm{Nrd}_{D/K}(x) = \mathrm{Nrd}_{D/K}(\sigma(x))$, we have $w(\sigma(x)) = w(x)$ for all $x \in D$. Let t_D be the *parameter* of (D, w) (see [Rei03, Th. 13.2]). We may choose $\pi_D \in D^*$ such that $w(\pi_D) \equiv w(t_D) \mod 2w(D^*)$ and $\sigma(\pi_D) = \pm\pi_D$ (see [Lar99, Prop. 2.7]). Let $R_w = \{x \in D \mid w(x) \geq 0\}$ and $\mathfrak{m}_w = \{x \in D \mid w(x) > 0\}$. Let $D(w) = R_w/\mathfrak{m}_w$ be the residue division algebra (see [Rei03, Th. 13.2]) of (D, w) over $k(v)$ with involution σ_w such that $\sigma_w(q_w(x)) = q_w(\sigma_w(x))$ for all $x \in R_w$, where $q_w(x) = x + \mathfrak{m}_w$.

Let (V, h) be an ε-hermitian space over (D, σ) for $\varepsilon \in \{1, -1\}$. Then there exists an orthogonal basis of V such that h has a diagonal form $\langle a_1, \ldots, a_m\rangle$, $a_i \in D$, $\sigma(a_i) = \varepsilon a_i$. If $w(a_i) = 0$ for all i, then $q_w(h) = \langle q_w(a_1), \ldots, q_w(a_m)\rangle \in \mathrm{Herm}^\varepsilon(D(w), \sigma_w)$. Up to isometry, we may assume that any $h \in \mathrm{Herm}^\varepsilon(D, \sigma)$ has diagonal entries with w-value either 0 or $w(t_D)$ [Lar99, Prop. 2.20].

Proposition 1.2.3 ([Lar06, Th. 3.4, Th. 3.6], [Lar99, Th. 3.27, Th. 3.29])**.** Suppose $\sigma(\pi_D) = \varepsilon'\pi_D$. There exists a unique decomposition $h_{K_v} \simeq h_1 \perp h_2\pi_D$, where $h_1 \in \mathrm{Herm}^\varepsilon(D \otimes_K K_v, \sigma \otimes_K \mathrm{Id}_{K_v})$, $h_2 \in \mathrm{Herm}^{\varepsilon\varepsilon'}(D \otimes_K K_v, \mathrm{Int}(\pi_D) \circ (\sigma \otimes_K \mathrm{Id}_{K_v}))$ and each diagonal entry of h_1 and h_2 has w-value 0. Furthermore, the following are equivalent:

(a) h is isotropic;

(b) h_1 or h_2 is isotropic;

(c) $q_w(h_1)$ or $q_w(h_2)$ is isotropic.

We have specified w in every notation because we will consider more than one valuation in chapter 2. In chapter 3 and chapter 4, we will use more friendly overlines for structures over residue fields.

1.3. Algebraic groups and Rationality

We refer readers to [Spr98; Bor91; Hum75] for details of algebraic groups over fields and [$\mathrm{SGA}_{3.\mathrm{I}}$; $\mathrm{SGA}_{3.\mathrm{II}}$; $\mathrm{SGA}_{3.\mathrm{III}}$] for details of group schemes. Let K be a field. Let K_{alg} be the algebraic closure of K. Let K_{sep} be the separable closure of K in K_{alg}. Let $\mathbf{Algebras}_K$ be the category of commutative associative unital algebras over K

and K-algebra homomorphisms. Let **Sets** be the category of sets and maps. In this dissertation, a *variety over* K means a geometrically reduced separated scheme of finite type over K (not necessarily irreducible). Let X be a variety over K. Let L be a commutative associative unital algebras over K (for example, L is a field extension of K). We denote $X_L = X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ the *scalar extension* of X to L. We also denote $X_{\mathrm{sep}} = X_{K_{\mathrm{sep}}}$ and $X_{\mathrm{alg}} = X_{K_{\mathrm{alg}}}$. We denote $X(L) = \mathrm{Hom}_{\mathrm{Spec}(K)}(\mathrm{Spec}(L), X)$ the set of L-*points* of X. By Yoneda's lemma [Yon54], a variety X over K is identified with its the *functor of points* $X\colon \mathbf{Algebras}_K^{\mathrm{op}} \to \mathbf{Sets}$.

Example 1.3.1. Let $\mathbb{P}^n$ be the projective space of dimension n over K [$\mathrm{EGA}_{\mathrm{II}}$, Def. 4.1.1]. A *projective scheme* over K is a closed subscheme of some $\mathbb{P}^n$. By [$\mathrm{EGA}_{\mathrm{II}}$, Th. 5.5.3], every projective scheme over K is a variety over K.

Let **Groups** be the category of groups and group homomorphisms. A variety G over K is called an *algebraic group over* K if its functor of points is from $\mathbf{Algebras}_K^{\mathrm{op}}$ to **Groups**. A *morphism* $f\colon G_1 \to G_2$ of two algebraic groups over K is a natural transformation of their functor of points.

Example 1.3.2. The *general linear group* over K is $\mathrm{GL}_n\colon \mathbf{Algebras}_K^{\mathrm{op}} \to \mathbf{Groups}$ such that $\mathrm{GL}_n(L) = \{n \times n$ invertible matrices with entries in $L\}$.

Example 1.3.3. The *multiplicative group* over K is $\mathbb{G}_m\colon \mathbf{Algebras}_K^{\mathrm{op}} \to \mathbf{Groups}$ such that $\mathbb{G}_m(L) = L^*$ for all $L \in \mathbf{Algebras}_K$.

Let G be an algebraic group over K. A subvariety H of G over K is a *subgroup* of G if $H(L)$ is subgroup of $G(L)$ for all $L \in \mathbf{Algebras}_K$. By [$\mathrm{SGA}_{3.\mathrm{I}}$, VI_{A}, 0.5.2], every subgroup H of G is closed. A subgroup N of G is a *normal* subgroup of G if $N(L)$ is a normal subgroup of $G(L)$ for all $L \in \mathbf{Algebras}_K$. By [$\mathrm{SGA}_{3.\mathrm{I}}$, VI_{A}, 3.3.2(v)], there exists a quotient algebraic group G/N over K and a canonical morphism $G \to G/N$. Since varieties are assumed to be geometrically reduced, by [$\mathrm{SGA}_{3.\mathrm{I}}$, VI_{A}, 1.3.1], G is *smooth*, i.e. all local rings of G_{sep} are regular.

Suppose K is a *perfect* field. Then $K_{\text{sep}} = K_{\text{alg}}$ and the structure of G is described by the following tower of normal subgroups and quotients. By [$\text{SGA}_{3.\text{I}}$, VI_{A}, 2.6.5], there exists a unique irreducible component G^0 that contains the identity element of G. Further, G^0 is a normal closed subgroup of G over K and also a connected component of G. By [$\text{SGA}_{3.\text{I}}$, VI_{A}, 5.5.1], G/G^0 is *étale* over K, i.e. its scalar extension to K_{alg} is a finite product of copies of $\text{Spec}(K_{\text{alg}})$. By [$\text{SGA}_{3.\text{I}}$, VI_{B}, 11.11], G is affine if and only if G is a closed subgroup of the general linear group GL_n over K. An affine algebraic group G is also called a *linear* algebraic group. By [Che60], there exists a unique maximal linear connected normal closed subgroup G^1 of G^0 such that G^0/G^1 is an *abelian variety* over K, i.e. it is a projective variety as well as an algebraic group. The *commutator* subgroup $[G, G]$ of G satisfies that $[G, G](L)$ is generated by $aba^{-1}b^{-1}$ for all $a, b \in G(L)$ and for all L/K. We have $[G, G]$ is a normal subgroup of G. Let $H_0 = G$, $H_{n+1} = [H_n, H_n]$ for all $n \geq 0$. The group G is called *solvable* if $H_n = \{e_G\}$ for some n, where e_G is the identity element of G. By [Che58, §9.4, prop. 2], there exists a unique maximal connected solvable normal subgroup $\text{Rad}(G^1_{\text{sep}})$ of G^1_{sep}. By [Spr98, Rem. 12.1.7], $\text{Rad}(G^1_{\text{sep}})$ is defined over K. Suppose $\text{Rad}(G^1)$ is an algebraic group over K such that $\text{Rad}(G^1)_{\text{sep}} \simeq \text{Rad}(G^1_{\text{sep}})$ and $\text{Rad}(G^1)$ is called the *radical* of G^1 over K. If $\text{Rad}(G^1) = \{e_G\}$, then G^1 is called *semisimple*. Let $G^2 = \text{Rad}(G^1)$ and $G^{\text{ss}} = G^1/G^2$. Then G^2 is solvable and G^{ss} is semisimple. Since G^1 is a linear algebraic group over K, we have $G^1 \hookrightarrow \text{GL}_n$ for some integer $n > 0$. By Jordan decomposition, $g = g_s g_u$ for all $g \in G^1$, where g_s is *semisimple*, i.e. g_s is represented by a diagonal matrix in GL_n; g_u is *unipotent*, i.e. $(g_u - I_n)^m = 0$ in GL_n for some integer $m > 0$. A linear algebraic group is called unipotent if every element of it is unipotent. Let $\text{Rad}_u(G^1_{\text{sep}}) = \{g \in \text{Rad}(G^1_{\text{sep}}) \mid g = g_u\}$. By [Che58, §12.3, Th. 1], $\text{Rad}(G^1_{\text{sep}})$ is a normal closed subgroup of G^2_{sep}. By [Spr98, Rem. 12.1.7], $\text{Rad}_u(G^1_{\text{sep}})$ is defined over K. Suppose $\text{Rad}_u(G^1)$ is an algebraic group over K such that $\text{Rad}_u(G^1)_{\text{sep}} \simeq \text{Rad}_u(G^1_{\text{sep}})$ and $\text{Rad}_u(G^1)$ is called the *unipotent radical* of G^1 over K. If $\text{Rad}_u(G^1) = \{e_G\}$, then G^1 is called *reductive*. Let $G^3 = \text{Rad}_u(G^1)$. Then

G^3 is unipotent and G^2/G^3 is a *torus*, i.e. its scalar extension to K_{alg} is a finite direct product of copies of $\mathbb{G}_m$. The following table summarizes main properties of normal subgroups and quotient groups.

Normal subgroups	G	G^0	G^1	G^2	G^3
Properties	algebraic	connected	linear	solvable	unipotent
Quotient groups	G/G^0	G^0/G^1	G^1/G^2	G^2/G^3	G^1/G^3
Properties	étale	projective	semisimple	torus	reductive

From now on, we focus on *connected linear algebraic groups.*

Suppose G is a connected linear algebraic group over a field K such that $G \hookrightarrow \mathrm{GL}_n$ for some integer $n > 0$. Let $M_n(K)$ be the group of $n \times n$ matrices over K and I_n the identity matrix. The *Lie algebra* of G is defined to be

$$\mathrm{Lie}(G) = \{M \in M_n(K) \mid I_n + Mt \in G\left(\frac{K[t]}{(t^2)}\right)\}.$$

with addition and scalar multiplication from $M_n(K)$ and Lie bracket $[M_1, M_2] = M_1M_2 - M_2M_1$ for all $M_1, M_2 \in \mathrm{Lie}(G)$. Here t is an indeterminate and $\frac{K[t]}{(t^2)}$ is called the K-algebra of *dual numbers*. Let $f\colon G_1 \to G_2$ be a morphism of connected linear algebraic groups over K such that $G_1 \hookrightarrow \mathrm{GL}_n$ and $G_2 \hookrightarrow \mathrm{GL}_n$ for some integer $n > 0$. The *differential* $df\colon \mathrm{Lie}(G_1) \to \mathrm{Lie}(G_2)$ is defined by

$$f(I_n + MT) = I_n + df(M)t$$

in $G_2\left(\frac{K[t]}{(t^2)}\right)$ for all $M \in \mathrm{Lie}(G_1)$. The *adjoint representation* of G is defined by

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathrm{Lie}(G)),\ g \mapsto d(\mathrm{Int}(g))$$

for all $g \in G$, where $\mathrm{Int}(g)\colon G \to G$ is the *interior automorphism* of G given by $\mathrm{Int}(g)(x) = gxg^{-1}$ for all $x \in G$.

Let K be a perfect field. An algebraic group T over K is a *torus* if $T_{\text{alg}} \simeq (\mathbb{G}_m)^n_{\text{alg}}$ for some integer $n > 0$. A torus T over K is *split* if $T \simeq \mathbb{G}_m^n$. If G contains a split torus, then we say that G is *split*. If T is a subgroup of a connected linear algebraic group G over K and T is a torus, then T is called a *subtorus* of G. A subtorus T of G is called a *maximal torus* of G if for all subtori T' of G such that $T \subseteq T'$, we have $T' = T$. Let T_{alg} be a maximal torus of G_{alg} and let Ad be the adjoint representation $\mathrm{Ad}\colon G_{\mathrm{alg}} \to \mathrm{Lie}(G_{\mathrm{alg}})$. Let $T^*_{\mathrm{alg}} = \mathrm{Hom}(T_{\mathrm{alg}}, (\mathbb{G}_m)_{\mathrm{alg}})$ be the set of morphisms of algebraic groups over K_{alg}. Denote $\mathfrak{g} = \mathrm{Lie}(G_{\mathrm{alg}})$. For $\chi \in T^*_{\mathrm{alg}}$ define

$$\mathfrak{g}_\chi = \{M \in \mathfrak{g} \mid \mathrm{Ad}(g)(M) = \chi(g)M \text{ for all } g \in G_{\mathrm{alg}}\}$$

If $\chi \neq 0$ and $\mathfrak{g}_\chi \neq 0$, then χ is called a *root* of G_{alg} with respect to T_{alg}. Let $\Phi(G_{\mathrm{alg}})$ be the set of all roots of G_{alg} with respect to T_{alg}. Then $\Phi(G_{\mathrm{alg}}) \subset T^*_{\mathrm{alg}} \otimes_{\mathbb{Z}} \mathbb{R}$. Since $T_{\mathrm{alg}} \simeq (\mathbb{G}_m)^n_{\mathrm{alg}}$, we have $T^*_{\mathrm{alg}} \simeq \mathbb{Z}^n$ and hence $\Phi(G_{\mathrm{alg}})$ is identified with a subset of $\mathbb{R}^n$. For $\alpha \in \Phi$ and $\alpha \neq 0$, define the *reflection* $s_\alpha(x) = x - 2\dfrac{(x,\alpha)}{(\alpha,\alpha)}\alpha$ for all $x \in \mathbb{R}^n$, where $(\cdot,\cdot)$ is the standard inner product of $\mathbb{R}^n$. A subset $\Phi \subset \mathbb{R}^n$ is called a *root system* [Bou$_{\text{LIE4-6}}$, VI, § 1, no. 1, Def. 1] of $\mathbb{R}^n$ if

(1) $0 \notin \Phi$, Φ is finite and Φ spans $\mathbb{R}^n$;
(2) For all $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$;
(3) For all $\alpha \in \Phi$, $s_\alpha(\Phi) = \Phi$;
(4) For all $\alpha, \beta \in \Phi$, there exists $n \in \mathbb{Z}$ such that $s_\alpha(\beta) - \beta = n\alpha$.

Then $\Phi(G_{\mathrm{alg}})$ is a root system [KMRT98, Th. 25.1].

Let Φ be a root system of $\mathbb{R}^n$. Let $\Phi^+ = \{\alpha \in \Phi \mid (\alpha, x) > 0\}$ for some $x \in \mathbb{R}^n$. There exists $\Delta \subset \Phi^+$ such that Δ is a basis of $\mathbb{R}^n$ and every element of Φ^+ is a linear combination of elements of Δ with positive integeral coefficients; every element of $\Phi^- = \Phi \setminus \Phi^+$ is a linear combination of elements of Δ with negative integeral coefficients. We draw the Dynkin diagram of $\mathrm{Dyn}(\Phi)$ be drawing $n = |\Delta|$ vertices, each vertex corresponds an element of Δ. For $\alpha, \beta \in \Delta$, define $\langle \alpha, \beta \rangle = 2\dfrac{(\beta,\alpha)}{(\alpha,\alpha)}$.

- If $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = 0$, we draw nothing between the vertex of α and the vertex of β;

- If $\langle\alpha,\beta\rangle = \langle\beta,\alpha\rangle = -1$, we draw an undirected edge between the vertex of α and the vertex of β;
- If $\langle\alpha,\beta\rangle = -1$ and $\langle\beta,\alpha\rangle = -2$, we draw a directed edge from the vertex of α to the vertex of β with multiplicity 2;
- If $\langle\alpha,\beta\rangle = -1$ and $\langle\beta,\alpha\rangle = -3$, we draw a directed edge from the vertex of α to the vertex of β with multiplicity 3.

A subset S of a root system is *closed* if any linear combination of roots of S with coefficients in $\mathbb{Z}$ is still in S. A subset of a root system is *irreducible* if it cannot be written as the disjoint union of two nonempty closed subsets. By [$\text{Bou}_{\text{LIE4-6}}$, VI, § 4, no. 2, Th. 3], Φ is irreducible if and only if $\text{Dyn}(\Phi)$ is connected; and every Dynkin diagrams of an irreducible root system is called one of the following A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4)$, E_6, E_7, E_8, F_4, G_2. An algebraic group G over K is *simple* if its normal closed subgroups are only $\{e_G\}$ and G. A semisimple algebraic group G over K is *almost simple* if $G/Z(G)$ is simple. An almost simple algebraic group G over K is *absolutely almost simple* if G_{alg} is almost simple. By [Che58, § 17, Prop. 1], G is absolutely almost simple if and only if $\Phi(G_{\text{alg}})$ is irreducible. A sujective morphism $f\colon G' \to G''$ of connected linear algebraic groups over K with finite kernel is called an *isogeny*. We say that G' and G'' are *strictly isogenous* if there exists a third group H with central isogenies $H \to G'$ and $H \to G''$. "Strictly isogenous" is an equivalence relation. If $\ker(f)$ is a subgroup of $Z(G')$, then f is called a *central* isogeny. When $\operatorname{char} K = 0$, all isogenies are central. If G is a semisimple connected linear algebraic group over K, by [BT65, 2.15(c)], there exists an isogeny over K from a finite product of absolutely almost simple groups to G. This fact is important to the classification of projective homogeneous spaces.

Definition 1.3.4. A connected linear algebraic group G over K is *rational* if its function field $K(G)$ is a purely transcendental extension of K.

Example 1.3.5. The general linear group GL_n over K is a rational connected linear algebraic group over K since it is open in $\mathbb{A}_K^{n^2}$. Similarly, the projective general linear group PGL_n over K is a rational connected linear algebraic group over K.

Let A be a central simple algebra over K. By the proof of [HHK09, Th. 5.1], $\mathrm{GL}_n(A)$ and $\mathrm{PGL}_n(A)$ are also rational connected linear algebraic group over K.

Suppose $\deg(A) = d$. By [KMRT98, Th. 25.9], $\mathrm{PGL}_1(A)$ has type A_{d-1}.

Example 1.3.6. Let K be a field of characteristic not 2. Let L be a quadratic field extension of K. Let A be a central division algebra over L. Let σ be an involution on A of the second kind such that $L^\sigma = K$. Let V be a finitely generated right A-module. Let $h\colon V \times V \to A$ be an ε-hermitian form for $\varepsilon \in \{1, -1\}$. The unitary group of is defined to be $\mathrm{U}(A, \sigma, h) = \{f \in \mathrm{End}_A(V)^* \mid h(f(x), f(y)) = h(x, y)\}$. Let ad_h be the adjoint involution of h in $\mathrm{End}_A(V)$. Let $\mathrm{U}(\mathrm{End}_A(V), \mathrm{ad}_h) = \{f \in \mathrm{End}_A(V)^* \mid f \circ \mathrm{ad}_h(f) = \mathrm{Id}_V\}$. Then $\mathrm{U}(A, \sigma, h) \simeq \mathrm{U}(\mathrm{End}_A(V), \mathrm{ad}_h)$. By [KMRT98, 23A], $\mathrm{U}(A, \sigma, h)$ is a connected linear algebraic group. Further, by Cayley-parametrization (see [CP98, Lem. 5] or [Mer96, p. 195, Lem. 1]), $\mathrm{U}(A, \sigma, h)$ is rational.

Suppose $\mathrm{rdim}(V) = r$, by [PR94, Prop. 2.15(3)], $\mathrm{U}(A, \sigma, h)$ has type A_{r-1}.

Example 1.3.7. Let K be a field of characteristic not 2. Let A be a central simple algebra over K. Let σ be an involution on A of the first kind. Let V be a finitely generated right A-module. Let $h\colon V \times V \to A$ be an ε-hermitian form for $\varepsilon \in \{1, -1\}$. The special unitary group of is defined to be $\mathrm{SU}(A, \sigma, h) = \{f \in \mathrm{End}_A(V)^* \mid h(f(x), f(y)) = h(x, y),\ \det(f) = 1\}$. By [KMRT98, 23A], $\mathrm{SU}(A, \sigma, h)$ is a connected linear algebraic group and $\mathrm{SU}(A, \sigma, h) = \mathrm{U}(A, \sigma, h)^0$. Further, by Cayley-parametrization (see [CP98, Lem. 5] or [Mer96, p. 195, Lem. 1]), $\mathrm{SU}(A, \sigma, h)$ is rational.

If $A = K$, $\sigma = \mathrm{Id}_K$, $\varepsilon = 1$, $h = q$ and $\dim_K(V) = 2n + 1$, then, by [PR94, Prop. 2.15(2)], $\mathrm{SU}(A, \sigma, h) = \mathrm{SO}_{2n+1}(q)$ has type B_n.

Suppose $\mathrm{rdim}_A(V) = 2n$. Let ad_h be the adjoint involution of h on $\mathrm{End}_A(V)$. If ad_h is symplectic (i.e. σ is orthogonal and $\varepsilon = -1$, or σ is symplectic and $\varepsilon = 1$), then, by [PR94, Prop. 2.15(1)], $\mathrm{SU}(A, \sigma, h)$ has type C_n. If ad_h is orthogonal (i.e. σ is orthogonal and $\varepsilon = 1$, or σ is symplectic and $\varepsilon = -1$), then, by [PR94, Prop. 2.15(2)], $\mathrm{SU}(A, \sigma, h)$ has type D_n.

1.4. Galois cohomology and Principal homogeneous spaces

We refer readers to [GS06; Ser02] form details of Galois cohomology.

Let G be an algebraic group over a field K. Suppose the absolute Galois group $\mathrm{Gal}(K_{\mathrm{sep}}/K)$ acts on $G(K_{\mathrm{sep}})$ by sending g to ${}^s g$ such that ${}^{s \circ t}(g) = {}^s g \cdot {}^t g$ for all $s, t \in \mathrm{Gal}(K_{\mathrm{sep}}/K)$ and $g \in G(K_{\mathrm{sep}})$, where $\cdot$ is the multiplication in $G(K_{\mathrm{sep}})$.

The *zero-th Galois cohomology group* is defined to be $H^0(K, G) = G_{\mathrm{sep}}^{\mathrm{Gal}(K_{\mathrm{sep}}/K)}$.

Next we define $H^1(K, G)$. A *1-cocycle* is a map $a\colon \mathrm{Gal}(K_{\mathrm{sep}}/K) \to G(K_{\mathrm{sep}})$ such that

$$a(st) = a(s) \cdot {}^s a(t)$$

for all $s, t \in \mathrm{Gal}(K_{\mathrm{sep}}/K)$. Two 1-cocyles a, b are *cohomologous* if there exists $g \in G(K_{\mathrm{sep}})$ such that

$$b(s) = g^{-1} \cdot a(s) \cdot {}^s g$$

Cohomologous is an equivalence relation in the set of 1-cocyles. The *first nonabelian Galois cohomology set* $H^1(K, G)$ is defined to be the set of equivalence classes of 1-cocyles. The equivalence class of $e\colon \mathrm{Gal}(K_{\mathrm{sep}}/K) \to G(K_{\mathrm{sep}})$ such that $e(s) = 1 \in G(K_{\mathrm{sep}})$ is called the *neutral* element of $H^1(K, G)$.

When $G(K_{\mathrm{sep}})$ is an abelian group, we define $H^2(K, G)$. A *2-cocycle* is a map $a\colon \mathrm{Gal}(K_{\mathrm{sep}}/K)^2 \to G(K_{\mathrm{sep}})$ such that

$${}^s a(t, u) \cdot a(st, u)^{-1} \cdot a(s, tu) \cdot a(s, t)^{-1} = 1$$

for all $s, t, u \in \mathrm{Gal}(K_{\mathrm{sep}}/K)$. The set of 2-cocycles form an abelian group. A map $a : \mathrm{Gal}(K_{\mathrm{sep}}/K)^2 \to G_{\mathrm{sep}}$ is *2-coboundary* if there exists a map $b\colon \mathrm{Gal}(K_{\mathrm{sep}}/K) \to G(K_{\mathrm{sep}})$ such that

$$a(s, t) = {}^s b(t) \cdot b(st)^{-1} \cdot b(s)$$

for all $s, t \in \mathrm{Gal}(K_{\mathrm{sep}}/K)$. The set of 2-coboundaries form a subgroup of the group of 2-cocycles. The *second Galois cohomology group* $H^2(K, G)$ is defined to be the quotient group of 2-cocycles by 2-coboundaries.

