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On the Design of Reflecting Systems with Virtual Sources

By

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Doctor of Philosophy

Mathematics

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B.A., Williams College, 2017
M.S., Emory University, 2022

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An abstract of
A dissertation submitted to the Faculty of the
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Abstract

On the Design of Reflecting Systems with Virtual Sources
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We consider the problem of the determination of a system of reflecting surfaces jointly transforming a given radiance distribution from a point source into an irradiance distribution appearing to an observer as produced by some virtual sources. Our work continues the work by Kochengin et al. [13] which dealt with the case when the required reflector is a single surface. Here, the reflector is allowed to consist of several disjoint surfaces.

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# Chapter 1

# Introduction

Let $\mathcal{O}$ be the origin of $\mathbb{R}^3$, and let $\mathbb{S}^2$ be the unit sphere centered at $\mathcal{O}$. We treat points on $\mathbb{S}^2$ as unit vectors with initial points at $\mathcal{O}$. Let an *aperture* be a connected open subset of $\mathbb{S}^2$. Physically, it makes sense to consider $\mathcal{O}$ as the location of an anisotropic point source of light such that rays of light are emitted in a set of directions defined by an aperture $D \subseteq \mathbb{S}^2$.

**Definition 1.1.** *Assume that we are given an aperture $D \subseteq \mathbb{S}^2$, and a continuous function $\rho : D \to (0, \infty)$. Consider the set $R = \{m\rho(m)|m \in D\} \subset \mathbb{R}^3$. If $m\rho(m)$ is a regular point for almost all $m \in D$, then the set $R = \{m\rho(m)|m \in D\} \subset \mathbb{R}^3$ is a* **reflector***.*

*If $\rho$ is a function such that $R = \{m\rho(m)|m \in D\} \subset \mathbb{R}^3$ is a regular surface (see Definition 2.2.1 in [5]), then we can call $R$ a* **smooth reflector***.*

Given an aperture $D$, assume that we have a continuous function $\rho : D \to (0, \infty)$ such that the corresponding set $R = \{m\rho(m)|m \in D\}$ is a reflector. Suppose that a ray originating from $\mathcal{O}$ in the direction $m \in D$ is incident on the reflector $R$ at the point $m\rho(m)$. If $m\rho(m)$ is regular, there is a unit vector, $n(m)$, normal to the reflector $R$ at $m\rho(m)$. Therefore, by the reflection law of geometric optics, a ray from $\mathcal{O}$ of direction $m$ reflects off of $R$ at the point $m\rho(m)$ in the direction

$$y(m) = m - 2\langle m, n(m)\rangle n(m) \tag{1.1}$$

where $\langle m, n(m)\rangle$ is the standard Euclidean inner product in $\mathbb{R}^3$ and $n(m)$ is oriented such that $\langle m, n(m)\rangle > 0$ [1].

The reflector $R$ is designed such that the ray described by the point $m\rho(m) \in R$ and the direction $y(m)$ corresponds to some element in a prespecified *target set* $T$. What one means by a 'target set' changes depending on the context. Also, the meaning of a 'correspondence between $y(m)$ and an element of $T$,' and the meaning of 'an element of the target set' will also vary depending on the specific problem being discussed. For example, if the target set $T$ is a subset of $\mathbb{S}^2$, then a possible correspondence can be $\frac{y(m)}{|y(m)|} \in T$; see [3]. Physically, in this case, $T$ can be considered as a set of directions for rays of light. If $T$ is a subset $\mathbb{R}^3 \setminus \{\mathcal{O}\}$, then, as another example, we can say that for every $m \in D$, there exists an $a(m) > 0$ such that $a(m)y(m)+m\rho(m) \in T$; see [20] and [11]. Physically, in this case, $T$ can be considered as a region that one wants to illuminate.

In this dissertation, we study the virtual source reflector problem. A *virtual source* is the collection of focus points made by extensions of diverging rays of light. In other words, a virtual source is found by tracing reflected rays that emerge from a reflector backward to the perceived origins of ray divergences. We study the case where our target set $T$ is a subset $\mathbb{R}^3 \setminus \{\mathcal{O}\}$ and for every $m \in D$, there exists an $a(m) > 0$ such that $-a(m)y(m) + m\rho(m) \in T$. Physically, $T$ can be viewed as a virtual source.

Assume that $g$ is an integrable and nonnegative function over an aperture

$D$, and $f$ is an integrable and nonnegative function over a target set $T$. Physically speaking, we say $g(m)$ for $m \in D$ is the radiance of the source at $\mathcal{O}$ in the directions $m \in D$, or that $g$ is a radiance distribution over $D$. We also say $f(x)$ for $x \in T$ is the irradiance of the target set at $x \in T$, or that $f$ is an irradiance distribution over $T$.

A *reflector system* consists of an aperture $D$, $\mathcal{O}$, a reflector $R$, an integrable and nonnegative function $g$ over $D$, and a target set $T$ with an integrable and nonnegative function $f$ over $T$. From a physical perspective: the light emitted from the source at $\mathcal{O}$ in directions defined by the aperture $D$, of radiance $g(m)$ for $m \in D$, is reflected off $R$, creating the irradiance $f(x)$ for $x \in T$. An example that can serve as an illustration is shown in Figure 1.

A *reflector problem* is, in short, an inverse problem that seeks to complete a reflector system by creating a reflector that fits the other information given. Specifically, suppose that we are given $\mathcal{O}$, an aperture $D$, an integrable and nonnegative function $g$ over $D$, and a target set $T$ with an integrable and nonnegative function $f$ over $T$. The aim of a reflector problem is to find a continuous and positive function $\rho$ over $D$ such that the corresponding set $R = \{m\rho(m) | m \in D\}$ is a reflector that produces the specified in advance irradiance distribution $f$ on $T$.

Reflector problems have been well studied due to their utility in physics and engineering. Such problems have found numerous applications in the construction of reflector antennas [21], mirror design [2], heat transfer [9], and beam shaping [7]. This work deals with exploring variants of the virtual source reflector problems; descriptions of which will be provided in their appropriate sections. All problems discussed in this dissertation are being considered in the high-frequency approximation, where the laws of geometric optics apply. Mathematical descriptions of these laws will be presented when appropriate. We now proceed with a general description and motivation for the reflector problem that we study.

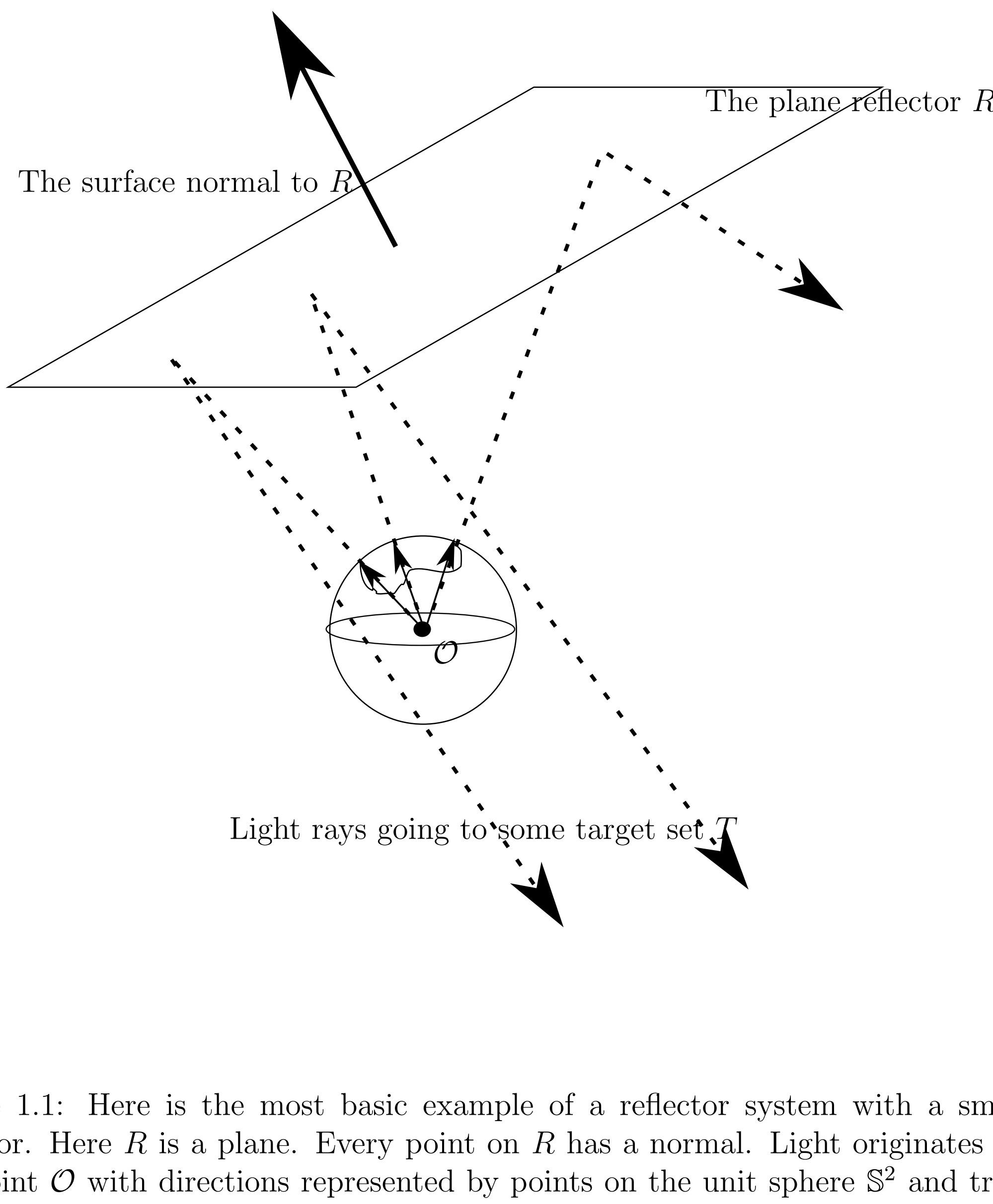


Figure 1.1: Here is the most basic example of a reflector system with a smooth reflector. Here $R$ is a plane. Every point on $R$ has a normal. Light originates from the point $\mathcal{O}$ with directions represented by points on the unit sphere $\mathbb{S}^2$ and travels according to some target set that is neither shown nor specified.