Let $1 \to G_1 \to G_2 \to G_3 \to 1$ be a short exact sequence of algebraic groups over K. By [Ser02, Prop. 36], there exists a long exact sequence

$$1 \to H^0(K, G_1) \to H^0(K, G_2) \to H^0(K, G_3) \to H^1(K, G_1) \xrightarrow{\delta^1} H^1(K, G_2),$$

where for all $x_3 \in H^0(K, G_3) = G_3(K_{\mathrm{sep}})^{\mathrm{Gal}(K_{\mathrm{sep}}/K)}$, if x is the image of $x_2 \in G_2(K_{\mathrm{sep}})$, then $\delta^1(x_3) = [a]$ is the cohomology class of the following 1-cocycle

$$a : \mathrm{Gal}(K_{\mathrm{sep}}/K) \to G_1(K_{\mathrm{sep}}),\ a(s) = x_2^{-1} \cdot {}^s x_2$$

for all $s \in \mathrm{Gal}(K_{\mathrm{sep}}/K)$. By [Ser02, Prop. 38], if G_1 is a normal subgroup of G_2, we can add one more term "$\to H^1(K, G_3)$" at the end of the long exact sequence. Further, by [Ser02, Prop. 38], if G_1 is a subgroup of $Z(G_2)$, we can add another term "$\xrightarrow{\delta^2} H^2(K, G_1)$". Suppose $y_3\colon \mathrm{Gal}(K_{\mathrm{sep}}/K) \to G_3(K_{\mathrm{sep}})$ is a 1-cocycle and it is lifted to $y_2\colon \mathrm{Gal}(K_{\mathrm{sep}}/K) \to G_2(K_{\mathrm{sep}})$. Then $\delta^2([y_3]) = [b]$ is the cohomology class of the following 2-cocycle

$$b\colon \mathrm{Gal}(K_{\mathrm{sep}}/K)^2 \to G_1(K_{\mathrm{sep}}),\ b(s,t) = {}^s y_2(t) \cdot y_2(st)^{-1} \cdot y_2(s).$$

Suppose $\mathrm{char}(K) \neq 2$. Let μ_2 be the group of second roots of unity in K_{sep}. By Kummer theory [GS06, Prop. 4.3.6], there exists an isomorphism

$$H^1(K, \mu_2) \simeq K^*/K^{*2}.$$

By [GS06, Cor. 4.4.9],

$$H^2(K, \mu_2) \simeq {}_2\,\mathrm{Br}(K).$$

Let $\mathscr{X}$ be a regular integral scheme with function field F. For every codimension one point x of $\mathscr{X}$, let $k(x)$ denote the residue field at x, $\mathrm{char}(k(x)) \neq 2$. Then there is a residue homomorphism

$$\partial_x \colon {}_2\operatorname{Br}(F) \simeq H^2(F, \mu_2) \to H^1(k(x), \mu_2) \simeq k(x)^*/k(x)^{*2}.$$

Suppose A is a central simple algebra over F of period 2. By a special case [Mer81] of the Merkurjev-Suslin theorem [MS82], A is Brauer equivalent to $H_1 \otimes \cdots \otimes H_n$ for some quaternion algebras $H_1, \dots, H_n$ over F. Let $(a, b)_F$ be a quaternion algebra over F for $a, b \in F^*$. Let v_x be the discrete valuation whose valuation ring is the local ring $\mathcal{O}_{\mathscr{X},x}$. Then the image of Brauer class of the quaternion algebra is defined to be

$$\partial_x([(a,b)_F]) = (-1)^{v_x(a)v_x(b)} a^{v_x(b)} b^{-v_x(a)} \in k(x)^*/k(x)^{*2}.$$

Further, $\partial_x([A]) = \prod_{i=1}^{n} \partial_x([H_i])$. We say that an element $\alpha \in {}_2\operatorname{Br}(F)$ is *ramified* at x if $\partial_x(\alpha) \neq 0$; we say that α is *unramified* at x if $\partial_x(\alpha) = 0$. The *ramification divisor* of α is defined as $\sum x$, where x runs over all codimension one points of $\mathscr{X}$ with $\partial_x(\alpha) \neq 0$.

Let X be an algebraic varietie over K. Suppose $\operatorname{Gal}(K_{\text{sep}}/K)$ acts on X_{sep} by ${}^s x$ for all $s \in \operatorname{Gal}(K_{\text{sep}}/K)$ and $x \in X_{\text{sep}}$. An algebraic variety Y over K is called a *K-form of X* if there exists an isomorphism $f \colon Y_{\text{sep}} \to X_{\text{sep}}$. Let $a \colon \operatorname{Gal}(K_{\text{sep}}/K) \to \operatorname{Aut}(X_{\text{sep}})$ be a 1-cocycle. A *K-form of X twisted by a* is denoted by ${}_aX$, where the underlying algebraic variety of ${}_aX$ is X and $\operatorname{Gal}(K_{\text{sep}}/K)$ acts on $({}_aX)_{\text{sep}}$ by $s * x = a(s) \cdot {}^s x$ for all $s \in \operatorname{Gal}(K_{\text{sep}}/K)$ and $x \in ({}_aX)_{\text{sep}}$. By [GS06, Th. 2.3.3], there exists a bijection between isomorphism classes of K-forms of X and $H^1(K, \operatorname{Aut}(X))$. Also Y is a K-form of X if and only if there exists a 1-cocycle $a \colon \operatorname{Gal}(K_{\text{sep}}/K) \to \operatorname{Aut}(X_{\text{sep}})$ such that $Y = (({}_aX)_{\text{sep}})^{\operatorname{Gal}(K_{\text{sep}}/K)} \simeq {}_aX$ and hence we identify Y with ${}_aX$.

Let G be a semisimple connected linear algebraic group over a field K. We say that G is *simply connected* if for all connected linear algebraic group H over K with a central isogeny $f \colon H \to G$, we have that f is an isomorphism. We say that G is *adjoint* if for all connected linear algebraic group H over K with a central isogeny $f \colon G \to H$, we have that f is an isomorphism. By [Tit66, §2.6.1, Prop. 2], there exists a simply connected group $\widetilde{G}$ over K with an isogeny $\widetilde{\pi} \colon \widetilde{G} \to G$, an adjoint group $\overline{G}$ over K with an isogeny $\overline{\pi}\colon G \to \overline{G}$ and they are unique up to isomorphism. Suppose $a\colon \operatorname{Gal}(K_{\text{sep}}/K) \to \operatorname{Aut}(G_{\text{sep}})(K_{\text{sep}})$ is a 1-cocycle. By [MPW96, Rem. 1.4], we have a short exact sequence

$$1 \to Z(G) \to G \xrightarrow{\text{Ad}} \overline{G} \to 1$$

By [GS06, Th. 2.3.3], if $\operatorname{Im}(a) \subseteq \operatorname{Im}(\overline{G}(K_{\text{sep}}) \to \operatorname{Inn}(G_{\text{sep}})(K_{\text{sep}}))$, then ${}_aG$ is called an *inner form* of G; otherwise ${}_aG$ is called an *outer form* of G [see MPW96, Rem. 1.4(ii)].

Let G be an algebraic group over a field K. A *Borel* subgroup of G over K is a maximal solvable connected linear closed subgroup of G. A subgroup P of G is called a *parabolic* subgroup if it contains some Borel subgroup. An algebraic group over K is *quasi-split* if it is reductive and contains a Borel subgroup over K.

Suppose G is semisimple connected linear and K is perfect field. By [Che58, §23.1, Prop. 1], there exists a maximal torus $\widetilde{T_{\text{sep}}}$ of $\widetilde{G_{\text{sep}}}$ such that isogeny $\widetilde{\pi}\colon \widetilde{G_{\text{sep}}} \to G_{\text{sep}}$ satisfies $f(\widetilde{T_{\text{sep}}}) = T_{\text{alg}}$ and it provides a bijection between $\Phi(\widetilde{G_{\text{sep}}})$ and $\Phi(G_{\text{sep}})$. By [MPW96, Prop. 1.10], for all semisimple connected linear algebraic group G, there exists a unique quasi-split group G^{qs} such that G is an inner form of G^{qs} [see also BT87, §1.3]. Two isogenies $f_1, f_2\colon G' \to G''$ are *conjugate* if there exists $g \in G'$ such that $f_2 = f_1 \circ \operatorname{Int}(g)$, it is an equivalence relation. By the isomorphism theorem [Spr98, Th. 9.6.2], there exists a bijection between conjugacy classes of isomorphisms $G_{\text{sep}} \to G^{\text{qs}}_{\text{sep}}$ and automorphisms of $\Phi(G^{\text{qs}})$. Let Δ be the set of simple roots of $\Phi(G_{\text{sep}}) = \Phi(G^{\text{qs}}_{\text{sep}})$. By [Tit62, §4.3], $\operatorname{Gal}(K_{\text{sep}}/K)$ acts on Δ and there exists a finite Galois extension K'/K such that $G^{\text{qs}}_{K'}$ contains a split maximal torus and $\operatorname{Gal}(K_{\text{sep}}/K') \simeq \operatorname{Aut}(\Delta)$. Let Z_n be the name of $\operatorname{Dyn}(\Phi(G_{\text{sep}}))$, we write and call

$$^{[\mathbf{K}':\mathbf{K}]}\mathbf{Z_n}$$

the *type* of G. When $[K' : K] = 1$, we omit it if no confusion is caused. We call A_n, B_n, C_n, ${}^1D_n$, ${}^2D_n$ *classical* types and ${}^3D_4$, ${}^6D_4$, E_6, E_7, E_8, F_4, G_2 *exceptional* types. In this dissertation, we are mainly interested in rational absolutely almost simple groups of classical types.

Example 1.4.1. Under assumptions of example 1.3.5, we have $\mathrm{PGL}_1(A)$ has type ${}^1A_{d-1}$, where $d = \deg(A)$. In fact, by $G^{\mathrm{qs}} = \mathrm{PGL}_d$ has maximal torus $\mathbb{G}_m^{d-1}$ over K. Conversely, since $\widetilde{\mathrm{PGL}_1(A)} = \mathrm{SL}_1(A) \simeq \mathrm{SL}_m(D)$ for some integer m such that $d = m\deg(D)$, it follows from [Tit66, Th. 1] that all absolutely almost simple group of type ${}^1A_{d-1}$ is strictly isogenous to some $\mathrm{PGL}_1(A)$ as this.

Example 1.4.2. Under assumptions of example 1.3.6, it follows from [Tit66, Table II] that $\mathrm{U}(A, \sigma, h)$ has type ${}^2A_{r-1}$, where $r = \mathrm{rdim}(V)$. Conversely, by [Tit66, Th. 1], all absolutely almost simple group of type ${}^2A_{r-1}$ is strictly isogenous to some $\mathrm{U}(A, \sigma, h)$ as this.

Example 1.4.3. Under assumptions of example 1.3.7, it follows from [Tit66, Table II] that $\mathrm{SO}_{2n+1}(q)$ has type ${}^1B_n$. Conversely, by [Tit66, Th. 1], all absolutely almost simple group of type B_n is strictly isogenous to some $\mathrm{SO}_{2n+1}(q)$ as this.

Suppose $\mathrm{rdim}(V) = 2n$, it also follows from [Tit66, Table II, Th. 1] that

If ad_h is symplectic, then $\mathrm{SU}(A, \sigma, h)$ has type ${}^1C_n$ and all absolutely almost simple group of type ${}^1C_n$ is strictly isogenous to some $\mathrm{SU}(A, \sigma, h)$ as this.

If ad_h is orthogonal and $\mathrm{disc}(h) = 1$, then $\mathrm{SU}(A, \sigma, h)$ has type ${}^1D_n$ and all absolutely almost simple group of type ${}^1D_n$ is strictly isogenous to some $\mathrm{SU}(A, \sigma, h)$ as this.

If ad_h is orthogonal and $\mathrm{disc}(h) \neq 1$, then $\mathrm{SU}(A, \sigma, h)$ has type ${}^2D_n$ and all semisimple groups of type ${}^2D_n$ is strictly isogenous to some $\mathrm{SU}(A, \sigma, h)$ as this.

Let G be an algebraic group over K and X an algebraic variety over K. If G acts on X on the left, then $G(L)$ acts on $X(L)$ on the left for all $L \in \mathbf{Algebras}_K$ and the action is defined as follows:

$$G(L) \times X(L) \to X(L),\ (gx)(l) = g(l)x(l)$$

for all $l \in \operatorname{Spec}(L)$, $g\colon \operatorname{Spec}(L) \to G$, $x\colon \operatorname{Spec}(L) \to X$.

For a K-algebra homomorphisms $\varphi\colon L_1 \to L_2$, we have $\varphi^{-1}\colon \operatorname{Spec}(L_2) \to \operatorname{Spec}(L_1)$ that sends a prime ideal l_2 of L_2 to its preimage $\varphi^{-1}(l_2)$ of L_1. Then it induces $G(L_1) \to G(L_2)$ and $X(L_1) \to X(L_2)$ defined by $- \circ \varphi^{-1}$. We have that the following diagram commutes

$$\begin{array}{ccc} G(L_1) \times X(L_1) & \longrightarrow & X(L_1) \\ \downarrow & & \downarrow \\ G(L_2) \times X(L_2) & \longrightarrow & X(L_2) \end{array}$$

for all K-algebra homomorphisms $L_1 \to L_2$.

Definition 1.4.4. Let G be an algebraic group over K and X an algebraic variety over K. We say that X *is a homogeneous space under* G if G acts on X on the left and $G(L)$ acts on $X(L)$ transitively for all $L \in \mathbf{Algebras}_K$, i.e.

$$G(L) \times X(L) \to X(L) \times X(L),\ (g, x) \mapsto (x, gx) \text{ for all } g \in G(L), x \in X(L)$$

is surjective for all $L \in \mathbf{Algebras}_K$.

We say that X is a *principal* homogeneous space (torsor) under G if the map above is bijective for all $L \in \mathbf{Algebras}_K$.

The *trivial* principal homogeneous space under G is G itself with left translation. By [Ser02, Prop. 33], there exists a bijection between the set of isomorphism classes of principal homogeneous spaces under G over K and $H^1(K, G)$, where the isomorphism class of the trivial principal homogeneous space under G corresponds the neutral element of $H^1(K, G)$. As a consequence:

Proposition 1.4.5. [Poo, Prop. 5.11.14]. Let G be a smooth algebraic group over a field K. Let X be a principal homogeneous space under G. Let $[X]$ be the cohomology class associated to X. Then $X(K) \neq \emptyset$ if and only if $[X]$ is the neutral element of $H^1(K, G)$.

Let F be a field. Let G be a connected linear algebraic group over F. Let X be a principal or projective homogeneous space under G. One is interested in knowing when does X have a F-rational point, i.e. $X(F) \neq \emptyset$. Then, for many examples of X, there are well known methods to verify whether X has a F-rational point or not. Let $\{F_v\}_{v\in\Omega}$ be a set of field extensions of F indexed by a set Ω. If $X(F) \neq \emptyset$, then clearly $X(F_v) \neq \emptyset$. We say that the *Hasse-principle* holds for X with respect to $\{F_v\}_{v\in\Omega}$ if

$$\prod_{v\in\Omega} X(F_v) \neq \emptyset \implies X(F) \neq \emptyset.$$

It is well-known that we can not expect the Hasse principle holds for F, Ω and $\{F_v\}_{v\in\Omega}$ in general. Next, we give a short survey on what is known about Hasse principle of principal homogeneous spaces.

Let F be a global field, i.e. a number field, or a function field of one variable over a finite field. A *place* of F is an equivalence class of absolute values of F. Let Ω be the set of all places of F, i.e. non-archimedean places which corresponds discrete valuations and archimedean places which are either real or complex. For $v \in \Omega$, let F_v be the completion of F at v. Let G be a semisimple, simply connected, linear algebraic group over F. Let X be a principal homogeneous space over G. By proposition 1.4.5, the Hasse principle for X is equivalent to the injectivity of

$$H^1(F,G) \to \prod_{v\in\Omega} H^1(F_v,G)$$

The Albert-Brauer-Hasse-Noether theorem [BNH32; AH32] states that if A is a central simple algebra over a global field F, then A splits iff A_{F_v} splits for all places v of F. By [GS06, Th. 2.4.3], there exists a bijection between isomorphism classes of central simple algebras over K of degree n and $H^1(F, \mathrm{PGL}_n)$. Hence

$$H^1(F,\mathrm{PGL}_n) \to \prod_{v\in\Omega} H^1(F_v,\mathrm{PGL}_n)$$

is injective and hence the Hasse principle holds for principal homogeneous spaces under PGL_n over global fields.

The Hasse-Minkowski theorem [Has23; Has24b; Has24a; Min90] states that if q_1 and q_2 are quadratic forms over a global field F, then $q_1 \simeq q_2$ if and only if $(q_1)_{F_v} \simeq (q_2)_{F_v}$ for all $v \in \Omega_F$. Let q be a quadratic space over F of rank n and let $\mathrm{O}_n(q)$ be the orthogonal group of q. By [KMRT98, Eq. 29.28], there exists a bijection between isomorphism classes of quadratic spaces of dimension n and $H^1(F, \mathrm{O}_n(q))$. Hence

$$H^1(F, \mathrm{O}_n(q)) \to \prod_{v \in \Omega} H^1(F_v, \mathrm{O}_n(q))$$

is injective and hence the Hasse principle holds for principal homogeneous spaces under $\mathrm{O}_n(q)$ over global fields.

Let Ω_∞ be the set of real places of a global field F. Let A be a central simple algebra over F. From the exact sequence $1 \to \mathrm{SL}_1(A) \to \mathrm{GL}_1(A) \xrightarrow{\mathrm{Nrd}} \mathbb{G}_m \to 1$, we have an exact sequence $A^* \xrightarrow{\mathrm{Nrd}_{A/F}} F^* \to H^1(F, \mathrm{SL}_1(A)) \to H^1(F, \mathrm{GL}_1(A))$. By Hilbert 90, $H^1(F, \mathrm{GL}_1(A)) = 1$ and hence $H^1(F, \mathrm{SL}_1(A)) = F^*/\mathrm{Nrd}_{A/F}(A^*)$. By a theorem of Hasse-Schilling-Maass [Rei03, Th. 33.15], $x \in \mathrm{Nrd}_{A/F}(A^*)$ if and only if $x_v > 0$ for all $v \in \Omega_\infty$ such that A is ramified at v. Then

$$H^1(F, \mathrm{SL}_1(A)) \to \prod_{v \in \Omega_\infty} H^1(F_v, \mathrm{SL}_1(A))$$

is injective.

If G is a semisimple, simply connected linear algebraic group over a global field F, then

$$H^1(F, G) \to \prod_{v \in \Omega_\infty} H^1(F_v, G)$$

is bijective. The case for G of classical types over a number field F is proved by Eichler, Kneser, Springer [Kne69, §5.1, Th. 1]; The case for G of non-E_8 types over a number field F is proved by [Har65; Har66]; The case for G of E_8 type over a number field F is proved by [Che89]; The case for G of any type over a function field F of a curve over a finite field is proved by [Har75].

See also [BP98], [COP02, Th. 5.2], [CGP04, Th. 5.2(b)], [CPS12, Th. 4.8], [HHK14, Th. 3.3.6], [Pre13], [Hu14] for Hasse principles for principal homogeneous spaces under other choices of F, Ω, $\{F_v\}_{v\in\Omega}$ and G.

1.5. Projective homogeneous spaces

We refer readers to [MPW96; MPW98] for details of projective homogeneous spaces.

Definition 1.5.1. Let G be an algebraic group over K and X an algebraic variety over K. We say that X is a *projective* homogeneous space under G if X is a homogeneous space under G and a projective variety over K.

Let G be an algebraic group over K and X an algebraic variety over K such that G acts on X. Then $G(L)$ acts on X_L for all $L \in \mathbf{Algebras}_K$ by

$$G(L) \times X_L \to X_L,\ g(x,l) = (g(l)x,l)$$

for all $g\colon \operatorname{Spec}(L) \to G$, $x \in X$, $l \in \operatorname{Spec}(L)$ such that $(x,l) \in X_L$. The action of $G(L)$ on X_L is well-defined.

Let G be a semisimple connected linear algebraic group over a field K and X an algebraic variety over K such that G acts on X. Then $G(K_{\mathrm{sep}})$ acts on X_{sep} and it gives a group homomorphism $\varphi\colon G(K_{\mathrm{sep}}) \to \operatorname{Aut}(X_{\mathrm{sep}})$. If $a\colon \operatorname{Gal}(K_{\mathrm{sep}}/K) \to G(K_{\mathrm{sep}})$ is a 1-cocycle, then the composition $\varphi \circ a$ is also a 1-cocycle. We write the K-form of X twisted by $\varphi \circ a$ as ${}_aX = {}_{\varphi\circ a}X$.

Lemma 1.5.2. [BS68, Prop. 8.4], [Dem77], [After MPW96, Prop. 1.3]. Let Δ be the set of simple roots of G_{sep} with respect to some maximal torus of G_{sep} and a choice of positive roots. There exists a bijection between the set of conjugacy classes of parabolic subgroups of G_{sep} and subsets of Δ. Further, for a fixed $\Theta \subseteq \Delta$, the set of all parabolic subgroups of G_{sep} from the cojugacy class corresponding to Φ form a variety defined over K. This variety over K is called the *Borel variety* of Θ and is denoted by $\mathscr{B}_\Theta(G)$.

Lemma 1.5.3. [HHK09, Rem. 3.9], [MPW96, Prop. 1.3, Prop. 1.5] Let G be a semisimple connected linear algebraic group over a field K and X an algebraic variety over K such that G acts on X. The following are equivalent:

(1) X is a projective homogeneous space under G;

(2) X is a projective variety and $G(K_{\text{alg}})$ acts on $X(K_{\text{alg}})$ transitively;

(3) X is a projective variety and $G(K_{\text{sep}})$ acts on $X(K_{\text{sep}})$ transitively;

(4) $X \simeq \mathscr{B}_\Theta(G)$ for some Θ as in lemma 1.5.2.

(5) there exists a quasi-split group G^{qs} such that G is an inner form of G^{qs} and a parabolic subgroup P of G^{qs} such that $X \simeq {}_a(G^{\text{qs}}/P)$, where $a\colon \text{Gal}(K_{\text{sep}}/K) \to G(K_{\text{sep}})$ is a 1-cocycle.

Because of (5), a projective homogeneous space is also called a *twisted flag variety*.

Lemma 1.5.4. [BT72, 2.20, (i)]. Let G, G' be two algebraic groups over a field K. Let $f\colon G \to G'$ be a central surjective morphism of algebraic groups over K.
(i) If P is a parabolic subgroup of G, then $f(P)$ is a parabolic subgroup of G'.
(ii) If P' is a parabolic subgroup of G', then $f^{-1}(P')$ is a parabolic subgroup of G.

Corollary 1.5.5. *Let G, G' be two semisimple connected linear algebraic groups over a field K and let X be an algebraic variety over K. If there exists a central isogeny $f\colon G \to G'$, then X is a projective homogeneous space under G if and only if X is a projective homogeneous space under G'.*

PROOF. It follows directly from lemma 1.5.3(4) and lemma 1.5.4. [See also MPW96, Rem. 1.4(i)]. □

Let F be an arbitrary field, $\text{char}(F) \neq 2$. Let A be a central simple algebra whose center $Z(A)$ is a field extension of F. Let σ be an involution on A such that

$Z(A)^\sigma = F$. Let V be a finitely generated right A-module and let $h\colon V \times V \to A$ be an ε-hermitan form over (A,σ) for $\varepsilon \in \{1,-1\}$. Suppose

$$G = G(A,\sigma,h) = \begin{cases} \mathrm{SU}(A,\sigma,h) & \text{if } \sigma \text{ is of the first kind;} \\ \mathrm{U}(A,\sigma,h) & \text{if } \sigma \text{ is of the second kind,} \end{cases}$$

By example 1.4.2 and example 1.4.3, G is a connected *rational* linear algebraic group of type ${}^2A_n$, B_n, C_n, ${}^1D_n$ or ${}^2D_n$, where $n = \mathrm{Rank}_F(G)$ such that

$$\mathrm{rdim}(V) = \begin{cases} n+1, & \text{if } \sigma \text{ is unitary;} \\ 2n+1, & \text{if } A = F, \sigma = \mathrm{Id}_F \text{ and } \dim_F(V) \text{ is odd;} \\ 2n, & \text{otherwise.} \end{cases}$$

Let $0 < n_1 < \cdots < n_r \leq n$ be an increasing sequence of integers. For every field extension L/F, let

$$\begin{aligned} X(n_1,\ldots,n_r)(L) = \; & \{(W_1,\ldots,W_r) \mid 0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_r,\ W_i \text{ is a totally} \\ & \text{isotropic subspace of } V \otimes_F L,\ \mathrm{rdim}_{A_L} W_i = n_i \text{ for all } 1 \leq i \leq r\}. \end{aligned}$$

Alternatively, by [KMRT98, p. 6.2] and [Kar00, p. 16.4],

$$\begin{aligned} X(n_1,\ldots,n_r)(L) = \{ & (I_1,\ldots,I_r) \mid 0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_r,\ I_j \text{ is a totally isotropic} \\ & \text{ideal of } \mathrm{End}_{A\otimes_F L}(V \otimes_F L),\ \mathrm{rdim}_{A_L} I_j = n_j \text{ for all } 1 \leq j \leq r\}. \end{aligned}$$

When $r = 1$, we denote $X(n_1)$ by X_{n_1}.