## 1.1 Virtual source Reflector Problem

In this section, we give a general formulation of the virtual source reflector problem. A more descriptive discussion and formulation for this problem are available in Chapter 3. This part is not required reading, as the mathematics introduced below will not be used again. However, this section can provide some good motivation for the general problem.

In this part, when we say *surface*, we mean a *regular surface*; see Definition 2.2.1 in [5]. Suppose that we are given a reflector system consisting of

1. $\mathcal{O}$,
2. an aperture $D \subset \mathbb{S}^2$,
3. a nonnegative $g \in L^1(D)$,
4. a bounded Borel set $T \subset \mathbb{R}^3 \setminus \{\mathcal{O}\}$ (typically a subset of a surface or a finite set),
5. a nonnegative and integrable function $f : T \to [0, \infty)$,
6. a smooth function $\rho : D \to (0, \infty)$ such that the set $R = \{m\rho(m) | m \in D\}$ is a smooth reflector.

Our reflector system can be described as follows. The light is emitted from the source at $\mathcal{O}$ in directions defined by the aperture $D$. Each ray of direction $m \in D$ has radiance $g(m)$ and is reflected off $R$ at the point $m\rho(m)$ in the direction $y(m)$ as described by (1.1). For every $m \in D$, there exists an $a(m) > 0$ such that $m\rho(m) - a(m)y(m) \in T$. This inspires a mapping from $D \to T$. We can therefore say that the elements of $D$ that map to $x \in T$ 'create' the irradiance $f(x)$. With this setup in mind, we proceed with a formulation of the virtual source reflector problem.

Let $u = (u^1, u^2)$ be smooth local coordinates on $\mathbb{S}^2$ such that $D$ lies in one coordinate patch. The position vector of a point $m \in D$ is $m = m(u)$. We choose the coordinates $u^1, u^2$ so that $\langle m, m_1 \times m_2 \rangle = 1$ in $D$; here, $\langle , \rangle$ denotes the scalar product in $\mathbb{R}^3$ and $m_i = \frac{\partial m}{\partial u^i}$, $i = 1, 2$. Observe that this implies that $\langle m, m_i \rangle = 0$, $i = 1, 2$. The first fundamental form of $\mathbb{S}^2$ is given by $e = e_{ij} du^i du^j$ where $e_{ij} = \langle m_i, m_j \rangle$.

Set $r(m) = m\rho(m)$, then $r(m)$ defines a surface $R = \{r(m) | m \in D\}$. Let $g = g_{ij} du^i du^j$ be the first fundamental form of $R$ where $g_{ij} = \langle r_i, r_j \rangle = \rho_i \rho_j + \rho^2 e_{ij}$, $r_i = \frac{\partial r}{\partial u_i}$, and $\rho_i = \frac{\partial \rho}{\partial u_i}$.

Let $n(m)$ be the normal vector field on $R$ such that $\langle n(m), m \rangle > 0$ everywhere on $R$. Then

$$n(m) = (\rho^2 + |\tilde{\nabla}\rho|^2)^{-1/2}(r - \tilde{\nabla}\rho) \tag{1.2}$$

where $|\tilde{\nabla}p|^2 = \rho_i \rho_j e^{ij}$. This combined with equation (1.1) determines the direction in which a ray will go after reflecting off $R$ [17].

The reflector $R = \{m\rho(m) | m \in D\}$ determines our reflector map $x : D \to T$ which, in turn, is determined by tracking the path of each ray described by the direction $m \in D$ to a point $x(m) \in T$. A ray, originating at $\mathcal{O}$ in direction $m$, hits the reflector $R$ at a point $m\rho(m)$. By the reflection law of geometric optics, the direction of reflection at $m\rho(m)$ can be described by the direction $y(m)$ in (1.1). However, since we are working on the virtual source reflector problem, we follow the ray of direction $-y(m)$ from $m\rho(m)$ until it reaches $T$ at some point $x(m)$. Thus, from a physical perspective, an irradiance $f(x(m))$ is created by the rays reflected away from $x(m)$. This defines a mapping $m \to x(m)$ that we call the *reflector map*; for convenience, we denote $x(m)$ as the image of $m$ under the reflector map.

If the reflector map is a diffeomorphism from $D$ to $T$ where $T$ is a subset of a surface, then one can introduce the first fundamental form of $T$ as $w = w_{ij} du^i du^j$, where $w_{ij} = \langle x_i, x_j \rangle, x_i = \frac{\partial x}{\partial u^i}$.

According to the differential form of the energy conservation law [1],

$$f(x(m))|J(x(m))| = g(m) \tag{1.3}$$

where $J$ is the Jacobian determinant of the map $x$. Note that

$$J(x(m)) = \pm \frac{d\nu(x(m))}{d\sigma(m)} = \pm \frac{\sqrt{\det(w_{ij})}}{\sqrt{\det(e_{ij})}} \tag{1.4}$$

where $d\sigma$ is the surface area element on $\mathbb{S}^2$, and $d\nu$ is the surface area element on $T$. We assign a $\pm$ sign to the Jacobian according to whether $x$ preserves the orientation or reverses it. Therefore, by integration of (1.3), for all Borel sets $\omega \subseteq T$,

$$\int_{x^{-1}[\omega]} g d\sigma = \int_{\omega} f d\nu \tag{1.5}$$

where $x^{-1}[\omega] = \{m \in D | x(m) \in \omega\}$ and $\int_D g d\sigma = \int_T f d\nu$.

With this motivation, we can now state the virtual source reflector problem. Assume that we are given $\mathcal{O}$, an aperture $D \subset \mathbb{S}^2$ with a nonnegative function $g \in L^1(D)$, and a bounded Borel set $T \subset \mathbb{R}^3 \setminus \{\mathcal{O}\}$ with a nonnegative, integrable function $f : T \to [0, \infty)$. The goal is to find a positive smooth function $\rho$ over $D$ such that the corresponding set $R = \{m\rho(m) | m \in D\}$ is a smooth reflector such that:

1. The ray originating from $\mathcal{O}$ in the direction $m \in D$ reflects off the reflector $R = \{m\rho(m) | m \in D\}$ in accordance with equation (1.1) such that the ray of direction $-y(m)$ from $m\rho(m)$ reaches the target set $T$.

2. $g(m)$ on $D$ is transformed by the reflector map into $f$ on $T$; i.e. for all Borel

subsets $\omega \subseteq T$,

$$\int_{x^{-1}[\omega]} g d\sigma = \int_{\omega} f d\nu \tag{1.6}$$

where $x : D \to T$ the reflector map corresponding to the reflector $R = \{m\rho(m) | m \in D\}$, $x^{-1}[\omega] = \{m \in D | x(m) \in \omega\}$, $d\sigma$ is the surface area element on $\mathbb{S}^2$, and $d\nu$ is the area element on $T$ ($\nu$ is typically some discrete or Lebesgue measure).

3. The law of total energy conservation is obeyed: $\int_D g d\sigma = \int_T f d\nu$.

## 1.2 Dissertation Synopsis

For this dissertation, we work only on the virtual source reflector problem. In Chapter 3, we continue the work in [13] and develop existence and uniqueness results. For the existence results, we focus on the rotationally symmetric case and the case where the points in the target set are sufficiently close together. In Chapter 4, we develop an expansion of the weak solution in [13] and we develop existence results. In this case, the reflector is allowed to consist of several disjoint surfaces.

# Chapter 2

# Notation and Terminology

Here we summarize some notations and terminology used in the upcoming chapters.

| | |
|---|---|
| $\mathbb{R}^3$: | The three-dimensional euclidean vector space. |
| $\mathcal{O}$: | The origin of $\mathbb{R}^3$. |
| $\mathbb{S}^2$: | The unit sphere centered at $\mathcal{O}$ of a Cartesian coordinate system in $\mathbb{R}^3$. |
| $\mathbb{N}$: | The set of all natural numbers. |
| $[n] = \{1, 2, \ldots, n\}$: | For some $n \in \mathbb{N}$. |
| $k_x = \frac{x}{\lvert x\rvert}$: | For some $x \in \mathbb{R}^3 \setminus \{\mathcal{O}\}$. |
| $\lvert Q\rvert$: | The cardinality of $Q$. |
| $2^Q$: | The powerset of $Q$. |
| $f[A] = \{f(x) \mid x \in A\}$: | Given a function $f : X \to Y$, then for $A \subseteq X$, $f[A]$ is the **image** of $A$ with respect to $f$. |

$\partial B$: The boundary of $B$.

$\overline{B}$: The closure of $B$.

$\mathrm{Int}(B)$: The interior of $B$.

$\sigma(B)$: The standard measure of $\mathbb{S}^2$ of a set $B \subset \mathbb{S}^2$.

$L^1(\mathbb{S}^2)$: The set of functions $g$ on $\mathbb{S}^2$ such that $\int_{\mathbb{S}^2} |g(m)| d\sigma(m) < \infty$.

# Chapter 3

# Convex Weak Solutions to the Virtual Source Reflector Problem

## 3.1 Introduction

We first recall the classical law of reflection that was previously introduced in the introduction. Assume that we are given an aperture $D$. Also, assume that we are given a continuous function $\rho : D \to (0, \infty)$ such that the corresponding set $R = \{m\rho(m) | m \in D\}$ is a reflector. Suppose that a ray originating from $\mathcal{O}$ in the direction $m \in D$ is incident on the reflector $R$ at the point $m\rho(m)$. If $m\rho(m)$ is regular, there is a unit vector, $n(m)$, normal to the reflector $R$ at $m\rho(m)$. Therefore, by the reflection law of geometric optics, a ray from $\mathcal{O}$ of direction $m$ reflects off $R$ at the point $m\rho(m)$ in the direction

$$y(m) = m - 2\langle m, n(m)\rangle n(m) \tag{3.1}$$

where $\langle m, n(m) \rangle$ is the standard Euclidean inner product in $\mathbb{R}^3$ and $n(m)$ is oriented such that $\langle m, n(m) \rangle > 0$ [1].