Lemma 1.5.6 ([MPW96; MPW98, sec. 5 and sec. 9])**.** Let $0 < n_1 < \cdots < n_r \leq n$, $\varepsilon \in \{+,-\}$ and L/F a field extension. Then

(1) $X(n_1,\ldots,n_r)(L) \neq \emptyset$ if and only if $X_{n_r}(L) \neq \emptyset$ and $\mathrm{ind}(A_L) \mid \gcd\{n_1,\ldots,n_r\}$.

(2) $X^\varepsilon(n_1,\ldots,n_r)(L) \neq \emptyset$ if and only if $X^\varepsilon_{n_r}(L) \neq \emptyset$ and $\mathrm{ind}(A_L) \mid \gcd\{n_1,\ldots,n_r\}$.

Example 1.5.7 (Type ${}^1A_n$)**.** Let $\mathrm{PGL}_1(A)$ be as in example 1.3.6 and example 1.4.1. A *generalized Severi-Brauer variety* $\mathrm{SB}_r(A)$ of A over K [Bla91; VS94] satisfies

$$\mathrm{SB}_r(A)(L) = \{I \mid I \text{ is a right ideal of } A_L,\ \mathrm{rdim}_{A_L}(I) = r\}$$

for all field extensions L/K. The action of $\mathrm{PGL}_1(A)$ on $\mathrm{SB}_r(A)$ is left multiplication, then $\mathrm{SB}_r(A)$ is a projective homogeneous space under $\mathrm{PGL}_1(A)$. The set of projective homogeneous spaces of $\mathrm{PGL}_1(A)$ is

$$\{X(n_1,\ldots,n_r) \mid 0 < n_1 < \cdots < n_r < n\}$$

where for all field extensions L/K,

$$\begin{aligned}&Y(n_1,\ldots,n_r)(L)\\&=\{(I_1,\ldots,I_r)\in \mathrm{SB}_{n_1}(A)(L)\times\cdots\times \mathrm{SB}_{n_r}(A)(L) \mid 0\subsetneq I_1\subsetneq\cdots\subsetneq I_r\}.\end{aligned}$$

By [KMRT98, Prop. 1.17], $\mathrm{SB}_r(A)(L)\neq\emptyset$ if and only if $\mathrm{ind}(A_L)|r$. Then

$$Y(n_1,\ldots,n_r)(L)\neq\emptyset \iff \mathrm{ind}(A_L)|\gcd\{n_1,\ldots,n_r\}.$$

In particular, $\mathrm{SB}_1(A)$ is called the *Severi-Brauer variety* associated to A. If $A = (a,b)_K$ is a quaternion algebra, then $\mathrm{SB}_1(A)(L)$ is the projective plane conic

$$\mathrm{Proj}\left(\frac{L[X_0,X_1,X_2]}{(aX_0^2+bX_1^2-abX_2^2)}\right).$$

Here A is split over L/K if and only if $aX_0^2+bX_1^2-abX_2^2$ has a nontrivial solution over L.

Example 1.5.8 (Type ^2A_n)**.** [MPW98, §9.I]. Let $\mathrm{U}(A,\sigma,h)$ be as in example 1.3.6 and example 1.4.2. The set of projective homogeneous spaces of $\mathrm{U}(A,\sigma,h)$ is

$$\{X(n_1,\ldots,n_r) \mid 0\le n_1<\cdots<n_r<\lfloor n/2\rfloor.\}$$

Example 1.5.9 (Type B_n)**.** [MPW96, §5.II]. Let $\mathrm{SO}_{2n+1}(q)$ be as in example 1.3.7 and example 1.4.3. Let $X_q = \mathrm{Proj}\left(\frac{\mathrm{Sym}(V^*)}{(q)}\right)$. Then for all L/F, q_L is isotropic over L if and only if $X_q(L)\neq\emptyset$. The set of projective homogeneous spaces of $\mathrm{SO}_{2n+1}(q)$ is

$$\{X(n_1,\ldots,n_r) \mid 0\le n_1<\cdots<n_r\le n.\}$$

Here when $r = 1$ and $n_1 = 1$, we have $X_q = X(1)$.

Example 1.5.10 (Type C_n)**.** [MPW96, §5.III]. Let $\mathrm{SU}(A, \sigma, h)$ be as in example 1.3.7 and example 1.4.3. If ad_h is symplectic (i.e. σ is symplectic and h is hermitian, or σ is orthogonal and h is skew-hermitian), then $\mathrm{SU}(A, \sigma, h)$ has type C_n. The set of projective homogeneous spaces of $\mathrm{SU}(A, \sigma, h)$ is

$$\{X(n_1, \ldots, n_r) \mid 0 \leq n_1 < \cdots < n_r \leq n.\}$$

Example 1.5.11 (Type ${}^2D_n$)**.** [MPW96, §5.IV]. Let $\mathrm{SU}(A, \sigma, h)$ be as in example 1.3.7 and example 1.4.3. If ad_h is orthogonal (i.e. σ is orthogonal and h is hermitian, or σ is symplectic and h is skew-hermitian) and $\mathrm{disc}(h) \neq 1$, then $\mathrm{SU}(A, \sigma, h)$ has type ${}^2D_n$. The set of projective homogeneous spaces of $\mathrm{SU}(A, \sigma, h)$ is

$$\{X(n_1, \ldots, n_r) \mid 0 \leq n_1 < \cdots < n_r < n.\}$$

Example 1.5.12 (Type ${}^1D_n$)**.** [MPW96, §5.IV]. Let $\mathrm{SU}(A, \sigma, h)$ be as in example 1.3.7 and example 1.4.3. If ad_h is orthogonal (i.e. σ is orthogonal and h is hermitian, or σ is symplectic and h is skew-hermitian) and $\mathrm{disc}(h) = 1$, then $\mathrm{SU}(A, \sigma, h)$ has type ${}^1D_n$. If ad_h is orthogonal, $\mathrm{disc}(h) = 1$, $r = 1$ and $n_1 = n$, then X_n has two connected components X_n^+ and X_n^-. In this case, for $\varepsilon \in \{+, -\}$, denote

$$X^\varepsilon(n_1, \ldots, n_r)(L) = \{(I_1, \ldots, I_r) \in X(n_1, \ldots, n_r)(L) \mid I_r \in X_n^\varepsilon(L)\}, \tag{1.5.13}$$

The set of projective homogeneous spaces of $\mathrm{SU}(A, \sigma, h)$ is

$$\{X(n_1, \ldots, n_r) \mid 0 \leq n_1 < \cdots < n_r < n.\} \cup X_n^+ \cup X_n^-$$

$$\cup \{X^\varepsilon(n_1, \ldots, n_r) \mid 0 \leq n_1 < \cdots < n_{r-1} < n-1,\ n_r = n, r > 1, \varepsilon \in \{+, -\}.\}$$

In particular, let K be a field of characteristic not 2, let $\mathbb{H}\colon K^2 \to K$ be the hyperbolic plane such that $\mathbb{H}(x_1, x_2) = x_1 x_2$ for all $x_1, x_2 \in K$. Then

$$
\begin{aligned}
&\mathrm{SO}_2(\mathbb{H})\\
=&\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K) \;\middle|\; \mathbb{H}\left((x_1,x_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \mathbb{H}(x_1,x_2),\ ad-bc=1\right\}\\
=&\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K) \;\middle|\; (ax_1+bx_2)(cx_1+dx_2) = x_1x_2,\ ad-bc=1\right\}\\
=&\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K) \;\middle|\; ac=bd=0,\ ad+bc=ad-bc=1\right\}\\
=&\left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{GL}_2(K) \;\middle|\; a \in K^*\right\}
\end{aligned}
$$

Here $n=1$, $X_1 = X_{\mathbb{H}} = \{(x_1 : x_2) \in \mathbb{P}^1_K \mid x_1x_2 = 0\}$ has two elements. Each singleton $X_1^+ = \{(1:0) \in \mathbb{P}^1_K\}$, $X_1^- = \{(0:1) \in \mathbb{P}^1_K\}$ is an orbit of the $\mathrm{SO}_2(\mathbb{H})$ action on X_1.

Summarizing example 1.5.7, example 1.5.8, example 1.5.9, example 1.5.10, example 1.5.12 and example 1.5.11, we have:

(1.5.14)

$$
X = \begin{cases}
X(n_1,\dots,n_r), n_r < \lfloor n/2 \rfloor, \ \text{if } \sigma \text{ is unitary;}\\
X(n_1,\dots,n_r), \ \text{if } A = F, \sigma = \mathrm{Id}_F \ \text{and} \ \dim_F(V) \text{ is odd;}\\
X(n_1,\dots,n_r), \ \text{if } \ \mathrm{ad}_h \ \text{is symplectic;}\\
X(n_1,\dots,n_r), n_r < n, \ \text{if } \ \mathrm{ad}_h \ \text{is orthogonal and } \ \mathrm{disc}(h) \neq 1;\\
\left\{\begin{array}{l} X(n_1,\dots,n_r), n_r < n \text{ or}\\ X^{\pm}(n_1,\dots,n_r),\\ \ n_{r-1} < n-1 (\text{if } r>1), n_r = n \end{array}\right\}, \ \text{if } \ \mathrm{ad}_h \ \text{is orthogonal and } \ \mathrm{disc}(h) = 1.
\end{cases}
$$

1.6. Morita invariance

Let K be a field. Let A be a central simple algebra over K with an involution σ. Let $k = K^\sigma$. Suppose $\operatorname{char} k \neq 2$. Let V be a finitely generated right A-module and $\varepsilon \in \{1,-1\}$. Let $h\colon V \times V \to A$ be an ε-hermitian space over (A,σ).

Suppose $A = M_m(D)$ for a central division algebra D over F. By [KMRT98, Th. 3.1, Rem. 3.11, Rem. 3.20], D has an involution τ of same kind as σ. Fix an ε_0-hermitian space (D^m, g) over (D, τ) for $\varepsilon_0 \in \{1, -1\}$. By Morita equivalence [Knu91, ch. I, 9.3.5], there exists an $\varepsilon\varepsilon_0$-hermitian space (V_0, h_0) over (D, τ) defined by

$$V_0 = V \otimes_A D^m, \ h_0(x \otimes a, y \otimes b) = g(a, h(x, y)b).$$

Lemma 1.6.1. $\mathrm{rdim}_A(V) = \mathrm{rdim}_D(V_0)$.

Proof. By the definition of the reduced dimension and $\dim_K(A) = m^2 \dim_K(D)$, we have

$$\mathrm{rdim}_D(V_0) = \frac{\dim_K(V_0)}{\deg(D)} = \frac{\dim_K(V \otimes_A D^m)}{\deg(D)} = \frac{m \dim_K(V) \dim_K(D)}{\dim_K(A) \deg(D)}$$
$$= \frac{\dim_K(V))}{m \deg(D)} = \frac{\dim_K(V))}{\deg(A)} = \mathrm{rdim}_A(V)$$

□

Lemma 1.6.2. $\mathrm{Rank}(h) = \mathrm{Rank}(h_0)$.

Proof. By the definition of the rank of an ε-hermitian space, we have

$$\mathrm{Rank}(h) = \frac{\mathrm{rdim}(V)}{\mathrm{ind}(A)} = \frac{\mathrm{rdim}(V_0)}{\mathrm{ind}(D)} = \mathrm{Rank}(h_0).$$

□

Lemma 1.6.3. [Knu91, Ch. 1, 9.3.5].

(1) h is isotropic if and only if h_0 is isotropic.

(2) h is hyperbolic if and only if h_0 is hyperbolic.

For $0 < n_1 < \cdots < n_r \leq n$, let X be the projective homogeneous space under $G(A, \sigma, h)$ and X_0 be the projective homogeneous space under $G(D, \tau, h_0)$.

Lemma 1.6.4. [Kar00, Prop. 16.10]. $X(n_1, \ldots, n_r) \simeq X_0(n_1, \ldots, n_r)$.

In fact, we only need $X(n_1,\dots,n_r)(L)\neq\emptyset \iff X_0(n_1,\dots,n_r)(L)\neq\emptyset$. This is true since Morita equivalence preserves isotropy [Knu91, ch. I, 9.3.5] and it preserves reduced dimension.

Lemma 1.6.5. Suppose $\operatorname{rdim}(V)=2n$, ad_h is orthogonal, $\operatorname{disc}(h)=1$, $n_{r-1}<n-1$ (if $r>1$) and $n_r=n$. If $\operatorname{ind}(A_L)\,|\,\gcd\{n_1,\dots,n_r\}$, then $X^\varepsilon(n_1,\dots,n_r)(L)\neq\emptyset$ if and only if $X_0^\varepsilon(n_1,\dots,n_r)(L)\neq\emptyset$, for $\varepsilon\in\{+,-\}$.

Proof. By lemma 1.5.6 and lemma 1.6.4, it suffices to show that for $\varepsilon\in\{+,-\}$,

$$X_n^\varepsilon(L)\neq\emptyset \iff (X_0)_n^\varepsilon(L)\neq\emptyset.$$

This is true by the definition of X_n^ε (see the paragraph at [MPW96, p.577, 5.41, 5.42]). □

Lemma 1.6.6. Suppose $\operatorname{rdim}(V)=2n$, ad_h is orthogonal, $\operatorname{disc}(h)=1$, $n_{r-1}<n-1$ (if $r>1$) and $n_r=n$. Let $X^\varepsilon=X^\varepsilon(n_1,\dots,n_r)$ for $\varepsilon\in\{+,-\}$. Then $X^+(L)\neq\emptyset$ and $X^-(L)\neq\emptyset$ if and only if A_L is split and h_L is hyperbolic.

Proof. Suppose that A_L is split and h_L is hyperbolic. Then h_L is Morita equivalent to a hyperbolic quadratic form q over L. Let $X_0^\pm$ be corresponding projective homogeneous spaces under $\mathrm{SO}_{2n}(q)$. Since the Witt index of q is n, we have $(X_0)_n^+(L)\neq\emptyset$ and $(X_0)_n^-(L)\neq\emptyset$. Since A_L is split, we have $\operatorname{ind}(A_L)=1\,|\gcd\{n_1,\dots,n_r\}$. By lemma 1.5.6(2), $X_0^+(L)\neq\emptyset$ and $X_0^-(L)\neq\emptyset$. By lemma 1.6.5, $X^+(L)\neq\emptyset$ and $X^-(L)\neq\emptyset$.

Conversely, suppose $X^+(L)\neq\emptyset$ and $X^-(L)\neq\emptyset$. Let $W^+\in X^+(L)$ and $W^-\in X^-(L)$. Since there exists a totally isotropic subspace of reduced dimension n, which is equal to the Witt index of h_L, we have that h_L is hyperbolic. By Witt's extension theorem [Bou$_{A9}$, § 4, no. 3, th. 1] there exists $\varphi\in\mathrm{U}(A,\sigma,h)$ such that $\varphi(W^+)=W^-$. Since $\mathrm{SU}(A,\sigma,h)$ sends $X^+(L)$ into $X^+(L)$ and $X^-(L)$ into $X^-(L)$, we obtain $\varphi\notin\mathrm{SU}(A,\sigma,h)$. Thus, by [Kne69, 2.6, lem. 1. a)], A_L is split. □

Lemma 1.6.7. Let K be a field. Let A be a central simple algebra over K with an involution σ. Let $k = K^\sigma$. Suppose $\operatorname{char} k \neq 2$. Suppose $A \simeq M_m(D)$ for a central division algebra D over K. Suppose σ is an involution on A and $\varepsilon \in \{1, -1\}$. Then there exists an involution τ on D and $\varepsilon_0 \in \{1, -1\}$ such that $u(A, \sigma, \varepsilon) = u(D, \tau, \varepsilon\varepsilon_0)$.

Furthermore, $u^+(A) = u^+(D)$, $u^-(A) = u^-(D)$ and $u^0(A) = u^0(D)$.

Proof. By [Knu91, ch. I, 9.3.5], there exists a fixed ε_0-hermitian space (D^m, g) over (D, τ) such that σ is the adjoint involution of g in $\operatorname{End}_D(D^m) \simeq A$. Any ε-hermitian form (V, h) over (A, σ) is Morita equivalent to an $\varepsilon\varepsilon_0$-hermitian form $(V \otimes_A D^m, h_0)$ over (D, τ) such that h is isotropic if and only if h_0 is isotropic.

By lemma 1.6.2, $\operatorname{Rank}(h) = \operatorname{Rank}(h_0)$ for all pairs (h, h_0), we have $u(A, \sigma, \varepsilon) = u(D, \tau, \varepsilon\varepsilon_0)$.

By [KMRT98, p. 4.2], σ is orthogonal if and only if τ is orthogonal and $\varepsilon_0 = 1$ or τ is symplectic and $\varepsilon_0 = -1$; σ is symplectic if and only if τ is orthogonal and $\varepsilon_0 = -1$ or τ is symplectic and $\varepsilon_0 = 1$; σ is unitary if and only if τ is unitary. Hence $u^+(A) = u^+(D)$, $u^-(A) = u^-(D)$ and $u^0(A) = u^0(D)$. $\square$

CHAPTER 2

Hasse principle of projective homogeneous spaces

This chapter and the next chapter are based on my preprint [Wu15a].

Let F be a field. Let G be a connected linear algebraic group over F. Let X be a principal or projective homogeneous space under G. Let $\{F_v\}_{v\in\Omega}$ be a set of field extensions of F indexed by a set Ω. If $X(F) \neq \emptyset$, then clearly $X(F_v) \neq \emptyset$. The *Hasse-principle* holds for X with respect to $\{F_v\}_{v\in\Omega}$ if

$$\prod_{v\in\Omega} X(F_v) \neq \emptyset \implies X(F) \neq \emptyset.$$

Next, we give a short survey on what is known about Hasse principle of projective homogeneous spaces.

Let q be a quadratic form over a global field F. Let X_q be projective quadric associated to q. Then $X_q(L) \neq \emptyset$ if and only if q_L is isotropic for L/F. Let Ω be the set of all places on F. The Hasse-Minkowski theorem [Has23; Has24b; Has24a; Min90] states that if $q\colon V \to F$ is a quadratic form over a global field F, then q is isotropic over F iff q_{F_v} is isotropic over F_v for all $v \in \Omega_F$. Suppose $X_q(F_v) \neq \emptyset$ for all $v \in \Omega$. Then q_{F_v} is isotropic for all $v \in \Omega$. By the Hasse-Minkowski theorem, q is isotropic over F and hence $X_q(F) \neq \emptyset$. The local-global principle holds for projective quadrics over global fields. This is also why local-global principles are called Hasse principles.

The Albert-Brauer-Hasse-Noether theorem [BNH32; AH32] states that if A is a central simple algebra over a global field F, then $\operatorname{ind}(A) = \operatorname{lcm}_{v\in\Omega}\{\operatorname{ind}(A_{F_v})\}$. Suppose $\mathrm{SB}_r(A)(F_v) \neq \emptyset$ for all $v \in \Omega$. By [KMRT98, Prop. 1.17], $\mathrm{SB}_r(A)(F_v) \neq \emptyset$ if and only if $\operatorname{ind}(A_{F_v})|r$. Then $\operatorname{ind}(A_{F_v})|r$ for all $v \in \Omega$. Then $\operatorname{ind}(A) = \operatorname{lcm}_{v\in\Omega}\{\operatorname{ind}(A_{F_v})\}|r$.

By [KMRT98, Prop. 1.17] again, $\mathrm{SB}_r(A)(F) \neq \emptyset$. Hence the Hasse principle holds for generalized Severi-Brauer varieties over global fields.

Let D be a quaternion division algebra over a global field F. Let σ be the canonical involution on D. Let h be a skew-hermitian space over (D, σ) of rank ≥ 3. Kneser [Kne69, p. V.5.10] and Springer [Kne69, App.] have proved that if h_{F_v} is isotropic for all $v \in \Omega$, then h is isotropic. Further, the Hasse principle holds for projective homogeneous spaces under $\mathrm{SU}(D, \sigma, h)$ over F.

Let D be a division algebra over a global field F. Let σ be an involution on D of the second kind. Let h be a ε-hermitian space over (D, σ). Landherr [Lan37] has proved that if h_{F_v} is isotropic for all $v \in \Omega$, then h is isotropic. Further, the Hasse principle holds for projective homogeneous spaces under $\mathrm{U}(D, \sigma, h)$ over F.

Let G be a connected linear algebraic group over a number field F. Harder [Har68] has proved that the Hasse principle holds for all projective homogeneous space under G. Later, Borovoi [Bor93, Cor. 7.5] provides a new proof for the same result.

Let T be a complete discrete valuation ring with residue field k. Let K be the field of fractions of T. Let F be the function field of a smooth, projective, geometrically integral curve $\mathscr{X}_0$ over K. Recently, such a field F has been called a *semi-global* field. Let Ω be the set of all rank one discrete valuations on F (or the set of all divisorial discrete valuations from all codimension one points of all regular projective models $\mathscr{X} \to \mathrm{Spec}(T)$ of the curve $\mathscr{X}_0$). For each $v \in \Omega$, let F_v be the completion of F at v. Let G be a connected linear algebraic group over F and let X be a projective homogeneous space under G over F. We fix the above hypotheses for the next three paragraphs.

Suppose the residue field of T is k and $\mathrm{char}(k) \neq 2$. Colliot-Thélène, Parimala and Suresh [CPS12, Th. 3.1] have proved the following: *Let q be a quadratic form over F of rank ≥ 3. If q_{F_v} is isotropic for all $v \in \Omega$, then q is isotropic.* Hence the Hasse principle holds for all projective homogeneous spaces under $\mathrm{SO}(q)$ for such q. In the same paper, they made the following

Conjecture 2.0.1. [CPS12, conj. 1]. Let K be a p-adic field and F a function field of a curve over K. Let G be a connected linear algebraic group over F and let X be a projective homogeneous space under G over F. Then the Hasse principle holds for X.

Reddy and Suresh [RS13, Prop. 2.6] have proved the following: *Let l be a prime such that $l \neq \mathrm{char}(k)$. Let A be a central simple F-algebra of index a power of l, Suppose K contains a primitive $\mathrm{ind}(A)$-th root of unity. Then $\mathrm{ind}(A) = \mathrm{ind}(A \otimes_F F_v)$ for some $v \in \Omega$.* Their proof only needs the fact that K contains a primitive $\mathrm{per}(A)$-th root of unity. Hence the Hasse principle holds for all projective homogeneous space under $\mathrm{PGL}_1(A)$ if roots of unity are there.

After [COP02, Th. 3.1] and [CGP04, Th. 5.7], Harbater, Hartmann and Krashen [HHK11, Th. 9.2] have proved that if k is algebraically closed and $\mathrm{char}\, k = 0$, then the Hasse principle holds for projective homogeneous spaces under connected *rational* groups.

In this chapter, we obtain partial answer to conjecture 2.0.1 in corollary 2.3.7 as a corollary of our main result theorem 2.3.6.

2.1. Maximal orders

In this section we recall a theorem of Larmour on Hermitian spaces over discretely valued fields and prove results concerning maximal orders.

Definition 2.1.1. Let R be a noetherian integral domain with field of fractions K. Let A be a finite dimensional algebra over K. A subring Λ of A is called an *R-order in A* if Λ is a finitely generated R-submodule of A and $K\Lambda = A$.

An R-order Λ in A is called *maximal* if for all R-order Λ' in A such that $\Lambda' \supseteq \Lambda$, we have $\Lambda' = \Lambda$.

Let (K, v) be a discrete valued field with valuation ring R_v and residue field $k(v)$, $\mathrm{char}(k(v)) \neq 2$. Let K_v be the completion of K at v. Let D be a finite-dimensional division algebra over K with an involution σ such that $Z(D)^\sigma = K$. If $D \otimes_K K_v$ is a division algebra over K_v, then v extends uniquely to a valuation w on D such that $w(\sigma(x)) = w(x)$ for all $x \in D$. Let $R_w = \{x \in D \mid w(x) \geq 0\}$ be the valuation ring of (D, w).

Lemma 2.1.2. Suppose that $D \otimes_K K_v$ is a division algebra over K_v. There exists a unique maximal R_v-order Λ in D and the following four sets are identical.

(1) the maximal R_v-order Λ in D;
(2) the valuation ring $R_w = \{x \in D \mid w(x) \geq 0\}$;
(3) $N = \{x \in D \mid N_{D/K}(x) \in R_v\}$;
(4) the integral closure S of R_v in D.

PROOF. *Existence*: By [Rei03, Cor. 10.4], there exists a maximal R_v-order Λ in D.

Uniqueness: If Λ and Λ' are two maximal R_v-orders in D, by [Rei03, Th. 11.5] $\Lambda \otimes \widehat{R_v}$ and $\Lambda' \otimes \widehat{R_v}$ are two maximal $\widehat{R_v}$-orders in $D \otimes K_v$. By [Rei03, Th. 12.8], the maximal $\widehat{R_v}$-order in $D \otimes K_v$ is unique. Then $\Lambda \otimes \widehat{R_v} = \Lambda' \otimes \widehat{R_v}$. Then by [Rei03, Th. 5.2], $\Lambda = (\Lambda \otimes \widehat{R_v}) \cap D = (\Lambda' \otimes \widehat{R_v}) \cap D = \Lambda'$.

Equalities: Let Λ be the unique maximal R_v-order in D. By [Rei03, Eq. 12.7, Th. 12.8], the following sets are equal

- the maximal $\widehat{R_v}$-order $\widehat{\Lambda} = \Lambda \otimes \widehat{R_v}$ in $D \otimes K_v$;
- the valuation ring $\widehat{R_w} = \{x \in D \otimes K_v \mid w(x) \geq 0\}$;
- $\widehat{N} = \{x \in D \otimes K_v \mid N_{D\otimes K_v/K_v}(x) \in \widehat{R_v}\}$;
- the integral closure $\widehat{S}$ of $\widehat{R_v}$ in $D \otimes K_v$.

The proof of (1) equals (2): For $x \in D$, $w(x \otimes 1) = w(x)$, then $\widehat{R_w} \cap D = R_w$. Then $\Lambda = \widehat{\Lambda} \cap D = \widehat{R_w} \cap D = R_w$.

The proof of (1) equals (3): For $x \in D$, by [$\text{Bou}_{\text{AC8-9}}$, § 17, no. 3, prop. 4, (30)], $\mathrm{Nrd}_{(D\otimes K_v)/K_v}(x \otimes 1) = \mathrm{Nrd}_{D/K}(x)$, then

$$N_{(D\otimes K_v)/K_v}(x \otimes 1) = \mathrm{Nrd}_{(D\otimes K_v)/K_v}(x \otimes 1)^{\deg(D)} = \mathrm{Nrd}_{D/K}(x)^{\deg(D)} = N_{D/K}(x)$$

and hence $\widehat{N} \cap D = N$. Then $\Lambda = \widehat{\Lambda} \cap D = \widehat{N} \cap D = N$.

The proof of (1) equals (4): By [Rei03, Th. 8.6], $\Lambda \subseteq S$. Also, $S \subseteq \widehat{S} \cap D = \widehat{\Lambda} \cap D = \Lambda$. Therefore $\Lambda = S$. □

The next lemma will be applied in lemma 2.2.7.

Lemma 2.1.3. Suppose $D = (a, b)$ is a quaternion division algebra given by $i^2 = a$, $j^2 = b$, $ij = -ji$, where $a, b \in K$. Suppose $D \otimes_K K_v$ is a division algebra over K_v. If $v(a) = 0$ and $v(b) \in \{0, 1\}$, then $\Lambda = R_v + R_v i + R_v j + R_v ij$ is the unique maximal R_v-order in D.

Proof. By lemma 2.1.2, Λ is the unique maximal order if and only if Λ is the integral closure of R_v in D. Since i and j are integral over R_v, every element of Λ is integral over R_v.

Let $x \in D$. Then

$$x = y(x_0 + x_1 i + x_2 j + x_3 ij)$$

for some $y \in K^*$ and $x_0, x_1, x_2, x_3 \in R_v$ with $\min_{0 \le l \le 3}\{v(x_l)\} = 0$ (i.e. $(\overline{x_0}, \overline{x_1}, \overline{x_2}, \overline{x_3}) \neq \vec{0}$ in $k(v)^4$).

Suppose that x is integral over R_v. We show that $y \in R_v$. By taking the reduced norm, we have

$$\mathrm{Nrd}_{D/K}(x) = y^2(x_0^2 - x_1^2 a - x_2^2 b + x_3^2 ab).$$

Since x is integral over R_v, $\mathrm{Nrd}_{D/K}(x) \in R_v$ and hence $v(\mathrm{Nrd}_{D/K}(x)) \ge 0$. Suppose that $y \notin R_v$. Then $v(y) < 0$ and

$$v(x_0^2 - x_1^2 a - x_2^2 b + x_3^2 ab) = v(\mathrm{Nrd}_{D/K}(x) y^{-2}) \ge 2. \tag{2.1.4}$$

Case 1: D is unramified at v. Then $v(a) = v(b) = 0$. By going modulo the maximal ideal of R_v and using eq. (2.1.4), we see that $(\overline{x_0}, \overline{x_1}, \overline{x_2}, \overline{x_3}) \in k(v)^4$ is an isotropic vector for $\langle \overline{1}, -\overline{a}, -\overline{b}, \overline{a}\overline{b} \rangle$. Since K_v is a complete discretely valued field, by a theorem of Springer, $\langle 1, -a, -b, ab\rangle$ is isotropic over K_v, which contradicts the fact that $D \otimes_K K_v$ is division. Hence $y \in R_v$.

Case 2: D is ramified at v. Then $v(a) = 0$ and $v(b) = 1$. Since $(\overline{x_0}, \overline{x_1}, \overline{x_2}, \overline{x_3}) \neq \vec{0}$, we have $(\overline{x_0}, \overline{x_1}) \neq \vec{0}$ or $(\overline{x_2}, \overline{x_3}) \neq \vec{0}$ in $k(v)^2$.

Suppose $(\overline{x_0}, \overline{x_1}) \neq \vec{0}$. Going modulo the maximal ideal of R_v and using eq. (2.1.4), we see that $(\overline{x_0}, \overline{x_1}) \in k(v)^2$ is an isotropic vector for $\langle \overline{1}, -\overline{a}\rangle$.

Suppose $(\overline{x_0}, \overline{x_1}) = \vec{0}$. Then $(\overline{x_2}, \overline{x_3}) \neq \vec{0}$. Since $v(x_0) = v(x_1) \geq 1$, $v(x_0^2 - x_1^2 a) \geq 2$. Then, by eq. (2.1.4), we have $v(x_2^2 b - x_3^2 ab) \geq 2$. Since $v(b) = 1$, $v(x_2^2 - x_3^2 a) \geq 1$. Once again going modulo the maximal ideal of R_v, we see that $(\overline{x_2}, \overline{x_3}) \in k(v)^2$ is an isotropic vector of $\langle \overline{1}, -\overline{a}\rangle$.

By a theorem of Springer, $\langle 1, -a\rangle$ is isotropic over K_v, which contradicts the fact that $D \otimes_K K_v$ is division. Hence $y \in R_v$.

In both cases, we have $y \in R_v$. Thus $x \in \Lambda$ and Λ is the unique maximal R_v-order in D. □

2.2. Complete regular local ring of dimension 2

We fix the following notation and assumption throughout this section.

- R is a complete regular noetherian local ring of dimension 2,
- K is the field of fractions of R,
- $\mathfrak{m} = (\pi, \delta)$ is the maximal ideal of R,
- $k = R/\mathfrak{m}$, $\operatorname{char} k \neq 2$,
- $L = K(\sqrt{\lambda})$, $\lambda \in R$ with $\lambda = w$, $w\pi$ or $w\delta$ for a unit $w \in R$,
- S is the integral closure of R in L.

By the assumption on λ and [PS14, Prop. 3.1, Prop. 3.2], S is a regular local ring of dimension 2 with maximal ideal (π', δ'), where

- if $\lambda = w$ is a unit in R, then $\pi' = \pi$ and $\delta' = \delta$;
- if $\lambda = w\pi$, then $\pi' = \sqrt{w\pi}$ and $\delta' = \delta$;
- if $\lambda = w\delta$, then $\pi' = \pi$ and $\delta' = \sqrt{w\delta}$.

Let D be a central division algebra over L which is unramified at all height one prime ideals of S except possibly at π' and δ'. Let $\mathfrak{p}$ be a height one prime ideal of S. By [Mor89, Th. 2], the valuation $v_{\mathfrak{p}}$ extends to D if and only if $D \otimes_L L_{\mathfrak{p}}$ is a division algebra. Suppose $\deg(D) = d$ and K contains a primitive d-th root of unity, by [RS13, Prop. 2.4], $D \otimes_L L_{(\pi')}$ and $D \otimes_L L_{(\delta')}$ are division. Let $w_{\pi'}$ and $w_{\delta'}$ be the unique extensions of $v_{(\pi')}$ and $v_{(\delta')}$ to $D \otimes_L L_{(\pi')}$ and $D \otimes_L L_{(\delta')}$, respectively.

Lemma 2.2.1. Suppose that $\deg(D) = d$, K contains a primitive d-th root of unity and D has an involution σ (of the first or the second kind) with $L^\sigma = K$. Suppose there exists a maximal S-order Λ in D with $\sigma(\Lambda) = \Lambda$ and $\pi_D, \delta_D \in \Lambda$ such that

(1) $\mathrm{Nrd}_{D/L}(\pi_D) = u_0 \pi'^{d/e_0}$, where $u_0 \in R^*$, $e_0 = [w_{\pi'}(D^*) : v_{\pi'}(L^*)]$ and e_0 is invertible in k; $\mathrm{Nrd}_{D/L}(\delta_D) = u_1 \delta'^{d/e_1}$, where $u_1 \in R^*$, $e_1 = [w_{\delta'}(D^*) : v_{\delta'}(L^*)]$ and e_1 is invertible in k.

(2) $\sigma(\pi_D) = \varepsilon_0 \pi_D$, $\sigma(\delta_D) = \varepsilon_1 \delta_D$ and $\pi_D \delta_D = \varepsilon_2 \delta_D \pi_D$, $\varepsilon_0, \varepsilon_1, \varepsilon_2 \in \{1, -1\}$.

Let $c \in \Lambda$ such that $\sigma(c) = \pm c$ and $\mathrm{Nrd}_{D/L}(c) = u_c \pi'^{dm/e_0} \delta'^{dn/e_1}$ for $u_c \in S^*$, $m, n \in \mathbb{Z}$. Then

$$\langle c \rangle \simeq \langle \theta \pi_D^{m'} \delta_D^{n'} \rangle$$

for $\theta \in \Lambda^*$ and $m', n' \in \{0, 1\}$.

Proof. Since $\mathrm{Nrd}_{D/L}(c) = u_c \pi'^{dm/e_0} \delta'^{dn/e_1}$, it follows that $w_{\pi'}(c) = m w_{\pi'}(\pi'_D)$ and $w_{\delta'}(c) = n w_{\delta'}(\delta'_D)$. Write $m = 2r + m'$, $n = 2s + n'$ with $m', n' \in \{0, 1\}$. Let $x = \pi_D^r \delta_D^s$. Then $\sigma(x) = \varepsilon_0^r \varepsilon_1^s (\varepsilon_2)^{rs} x = \varepsilon_c x$, where $\varepsilon_c = \varepsilon_0^r \varepsilon_1^s (\varepsilon_2)^{rs} \in \{1, -1\}$. By the choice of π_D and δ_D, we have $\mathrm{Nrd}_{D/L}(x) = u_0^r u_1^s \pi'^{dr/e_0} \delta'^{ds/e_1}$.

Let $\theta = \varepsilon_c x^{-1} c x^{-1} (\pi_D^{m'} \delta_D^{n'})^{-1}$. Then $c = \sigma(x)(\theta \pi_D^{m'} \delta_D^{n'}) x$. In particular we have

$$\langle c \rangle \simeq \langle \theta \pi_D^{m'} \delta_D^{n'} \rangle.$$

Thus it is enough to show that $\theta \in \Lambda^*$.

Since $\Lambda = \bigcap_{\mathfrak{p}} \Lambda_{\mathfrak{p}}$, where $\mathfrak{p}$ runs through all height one prime ideals of S, we have $\Lambda^* = \bigcap_{\mathfrak{p}} \Lambda_{\mathfrak{p}}^*$. It suffices to show that $\theta \in \Lambda_{\mathfrak{p}}^*$ for all height one prime ideals $\mathfrak{p}$ of S. We have that $\mathrm{Nrd}_{D/L}(\theta) = \mathrm{Nrd}_{D/L}(x)^{-2}\,\mathrm{Nrd}_{D/L}(c)\,\mathrm{Nrd}_{D/L}(\pi_D^{m'}\delta_D^{n'})^{-1} = u_c$ is a unit in S.
Case 1: Suppose $\mathfrak{p} \neq (\pi'), (\delta')$. Since $\pi_D, \delta_D \in \Lambda$ and $\mathrm{Nrd}_{D/L}(\pi_D)$, $\mathrm{Nrd}_{D/L}(\delta_D)$ are units at $\mathfrak{p}$, by [Sal99, 4.3(c)], π_D and δ_D are units in $\Lambda_{\mathfrak{p}}$. Since $x \in \Lambda_{\mathfrak{p}}^*$ and $c \in \Lambda$, we have $\theta \in \Lambda_{\mathfrak{p}}$. Since $\mathrm{Nrd}_{\Lambda_{\mathfrak{p}}/S_{\mathfrak{p}}}(\theta) = \mathrm{Nrd}_{D/L}(\theta) \in S^*$, by [Sal99, 4.3(c)], $\theta \in \Lambda_{\mathfrak{p}}^*$.
Case 2: Suppose $\mathfrak{p} = (\pi')$. Since $w_{\pi'}(\theta) = 0$, by lemma 2.1.2, $\theta \in \Lambda_{(\pi')}^*$.
Case 3: Suppose $\mathfrak{p} = (\delta')$. The proof of $\theta \in \Lambda_{(\delta')}^*$ is similar to Case 2. □

Corollary 2.2.2. *Let D, σ, Λ, π_D and δ_D be as in lemma 2.2.1. Let h be a non-degenerate ε-hermitian form over (D, σ). Suppose that $h = \langle a_1, \ldots, a_n\rangle$ with $a_i \in \Lambda$ and $\mathrm{Nrd}_{D/L}(a_i)$ is a unit of S times a power of π' and a power of δ' for all $1 \le i \le n$. Then*

$$h \simeq \langle u_1, \ldots, u_{n_0}\rangle \perp \langle v_1, \ldots, v_{n_1}\rangle\pi_D \perp \langle w_1, \ldots, w_{n_2}\rangle\delta_D \perp \langle \theta_1, \ldots, \theta_{n_3}\rangle\pi_D\delta_D$$

with $u_i, v_i, \theta_i \in \Lambda^*$ and $n_0 + n_1 + n_2 + n_3 = n$.

Proof. Follows from lemma 2.2.1. □

Corollary 2.2.3. *Under all hypotheses of corollary 2.2.2, if $h \otimes_K 1_{K_\pi}$ is isotropic over $(D \otimes_K K_\pi, \sigma \otimes_K \mathrm{Id}_{K_\pi})$ or $h \otimes_K 1_{K_\delta}$ is isotropic over $(D \otimes_K K_\delta, \sigma \otimes_K \mathrm{Id}_{K_\delta})$, then h is isotropic over (D, σ).*

Proof. By corollary 2.2.2, we have

$$h \simeq h_{00} \perp h_{10}\delta_D \perp h_{01}\pi_D \perp h_{11}\delta_D\pi_D$$

where diagonal entries of h_{ij} are in Λ^*. Applying Larmour's result proposition 1.2.3 to $h_{K_{\pi'}}$, we have $q_{\pi'}(h_{00} \perp h_{10}\delta_D)$ or $q_{\pi'}(h_{01} \perp h_{11}\delta_D)$ is isotropic over $D(\pi')$. Applying proposition 1.2.3 again, we obtain that one of $q_{\bar{\delta}'}(q_{\pi'}(h_{ij}))$ is isotropic over $D(\pi')(\bar{\delta}')$. Since the diagonal entries of h_{ij} are in Λ^*, $(D, \mathrm{Int}(\delta_D^i\pi_D^j) \circ \sigma, h_{ij})$ is defined over the maximal R-order Λ in D. By [Knu91, ch. II, 4.6.1 and 4.6.2], one of h_{ij} is isotropic over $(\Lambda, \mathrm{Int}(\delta_D^i \pi_D^j) \circ \sigma|_\Lambda)$. Then one of $h_{ij}\delta_D^i\pi_D^j$ is isotropic over $(\Lambda, \sigma|_\Lambda)$. Then h is isotropic over $(\Lambda, \sigma|_\Lambda)$ and hence over (D, σ). □

Corollary 2.2.4. *Let R, K, S and L be as before and let ι be an automorphism of L such that $\iota|_K = \mathrm{Id}_K$. Let $h = \langle a_1, \dots, a_n\rangle$ be an ε-hermitian space over (L, ι) for $\varepsilon \in \{1, -1\}$. Suppose that each divisor of a_i is supported only along π' and δ'. If $h \otimes_K 1_{K_\pi}$ is isotropic over $(L \otimes_K K_\pi, \iota \otimes_K \mathrm{Id}_{K_\pi})$ or $h \otimes_K 1_{K_\delta}$ is isotropic over $(L \otimes_K K_\delta, \iota \otimes_K \mathrm{Id}_{K_\delta})$, then h is isotropic over $(L/K, \iota)$.*

Proof. Let $D = L$, $\sigma = \iota$, $\Lambda = S$, $\pi_D = \pi'$ and $\delta_D = \delta'$ in corollary 2.2.3. □

Suppose D is a quaternion algebra. The aim of the rest of the section is to show that there exists a maximal order Λ, π_D and δ_D as in lemma 2.2.1.

We begin with Saltman's classification.

Proposition 2.2.5. [Sal97; Sal98, Prop. 1.2], [Sal07, Prop. 2.1] Suppose $\alpha \in {}_2\mathrm{Br}(K)$. If α is unramified at all height one prime ideals of R except possibly at (π) and (δ), then α is of the form $\alpha = \alpha' + \alpha''$, where $\alpha' \in \mathrm{Br}(R)$ and α is described as follows:

(i) If α is unramified at all height one prime ideals of R, then $\alpha = \alpha'$;

(ii) If α is ramified only at (π), then $\alpha = \alpha' + (u, \pi)$ for some $u \in R^* \setminus R^{*2}$;

(iii) If α is ramified only at (π) and (δ), then there exists $u, v \in R^*$ such that

(a) $\alpha = \alpha' + (u\pi, v\delta)$; or

(b) $\alpha = \alpha' + (u, \pi) + (v, \delta)$, where u, v and uv are not squares and u, v are in different square classes; or

(c) $\alpha = \alpha' + (u, \pi\delta)$, where u is not a square.

Lemma 2.2.6. Let D be a quaternion division algebra over K which is unramified at all height one prime ideals of R except possibly at (π) and (δ). Then D is isomorphic to one of the following over K.

(1) (u, v), $u, v \in R^*$;

(2) $(u, v\pi)$, $u \in R^*$ is not a square;

(3) $(u, v\delta)$, $u \in R^*$ is not a square;

(4) $(u\pi, v\delta)$, $u, v \in R^*$;

(5) $(u, v\pi\delta)$, $u \in R^*$ is not a square and $v \in R^*$.

Proof. (1) Suppose D is unramified on R. By [AG60a, Th. 7.4], there exists an Azumaya algebra $\mathcal{D}$ over R with $\mathcal{D} \otimes_R K \simeq D$. Since D is a quaternion algebra over K, $\mathcal{D} \otimes_R k$ is a quaternion algebra over k. Hence $\mathcal{D} \otimes_R k = (a, b)$ for $a, b \in k^*$. Let $u, v \in R^*$ be lifts of $a, b \in k$. Since R is complete, by [AG60a, Th. 6.5], $D \simeq (u, v)$.

(2) Let α be the class of D in ${}_2\operatorname{Br}(K)$. Suppose that D is ramified on R only at (π). Then, by proposition 2.2.5, $\alpha = \alpha' + (u, \pi)$ for $\alpha' \in \operatorname{Br}(R)$ and $u \in R^*$. As in the proof of [RS13, Prop. 2.4], we have $\operatorname{ind}(D) = \operatorname{ind}(D \otimes K_\pi) = 2(\operatorname{ind}(\alpha' \otimes K(\sqrt{u})))$. Since D is a quaternion algebra, $\alpha \otimes K(\sqrt{u})$ is split. Then $\alpha' = (u, v)$ for some $v \in K^*$. Since α' is unramified on R, we may assume that $v \in R^*$. Thus $\alpha = \alpha' \otimes (u, \pi) = (u, v) \otimes (u, \pi) = (u, v\pi)$ in $\operatorname{Br}(K)$. Then $D = (u, v\pi)$.

(3) Similarly, if D is ramified only at δ, then $D = (u, v\delta)$.

(4) and (5). Suppose that D only ramifies at π and δ. Then, by proposition 2.2.5, we have $\alpha = \alpha' + \alpha''$ with $\alpha' \in \operatorname{Br}(R)$ and $\alpha'' = (u\pi, v\delta)$ or $(u, \pi) + (v, \delta)$ or $(u, \pi\delta)$ with $u, v \in R^*$.

(i) Suppose that $\alpha'' = (u, \pi\delta)$. Then as above, it follows that $D = (u, v\pi\delta)$.

(ii) Suppose that $\alpha'' = (u\pi, v\delta)$. Then, as above, we have that $\alpha'' \otimes K(\sqrt{\delta})$ is trivial. Since α' is unramified on R, as in the proof of [RS13, Prop. 2.4], α' is trivial. Thus $\alpha = (u\pi, v\delta)$.

(iii) Suppose that $\alpha'' = (u, \pi) + (v, \delta)$. As in the proof of [RS13, Prop. 2.4], we have $\operatorname{ind}(\alpha) = \operatorname{ind}(\alpha' \otimes K(\sqrt{u}, \sqrt{v})) \cdot [K(\sqrt{u}, \sqrt{v}) : K]$. Since $\operatorname{ind}(\alpha) = 2$, we have $[K(\sqrt{u}, \sqrt{v}) : K] \leq 2$. Since u and v are non-squares in K, u and v are in the same square class, a contradiction to proposition 2.2.5(iii)(b). Thus this case does not happen. □

Next, we consider maximal-orders of certain quaternion algebras.

Lemma 2.2.7. Let $D = (a, b)$ be a quaternion division algebra over K given by i, j such that $i^2 = a$, $j^2 = b$ and $ij = -ji$. Let Λ be the R-algebra generated by $\{1, i, j, ij\}$. If D has one of the forms of lemma 2.2.6, then Λ is a maximal R-order in D.

Proof. By definition, Λ is an order in D. By [AG60b, Th. 1.5], an order of a noetherian integrally closed domain is maximal if and only if it is reflexive and its localization at all height one prime ideals are maximal orders. Since R is a regular local ring, it is a noetherian integrally closed domain. Since Λ is a finitely generated free R-module, it is reflexive. We show that $\Lambda_{\mathfrak{p}}$ is a maximal $R_{\mathfrak{p}}$-order for all height one prime ideals $\mathfrak{p}$ of R.

Case 1: Suppose $\mathfrak{p} \neq (\pi)$ and $\mathfrak{p} \neq (\delta)$. Then $a, b \in R_{\mathfrak{p}}^*$ and hence $\Lambda_{\mathfrak{p}}$ is an Azumaya algebra over $R_{\mathfrak{p}}$. In particular $\Lambda_{\mathfrak{p}}$ is a maximal $R_{\mathfrak{p}}$-order in D.

Case 2: Suppose $\mathfrak{p} = (\pi)$. Then, by [RS13, Prop. 2.4], $D \otimes_K K_\pi$ is a quaternion division algebra over K_π. By lemma 2.1.3, $\Lambda_{(\pi)}$ is a maximal $R_{(\pi)}$-order in D.

Case 3: Suppose $\mathfrak{p} = (\delta)$. Similar to case 2, we can show that $\Lambda_{(\delta)}$ is a maximal $R_{(\delta)}$-order in D. □

Next, we construct parameters for certain quaternions with involutions of the first kind.

Lemma 2.2.8. Let D be a quaternion division algebra over K having one of the forms of lemma 2.2.6 *except* (5) and let σ be the canonical involution on D. Let Λ be the maximal order as in lemma 2.2.7.

Then there exists $\pi_D, \delta_D \in \Lambda$ such that

(1) $\mathrm{Nrd}_{D/K}(\pi_D) = u_0 \pi^{2/e_0}$ and $\mathrm{Nrd}_{D/K}(\delta_D) = u_1 \delta^{2/e_1}$, where $u_0, u_1 \in R^*$, $e_0 = [w_\pi(D^*) : v_\pi(K^*)]$, $e_1 = [w_\pi(D^*) : v_\delta(K^*)]$ and $e_0, e_1 \in \{1, 2\}$;

(2) $\sigma(\pi_D) = \pm\pi_D$, $\sigma(\delta_D) = \pm\delta_D$, $\sigma(\pi_D\delta_D) = \pm\pi_D\delta_D$ and $\pi_D\delta_D = \pm\delta_D\pi_D$.