**Definition 3.1.1.** *Assume that we are given an aperture $D \subseteq \mathbb{S}^2$. Let $U$ be an open subset of $\mathbb{S}^2$ such that $D \subseteq U$. Consider a continuous function $\rho : U \to (0, \infty)$. If $m\rho(m)$ is regular for almost all $m \in D$, then the set $R = \{m\rho(m) | m \in U\} \subset \mathbb{R}^3$ is a* **refractor**.

Note that $R = \{m\rho(m) | m \in D\}$ can be considered as either a reflector or a refractor. If $R = \{m\rho(m) | m \in D\}$ is considered as a refractor, the refracted direction $\hat{y}$ is determined by Snell's law and is given as

$$\hat{y}(m) = c_f m - \left( \sqrt{1 - c_f^2 (1 - \langle m, n(m) \rangle^2)} - c_f \langle m, n(m) \rangle \right) n(m) \tag{3.2}$$

where $c_f$ denotes the refraction index.

We borrow the following motivation from [13]. Consider a two-sheeted hyperboloid of revolution with sheets $B$ and $H$. Let $\mathcal{O}$ be the focus inside the convex body bounded by the first sheet $B$ and $x$ be the focus inside the convex body bounded by the sheet $H$. Suppose that a point source of light is positioned at $\mathcal{O}$ and the sheet $H$ is a reflector. $H$ has very special and important reflecting properties. Specifically, if a ray of direction $m$ from $\mathcal{O}$ is incident on a point $z \in H$ and is reflected in the direction $y(m)$ as defined by (3.1), then the ray from $z$ of direction $y(m)$ coincides with a ray from $x$ of direction $y(m)$. This means that the focus $x$ can be viewed, from a physical perspective, as a virtual source of rays reflected off $H$. A two-dimensional analog of this situation is illustrated in Figure 3.1.

This same situation can also be interpreted from a different point of view allowing us to treat it geometrically as a refraction problem, rather than a reflection problem. Now suppose a light ray of direction $m$ from $\mathcal{O}$ strikes $H$ and 'refracts' such

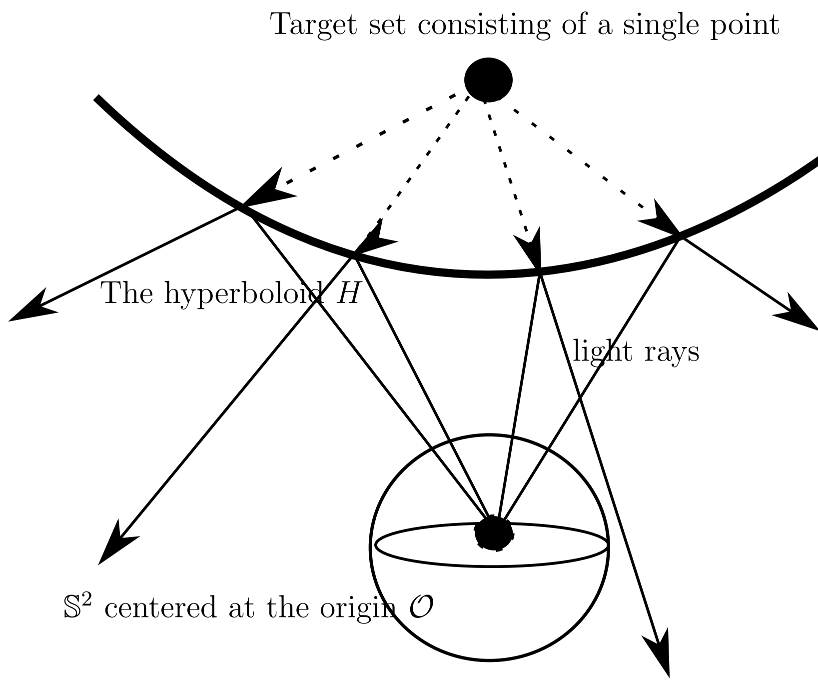


Figure 3.1: Here is an illustration of a virtual source reflector system where the target set is a single point. Note that all the rays of light reflect off of the hyperboloid $H$ such that it appears that the light is originating from the target point.

that the refracted direction is given by

$$\hat{y} = -y = -m + 2\langle m, n(m)\rangle n(m). \tag{3.3}$$

Then since the refracted direction is the opposite of the reflected direction, every ray of direction $m$ that strikes the refractor $H$ will cross the focus $x$. Equation (3.3) can also be considered as the version equation of (3.2) where $c_f = -1$. Under the law of total energy conservation, the total energy 'delivered' by the refractor $H$ to the point $x$ will be equal to the total energy produced by the source $\mathcal{O}$. We will only discuss this type of refraction, where $c_f = -1$, for the rest of the dissertation.

This interpretation of the reflection with a virtual source as a particular case of refraction is convenient from a geometric point of view and we use this terminology throughout this chapter and the next. Physically, however, it is more natural to treat the point $x$ as a virtual source. This would also be consistent with the case of a distributed virtual source; which we focus on.