Proof. We discuss every case of lemma 2.2.6 except (5). In the following, u, v are units and we assume them nonsquare if necessary (to make D a division algebra). We assume that for a quaternion algebra (a,b), $i^2=a$, $j^2=b$, $ij=-ji$. If $D=(u,v)$, take $\pi_D=\pi$ and $\delta_D=\delta$; otherwise take π_D and δ_D as follows.

D	π_D	δ_D	$\mathrm{Nrd}(\pi_D)$	$\mathrm{Nrd}(\delta_D)$	$\sigma(\pi_D)$	$\sigma(\delta_D)$	$\sigma(\pi_D\delta_D)$
$(u,v\pi)$	j	δ	$-v\pi$	δ^2	$-\pi_D$	δ_D	$-\pi_D\delta_D$
$(u,v\delta)$	π	j	π^2	$-v\delta$	π_D	$-\delta_D$	$-\pi_D\delta_D$
$(u\pi,v\delta)$	i	j	$-u\pi$	$-v\delta$	$-\pi_D$	$-\delta_D$	$-\pi_D\delta_D$

Then π_D and δ_D have required properties. □

Next, we construct parameters for certain quaternions with involutions of the second kind. Suppose that L/K is a degree 2 extension and D/L a quaternion algebra with an involution σ of second kind. Then, by a theorem of Albert (see [KMRT98, Th. 2.22]), there exists a quaternion algebra D_0 over K such that $D\simeq D_0\otimes_K L$ and the involution σ maps to the involution $\sigma\otimes\iota$ where σ_0 is the canonical involution of D_0 and ι is the non-trivial automorphism of L/K.

Lemma 2.2.9. Let $L=K(\sqrt{\lambda})$, S and (π',δ') as before. Let D_0 be a quaternion division algebra over K which is unramified at all height one prime ideals of R except possibly at (π) and (δ). If $D_0=(u,v\pi\delta)$, we suppose that λ is not a unit in R. Let $D=D_0\otimes_K L$. Let σ_0 the canonical involution of D_0, ι be the non-trivial automorphism of L/K and $\sigma=\sigma_0\otimes_K\iota$. If D is division, then there exist a maximal S-order Λ in D which is invariant under σ and $\pi_D,\delta_D\in\Lambda$ such that

(1) $\mathrm{Nrd}_{D/L}(\pi_D)=u_0\pi'^{2/e_0}$ and $\mathrm{Nrd}_{D/L}(\delta_D)=u_1\delta'^{2/e_1}$, where $u_0,u_1\in S^*$, $e_0=[w_{\pi'}(D^*):v_{\pi'}(L^*)]$, $e_1=[w_{\delta'}(D^*):v_{\delta'}(L^*)]$ and $e_0,e_1\in\{1,2\}$;

(2) $\sigma(\pi_D)=\pm\pi_D$, $\sigma(\delta_D)=\pm\delta_D$, $\sigma(\pi_D\delta_D)=\pm\pi_D\delta_D$ and $\pi_D\delta_D=\pm\delta_D\pi_D$.

Proof. By lemma 2.2.6, $D_0=(u,v)$, $(u,v\pi)$, $(u,v\delta)$, $(u\pi,v\delta)$ or $(u,v\pi\delta)$ for some $u,v\in R^*$. If $D_0=(a,b)$, then let $i_0,j_0\in D_0$ with $i_0^2=a$, $j_0^2=b$ and $i_0j_0=-j_0i_0$.

There are 3 possible shapes for λ, i.e. w, $w\pi$, $w\delta$ with w a unit. By the assumption that if $\lambda=w$, then D_0 is not of the form $(u,v\pi\delta)$. Since there are 5 possible shapes of D_0, we have $3 * 5 - 1 = 14$ possible combinations. In each of the cases, choose i and j as in the following two tables.

λ	w	w	w	w	$w\pi$	$w\pi$	$w\delta$	$w\delta$
D_0	(u,v)	$(u,v\pi)$	$(u,v\delta)$	$(u\pi,v\delta)$	(u,v)	$(u,v\delta)$	(u,v)	$(u,v\pi)$
D	(u,v)	$(u,v\pi')$	$(u,v\delta')$	$(u\pi',v\delta')$	(u,v)	$(u,v\delta')$	(u,v)	$(u,v\pi')$
i	$i_0 \otimes 1$							
j	$j_0 \otimes 1$							

λ	$w\pi$	$w\pi$	$w\pi$	$w\delta$	$w\delta$	$w\delta$
D_0	$(u,v\pi)$	$(u\pi,v\delta)$	$(u,v\pi\delta)$	$(u,v\delta)$	$(u\pi,v\delta)$	$(u,v\pi\delta)$
D	(u,vw)	$(uw,v\delta')$	$(u,vw\delta')$	(u,vw)	$(u\pi',vw)$	$(u,vw\pi')$
i	$i_0 \otimes 1$	$\frac{1}{\pi}(i_0 \otimes \sqrt{\lambda})$	$i_0 \otimes 1$	$i_0 \otimes 1$	$i_0 \otimes 1$	$i_0 \otimes 1$
j	$\frac{1}{\pi}(j_0 \otimes \sqrt{\lambda})$	$j_0 \otimes 1$	$\frac{1}{\pi}(j_0 \otimes \sqrt{\lambda})$	$\frac{1}{\delta}(j_0 \otimes \sqrt{\lambda})$	$\frac{1}{\delta}(j_0 \otimes \sqrt{\lambda})$	$\frac{1}{\delta}(j_0 \otimes \sqrt{\lambda})$

Then it can be checked that π' and δ' are the only primes in S which might divide $i^2, j^2 \in L$. Let $\Lambda = S + Si + Sj + Sij$. Then, by lemma 2.2.7, Λ is a maximal S-order of D. By the choice if i and j we have $\sigma(i) = \pm i$ and $\sigma(j) = \pm j$. Since $\sigma(S) = S$, $\sigma(\Lambda) = \Lambda$.

Let $\pi_D, \delta_D \in \Lambda$ be as in the proof of lemma 2.2.8. Then Λ, π_D and δ_D satisfy required properties (1) and (2). □

Corollary 2.2.10. *Let D be a quaternion division algebra over K with σ the canonical involution and h an ε-hermitian space over (D,σ). Suppose that D is unramified at all height one prime ideals of R except possibly at (π), (δ) and D is not of the shape of lemma 2.2.6(5). Let Λ be the maximal order as in lemma 2.2.8. Suppose h has a diagonal form $\langle a_1, \ldots, a_n\rangle$ such that $a_i \in \Lambda$ and $\mathrm{Nrd}_{D/K}(a_i)$ is a unit of R times a power of π and a power of δ. If $h \otimes_K 1_{K_\pi}$ is isotropic over $(D \otimes_K K_\pi, \sigma \otimes_K \mathrm{Id}_{K_\pi})$ or $h \otimes_K 1_{K_\delta}$ is isotropic over $(D \otimes_K K_\delta, \sigma \otimes_K \mathrm{Id}_{K_\delta})$, then h is isotropic over (D,σ).*

Proof. Follows from lemma 2.2.8 and corollary 2.2.3. □

Corollary 2.2.11. *Let $L = K(\sqrt{\lambda})$, $\lambda = w$, $w\pi$ or $w\delta$ for $w \in R^*$. Let S be the integral closure of R in L and the maximal ideal $m' = (\pi', \delta')$ of S as above. Let D_0 be a quaternion division algebra over K having one of the forms of lemma 2.2.6 and σ_0 the canonical involution on D_0. When $D_0 = (u, v\pi\delta)$, we suppose that λ is not a unit in R. Let ι be the non-trivial automorphism of L/K. Let $D = D_0 \otimes_K L$ and $\sigma = \sigma_0 \otimes_K \iota$. Suppose that D is division. Let Λ be the maximal order as in lemma 2.2.9. Let h be an ε-hermitian space over (D, σ). Suppose h has a diagonal form $\langle a_1, \dots, a_n\rangle$ such that $a_i \in \Lambda$ and $\mathrm{Nrd}_{D/L}(a_i)$ is a unit of S times a power of π' and a power of δ'. If $h \otimes_K 1_{K_\pi}$ is isotropic over K_π or $h \otimes_K 1_{K_\delta}$ is isotropic over K_δ, then h is isotropic over K.*

Proof. Follows from lemma 2.2.9 and corollary 2.2.3. □

The next corollary is for σ of the first kind.

Corollary 2.2.12. *Under the hypotheses of corollary 2.2.10, let X be a projective homogeneous space under $G = \mathrm{SU}(D, \sigma, h)$ over K. If $X(K_\pi) \neq \emptyset$ or $X(K_\delta) \neq \emptyset$, then $X(K) \neq \emptyset$.*

Proof. First we assume that X is of the shape of the 2nd, 3rd, 4th or first part of the 5th case of eq. (1.5.14).

By [RS13, p. 2.4], $\mathrm{ind}(D) = \mathrm{ind}(D \otimes_K K_\pi) = \mathrm{ind}(D \otimes_K K_\delta)$. Then $\mathrm{ind}(D \otimes_K K_\pi)|g$ iff $\mathrm{ind}(D \otimes_K K_\delta)|g$ iff $\mathrm{ind}(D)|g$, where $g = \gcd\{n_1, \dots, n_r\}$. Let $t = n_r$. By lemma 1.5.6, it suffices to show that if $X_t(K_\pi) \neq \emptyset$ or $X_t(K_\delta) \neq \emptyset$, then $X_t(K) \neq \emptyset$. Suppose $X_t(K_\pi) \neq \emptyset$. Then h_{K_π} has a totally isotropic subspace of reduced dimension t, where t is even. Then h_{K_π} is isotropic over D. So, by corollary 2.2.10, $h\colon V \times V \to D$ is isotropic over D. Let $x \in V$, $x \neq 0$ be an isotropic vector of h. Let i_W denote the Witt index. Then $2 \leq \mathrm{rdim}_D(xD) \leq t \leq 2i_W(h_{K_\pi})$ and $\mathrm{rdim}_D(xD)$ is even.

We induct on t. If $t = 2$, then $\mathrm{rdim}_D(xD) = 2$, we have $xD \in X_t(K)$ and hence $X_t(K) \neq \emptyset$.

Now we suppose $t > 2$. If $\mathrm{rdim}_D(xD) = t$, then $xD \in X_t(K)$ and hence $X_t(K) \neq \emptyset$. If $\mathrm{rdim}_D(xD) < t$, by [Knu91, ch.1, 3.7.4], there exists a hyperbolic plane $\mathbb{H} \subseteq (V, h)$ such that $x \in \mathbb{H}$ and $h = h' \perp \mathbb{H}$. Then by [KMRT98, p.73],

$$2i_W(h'_{K_\pi}) = 2i_W(h_{K_\pi}) - 2 \geq 2i_W(h_{K_\pi}) - \mathrm{rdim}_D(xD) \geq t - \mathrm{rdim}_D(xD) > 0.$$

Write X'_t for the corresponding projective homogeneous variety under $\mathrm{SU}(D, \sigma, h')$ over K. Then $X'_{t-\mathrm{rdim}_D(xD)}(K_\pi) \neq \emptyset$. Since $t - \mathrm{rdim}_D(xD) < t$, by induction, we have $X'_{t-\mathrm{rdim}_D(xD)}(K) \neq \emptyset$. Suppose $N \in X'_{t-\mathrm{rdim}_D(xD)}(K)$. Then $N \oplus xD \in X_t(K)$. Hence $X_t(K) \neq \emptyset$.

Therefore $X_t(K_\pi) \neq \emptyset$ implies $X_t(K) \neq \emptyset$. Similarly, $X_t(K_\delta) \neq \emptyset$ implies $X_t(K) \neq \emptyset$.

Next we assume that X is of the shape of the second part of the 5th case of eq. (1.5.14), now $t = n = n_r$. We need to prove the following

Subcase $(+)$: If $X_n^+(K_\pi) \neq \emptyset$ or $X_n^+(K_\delta) \neq \emptyset$, then $X_n^+(K) \neq \emptyset$;

Subcase $(-)$: If $X_n^-(K_\pi) \neq \emptyset$ or $X_n^-(K_\delta) \neq \emptyset$, then $X_n^-(K) \neq \emptyset$.

Suppose $X_n^+(K_\pi) \neq \emptyset$. Then h_{K_π} is hyperbolic. By corollary 2.2.10 with Witt decomposition, Witt cancellation and induction, h is hyperbolic. Then $X_n(K) = X_n^+(K) \sqcup X_n^-(K) \neq \emptyset$. If $X_n^+(K) \neq \emptyset$ we are done. If $X_n^-(K) \neq \emptyset$, then $X_n^-(K_\pi) \neq \emptyset$. Then both $X_n^+(K_\pi) \neq \emptyset$ and $X_n^-(K_\pi) \neq \emptyset$. By lemma 1.6.6, we have D_{K_π} is split. By [RS13, Prop. 2.4], D is split over K, a contradiction to our assumption that D is division. Hence, $X_n^+(K) \neq \emptyset$ and $X_n^-(K) = \emptyset$.

The proof for the subcase $(-)$ is similar. □

The next corollary is for σ of the second kind.

Corollary 2.2.13. *Under the hypotheses of corollary 2.2.11, let X be a projective homogeneous space under $G = \mathrm{U}(D, \sigma, h)$ over K (see the first case of eq. (1.5.14)). If $X(K_\pi) \neq \emptyset$ or $X(K_\delta) \neq \emptyset$, then $X(K) \neq \emptyset$.*

PROOF. The proof is similar to the first half of corollary 2.2.12 (for the 2nd, 3rd, 4th and the first part of the 5th cases of eq. (1.5.14)), using corollary 2.2.11. □

Corollary 2.2.14. *Under the hypotheses of corollary 2.2.4, let X be a projective homogeneous space under $G = \mathrm{U}(L, \iota, h)$ over K. If $X(K_\pi) \neq \emptyset$ or $X(K_\delta) \neq \emptyset$, then $X(K) \neq \emptyset$.*

PROOF. The proof is similar to the first half of corollary 2.2.12, using corollary 2.2.4. □

2.3. Patching and Hasse principle

In this section, we prove theorem 2.3.6.

Let T be a complete discrete valuation ring with a parameter t. Suppose $\mathrm{char}(T/tT) \neq 2$. Let $\mathscr{X}$ be a regular projective T-curve with function field F and special fiber $\mathscr{X}_1$.

For the patching data, we adopt notations as in [HHK09, Notation 3.3]. For every closed point P of $\mathscr{X}_1$, let $\widehat{R_P}$ be the completion of the local ring R_P of $\mathscr{X}$ at P and $F_P = \mathrm{Frac}(\widehat{R_P})$. Let $\mathscr{X}_\eta$ be an irreducible component of $\mathscr{X}_1$ and U be a non-empty open subset of $\mathscr{X}_\eta$ containing only smooth points. Let R_U be the set of elements in F which are regular at every closed point of U. Let $\widehat{R}_U$ be the (t)-adic completion of R_U and $F_U = \mathrm{Frac}(\widehat{R}_U)$.

Lemma 2.3.1. [HHK09, Th. 3.7] Let G be a rational connected linear algebraic group over F and let X be a projective homogeneous space under G. Let $\mathcal{P}$ be a nonempty finite subset of $\mathscr{X}_1$. Let $\mathcal{U}$ be the set of connected components of $\mathscr{X}_1 \setminus \mathcal{P}$. Then

$$\prod_{P \in \mathcal{P}} X(F_P) \times \prod_{U \in \mathcal{U}} X(F_U) \neq \emptyset \implies X(K) \neq \emptyset.$$

The next lemma deals with the last case of lemma 2.2.6 to make it possible to apply lemma 2.2.8 in the proof of theorem 2.3.5.

Lemma 2.3.2. Let R be a regular local ring with field of fractions K, maximal ideal (π, δ) and residue field k with $\operatorname{char} k \neq 2$. Suppose $\alpha = (u, v\pi\delta) \in {}_2\operatorname{Br}(K)$. Let $\mathscr{X} = \operatorname{Proj}(R[x, y]/(\pi x - \delta y)) \to \operatorname{Spec}(R)$ be the blow-up of $\operatorname{Spec}(R)$ at its maximal ideal. For every closed point Q of $\mathscr{X}$, let $\mathfrak{m}_Q$ be the maximal ideal of $\mathscr{O}_{\mathscr{X},Q}$. Then $\alpha = (u, t)$ for $t \in \mathscr{O}_{\mathscr{X},Q}$ such that t is either a unit or a regular parameter (i.e. $t \notin \mathfrak{m}_Q \setminus \mathfrak{m}_Q^2$).

Proof. Let Q_1 be the closed point given by the homogeneous ideal (π, δ, x) and Q_2 the closed point given by the homogeneous ideal (π, δ, y). Let $t = \dfrac{x}{y} \in K$. Then $\delta = t\pi$ in K. Hence at Q_1, t is a regular parameter and $\alpha = (u, v\pi\delta) = (u, vt\pi^2) = (u, t)$. Similarly, at Q_2, $1/t$ is a regular parameter and $\alpha = (u, 1/t)$. Let Q be a closed point of $\mathscr{X}$ that is neither Q_1 nor Q_2. Then at Q, t is a unit and $\alpha = (u, t)$. $\square$

The next lemma deals with λ from lemma 2.2.9 to make it possible to apply lemma 2.2.8 in the proof of theorem 2.3.5.

Lemma 2.3.3. Let R be a regular local ring of dimension 2 with field of fractions K and residue field k with $\operatorname{char} k \neq 2$. Let $\lambda \in K$ and $\alpha \in {}_2\operatorname{Br}(K)$. Then there exists a finite sequence of blow-ups $\mathscr{X} \to \operatorname{Spec}(R)$ such that for every closed point P of $\mathscr{X}$, the maximal ideal $\mathfrak{m}_P$ of $\mathscr{O}_{\mathscr{X},P}$ is given by $\mathfrak{m}_P = (\pi, \delta)$, $\lambda = w$, $w\pi$ or $w\delta$, up to squares for $u \in \mathscr{O}^*_{\mathscr{X},P}$ and $\alpha = \alpha' + \alpha''$ with α' and α'' as in proposition 2.2.5. Furthermore, if $\alpha'' = (u, v\pi\delta)$ for units $u, v \in R^*$, then $\lambda \notin \mathscr{O}^*_{\mathscr{X},P}$, up to squares.

Proof. By choosing a finite sequence of blow-ups $\mathscr{X} \to \operatorname{Spec}(R)$, we may assume that for every closed point P of $\mathscr{X}$, $\mathfrak{m}_P = (\pi, \delta)$, $\lambda = w$, $w\pi$, $w\delta$ or $w\pi\delta$, up to squares, for $w \in \mathscr{O}^*_{\mathscr{X},P}$ and α is unramified at P except possibly at π and δ. In fact, let P be a closed point of $\mathscr{X}$ such that $\mathfrak{m}_P = (\pi, \delta)$ and $\lambda = w\pi\delta$ for some unit w of $\mathscr{O}_{\mathscr{X},P}$. Let $\mathscr{X}'$ be the blowup of $\mathscr{X}$ at P and Q a closed point on the exceptional curve. By lemma 2.3.2, $\lambda = wt$ or w', up to squares, for units w and w' and t is either a unit or regular parameter. Since there are only finitely many closed points on $\mathscr{X}$ with $\lambda = w\pi\delta$, we have a finite sequence of blowups $\mathscr{X}' \to \mathscr{X}$ such that for every closed point P' of $\mathscr{X}'$, $\mathfrak{m}_{P'}$, λ and α has the desired property at P'. In particular, $\alpha = \alpha' + \alpha''$ with α' and α'' as in proposition 2.2.5.

Suppose there exists a closed point P of $\mathscr{X}'$ such that $\alpha'' = (u, v\pi\delta)$ and $\lambda = w$, for $u, v, w \in \mathscr{O}^*_{\mathscr{X}',P}$. Let $\mathscr{X}'' \to \mathscr{X}'$ be the blow-up at P as in lemma 2.3.2. Then for every closed point Q of the exceptional curve of $\mathscr{X}''$, by lemma 2.3.2, we have $\alpha'' = (u, v)$ or (u, t) for a regular parameter t at Q and $u, v \in \mathscr{O}^*_{\mathscr{X}'',Q}$. Since $\lambda = w \in \mathscr{O}^*_{\mathscr{X}',P}$, it remains a unit in $\mathscr{O}^*_{\mathscr{X}'',Q}$. Since there are only finitely many closed points with $\alpha'' = (u, v\pi\delta)$, we have the required sequence of blow-ups $\mathscr{X}'' \to \operatorname{Spec}(R)$. □

Lemma 2.3.4. Let R be a regular local ring of dimension 2 with field of fractions K and residue field k. Suppose $\operatorname{char} k \neq 2$. Let $\widehat{R}$ be the completion of R at its maximal ideal and $\widehat{K}$ the field of fractions of $\widehat{R}$. Let $\mu \in \widehat{K}^*$. Then there is a finite sequence of blow-ups $\mathscr{X} \to \operatorname{Spec}(R)$ such that for every closed point Q of $\mathscr{X} \times_{\operatorname{Spec} R} \operatorname{Spec}(\widehat{R})$, the maximal ideal at Q is given by (π, δ) with the support of μ at Q is at most (π) and (δ). Also, either (π) or (δ) corresponds to an exceptional curve in $\mathscr{X}$.

Proof. Since $\widehat{R}$ is a regular local ring of dimension 2, there exists a finite sequence of blow-ups $\widehat{\mathscr{X}} \to \operatorname{Spec} \widehat{R}$ at the closed point of $\operatorname{Spec}(\widehat{R})$ and closed points on the exceptional curves such that the support of μ on $\widehat{\mathscr{X}}$ is a union of regular curves with normal crossings [Abh69] or [Lip75]. Since any exceptional curve is the projective line over a finite extension of k, there exists a finite sequence of blow-ups $\mathscr{X} \to \operatorname{Spec}(R)$ such that $\mathscr{X} \times_{\operatorname{Spec}(R)} \operatorname{Spec} \widehat{R} = \widehat{\mathscr{X}}$ (see [HHK15, prop. 3.6]).

Let Q be a closed point of $\widehat{\mathscr{X}}$. Then, by the choice of $\widehat{\mathscr{X}}$, the maximal ideal at Q is given by (π, δ) and the support of μ at Q is at most (π) and (δ). Suppose that neither (π) nor (δ) is an exceptional curve. Then blow-up Q. The resulting sequence of blow-ups has required properties. □

Theorem 2.3.5. *Let K be a complete discrete valued field with residue field k, $\operatorname{char} k \neq 2$. Let F be the function field of a smooth, projective, geometrically integral curve over K. Let L/F be an extension of degree at most 2 and A a finite-dimensional* *simple F-algebra with center L. Let σ be an involution on A such that $F = L^\sigma$. Let $h\colon V \times V \to A$ be an ε-hermitian space over (A, σ) for $\varepsilon \in \{1, -1\}$. Let*

$$G(A,\sigma,h) = \begin{cases} \mathrm{SU}(A,\sigma,h) & \text{if } \sigma \text{ is of the first kind;} \\ \mathrm{U}(A,\sigma,h) & \text{if } \sigma \text{ is of the second kind.} \end{cases}$$

Suppose that for any regular proper model $\mathscr{X}$ of F and for any closed point P of $\mathscr{X}$ $\mathrm{ind}(A \otimes F_P) \leq 2$. Then the Hasse principle holds for any projective homogeneous space under $G(A, \sigma, h)$.

Proof. Let X be a projective homogeneous space under $G(A, \sigma, h)$. Suppose that $X(F_v) \neq \emptyset$ for all divisorial discrete valuations of F. We use [HHK09, Th. 3.7] to show that $X(F) \neq \emptyset$. Since σ is arbitrary, we assume that $\varepsilon = 1$.

Write $L = F(\sqrt{\lambda})$ for $\lambda \in F^*$. Let $\mathscr{X}$ be a regular proper model of F such that the union of the support of λ and the special fiber $\mathscr{X}_1$ of $\mathscr{X}$ is a union of regular curves with normal crossings. Let η be a codimension zero point of $\mathscr{X}_1$. Since $X(F_\eta) \neq \emptyset$, by [HHK11, Th. 5.8], there exists a non-empty open subset U_η of the closure of η such that $X(F_{U_\eta}) \neq \emptyset$ and U_η does not meet other regular curves in the special fiber $\mathscr{X}_1$.

Let $\mathcal{P}$ be the finite set of closed points of $\mathscr{X}_1$ which are not on U_η for any codimension zero point η of $\mathscr{X}_1$. For $P \in \mathcal{P}$, let D_P be the central division algebra over $L_P = L \otimes F_P$ which is Brauer equivalent to $A \otimes F_P$. By Morita equivalence [Knu91, ch. I, 9.3.5], there exists an involution σ_P on D_P and h corresponds to a hermitian form h_P over (D_P, σ).

Since for any closed point P of $\mathscr{X}$, $\deg(D_P) \leq 2$, either $D_P = L_P$ or D_P is a quaternion division algebra. If $[L : F] = 2$, since $L^\sigma = F$, $L_P^{\sigma_P} = F_P$ and by a theorem of Albert [KMRT98, Th. 2.22], there exists a central division algebra $(D_P)_0$ over F_P such that $\deg((D_P)_0) \leq 2$ and $D_P \simeq (D_P)_0 \otimes L_P$. If $\deg((D_P)_0) = 2$, then write $(D_P)_0 = (a_P, b_P)$ for some $a_P, b_P \in F_P$.

By lemma 2.3.4, there exists a finite sequence of blow-ups $\phi\colon \mathscr{X}' \to \mathscr{X}$ such that for each $P \in \mathcal{P}$ and $Q \in \phi^{-1}(P)$, the support of a_P and b_P at Q have normal crossings.

In particular the ramification divisor of $(D_P)_0$ at Q has normal crossings. Let η be an exceptional curve in $\mathscr{X}'$. Since $X(F_\eta) \neq \emptyset$, as above there exists a non-empty open set U_η of the closure of η such that $X(F_{U_\eta}) \neq \emptyset$. Let $Q \in \mathscr{X}'$ be in the closure of η. Suppose $D \otimes F_Q$ is non-split. Since $\phi(Q) = P$ and $D \otimes F_Q$ is Brauer equivalent to $D_P \otimes F_Q = (D_P)_0 \otimes L \otimes F_Q$. In particular the support of the ramification divisor of $(D_P)_0 \otimes F_Q$ has normal crossings. Thus, replacing $\mathscr{X}$ by $\mathscr{X}'$, we assume that if $P \in \mathscr{P}$, then $D_P = (D_P)_0 \otimes L_P$ and the ramification divisor of $(D_P)_0$ has normal crossings at P. Further, replacing $\mathscr{X}$ by a finite sequence of blow-ups at the points of $\mathcal{P}$, using lemma 2.3.3, we assume that for $P \in \mathcal{P}$, D_P and λ are as in lemma 2.3.3.

Let $P \in \mathscr{P}$. If $D_P = L_P$, let Λ_P be the integral closure of $\widehat{R_P}$ in L_P. If $D_P \neq L_P$, then $D_P = (D_P)_0 \otimes L_P$ with $(D_P)_0$ a quaternion algebra and $(D_P)_0$, λ are as in lemma 2.3.3. Let Λ_P be the order as in lemma 2.2.8 or lemma 2.2.9. Since D_P is division, $h_P = \langle a_1^P, \ldots, a_m^P \rangle$ with $a_i^P \in \Lambda_P$ and $\sigma_P(a_i^P) = a_i^P$. Let $f_i^P = \mathrm{Nrd}_{D_P}(a_i^P) \in F_P \subseteq L_P$. Since $\sigma_P(a_i^P) = a_i^P$, $f_i^P \in F_P$. Once again, using lemma 2.3.4, replacing $\mathscr{X}$ by a finite sequence of blow-ups of $\mathscr{X}$ at the points of $\mathcal{P}$, we assume that for every $P \in \mathcal{P}$, the maximal ideal at P is given by (π_P, δ_P), the support of f_i^P is at most π_P and δ_P and at least one of π_P and π_P is an exceptional curve.

Let X^P be the projective homogeneous space under $G(D_P, \sigma_P, h_P)$. The maximal ideal at P is given by (π_P, δ_P) and either π_P or δ_P, say π_P, gives an exceptional curve. Since the valuation given by an exceptional curve is a divisorial discrete valuation, $X(F_{\pi_P}) \neq \emptyset$. Thus, by lemma 1.6.4 or lemma 1.6.5, $X^P((F_P)_{\pi_P}) \neq \emptyset$. If $D_P = L_P$, then, by [CPS12, Th. 3.1] or corollary 2.2.14, $X(F_P) \neq \emptyset$. If D_P is a quaternion algebra, then, by corollary 2.2.12 or corollary 2.2.13, $X^P(F_P) \neq \emptyset$. By lemma 1.6.4 or lemma 1.6.5 again, $X(F_P) \neq \emptyset$ for all $P \in \mathcal{P}$.

Therefore, by [HHK09, Th. 3.7], $X(F) \neq \emptyset$. □

Now we state and prove the main result of chapter 2.

Theorem 2.3.6. *Let K be a complete discrete valued field with residue field k, $\operatorname{char} k \neq 2$. Let F be the function field of a smooth, projective, geometrically integral curve over K. Let Ω be the set of all rank one discrete valuations on F. For each $v \in \Omega$, let F_v be the completion of F at v. Let A be a finite-dimensional simple F-algebra with an involution σ such that $F = Z(A)^\sigma$. Suppose that at least one of the following is satisfied.*

(1) $\operatorname{ind}(A) \leq 2$*;*

(2) $\operatorname{per}(A) = 2$*,* $|l^*/l^{*2}| \leq 2$ *and* ${}_2\operatorname{Br}(l) = 0$ *for all finite extensions* l/k*.*

Let $\varepsilon \in \{1, -1\}$ and $h\colon V \times V \to A$ an ε-hermitian space over (A, σ). Let X be a projective homogeneous space under

$$G = \begin{cases} \mathrm{SU}(A, \sigma, h) & \textit{if } \sigma \textit{ is of the first kind;} \\ \mathrm{U}(A, \sigma, h) & \textit{if } \sigma \textit{ is of the second kind.} \end{cases}$$

If $X(F_v) \neq \emptyset$ for all $v \in \Omega$, then $X(F) \neq \emptyset$.

Remark. In case (1), the underlying division algebra of A is F, or a quadratic field extension of F, or a quaternion division algebra with center F, or a quaternion division algebra whose center is a quadratic extension of F.

In case (2), if σ is of the first kind, then $\operatorname{per}(A) = 2$ since $A \simeq A^{\mathrm{op}}$; if σ is of the second kind, in general we do not have $\operatorname{per}(A) = 2$. By [Ser79, XIII, §2], examples of such k in (2) are finite fields or fields of Laurent series with coefficients in an algebraically closed field of characteristic 0, for example $\mathbb{C}((t))$.

Proof. Let $L = Z(A)$. Let $\mathscr{X}$ be a regular proper model of L with ramification locus of A a union of regular curves with normal crossings and P a closed point of $\mathscr{X}$. Let k_P be the residue field of $\widehat{R_P}$ and $L_P = L \otimes F_P$.

(1) If $\operatorname{ind}(A) \leq 2$, we have $\operatorname{ind}(A \otimes L_P) \leq 2$ for all closed points P of $\mathscr{X}$.

(2) Suppose $\operatorname{per}(A) = 2$, $|l^*/l^{*2}| \leq 2$ and ${}_2\operatorname{Br}(l) = 0$ for all finite extensions l/k. Then k_P^* has at most two square classes and ${}_2\operatorname{Br}(k_P) = 0$. Then by [AG60a, p. 6.2], ${}_2\operatorname{Br}(\widehat{R_P}) = 0$. Then, by proposition 2.2.5, $\operatorname{ind}(A \otimes L_P) \leq 2$.

Hence the Hasse principle is a consequence of theorem 2.3.5. □

Next, we prove corollary 2.3.7, which partially answers conjecture 2.0.1.

Corollary 2.3.7. *Let p be an odd prime. Let K be a p-adic field. Let F a function field in one variable over K. Let Ω be the set of all discrete valuations on F. Let G be a connected linear algebraic group such that there exists an isogeny from a product of almost simple groups of one of the following types to the semisimple group $G/\operatorname{Rad}(G)$.*

$$^1A_n, \quad ^2A_n^*, \quad B_n, \quad C_n, \quad ^1D_n, \quad ^2D_n,$$

where $^2A_n^*$ means that the almost simple factor is isogenous to a unitary group $\mathrm{U}(A,\sigma,h)$ such that σ is of the second kind and $\operatorname{per}(A)=2$. Let X be a projective homogeneous space under G. Then

$$\prod_{v\in\Omega} X(F_v)\neq\emptyset \implies X(F)\neq\emptyset.$$

Proof. Let G^{ss} be the semisimple group $G/\operatorname{Rad}(G)$. By [CGP04, Cor. 5.7], X is a projective homogeneous space under G^{ss}. By [Bor91, 14.10(2)], there exists an isogeny $G_1\times\cdots\times G_r\to G^{ss}$ where G_i are almost simple groups. Since char $F=0$, all isogenies of algebraic groups over F are central. By [BT72, 2.20, (i)], central sujective morphisms of algebraic groups give isomorphisms of their projective homogeneous spaces. Then X is a projective homogeneous space under $G_1\times\cdots\times G_r$. By [MPW98, 6.10(e)], $X\simeq X_1\times\cdots\times X_r$ where X_i is a projective homogeneous space under G_i for each $1\le i\le r$. Then $X(F)\neq\emptyset$ if and only if $X_i(F)\neq\emptyset$ for all $1\le i\le r$. By assumption, G_i has one of the types ^1A_n, $^2A_n^*$, B_n, C_n, ^1D_n, ^2D_n. The type ^1A_n case has been proved by Reddy and Suresh [RS13, Th. 2.6]. The type B_n case has been proved by Colliot-Thélène, Parimala and Suresh [CPS12, Th. 3.1]. By [Tit66, Table 1], if G_i has type $^2A_n^*$, then G_i is isogenous to $\mathrm{U}(A,\sigma,h)$; if G_i has type B_n, C_n or D_n, then G_i is isogenous to $\mathrm{SU}(A,\sigma,h)$. By [BT72, 2.20, (i)] again, we may assume that G_i is the unitary group or the special unitary group as above and hence X is as in eq. (1.5.14). The rest follow from theorem 2.3.6. □

CHAPTER 3

Springer's problem for odd degree extensions

Let F be a field of characteristic not 2. Let q be a quadratic form over F. Let M be an odd degree extension of F. Springer [Spr52] has proved that if q_M is isotropic, then q is isotropic.

We could ask a similar question about Hermitian forms. Let A be a central simple algebra over F with an involution σ. Let $h\colon V \times V \to A$ be an ε-hermitian form over (A, σ) for $\varepsilon \in \{1, -1\}$. Let M be an odd degree extension of F. It is natural to ask whether the isotropy of h_M implies the isotropy of h. This question has been studied by many mathematicians and they have obtained partial answers.

Bayer-Fluckiger and Lenstra [BL90] have proved that if h_M is hyperbolic, then h is hyperbolic.

Suppose h_1 and h_2 are two ε-hermitian spaces over (A, σ). Lewis [Lew00] has proved that when σ is of the first kind, if $(h_1)_M \simeq (h_2)_M$, then $h_1 \simeq h_2$. Barquéro-Salavert [Bar06] has proved that when σ is of the second kind, if $(h_1)_M \simeq (h_2)_M$, then $h_1 \simeq h_2$.

Parimala, Sridharan and Suresh [PSS01] have proved that if A is a quaternion algebra and σ is of the first kind, if h_M is hyperbolic, then h is hyperbolic. They have also provided an example to show that this is not true in general if $\operatorname{ind}(A)$ is odd and σ of the second kind.

Let $E = \operatorname{End}_A(V)$ and let τ be the adjoint involution of h. Black and Quéguiner-Mathieu [BQ14] proved that when $\deg E = 12$ and τ is orthogonal, if τ_M is hyperbolic, then τ is hyperbolic. They have also proved that when $\deg E = 6$, $\operatorname{per} E = 2$ and τ is unitary, if τ_M is hyperbolic, then τ is hyperbolic.

3.1. Reduction to the residue field

We begin with the following.

Lemma 3.1.1. Let (L, v) be a complete discrete valued field and k_L the residue field of L with char $k_L \neq 2$. Let M be an odd degree extension of L, with residue field k_M. We make the following assumption on residue fields:

Let E be a central division algebra E over k_L with an involution τ. Let $\varepsilon' \in \{1, -1\}$. Let φ be an ε'-hermitian form over (E, τ). If φ_{k_M} is isotropic, then φ is isotropic, for all tuples (E, τ, ε').

Let D be a central division algebra over L and $\operatorname{per}(D) = 2$. Let σ be an involution on D. Let $\varepsilon \in \{1, -1\}$. Let h be an ε-hermitian form over (D, σ). If h_M is isotropic, then h is isotropic.

Proof. Since L is complete, the valuation v on L extends to a discrete valuation v' on M. Let t be a uniformizer of L, t' a uniformizer of M such that $(t')^e = t$ where $e = e(M/L)$. By [GS06, Prop. 4.5.11, 2.], $D' = D \otimes_L M$ is a division algebra. Let w be the extension of v to D and w' the extension of v' to D'. Let π be a uniformizer of D and π' a uniformizer of D'. By [Lar99, Prop. 2.7], there exists $x \in D$ such that

$$w(x) \equiv w(\pi) \mod 2w(D^*), \quad \sigma(x) = \varepsilon x, \varepsilon \in \{1, -1\}. \tag{3.1.2}$$

By the second to the last paragraph of [Wad02, p. 393], $e(D'/D)$ is a factor of $[M : L]$. Since $[M : L]$ is odd, $e(D'/D)$ is odd. Then $w'(\pi \otimes_L 1_M) \equiv w'(\pi') \mod 2w'(D'^*)$. Let $x' = x \otimes 1 \in D'$ and $\sigma(x') = \varepsilon x'$. By Larmour's theorem, proposition 1.2.3,

$$h \simeq h_1 \perp h_2 x \tag{3.1.3}$$

where all diagonal entries of h_1 and h_2 have valuation 0 in D. Thus

$$h_M \simeq (h_1)_M \perp (h_2)_M (x \otimes_L 1_M) = (h_1)_M \perp (h_2)_M x' \tag{3.1.4}$$

In the following, an overline means "over the residue field". We have

$$\begin{aligned} & h_M \text{ is isotropic,} \\ \iff & \text{one of } \overline{(h_i)_M} \text{ is isotropic over } (\overline{D \otimes_L M}, \overline{\sigma \otimes_L \mathrm{Id}_M}), \\ & \text{by applying proposition 1.2.3 to eq. (3.1.4).} \\ \iff & \text{one of } (\overline{h_i})_{k_M} \text{ is isotropic over } (\overline{D} \otimes_{k_L} k_M, \overline{\sigma} \otimes_{k_L} \mathrm{Id}_{k_M}). \\ \iff & \text{one of } \overline{h_i} \text{ is isotropic over } (\overline{D}, \overline{\sigma}) \text{, by the given condition on } k_M/k_L. \\ \iff & h \text{ is isotropic over } (D, \sigma) \text{, by applying proposition 1.2.3 to eq. (3.1.3).} \end{aligned}$$

where $i \in \{1, 2\}$. □

3.2. Springer's theorem over local or global fields

3.2.1. Let L be an arbitrary field of characteristic not 2. Let M be an odd degree extension of L. For each discrete valuation v of L with valuation ring R_v and maximal ideal $\mathfrak{p}_v$, let $\widehat{R_v}$ be its completion and $L_v = \mathrm{Frac}(\widehat{R_v})$. Let S be the integral closure of R_v in M and $\mathfrak{P}_i, 1 \le i \le n$ be prime ideals of S lying over $\mathfrak{p}_v$. Let $\widehat{S_i}$ be the completion of S at $\mathfrak{P}_i$ and $M_i = \mathrm{Frac}(\widehat{S_i})$. By [CF67, p. 15, (2)],

$$M \otimes_L L_v \simeq \prod_{i=1}^{n} M_i.$$

Since $[M : L] = [M \otimes_L L_v : L_v] = \sum_{i=1}^{n} [M_i : L_v]$ is odd, there exists some j, $1 \le j \le n$ such that $[M_j : L_v]$ is odd.

Lemma 3.2.2. Let L be a non-archimedean *local* field of characteristic not 2. Let M be an odd degree extension of L. Let D be a division algebra over L such that $D \neq L$. Let σ be an involution of D. Let h be an ε-hermitian form over (D, σ). If h_M is isotropic, then h is isotropic.

Proof. Let σ be of the first kind. By [Sch85, ch. 10, 2.2(i)], D is the unique quaternion division algebra over L, and it suffices to apply [PSS01, Th. 3.5].

Let σ be of the second kind. If $\varepsilon = -1$, by Hilbert 90 [Bou$_{\text{A4-7}}$, ch. V, § 11, no. 6, th. 3, a)], there exists $\mu \in Z(D) \setminus L$ such that $\sigma(\mu) = -\mu$. By scaling [Knu91, ch. I, 5.8], h is isotropic over (D, σ) if and only if $\mu^{-1}h$ is isotropic over (D, σ), where

$\mathrm{Int}(\mu) \circ \sigma = \sigma$ and $\mu^{-1}h$ is a hermitian form. Hence we may assume that $\varepsilon = 1$. By [Sch85, ch. 10, 2.2(ii)], D/L is a quadratic field extension. Also D_M/M is a quadratic field extension. Let h be a hermitian form over (D, σ), q is the quadratic form over L associated to $h(x, x)$. By definition, q_M is the quadratic form over L associated to $h_M(x, x)$. Then

$$\begin{aligned} & h_M \text{ is isotropic over } D_M, & \\ \Longleftrightarrow\ & q_M \text{ is isotropic over } M, & \text{by [Sch85, ch. 10, 1.1(i)];} \\ \Longleftrightarrow\ & q \text{ is isotropic over } L, & \text{by Springer's theorem [Spr52];} \\ \Longleftrightarrow\ & h \text{ is isotropic over } D, & \text{by [Sch85, ch. 10, 1.1(i)].} \end{aligned}$$

□

Lemma 3.2.3. Let L be a *global* field of characteristic not 2. Let M be an odd degree extension of L. Let D be a division L-algebra with an involution σ such that $D \neq L$ and $\mathrm{per}(D) = 2$. Let h be an ε-hermitian form over (D, σ). If h_M is isotropic, then h is isotropic.

Proof. If σ is of the first kind, by [Sch85, ch. 10, 2.3(vi)], D is a quaternion division algebra and the result follows from [PSS01, Th. 3.5].

Now suppose σ is of the second kind. Suppose $Z(D) = L(\sqrt{\lambda})$. Let Ω_L be all the places of L and Ω_M all the places of M. If $v \in \Omega_L$ such that λ is a square in L_v, by [Sch85, ch. 10, 6.3] h_{L_v} is hyperbolic over $(D \otimes_L L_v, \sigma \otimes_L \mathrm{Id}_{L_v})$.

Suppose $v \in \Omega_L$ is such that λ is not a square in L_v. by 3.2.1 we have an odd degree extension M_j/L_v.

Case 1: v is non-archimedean and $D \otimes_L L_v$ is not split. Since h_M is isotropic, h_{M_j} is isotropic. By lemma 3.2.2, h_{L_v} is isotropic.

Case 2: v is non-archimedean and $D \otimes_L L_v$ is split. Then $D \otimes M_j$ is split. Since h_M is isotropic, h_{M_j} is isotropic. Suppose h_{L_v} is Morita equivalent to a quadratic form q over L_v. Then h_{M_j} is Morita equivalent to the quadratic form q_{M_j}. Then q_{M_j}

is isotropic. By [Spr52], q is isotropic over L_v. By Morita equivalence again, h_{L_v} is isotropic.

Case 3: v is archimedean. Any place $w \in \Omega_M$ that lies over v is still archimedean. Since $[M_j : L_v]$ is odd, $M_j = L_v \simeq \mathbb{R}$ or $\mathbb{C}$. Since h_M is isotropic, $h_{M_w} = h_{L_v}$ is isotropic.

By three cases above, h_{L_v} is isotropic for all $v \in \Omega_L$. Finally, by Landherr's local-global principle over L (see [Lan37] or [Sch85, ch. 10, 6.2]), h is isotropic. □

3.3. Springer's theorem over function fields of p-adic curves

The next theorem is our main theorem of chapter 3.

Theorem 3.3.1. *Let p be an odd prime. Let K be a p-adic field. Let F be the function field of a smooth, projective, geometrically integral curve over K. Let Ω be the set of all rank one discrete valuations on F. Let A be a finite-dimensional central simple F-algebra with an involution σ of the first kind. Let $h\colon V \times V \to A$ be an ε-hermitian space over (A, σ) for $\varepsilon \in \{1, -1\}$.*

Let M be an odd degree extension of F. If h_M is isotropic, then h is isotropic.

Proof. In fact, by Morita equivalence [Knu91, ch. I, 9.3.5], we assume that $A = D$ is a central division F-algebra. Suppose that h_M is isotropic. Let $\deg D = d$, $\dim_D(V) = m$ and $i_W(h_M)$ the Witt index of h_M. Then $1 \leq i_W(h_M) \leq \frac{m}{2}$ and $X_d(M) \neq \emptyset$, where X_d is as in eq. (1.5.14).

Suppose $i_W(h_M) = \frac{m}{2}$. Then h_M is hyperbolic. By [BL90], h is hyperbolic.

Suppose that $i_W(h_M) < \frac{m}{2}$. Let $v \in \Omega$. By 3.2.1, we have an extension M_j/F_v such that $[M_j : F_v]$ is odd. Let k_j be the residue field of M_j and $k(v)$ the residue field of F_v. Since $e(M_j/F_v)f(M_j/F_v) = [M_j : F_v]$ is odd, $[k_j : k(v)] = f(M_j/F_v)$ is odd. Since $X_d(M) \neq \emptyset$, we have $X_d(M \otimes F_v) \neq \emptyset$. In particular, $X_d(M_j) \neq \emptyset$. Since the residue fields are either local or global (see [Par14, §8.1]), $[k_j : k(v)]$ is odd and $\mathrm{per}(D \otimes_F F_v) | 2$, by lemma 3.2.2 and lemma 3.2.3, the conditions in lemma 3.1.1 are satisfied. By Morita equivalence and lemma 3.1.1, $X_d(F_v) \neq \emptyset$ for all v. Finally by the Hasse principle theorem 2.3.6, $X_d(F) \neq \emptyset$, so h is isotropic. □

CHAPTER 4

Hermitian u-invariants

This chapter is based on my preprint [Wu15b].

Let p be an odd prime number. Let F be the function field of a smooth projective geometrically integral curve over a p-adic field. Let D be a central division F-algebra with an involution σ of the first kind. We are interested in finding $u^+(D)$ and $u^-(D)$.

If $D = F$, then $u^+(D) = u(F)$ and $u^-(D) = 0$. Here $u(F)$ is the u-invariant for quadratic forms over F. Merkurjev has shown that $u(F) \leq 26$. Hoffman and Van Geel [HV98] have shown that $u(F) \leq 22$. Parimala and Suresh [PS98] have shown that $u(F) \leq 10$. Recently, Parimala and Suresh [PS10] have shown that $u(F) = 8$ for $\mathrm{char}(F) \neq 2$. Leep [Lee13] has shown that $u(F) = 8$ including $\mathrm{char}(F) = 2$ using a result of [Hea10]. Harbater, Hartmann and Krashen re-proved $u(F) = 8$ for $\mathrm{char}(F) \neq 2$ using patching in [HHK09, Cor. 4.15].

Since the case $D = F$ is settled, for the rest of the chapter, we suppose $D \neq F$. Mahmoudi [Mah05, Prop. 3.6] has proved an inequality of Hermitian u-invariants:

$$u(D, \sigma, \varepsilon) \leq \frac{r(r+1)}{2\dim_F(D)} u(F)$$

where $r = \dim_F\{x \in D \mid \sigma(x) = \varepsilon x\}$ and r is increasing with respect to $\deg(D)$. By [Sal97, Th. 3.4], $\deg(D) \in \{2, 4\}$. Suppose $d = 4$. If σ is orthogonal and $\varepsilon = 1$ or σ is symplectic and $\varepsilon = -1$, we have $r = \dfrac{4(4+1)}{2} = 10$, then

$$u^+(D) \leq \frac{10 * 11}{2 * 4^2} * 8 = \frac{55}{2}.$$

If σ is orthogonal and $\varepsilon = -1$ or σ is symplectic and $\varepsilon = 1$, we have $r = \dfrac{4(4-1)}{2} = 6$, then

$$u^-(D) \leq \frac{6 * 7}{2 * 4^2} * 8 = \frac{21}{2}.$$

Since u-invariants are integers, we have

$$u^+(D) \leq 27, \text{ and } u^-(D) \leq 10.$$

Parihar and Suresh [PS13, Cor. 4.8] have obtained sharper bounds

$$u^+(D) \leq 14 \text{ and } u^-(D) \leq 8$$

using their inequality from exact sequence of Witt groups [PS13, Cor. 3.3].

In this chapter, we obtain exact values of Hermitian u-invariants in theorem 4.3.2.

Let A be a central simple algebra over a field k. Suppose $\operatorname{char} k \neq 2$ and $\operatorname{per}(A) = 2$. Then, by a special case [Mer81] of the Merkurjev-Suslin theorem [MS82], A is Brauer equivalent to $H_1 \otimes \cdots \otimes H_n$ for some quaternion algebras $H_1, \ldots, H_n$ over k. Let K/k be a quadratic extension. In [PS13, Cor. 4.11], upper bounds for $u^+(A)$, $u^-(A)$, $u^0(A \otimes K)$ are given and they depend only on $u(k)$ and n. We obtain sharper upper bounds for these Hermitian u-invariants in theorem 4.4.2.

4.1. Hermitian u-invariants over complete discrete valued fields

Since Hermitian u-invariants are preserved by Morita invariance lemma 1.6.7, we mostly focus on central division algebras.