To quote [13]: with this terminology, the problem studied in this chapter can now be described as a problem of finding a convex refractor $R$ which will refract a given anisotropic bundle of rays from a source $\mathcal{O}$ in such a way that the refracted rays are incident on a specified set in space and produce there, a given-in-advance intensity distribution. More specifically, suppose that we have a system consisting of an anisotropic point source at $\mathcal{O}$, an aperture $D$, a nonnegative $g(m) \in L^1(\mathbb{S}^2)$, a target set $T \subset \mathbb{R}^3 \setminus \{\mathcal{O}\}$, and nonnegative integrable function $f$ defined on $T$. The problem consists of finding a refractor $R$ which produces the specified in advance $f$ on $T$. Henceforth, we call this problem the *refractor problem*.

The only previous work available with respect to this problem can be found in [13]. In the paper, they develop a definition of a weak solution to a PDE of Monge

Ampère type; specifically, the PDE described by equation (4) in [13] (the formulation of said PDE will be reproduced in Appendix A for the convenience of the reader). They detail the construction of convex refractors and provide an existence theorem for the case where the target set is discrete (Theorem 3.5.1). Due to the weak convergence of Dirac measures to Lebesgue measures, one can create refractors that produce discrete irradiance distributions that are arbitrarily close to a continuous distribution, like pixels in a photo. However, this does not imply the existence of a refractor that produces a continuous intensity distribution at the limit. This is expected for problems that can be described by a fully nonlinear PDE of Monge-Ampére type [18]. However, if the refractors are convex, due to the unique properties that convexity provides(see [19], Appendix C, Appendix D), one can use the weak convergence of Dirac measures to Lebesgue measures to obtain a refractor that produces a continuous irradiance distribution; see [11],[3].

In this chapter, we work on the weak formulation developed in [13], where we develop existence and uniqueness results. Due to a mistake in [13] (see Appendix B), Theorem 9 in [13] (Theorem 3.5.1) is the only available existence theorem for the refractor problem. Specifically, Theorem 9 in [13] gives sufficient conditions such that there exists a weak solution; as [13] points out, these conditions are not easy to verify, and it is desirable to provide more explicit, sufficient conditions. We aim to make progress towards that goal in this chapter. We find that Theorem 9 in [13] can be used to prove another existence theorem for the discrete case (Theorem 3.5.2) that, in turn, can be extended to the continuous case (Theorem 3.6.1). We use this result to then prove the existence of solutions for the rotationally symmetric case (Theorem 3.7.1). Additionally, we prove a uniqueness theorem for the case where the target set is finite (Theorem 3.4.1) and for the general case (Theorem 3.4.2).

## 3.2 Hyperboloids of Revolution

We do all our work in $\mathbb{R}^3$. We denote $\mathbb{S}^2$ to be the unit sphere with the center at $\mathcal{O}$ and $k_x = x/|x|$ for all $x \in \mathbb{R}^3 \setminus \{\mathcal{O}\}$. We borrow much of this geometric setup from [13]. Hyperboloids of revolution are of paramount importance when solving the virtual source reflector problem due to their unique optical properties.

Consider the rotationally symmetric hyperboloid of two sheets in $\mathbb{R}^3$ such that one focus is $\mathcal{O}$ and the other is $x$; let $H(x)$ be the branch of the hyperboloid that has $x$ as a focus. From now on, when we use the term *hyperboloid*, we are only referring to this branch.

With each hyperboloid $H(x)$ we associate its radial projection by rays from the origin onto an open spherical disk $D(x) \subset \mathbb{S}^2$ and its polar radius

$$\hbar_\epsilon(m) = \frac{|x|(1-\epsilon^2)}{2\epsilon(1-\epsilon\langle m, k_x\rangle)},\ m \in D(x) \tag{3.4}$$

where $\epsilon$ is the eccentricity of the hyperboloid. Be aware that $\epsilon > 1$ since we are describing a hyperboloid.

Define $H_\epsilon(x)$ to be the hyperboloid with eccentricity $\epsilon$ and focus $x$. We now introduce a similar function $\hbar_{x,\epsilon}(m)$ which introduces $x \in \mathbb{R}^3 \setminus \{\mathcal{O}\}$ as a variable. In this and the following chapters, we define $h_{x,\epsilon}(m) = m\hbar_{x,\epsilon}(m)$ for $m \in D(x)$ and $x \in \mathbb{R}^3 \setminus \{\mathcal{O}\}$. Let $D_\epsilon(x) \subset \mathbb{S}^2$ be the preimage of $H_\epsilon(x)$ under $h_{x,\epsilon}$, then $D_\epsilon(x) = \{m \in \mathbb{S}^2 | \frac{1}{\epsilon} < \langle m, k_x\rangle\}$. Thus we can easily verify that $H_\epsilon(x) = \{h_{x,\epsilon}(m) | m \in D_\epsilon(x)\}$.

From a physical perspective, $x$ being the focus means that all light from the origin reflected off of the reflector $H(x)$ appears to be originating from $x$, making $x$ a virtual source.

From the above work, we see by taking the eccentricity $\epsilon$ to infinity that the shape of the hyperboloid becomes a plane, which is the *directrix* of the hyperboloid, and $D_\epsilon(x)$ and goes to the hemisphere oriented towards $x$. The following two propositions summarize what I say precisely.

**Proposition 3.2.1.** *As the eccentricity $\epsilon$ of $H_\epsilon(x)$ goes to infinity, $D_\epsilon(x)$ goes to $\{m \in \mathbb{S}^2 | \langle m, k_x \rangle \geq 0\}$.*

**Proposition 3.2.2.** *As the eccentricity $\epsilon$ of $H_\epsilon(x)$ goes to infinity, the resultant set is a plane represented by the equation $\langle x, y - \frac{x}{2} \rangle = 0$ where $y \in \mathbb{R}^3$, or equivalently by the polar radius equation $r(m) = \frac{|x|}{2\langle m, k_x \rangle}$ where $m \in \{m \in \mathbb{S}^2 | \langle m, k_x \rangle > 0\}$.*

Observe that as the eccentricity $\epsilon$ goes to 1, we obtain a ray originating at $x$ going in the direction described by the vector $k_x$. We call this a *degenerate hyperboloid.*

An important property of hyperboloids can be described by the following proposition.

**Proposition 3.2.3.** *Let $c > 0$ and $\epsilon > 1$ such that $c\epsilon > 1$. Then the hyperboloids $H_{c\epsilon}(x)$ and $H_\epsilon(x)$ have the same foci: $\mathcal{O}$ and $x$.*

The aforementioned property is important because a reflector $H_\epsilon(x)$ will reflect the light emitted from $\mathcal{O}$ so that the light appears to be emitted from $x$; thus, making $x$ a virtual source. Alternatively, a refractor $H_\epsilon(x)$ will refract the light emitted from $\mathcal{O}$ so that the light is delivered to $x$. These properties are true no matter how large or small the eccentricity is; all that matters is the location of the foci.

Let $A_x = \{m \in \mathbb{S}^2 | \langle m, k_x \rangle \geq 0\}$ and $A_x^\delta = \{m \in \mathbb{S}^2 | \langle m, k_x \rangle \geq \delta\}$ for $\delta \in \mathbb{R}$. Observe that $A_x = A_x^0$, $A_x^1 = \{k_x\}$, $A_x^\delta = \varnothing$ for $\delta > 1$, and $A_x^\delta = A_x^{-1} = \mathbb{S}^2$ for $\delta \leq -1$. It is also clear that if $\delta_1 \leq \delta_2$, then $A_x^{\delta_1} \subseteq A_x^{\delta_2}$ with a strict inclusion if $\delta_1 < \delta_2$ and $\delta_1, \delta_2 \in [-1, 1]$. So while $\delta$ only has practical significance while taking values in $[-1, 1]$, allowing it to take all values in $\mathbb{R}$ makes some of the upcoming proofs easier.

By Propositions 3.2.1 and 3.2.2, we have the following statement.

**Proposition 3.2.4.** *Let* $0 < \delta < 1$ *and* $B \subseteq A_x^\delta$. *Then if* $\epsilon > \frac{1}{\delta}$, *then* $h_{x,\epsilon}[B] \subset H_\epsilon(x)$. *In particular:*

1. $\frac{1}{\epsilon} < \delta$ *implies that* $A_x^\delta \subset D_\epsilon(x)$,
2. $\frac{1}{\epsilon} > \delta$ *implies that* $D_\epsilon(x) \subset A_x^\delta$,
3. $\frac{1}{\epsilon} = \delta$ *implies that* $\mathrm{Int}(A_x^\delta) = D_\epsilon(x)$.

## 3.3 Convex Weak Solutions

We now have the background to construct and proceed with our discussion of the weak solution. Keep in mind that this weak solution definition, apart from some minor differences in notation, is identical to the weak solution defined in [13].

Let $c = \min_{x,y\in T}\langle k_x, k_y\rangle$, $\ell = \min_{x\in T}|x|$, and $L = \max_{x\in T}|x|$. Assume that we are given a set $T \subseteq \mathbb{R}^3$. We say that $T$ satisfies Hypothesis H1 if the following condition is met.

**Hypothesis H1** (Hypothesis H1 in [13]). *$T$ is a compact subset of* $\mathbb{R}^3$ *contained in a half space of* $\mathbb{R}^3$, $\ell > 0$, *and* $2\ell c > L$.

We also define a constant,

$$\epsilon_0 = \frac{\ell + \sqrt{\ell^2 - 2L\ell c + L^2}}{2\ell c - L}, \tag{3.5}$$

that depends only on $T$.

We first assume we are given a target set $T$ that satisfies Hypothesis H1. Let $\tilde{H}_\epsilon(x)$ be the convex body bounded by $H_\epsilon(x)$. Consider the aperture $D_T^{\delta_\gamma} = \mathrm{Int}\left(\bigcap_{x\in T} A_x^{\delta_\gamma}\right)$ where $\delta_\gamma = \frac{1}{\epsilon_0+\gamma}$ for some $\gamma > 0$. We then define a simply connected refractor over $D_T^{\delta_\gamma}$ as the boundary of the intersection of the convex bodies bounded by hyperboloids. Specifically,

$$R = \partial h \text{ where } h = \bigcap_{x\in T} \tilde{H}_{\epsilon_x}(x) \tag{3.6}$$

where each $\epsilon_x \geq \epsilon' \geq \epsilon_0 + \gamma = \frac{1}{\delta_\gamma}$. Observe that

$$R = \left\{ m \sup_{x\in T} \hbar_{x,\epsilon_x}(m) \,\middle|\, m \in \mathrm{Int}\left(\bigcap_{x\in T} A_x^{\frac{1}{\epsilon_x}}\right)\right\}. \tag{3.7}$$