Lemma 4.1.1. Let D be a central division algebra over a field K with an involution σ. Let $k = K^\sigma$, $\operatorname{char} k \neq 2$. Suppose k is a non-archimedean local field.
(1) If σ is of the first kind and $D \neq k$, then $u^+(D) = 3$, $u^-(D) = 1$.
(2) If σ is of the second kind, then $u^0(D) = 2$.

Proof. (1) Suppose σ is of the first kind. By [Sch85, ch. 10, Th. 2.2] and that $D \neq k$, D is a quaternion algebra. Suppose σ is the canonical symplectic involution and $\varepsilon = -1$. By [Tsu61, Th. 1], every skew-hermitian space of rank > 3 over (D, σ) is isotropic. By [Tsu61, Th. 3], every skew-hermitian space of rank $= 3$ and discriminant 1 over (D, σ) is anisotropic. Hence $u^+(D) = 3$.

By [Sch85, ch. 10,1.7], $h(x,x)$ is identified with a quadratic space q_h over K such that h is isotropic if and only if q_h is isotropic and $\operatorname{Rank}(q_h) = 4\operatorname{Rank}(h)$. Since $u(k) = 4$, we have $u^-(D) \leq 1$ and hence $u^-(D) = 1$.

(2) Suppose σ is of the second kind, by [Sch85, ch. 10, 2.2], $D = K$. Then $u^0(D) \leq \frac{1}{2}u(k) = 2$. Suppose $K = k(\sqrt{\lambda})$, where $\lambda \in k^* \setminus k^{*2}$ and $\sigma(\sqrt{\lambda}) = -\sqrt{\lambda}$. Assume that k has a discrete valuation v and a parameter π. Up to a square, we may assume that $v(\lambda) \in \{0,1\}$.

If $v(\lambda) = 0$, then, since λ is not a square in k, by a theorem of Springer, $\langle 1, -\lambda, \pi, -\lambda\pi\rangle$ is anisotropic over k. Then the Hermitian form $\langle 1, \pi\rangle$ is anisotropic over (K,σ) and hence $u^0(D) = u(K,\sigma,1) \geq 2$.

Since the residue field of k is a finite field with two square classes, by Hensel's lemma, there exists $u \notin k^{*2}$ such that $v(u) = 0$. If $v(\lambda) = 1$, then $\langle 1, -\lambda, -u, \lambda u\rangle$ is anisotropic over k, by a theorem of Springer, $\langle 1, -u\rangle$ is anisotropic over (K,σ) and hence $u^0(D) = u(K,\sigma,1) \geq 2$.

We have shown that $u^0(D) \geq 2$ and hence $u^0(D) = 2$. $\square$

We fix the following notation for the rest of this section. Let (k, v) be a complete discrete valued field with residue field $\overline{k}$, $\operatorname{char}\overline{k} \neq 2$. Let D be a finite-dimensional division k-algebra with center K with an involution σ such that $K^\sigma = k$. By [CF67, ch. II, 10.1], v extends to a valuation v' on K and by [Wad86], v' extends to a valuation w on D such that

$$w(x) = \frac{1}{\operatorname{ind}(D)} v(\operatorname{Nrd}_{D/K}(x))$$

for all $x \in D^*$. Since $\operatorname{Nrd}_{D/K}(x) = \operatorname{Nrd}_{D/K}(\sigma(x))$, we have $w(\sigma(x)) = w(x)$ for all $x \in D$. Let $R_w = \{x \in D \mid w(x) \geq 0\}$ and $\mathfrak{m}_w = \{x \in D \mid w(x) > 0\}$. Let $\overline{D} = R_w/\mathfrak{m}_w$ be the residue division algebra (see [Rei03, Th. 13.2]) of (D,w) over $\overline{k}$ with involution $\overline{\sigma}$ such that $\overline{\sigma}(\overline{x}) = \overline{\sigma(x)}$ for all $x \in R_w$, where $\overline{x} = x + \mathfrak{m}_w$. Let h be a nondegenerate ε-hermitian form over (D,σ). Then $h = \langle a_1, \ldots, a_n\rangle$, for some $a_i \in D$ with $\sigma(a_i) = \varepsilon a_i$. If $w(a_i) = 0$ for all $1 \leq i \leq n$, then $\overline{h} = \langle \overline{a}_1, \ldots, \overline{a}_n\rangle \in \operatorname{Herm}^\varepsilon(\overline{D}, \overline{\sigma})$.

Let t_D be a parameter of (D, w). By [Lar99, Prop. 2.7], there exists $\pi_D \in D$ such that $w(\pi_D) \equiv w(t_D) \mod 2w(D^*)$ and $\sigma(\pi_D) = \varepsilon' \pi_D$ for some $\varepsilon' \in \{1, -1\}$. Larmour's hermitian analogue (proposition 1.2.3) of a theorem of Springer can be rephrased as follows: there exist $h_1 \in \mathrm{Herm}^{\varepsilon}(D, \sigma)$, $h_2 \in \mathrm{Herm}^{\varepsilon\varepsilon'}(D, \mathrm{Int}(\pi_D) \circ \sigma)$, with $h \simeq h_1 \perp h_2 \pi_D$, with each diagonal entries of h_1 and h_2 have w-value 0. Further, h is isotropic if and only if h_1 or h_2 is isotropic, if and only if $\overline{h}_1$ or $\overline{h}_2$ is isotropic.

Corollary 4.1.2. $u(D, \sigma, \varepsilon) = u(\overline{D}, \overline{\sigma}, \varepsilon) + u(\overline{D}, \overline{\mathrm{Int}(\pi_D) \circ \sigma}, \varepsilon\varepsilon')$.

Proof. Suppose $h \in \mathrm{Herm}^{\varepsilon}(D, \sigma)$ and $h \simeq h_1 \perp h_2 \pi_D$ as in proposition 1.2.3. Since $\mathrm{Rank}(h) = \mathrm{Rank}(h_1) + \mathrm{Rank}(h_2) = \mathrm{Rank}(\overline{h_1}) + \mathrm{Rank}(\overline{h_2})$, if $\mathrm{Rank}(h) > u(\overline{D}, \overline{\sigma}, \varepsilon) + u(\overline{D}, \overline{\mathrm{Int}(\pi_D) \circ \sigma}, \varepsilon\varepsilon')$, then

$$\mathrm{Rank}(\overline{h_1}) > u(\overline{D}, \overline{\sigma}, \varepsilon) \text{ or } \mathrm{Rank}(\overline{h_2}) > u(\overline{D}, \overline{\mathrm{Int}(\pi_D) \circ \sigma}, \varepsilon\varepsilon').$$

Then $\overline{h}_1$ or $\overline{h}_2$ is isotropic. By proposition 1.2.3, h is isotropic. Hence $u(D, \sigma, \varepsilon) \leq u(\overline{D}, \overline{\sigma}, \varepsilon) + u(\overline{D}, \overline{\mathrm{Int}(\pi_D) \circ \sigma}, \varepsilon\varepsilon')$.

Conversely, suppose $g_1 = \langle a_1, \ldots, a_m \rangle \in \mathrm{Herm}^{\varepsilon}(\overline{D}, \overline{\sigma})$ such that $\overline{\sigma}(a_i) = \varepsilon a_i$, $m = u(\overline{D}, \overline{\sigma}, \varepsilon)$ and g_1 is anisotropic. Since $a_i \neq 0$, there exists $b_i \in R_w$, $w(b_i) = 0$ such that $\overline{b_i} = a_i$. Let $c_i = \dfrac{1}{2}(b_i + \varepsilon\sigma(b_i))$. Then $\sigma(c_i) = \varepsilon c_i$ and $\overline{c_i} = a_i$. Let $h_1 = \langle c_1, \ldots, c_m \rangle \in \mathrm{Herm}^{\varepsilon}(D, \sigma)$. Then $\overline{h_1} = g_1$ and by [Lar06, Prop. 2.3], h_1 is anisotropic.

Suppose $g_2 = \langle a_{m+1}, \ldots, a_{m+n} \rangle \in \mathrm{Herm}^{\varepsilon\varepsilon'}(\overline{D}, \overline{\mathrm{Int}(\pi_D) \circ \sigma})$ is anisotropic. Similar to the previous paragraph, there exists $h_2 \in \mathrm{Herm}^{\varepsilon\varepsilon'}(D, \mathrm{Int}(\pi_D) \circ \sigma)$ such that $\overline{h_2} = g_2$ and h_2 is anisotropic.

By proposition 1.2.3, $h = h_1 \perp h_2 \pi_D$ is anisotropic and $\mathrm{Rank}(h) = m + n$. Therefore $u(D, \sigma, \varepsilon) \geq u(\overline{D}, \overline{\sigma}, \varepsilon) + u(\overline{D}, \overline{\mathrm{Int}(\pi_D) \circ \sigma}, \varepsilon\varepsilon')$. □

Lemma 4.1.3. Suppose D is ramified at the discrete valuation v of k. Then there exist an involution σ on D of first kind and elements $\alpha, \pi_d \in D$ such that

(a) $\overline{\sigma}$ is an involution of the second kind;

(b) $\alpha^2 \in k$, $v(\alpha^2) = 0$ and $Z(\overline{D}) = \overline{k}(\overline{\alpha})$;

(c) $\pi_D \in D$ a parameter such that $\sigma(\pi_D) = \pm\pi_D$ and $\overline{\mathrm{Int}(\pi_D) \circ \sigma}$ is of the first kind.

Proof. Suppose D is ramified at v and $Z(D) = k$. Then D is Brauer equivalent to $D_0 \otimes (u, \pi)$ where D_0 is a central division algebra over k unramified at v, $\pi \in k^*$ is a parameter of v and $u \in k^* \setminus k^{*2}$, $v(u) = 0$. Furthermore, by [TW15, Th. 8.77], $\overline{D}$ is Brauer equivalent to $\overline{D_0} \otimes \overline{k}(\sqrt{\overline{u}})$ and $Z(\overline{D}) \simeq \overline{k}(\sqrt{\overline{u}})$.

(a) By [Cha+95, Prop. 4], the nontrivial automorphism of $Z(\overline{D})/\overline{k}$ extends to an involution on $\overline{D}$ of the second kind and it can be lifted to an involution σ on D of the first kind.

(b) Since k is complete, by [Cha+95, p. 53, Lem. 1], there exists $\alpha \in D$ such that $\alpha^2 \in Z(D)$, $\overline{\alpha} \in Z(\overline{D})$ corresponds $\sqrt{\overline{u}}$ in the isomorphism $Z(\overline{D}) \simeq \overline{k}(\sqrt{\overline{u}})$ and $\sigma(\alpha) = -\alpha$.

(c) By [JW90, Prop. 1.7], there exists a parameter $t_D \in D$ such that $\overline{\mathrm{Int}(t_D)}$ is the non-trivial $Z(\overline{D})/\overline{k}$-automorphism, i.e.

$$\overline{t_D \alpha t_D^{-1}} = -\overline{\alpha}.$$

Since $\overline{\sigma}$ is of the second kind and $\overline{\mathrm{Int}(t_D)}$ induces the non-trivial automorphims of $Z(\overline{D})$, we have $\overline{\mathrm{Int}(t_D) \circ \sigma}$ is of the first kind. Since σ is an involution, $w(t_D) = w(\sigma(t_D))$ and hence $\overline{\sigma(t_D) t_D^{-1}} \neq 0 \in \overline{D}$.

Case 1: Suppose that $\overline{\sigma(t_D) t_D^{-1}} = 1$. Let $\pi_D = t_D + \sigma(t_D)$. Then $\sigma(\pi_D) = \pi_D$. Since $\pi_D t_D^{-1} = 1 + \sigma(t_D) t_D^{-1}$ and $\mathrm{char}(\overline{k}) \neq 2$, we have

$$\overline{\pi_D t_D^{-1}} = 1 + \overline{\sigma(t_D) t_D^{-1}} = 1 + 1 = 2 \neq 0.$$

Hence $w(\pi_D) = w(t_D)$. Since $\overline{\pi_D t_D^{-1}} = 2 \in \overline{k}^*$, $\overline{\mathrm{Int}(\pi_D) \circ \sigma} = \overline{\mathrm{Int}(t_D) \circ \sigma}$ and hence $\overline{\mathrm{Int}(\pi_D) \circ \sigma}$ is of the first kind. Thus π_D satisfies condition (c).

Case 2: Suppose that $\overline{\sigma(t_D)t_D^{-1}} \neq 1$. Let $\pi_D = \alpha t_D - \sigma(\alpha t_D)$. Then $\sigma(\pi_D) = -\pi_D$. We have $\pi_D t_D^{-1} = \alpha - \sigma(t_D)\sigma(\alpha)t_D^{-1}$. Since $\overline{\sigma(\alpha)} = -\overline{\alpha}$ and $\overline{t_D\alpha t_D^{-1}} = -\overline{\alpha}$, we have

$$\begin{aligned}
\overline{\pi_D t_D^{-1}} &= \overline{\alpha} - \overline{\sigma(t_D)\sigma(\alpha)t_D^{-1}} \\
&= \overline{\alpha} - \overline{\sigma(t_D)t_D^{-1}} \cdot \overline{t_D\sigma(\alpha)t_D^{-1}} \\
&= \overline{\alpha} - \overline{\sigma(t_D)t_D^{-1}} \cdot \overline{(-t_D\alpha t_D^{-1})} \\
&= \overline{\alpha} - \overline{\sigma(t_D)t_D^{-1}} \cdot \overline{\alpha} \\
&= (1 - \overline{\sigma(t_D)t_D^{-1}})\overline{\alpha} \\
&\neq 0.
\end{aligned}$$

Hence $w(\pi_D) = w(t_D)$. Since $\overline{\sigma(\alpha)} = -\overline{\alpha}$, $\alpha^2 \in k$ and $\overline{t_D\alpha t_D^{-1}} = -\overline{\alpha}$, we have $\overline{\sigma(t_D)\alpha\sigma(t_D)^{-1}} = -\overline{\alpha}$ and

$$\begin{aligned}
&\overline{(\pi_D\alpha\pi_D^{-1} + \alpha)\pi_D t_D^{-1}} \\
= \;& \overline{\pi_D\alpha t_D^{-1}} + \overline{\alpha\pi_D t_D^{-1}} \\
= \;& \overline{(\alpha t_D - \sigma(t_D)\sigma(\alpha))\alpha t_D^{-1}} + \overline{\alpha(\alpha t_D - \sigma(t_D)\sigma(\alpha))t_D^{-1}} \\
= \;& \overline{\alpha t_D\alpha t_D^{-1}} - \overline{\sigma(t_D)\sigma(\alpha)\alpha t_D^{-1}} + \overline{\alpha^2} + \overline{\alpha\sigma(t_D)\alpha t_D^{-1}} \\
= \;& -\overline{\alpha^2} + \overline{\sigma(t_D)\alpha^2 t_D^{-1}} + \overline{\alpha^2} + \overline{\alpha(\sigma(t_D)\alpha\sigma(t_D)^{-1})\sigma(t_D)t_D^{-1}} \\
= \;& -\overline{\alpha^2} + \overline{\alpha^2\sigma(t_D)t_D^{-1}} + \overline{\alpha^2} - \overline{\alpha^2\sigma(t_D)t_D^{-1}} \\
= \;& 0.
\end{aligned}$$

Since $\overline{\pi_D t_D^{-1}} \neq 0$, $\overline{\pi_D\alpha\pi_D^{-1} + \alpha} = 0$ and hence $\overline{(\mathrm{Int}(\pi_D) \circ \sigma)}(\overline{\alpha}) = \overline{\alpha}$. Thus π_D satisfies (c).

In conclusion, σ, α and π_D satisfy required properties (a), (b) and (c). □

Corollary 4.1.4. *Suppose σ is of the first kind, i.e. $K = k$.*

(1) If D is unramified at the discrete valuation of k, then

$$u^+(D) = 2u^+(\overline{D}) \text{ and } u^-(D) = 2u^-(\overline{D}).$$

(2) If D is ramified at the discrete valuation of k, then

$$u^+(D) = u^0(\overline{D}) + u^+(\overline{D}) \text{ and } u^-(D) = u^0(\overline{D}) + u^-(\overline{D}).$$

Proof. Suppose D is unramified. Then we can take $\pi_D = \pi$, where π is a parameter of k. Since $\sigma(\pi) = \pi$, we have $\varepsilon' = 1$ and $\mathrm{Int}(\pi_D) \circ \sigma = \sigma$. Hence, by corollary 4.1.2, we have

$$u(D, \sigma, \varepsilon) = u(\overline{D}, \overline{\sigma}, \varepsilon) + u(\overline{D}, \overline{\mathrm{Int}(\pi_D) \circ \sigma}, \varepsilon\varepsilon') = 2u(\overline{D}, \overline{\sigma}, \varepsilon).$$

Then $u^+(D) = 2u^+(\overline{D})$ and $u^-(D) = 2u^-(\overline{D})$.

Suppose D is ramified. Then choose σ and π_D as in lemma 4.1.3. Then $\overline{\sigma}$ is of the second kind and $\overline{\mathrm{Int}(\pi_D) \circ \sigma}$ is of the first kind. By [Cha+95, Prop. 3], $\overline{\mathrm{Int}(\pi_D) \circ \sigma}$ and $\mathrm{Int}(\pi_D) \circ \sigma$ are of the same type. Then, by corollary 4.1.2, we have

$$u(D, \sigma, \varepsilon) = u(\overline{D}, \overline{\sigma}, \varepsilon) + u(\overline{D}, \overline{\mathrm{Int}(\pi_D) \circ \sigma}, \varepsilon\varepsilon')$$

Further, by [KMRT98, Prop. 2.7] if $\varepsilon' = 1$, then $\mathrm{Int}(\pi_D) \circ \sigma$ and σ are of the same type; if $\varepsilon' = -1$, then $\mathrm{Int}(\pi_D) \circ \sigma$ and σ are of different types. Then $u^+(D) = u^0(\overline{D}) + u^+(\overline{D})$ and $u^-(D) = u^0(\overline{D}) + u^-(\overline{D})$. □

Let K/k be a quadratic extension and $\overline{K}$ the residue field of K. Let D be a central division algebra over k with an involution σ of the first kind. Then $\sigma \otimes \iota$ is an involution on $D \otimes_k K$ of the second kind with ι being the non-trivial automorphism of K/k.

Suppose $D \otimes K$ is division and ramified at the discrete valuation of K. Then D is ramified at the discrete valuation of k and $Z(\overline{D \otimes K}) = Z(\overline{D}) \otimes \overline{K}$.

Suppose K/k is unramified. Then $\overline{K}/\overline{k}$ is a quadratic extension. We have $\overline{K} = \overline{k}(\sqrt{\lambda})$ and $Z(\overline{D}) = \overline{k}(\sqrt{u})$ for some $u, \lambda \in k$ units at the discrete valuation of k. Let π be a parameter of (k, v). Then $D \otimes_k K \simeq D_0 \otimes (u, \pi) \otimes_k K$ is a division algebra implies that $uk^{*2} \neq \lambda k^{*2}$. In particular, $Z(\overline{D \otimes K}) = \overline{k}(\sqrt{u}, \sqrt{\lambda})$ is a degree

4 extension of $\overline{k}$. Since $\overline{D\otimes K} = \overline{D}\otimes\overline{K} = \overline{D}\otimes\overline{k}(\sqrt{\overline{u}}, \sqrt{\overline{\lambda}})$ and $\overline{D}$ has an involution of the first kind, $\overline{D\otimes K}$ has three possible types of involutions of second kind with fixed fields $\overline{k_1} = \overline{k}(\sqrt{\overline{u}})$, $\overline{k_2} = \overline{k}(\sqrt{\overline{\lambda}})$ and $\overline{k_3} = \overline{k}(\sqrt{\overline{u\lambda}})$ respectively. The corresponding $u^0(\overline{D\otimes K})$ are defined by $u^0(\overline{D\otimes K}/\overline{k}_1)$, $u^0(\overline{D\otimes K}/\overline{k}_2)$ and $u^0(\overline{D\otimes K}/\overline{k}_3)$.

Corollary 4.1.5. *Let K/k be a quadratic extension and let ι be the non-trivial automorphism of K/k. Let D be a central division algebra over k with an involution σ of first kind such that $D\otimes_k K$ is division.*

(1) If $D\otimes K$ is unramified at the discrete valuation of K and K/k is unramified, then

$$u^0(D\otimes K) = 2u^0(\overline{D}\otimes\overline{K}).$$

(2) If $D\otimes K$ is ramified at the discrete valuation of K and K/k is unramified, then

$$u^0(D\otimes K) = u^0(\overline{D}\otimes\overline{K}/\overline{k}_2) + u^0(\overline{D}\otimes\overline{K}/\overline{k}_3).$$

(3) If K/k is ramified, then

$$u^0(D\otimes K) = u^+(\overline{D_0}) + u^-(\overline{D_0})$$

for some central division algebra D_0 unramified over k with $\deg(D) = \deg(D_0)$.

Proof. (1) Suppose D is unramified and K/k is unramified. Then $\overline{D\otimes K} = \overline{D}\otimes\overline{K}$ and $\overline{K}/\overline{k}$ is a quadratic extension. Let π be a parameter of k. Take $\pi_D = \pi$. Then $\sigma(\pi_D) = \pi_D$ and $\overline{\mathrm{Int}(\pi_D)\circ(\sigma\otimes\iota)} = \overline{\sigma\otimes\iota}$. By corollary 4.1.2,

$$u^0(D\otimes K) = 2u^0(\overline{D}\otimes\overline{K}).$$

(2) Suppose D is ramified and K/k is unramified. Suppose σ, $\alpha = \sqrt{u}$ and π_D are as in lemma 4.1.3. Then $Z(\overline{D\otimes K}) = \overline{k}(\sqrt{\overline{u}}, \sqrt{\overline{\lambda}})$ and the fixed field of $\overline{\sigma\otimes\iota}$ is $\overline{k}_3 = \overline{k}(\sqrt{\overline{u\lambda}})$ and the fixed field of $\overline{\mathrm{Int}(\pi_D)\circ(\sigma\otimes\iota)}$ is $\overline{k}_2 = \overline{k}(\sqrt{\overline{\lambda}})$. Thus, by

corollary 4.1.2, we have

$$u^0(D\otimes K) = u^0(\overline{D}\otimes \overline{K}/\overline{k}_2) + u^0(\overline{D}\otimes \overline{K}/\overline{k}_3).$$

(3) Suppose K/k is ramified. Then $K = k(\sqrt{\pi})$ for some parameter $\pi \in k$ and $\overline{K} = \overline{k}$. We have $D = D_0 \otimes (u, \pi)$ for some D_0 unramified on k and $u \in k$ a unit at the valuation of k [TW15, Th. 8.77]. Thus $D\otimes K = D_0 \otimes K$. Since $D \otimes K$ is division, $D\otimes K \simeq D_0 \otimes K$ and $\deg(D) = \deg(D_0)$. Let σ_0 be an involution of the first kind on D_0 and $\sigma \simeq \sigma_0 \otimes \gamma$, where γ is the canonical involution of (u, π). Since D_0 is unramified and K/k is ramified, we have $\overline{D\otimes K} = \overline{D_0}$ and $\overline{\sigma\otimes\iota} = \overline{\sigma_0}$. Let $\pi_D = \sqrt{\pi} \in K \subset D\otimes K$. Then $\overline{\mathrm{Int}(\pi_D)\circ(\sigma\otimes\iota)} = \overline{\sigma_0}$. Thus, by corollary 4.1.2,

$$u(D\otimes K, \sigma, \varepsilon) = u(\overline{D\otimes K}, \overline{\sigma_0}, \varepsilon) + u(\overline{D\otimes K}, \overline{\sigma_0}, -\varepsilon).$$

Hence $u^0(D\otimes K) = u^+(\overline{D_0}) + u^-(\overline{D_0})$. □

We end this section with the following well known

Lemma 4.1.6. Let k be a discrete valued field with residue field $\overline{k}$ and completion $\widehat{k}$. Suppose $\mathrm{char}(\overline{k}) \neq 2$. Let D be a division algebra over k with center K. Let σ be an involution on D such that $K^\sigma = k$. If $D\otimes \widehat{k}$ is division, then

$$u(D, \sigma, \varepsilon) \geq u(D\otimes\widehat{k}, \sigma\otimes \mathrm{Id}, \varepsilon).$$

Proof. Let v be the discrete valuation on k and $\pi \in k$ be a parameter. Since $D\otimes\widehat{k}$ is division, v extends to a valuation w on D. Let $\varepsilon = \pm 1$ and $\mathrm{Sym}^\varepsilon(D,\sigma) = \{x \in D \mid \sigma(x) = \varepsilon x\}$. Let $e_1, \ldots, e_r$ be a k-basis of $\mathrm{Sym}^\varepsilon(D,\sigma)$. Let $a \in \mathrm{Sym}^\varepsilon(D,\sigma)\otimes\widehat{k}$ and write $a = a_1e_1 + \cdots + a_re_r$ with $a_i \in \widehat{k}$. Let $b_i \in k$ be such that $a_i \equiv b_i$ modulo $\pi^{ew(a)+1}$ and $b = b_1e_1 + \cdots + b_re_r \in \mathrm{Sym}^\varepsilon(D,\sigma)$, where e is the ramification index $[w(D^*) : v(k^*)]$. Then $w(a) = w(b)$ and $\overline{ab^{-1}} = 1 \in \overline{D\otimes\widehat{k}}$. In particular, by proposition 1.2.3, $\langle a\rangle \simeq \langle b\rangle \otimes \widehat{k}$ as ε-hermitian forms over $D\otimes\widehat{k}$.

Let h be an ε-hermitian forms over $(D \otimes \widehat{k}, \sigma)$. Since $D \otimes \widehat{k}$ is division, $h = \langle \alpha_1, \ldots, \alpha_n \rangle$ for some $\alpha_i \in \mathrm{Sym}^{\varepsilon}(D,\sigma) \otimes \widehat{k}$. For each α_i, let $\beta_i \in \mathrm{Sym}^{\varepsilon}(D,\sigma)$ be such that $\langle \alpha_i \rangle \simeq \langle \beta_i \rangle \otimes \widehat{k}$ and $h_0 = \langle \beta_1, \ldots, \beta_n \rangle$. Then h_0 is an ε-hermitian form over (D,σ) and $h_0 \otimes \widehat{k} \simeq h$. If h is anisotropic over $\widehat{k}$, then, by proposition 1.2.3 again, h_0 is anisotropic. In particular, $u(D,\sigma,\varepsilon) \geq u(D \otimes \widehat{k}, \sigma \otimes \mathrm{Id}, \varepsilon)$. □

4.2. Division algebras over $\mathscr{A}_i(2)$-fields

Suppose i and m are two positive integers. A field k is called an *$\mathscr{A}_i(m)$-field* [Lee13, Def. 2.1] if every system of r homogeneous forms of degree m in more than rm^i variables over k has a nontrivial simutaneous zero over a field extension L/k such that $\gcd(m, [L:k]) = 1$ for all integers $r > 0$.

Let A be a central simple algebra over a field k. We say that A satisfies the *Springer's property* if for any involution σ on A of the first kind, $\varepsilon \in \{1, -1\}$ and for any odd degree extension L/k, if h is an anisotropic ε-hermitian space over (A,σ), then $h \otimes L$ is anisotropic.

Theorem 4.2.1. *Let k be an $\mathscr{A}_i(2)$-field. Let D be a central division algebra over k with an involution of the first kind. If D satisfies the Springer's property, then*

$$u^+(D) \leq (1 + \frac{1}{d})2^{i-1} \text{ and } u^-(D) \leq (1 - \frac{1}{d})2^{i-1},$$

where $d = \deg(D)$.

Proof. Let σ be an orthogonal involution on D. Let

$$\mathrm{Sym}^{\varepsilon}(D,\sigma) = \{x \in D \mid \sigma(x) = \varepsilon x\}$$

and $r = \dim_k(\mathrm{Sym}^{\varepsilon}(D,\sigma))$. Then $r = d(d+\varepsilon)/2$ [KMRT98, Prop. 2.6]. Let $e_1, \ldots, e_r$ be a k-basis of $\mathrm{Sym}^{\varepsilon}(D,\sigma)$. Let h be an ε-hermitian form over (D,σ) of rank $n > (1 + \frac{\varepsilon}{d})2^{i-1}$. Then for $x \in D^n$, we have

$$h(x,x) = q_1(x,x)e_1 + \cdots + q_r(x,x)e_r,$$

with each q_i a quadratic form over k in d^2n variables [Mah05, proof of prop. 3.6].

Since k is an $\mathscr{A}_i(2)$-field and $d^2n > d(d+\varepsilon)2^{i-1} = r2^i$, there exists an odd degree extension L/k such that $\{q_1, \ldots, q_r\}$ have a simultaneous nontrivial zero over L. Then h_L is isotropic over D_L. By Springer's property, h is isotropic over D. Hence $u(D, \sigma, \varepsilon) \leq (1 + \frac{\varepsilon}{d})2^{i-1}$.

Similarly, if σ is a symplectic involution on D, then $r = d(d-\varepsilon)/2$ and hence $u(D, \sigma, \varepsilon) \leq (1 - \frac{\varepsilon}{d})2^{i-1}$. □

Theorem 4.2.2. *Let k be an $\mathscr{A}_i(2)$-field. Let K/k be a quadratic extension. Let D be a central division algebra over K with an involution σ of the second kind with* $\sigma|_k = \mathrm{Id}$. *Suppose that D satisfies the Springer's property. Then $u^0(D) \leq 2^{i-1}$.*

Proof. Let σ be an involution on D of the second kind. Let $\mathrm{Sym}(D) = \{x \in D \mid \sigma(x) = x\}$. Then $\mathrm{Sym}(D)$ is vector space over k and $\dim_k \mathrm{Sym}(D) = d^2$, where $d^2 = \dim_K(D)$. Let $e_1, \ldots, e_{d^2}$ be a k-basis of $\mathrm{Sym}(D)$. Let h be a hermitian form over (D, σ) of rank $n > 2^{i-1}$. Then, for $x \in D^n$, $h(x, x) \in \mathrm{Sym}(D)$ and we have

$$h(x, x) = q_1(x, x)e_1 + \cdots + q_{d^2}(x, x)e_{d^2},$$

with each q_i a quadratic form over k in $2d^2n$ variables.

Since k is an $\mathscr{A}_i(2)$-field and $2d^2n > 2d^2 2^{i-1} = d^2 2^i$, there exists an odd degree extension L/k such that $\{q_1, \ldots, q_{d^2}\}$ have a simultaneous nontrivial zero over L. In particular, h_L is isotropic over D_L. By Springer's property, h is isotropic over D. Hence $u^0(D) \leq 2^{i-1}$. □

Corollary 4.2.3. *If D is a quaternion division algebra over an $\mathscr{A}_i(2)$-field k and σ is of the first kind, then $u^+(D) \leq 3 \cdot 2^{i-2}$ and $u^-(D) \leq 2^{i-2}$;*

Proof. Since D is a quaternion algebra, by [PSS01, Th. 3.5], (D, σ, ε) satisfies Springer's property. By theorem 4.2.1,

$$u^+(D) \leq (1 + \frac{1}{2})2^{i-1} = 3 \cdot 2^{i-2},$$

$$u^-(D) \leq (1-\frac{1}{2})2^{i-1} = 2^{i-2}.$$

□

Corollary 4.2.4. *If D is a quaternion division algebra over a global function field k, then $u^+(D) = 3$, $u^-(D) = 1$, and $u^0(D) = 2$.*

Proof. By Chevalley-Warning theorem [Che35; War35], every finite field is a C_1-field. By Tsen-Lang theorem [Lan52], every global function field is a C_2-field. Since every C_2-field is an $\mathscr{A}_2(2)$-field [Lee13, between 2.1 and 2.2], by corollary 4.2.3,

$$u^+(D) \leq 3 \text{ and } u^-(D) \leq 1.$$

By theorem 4.2.2, $u^0(D) \leq 2$. The equality follows from lemma 4.1.6 and lemma 4.1.1.

□

Corollary 4.2.5. *Let F the function field of an integral variety X over a p-adic field with $p \neq 2$. Let D be a quaternion algebra over F. If $\dim(X) = n$, then*

$$u^+(D) \leq 3 \cdot 2^n \text{ and } u^-(D) \leq 2^n.$$

Proof. Since D is a quaternion algebra, by [PSS01, Th. 3.5], D satisfies the Springer's property. Since $\dim(X) = n$, by [Hea10] and [Lee13], F is a $\mathscr{A}_{n+2}(2)$-field. Hence the corollary follows from corollary 4.2.3. □

Corollary 4.2.6. *Let F be a the function field of a p-adic curve. Let D be a division algebra over F with an involution of the first kind.*

(1) If D is a quaternion division algebra, then $u^+(D) \leq 6$ and $u^-(D) \leq 2$.

(2) If D is a biquaternion division algebra, then $u^+(D) \leq 5$ and $u^-(D) \leq 3$.

Proof. (1) By [Sal97; Sal98, Th. 3.4], $\deg(D) = d = 2$ or 4. If $d = 2$, then D is a quaternion algebra and by corollary 4.2.5, we have

$$u^+(D) \leq 3 \cdot 2^{3-2} = 6 \text{ and } u^-(D) \leq 2^{3-2} = 2.$$

(2) Suppose $d = 4$. By theorem 3.3.1, D satisfies Springer's property. Since F is a $\mathscr{A}_3(2)$-field, by theorem 4.2.1, we have

$$u^+(D) \leq (1+\frac{1}{4}) \cdot 2^{3-1} = 5 \text{ and } u^-(D) \leq (1-\frac{1}{4}) \cdot 2^{3-1} = 3.$$

□

Corollary 4.2.7. *Let F the function field of a p-adic curve. Let L/F be a quadratic extension. Let D a division algebra over F with an involution of the first kind. Then $u^0(D \otimes_F L) \leq 4$.*

Proof. By theorem 3.3.1, D satisfies Springer's property. Since F is a $\mathscr{A}_3(2)$-field, by theorem 4.2.2, we have $u^0(D \otimes_F L) \leq 2^{3-1} = 4$. □

4.3. Division algebras over semi-global fields

Let p be an odd prime number. Let F be the function field of a curve over a p-adic field. Let D is a division algebra over F with an involution σ. In this section, we show that the bounds in corollary 4.2.6 for u-invariants of hermitian of forms over central simple algebras over F are in fact exact values. We also compute $u^0(D)$ if D is a quaternion division algebra with an involution of the second kind over F.

Lemma 4.3.1. Let k be a complete discrete valued field with residue field $\overline{k}$. Suppose $\overline{k}$ is a non-archimedean local field or a global function field with $\text{char}(\overline{k}) \neq 2$. Let D be a division algebra over k with an involution of the first kind and K/k a quadratic extension.

(1) If D is a quaternion division algebra, then $u^+(D) = 6$ and $u^-(D) = 2$.

(2) If D is a biquaternion algebra, then $u^+(D) = 5$ and $u^-(D) = 3$.

(3) If $D \otimes_k K$ is a division algebra, then $u^0(D \otimes_k K) = 4$.

Proof. (1) Suppose D is an unramified quaternion algebra. Then $\overline{D}$ is a quaternion algebra. Since $\overline{k}$ is either a local field or a global function field, by lemma 4.1.1

and corollary 4.2.4, we have $u^+(\overline{D}) = 3$, $u^-(\overline{D}) = 1$ and $u^-(\overline{D}) = 2$. Thus, by corollary 4.1.4(1), $u^+(D) = 2 * 3 = 6$ and $u^-(D) = 2 * 1 = 2$.

Suppose D is a ramified quaternion algebra. Then $\overline{D}$ is a quadratic extension of $\overline{k}$ and by lemma 4.1.1 and corollary 4.1.4(2) $u^+(D) = 2+4 = 6$ and $u^-(D) = 2+0 = 2$.

(2) Suppose D is a biquaternion algebra. Since k is a complete discrete valued field with $\overline{k}$ is a global field or local field, D is ramified by a theorem of Albert [Lam05, Ch. III, 4.8] and a theorem of Springer [Lam05, Ch. VI, 1.9]. Thus $\overline{D}$ is a quaternion algebra and hence by lemma 4.1.1 and corollary 4.1.4(2), $u^+(D) = 2 + 3 = 5$ and $u^-(D) = 2 + 1 = 3$.

Suppose $D \otimes_k K \simeq D_0 \otimes (u, \pi) \otimes_k K$ is a division algebra. Recall that $\overline{k_1} = \overline{k}(\sqrt{u})$, $\overline{k_2} = \overline{k}(\sqrt{\lambda})$ and $\overline{k_3} = \overline{k}(\sqrt{u\lambda})$. By corollary 4.1.5, we have either $u^0(D \otimes K) = 2u^0(\overline{D \otimes K})$ or $u^0(D \otimes K) = u^0(\overline{D} \otimes \overline{K}/\overline{k}_2) + u^0(\overline{D} \otimes \overline{K}/\overline{k}_3)$ or $u^0(D \otimes K) = u^+(\overline{D_0}) + u^-(\overline{D_0})$ for some central division algebra D_0 unramified over k with $\deg(D) = \deg(D_0)$. By corollary 4.2.4, we have $u^+(\overline{D}) = 3$, $u^-(\overline{D}) = 1$ and $u^0(\overline{D}) = 2$.

In the case of corollary 4.1.5(1), $u^0(D \otimes K) = 2u^0(\overline{D \otimes K}) = 2 * 2 = 4$;

In the case of corollary 4.1.5(2), $u^0(D \otimes K) = u^0(\overline{D} \otimes \overline{K}/\overline{k}_2) + u^0(\overline{D} \otimes \overline{K}/\overline{k}_3)$. Since $\overline{k}$ is a p-adic field or a global field, so are $\overline{k_2}$ and $\overline{k_3}$. We have $u(\overline{k_2}) = u(\overline{k_2}) = 4$. Since $\overline{D} \otimes \overline{K}$ is a quadratic extension of $\overline{k_2}$ and $\overline{k_3}$, $u^0(\overline{D} \otimes \overline{K}/\overline{k_2}) = \frac{1}{2}u(\overline{k_2}) = 2$, $u^0(\overline{D} \otimes \overline{K}/\overline{k_3}) = \frac{1}{2}u(\overline{k_3}) = 2$. Thus, we also have $u^0(D \otimes K) = 4$.

In the case of corollary 4.1.5(3), $u^0(D \otimes K) = u^+(\overline{D_0}) + u^-(\overline{D_0}) = 3 + 1 = 4$. □

The next theorem is our main result of chapter 4.

Theorem 4.3.2. *Let F be the function field of a p-adic curve with $p \neq 2$ and D a division algebra over F with an involution of the first kind. Let L/F be a quadratic extension.*

(1) If D is quaternion, then

$$u^+(D) = 6 \text{ and } u^-(D) = 2.$$

(2) If D is quaternion and $D \otimes_F L$ is division, then

$$u^0(D \otimes_F L) = 4.$$

(3) If D is biquaternion, then

$$u^+(D) = 5 \text{ and } u^-(D) = 3.$$

PROOF. Since D is a division algebra. By [RS13, Th. 2.6], there exists a divisorial discrete valuation v of F such that $D \otimes F_v$ is division. Since v is a divisorial discrete valuation, the residue field at v is either a p-adic field or a global function field.

(1) and (3) follow from corollary 4.2.6, lemma 4.3.1(1)(2) and lemma 4.1.6.

(2) By [RS13, Th. 2.6], there exists a divisorial discrete valuation v of F such that $D \otimes L \otimes F_v$ is division. Thus, the result follows from corollary 4.2.7, lemma 4.3.1(3) and lemma 4.1.6. □

4.4. Tensor product of quaternions over arbitrary fields

In this section, we prove theorem 4.4.2. We begin with the following

Lemma 4.4.1. For $n \geq 1$, let $a_n = \frac{4}{5} + \frac{1}{5}\left(\frac{9}{4}\right)^n$, $b_n = -\frac{1}{5} + \frac{1}{5}\left(\frac{9}{4}\right)^n$ and $c_n = \frac{1}{5} + \frac{3}{10}\left(\frac{9}{4}\right)^n$. Then

$$a_{n+1} = \frac{3}{4}a_n + c_n, \; b_{n+1} = \frac{3}{2}b_n + \frac{1}{2}c_n, \; c_n = \frac{1}{2}a_n + b_n, \; \frac{3}{2}a_n \geq c_n \geq \frac{3}{2}b_n$$

for all $n \geq 1$.

PROOF.

$$\begin{aligned}
\frac{3}{4}a_n + c_n &= \frac{3}{4}\left(\frac{4}{5}+\frac{1}{5}\left(\frac{9}{4}\right)^n\right)+\frac{1}{5}+\frac{3}{10}\left(\frac{9}{4}\right)^n \\
&= \frac{3}{5}+\frac{3}{20}\left(\frac{9}{4}\right)^n+\frac{1}{5}+\frac{3}{10}\left(\frac{9}{4}\right)^n \\
&= (\frac{3}{5}+\frac{1}{5})+(\frac{3}{20}+\frac{3}{10})\left(\frac{9}{4}\right)^n \\
&= \frac{4}{5}+\frac{9}{20}\left(\frac{9}{4}\right)^n \\
&= \frac{4}{5}+\frac{1}{5}\left(\frac{9}{4}\right)^{n+1} = a_{n+1}.
\end{aligned}$$

$$\begin{aligned}
\frac{3}{2}b_n + \frac{1}{2}c_n &= \frac{3}{2}\left(-\frac{1}{5}+\frac{1}{5}\left(\frac{9}{4}\right)^n\right)+\frac{1}{2}\left(\frac{1}{5}+\frac{3}{10}\left(\frac{9}{4}\right)^n\right) \\
&= -\frac{3}{10}+\frac{3}{10}\left(\frac{9}{4}\right)^n+\frac{1}{10}+\frac{3}{20}\left(\frac{9}{4}\right)^n \\
&= (-\frac{3}{10}+\frac{1}{10})+(\frac{3}{10}+\frac{3}{20})\left(\frac{9}{4}\right)^n \\
&= -\frac{1}{5}+\frac{9}{20}\left(\frac{9}{4}\right)^n \\
&= -\frac{1}{5}+\frac{1}{5}\left(\frac{9}{4}\right)^{n+1} = b_{n+1}.
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}a_n + b_n &= \frac{1}{2}\left(\frac{4}{5}+\frac{1}{5}\left(\frac{9}{4}\right)^n\right)-\frac{1}{5}+\frac{1}{5}\left(\frac{9}{4}\right)^n \\
&= \frac{2}{5}+\frac{1}{10}\left(\frac{9}{4}\right)^n-\frac{1}{5}+\frac{1}{5}\left(\frac{9}{4}\right)^n \\
&= (\frac{2}{5}-\frac{1}{5})+(\frac{1}{10}+\frac{1}{5})\left(\frac{9}{4}\right)^n \\
&= \frac{1}{5}+\frac{3}{10}\left(\frac{9}{4}\right)^n = c_n.
\end{aligned}$$

Finally, since $\frac{3}{2}a_n = \frac{6}{5}+\frac{3}{10}\left(\frac{9}{4}\right)^n$, $\frac{3}{2}b_n = -\frac{3}{10}+\frac{3}{10}\left(\frac{9}{4}\right)^n$ and $\frac{6}{5} \geq \frac{1}{5} \geq -\frac{3}{10}$, we have $\frac{3}{2}a_n \geq c_n \geq \frac{3}{2}b_n$. □

Theorem 4.4.2. *Let A be a central simple algebra over a field k. Suppose* $\operatorname{char} k \neq 2$ *and* $\operatorname{per}(A) = 2$. *Suppose A is Brauer equivalent to $H_1 \otimes \cdots \otimes H_n$ for some quaternion algebras $H_1, \ldots, H_n$ over k. Then*

(1) $u^+(A) \leq (\frac{4}{5}+\frac{1}{5}\left(\frac{9}{4}\right)^n)u(k)$;

(2) $u^-(A) \leq (-\frac{1}{5}+\frac{1}{5}\left(\frac{9}{4}\right)^n)u(k)$;

(3) $u^0(A\otimes_k K)\leq (\frac{1}{5}+\frac{3}{10}\left(\frac{9}{4}\right)^n)u(k)$ *for all quadratic extension* K/k.

Proof. By lemma 1.6.7, we may assume that $A = H_1\otimes\cdots\otimes H_n$. Let $\sigma = \tau_1\otimes\cdots\otimes\tau_n$, where τ_i is the canonical involutions of H_i for $1\leq i\leq n$. For $n\geq 1$, let $a_n=\frac{4}{5}+\frac{1}{5}\left(\frac{9}{4}\right)^n$, $b_n=-\frac{1}{5}+\frac{1}{5}\left(\frac{9}{4}\right)^n$ and $c_n=\frac{1}{5}+\frac{3}{10}\left(\frac{9}{4}\right)^n$.

We proceed by induction. For $n = 1$, by [Mah05, Prop. 3.4] and [Lee84, Prop. 2.10] we have $u^+(H_1)\leq a_1u(k)$, by [Sch85, Ch. 10, 1.7], we have $u^-(H_1)\leq b_1u(k)$ and by [PS13, Prop. 4.4], we have $u^0(H_1)\leq c_1u(k)$.

Suppose $u^+(H_1\otimes_k\cdots\otimes_k H_n)\leq a_nu(k)$, $u^-(H_1\otimes_k\cdots\otimes_k H_n)\leq b_nu(k)$ and $u^0(H_1\otimes_k\cdots\otimes_k H_n)\leq c_nu(k)$.

Let $H_1,\ldots,H_{n+1}$ be quaternion algebas over k, τ_i the canonical involution of H_i and $\sigma=\tau_1\otimes\cdots\otimes\tau_{n+1}$ on $A = H_1\otimes\cdots\otimes H_{n+1}$. Since H_{n+1} is a quaternion algebra and τ_{n+1} is the canonical involution, there exist $\lambda_{n+1},\mu_{n+1}\in H_{n+1}^*$ such that $\tau_{n+1}(\lambda_{n+1})=-\lambda_{n+1}$, $\tau_{n+1}(\mu_{n+1})=-\mu_{n+1}$, $\lambda_{n+1}\mu_{n+1}=-\mu_{n+1}\lambda_{n+1}$ and $k(\lambda_{n+1})/k$ is a quadratic extension. Let $\lambda=1\otimes\cdots\otimes 1\otimes\lambda_{n+1}\in A$, $\mu=1\otimes\cdots\otimes1\otimes\mu_{n+1}\in A$ and $\tilde{A}$ be the centralizer of $k(\lambda)$ in A. Then $\tilde{A}=H_1\otimes\cdots\otimes H_n\otimes k(\lambda)$. Let $\sigma_1=\sigma|_{\tilde{A}}$ and $\sigma_2=\mathrm{Int}(\mu^{-1})\circ\sigma_1$. By [Mah05, Prop. 3.1, Prop. 3.2], we have σ_1 is unitary, σ_2 and σ are of the same type and

$$\begin{aligned}u(A,\sigma,\varepsilon) \;\leq\; & \min\{u(\tilde{A},\sigma_1,\varepsilon)+\frac{1}{2}u(\tilde{A}\otimes k(\lambda),\sigma_2,-\varepsilon),\\ & \frac{1}{2}u(\tilde{A}\otimes k(\lambda),\sigma_1,\varepsilon)+u(\tilde{A}\otimes k(\lambda),\sigma_2,-\varepsilon)\}.\end{aligned}$$

Since σ_1 is unitary and $\tilde{A}=H_1\otimes_k\cdots\otimes_k H_n\otimes k(\lambda)$, by the induction hypothesis, we have $u(\tilde{A},\sigma_1,\varepsilon)\leq c_nu(k)$. By [PS13, Prop. 4.2], $u(\tilde{A},\sigma_2,-\varepsilon)=u(H_1\otimes_k\cdots\otimes_k H_n\otimes k(\lambda),\sigma_2,-\varepsilon)\leq\frac{3}{2}u(H_1\otimes_k\cdots\otimes_k H_n,\tau_1\otimes\cdots\otimes\tau_n,-\varepsilon)$.

Since both σ and $\tau_1\otimes\cdots\otimes\tau_n$ are of the first kind and of different types, we have

$$u^+(H_1\otimes_k\cdots\otimes_k H_{n+1})\leq\min\{\frac{1}{2}(\frac{3}{2}a_n)+c_n,\frac{3}{2}a_n+\frac{1}{2}c_n\}u(k)=\frac{3}{4}a_n+c_n=a_{n+1}u(k),$$

$$u^-(H_1\otimes_k\cdots\otimes_k H_{n+1})\leq\min\{\frac{1}{2}(\frac{3}{2}b_n)+c_n,\frac{3}{2}b_n+\frac{1}{2}c_n\}u(k)=\frac{3}{2}b_n+\frac{1}{2}c_n=b_{n+1}u(k).$$

Finally by [PS13, Prop. 4.3],

$$u^0(H_1 \otimes_k \cdots \otimes_k H_{n+1} \otimes_k K) \leq \min\{\frac{1}{2}a_{n+1} + b_{n+1}, a_{n+1} + \frac{1}{2}b_{n+1}\}u(k)$$

$$= \frac{1}{2}a_{n+1} + b_{n+1} = c_{n+1}u(k).$$

Here lemma 4.4.1 was used in all three calculations. □

Remark. When $n = 2$, $a_2 = \frac{29}{16}$ is the same as that of [PS13, Cor. 4.5], $b_2 = \frac{13}{16}$ is smaller than the bound $\frac{17}{16}$ of [PS13, Cor. 4.6, Cor. 4.7]. When k is a semi-global field, $u^-(D) \leq \lfloor\frac{13}{2}\rfloor = 6$ is smaller than the bound 8 of [PS13, Cor. 4.8].

When $n \geq 3$, a_n is smaller than the bound $\frac{3^{2n-6}}{4^n} \cdot 213$ of [PS13, Cor. 4.10, Cor. 4.11].

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