Observe that $D_T^{\delta_\gamma} \subseteq \mathrm{Int}\left(\bigcap_{x\in T} A_x^{\frac{1}{\epsilon_x}}\right)$ and, by Theorem D.1 (see [19], Appendix D), almost every point on $R$ is regular; thus $R$ may be considered a refactor per Definition 3.1.1. Let

$$\mathcal{R}^{\epsilon'}_{convex}(T) \tag{3.8}$$

be the set of all such refractors. Please note that by Lemma 1 in [13], the set $\mathcal{R}^{\epsilon'}_{convex}(T)$ is nonempty.

Note the following definition.

**Definition 3.3.1.** *A hyperboloid $H(x)$ is said to be* ***supporting*** *to a set $Q \subset \mathbb{R}^3$ at a point $z \in \partial Q$ if the convex body $\tilde{H}(x)$ bounded by $H(x)$ contains $Q$ and $z \in H(x) \cap \partial Q$.*

For a subset $\omega \subseteq T$ and a refractor $R \in \mathcal{R}^{\epsilon'}_{convex}(T)$ put

$$M(\omega) = \{z \in R | \text{ there exists } x \in \omega \text{ such that } H(x) \text{ is supporting to } R \text{ at } z\}. \tag{3.9}$$

The intersection of $\overline{D_T^{\delta_\gamma}}$ with the image of the set $M(\omega)$ under radial projection on $\mathbb{S}^2$ we call the *visibility set of* $\omega$ and denote it by $V_{convex}(\omega)$. By Lemma 4 of [13], this set $V_{convex}(\omega)$ is measurable for all Borel sets $\omega \subseteq T$.

For $m \in D_T^{\delta_\gamma}$ let $r(m)$ be the set of points of intersection between the refractor $R$ and the ray of direction $m$ originating at $\mathcal{O}$. The possibly multivalued map $\alpha_{convex}$ : $D_T^{\delta_\gamma} \to T$,

$$\alpha_{convex}(m) = \{x \in T| \text{ there exists } H(x) \text{ supporting to } R \text{ at } r(m)\} \tag{3.10}$$

is called the *refractor map.*

Assume we are given a nonnegative $g \in L^1(\mathbb{S}^2)$. Let us define for measurable $X \subseteq \mathbb{S}^2$

$$\mu_g(X) = \int_X g(m) d\sigma(m) \tag{3.11}$$

where $\sigma$ denotes the standard measure on $\mathbb{S}^2$. Assume that $g \equiv 0$ outside of $D_T^{\delta_\gamma}$.

In order to formulate and solve the refractor problem (in the framework of weak solutions to be defined below), we need to define a measure representing the energy generated by $g$ and redistributed by a refractor $R \in \mathcal{R}^{\epsilon'}_{convex}(T)$.

Define for any refractor $R \in \mathcal{R}^{\epsilon'}_{convex}(T)$,

$$G_{convex}(\omega) = \mu_g(V_{convex}(\omega)) \tag{3.12}$$

which we will deem *the energy function.* It can be shown that $G$ is a finite measure on the Borel $\sigma$-algebra of $T$.

Let $F$ be a nonnegative, finite, Borel measure on Borel subsets of $T$. We say that a refractor $R \in \mathcal{R}^{\epsilon'}_{convex}(T)$ is a *convex weak solution to the refractor problem* if the refractor map $\alpha_{convex}$ determined by $R$ is such that $\alpha_{convex}(m) \subseteq T$ for all $m \in D_T^{\delta_\gamma}$, and

$$F(\omega) = G_{convex}(\omega) \text{ for any Borel set } \omega \subseteq T. \tag{3.13}$$

## 3.4 Uniqueness Theorems

We start with some uniqueness results. Note that Theorem 3.4.1 can be considered as a direct corollary to Theorem 3.4.2. However, we include both as separate statements and proofs; as the discrete versions of uniqueness theorems proved to be of special interest in related problems. For example, Theorem 12 in [11] is a uniqueness theorem for the discrete case of the near-field reflector problem, and [12] demonstrates a concrete algorithm to construct reflectors in the discrete case of the near-field reflector problem. We proceed with the following lemma, which is shown in the proof of Lemma 2 in [13].

**Lemma 3.4.1.** *Let $T$ be a target set that satisfies Hypothesis H1. Suppose we are given positive real numbers $\gamma$ and $\epsilon'$ such that $\epsilon' \geq \epsilon_0 + \gamma$ where $\epsilon_0$ is defined by (3.5). Let $R \in \mathcal{R}^{\epsilon'}_{convex}(T)$. Then $V_{convex}(\omega)$ is closed for all closed $\omega \subseteq T$.*

We now introduce some notation. If we write $V_{convex}(R; \omega)$ for some Borel set $\omega \subseteq T$ and some refractor $R \in \mathcal{R}^{\epsilon'}_{convex}(T)$, this is specifically the visibility set for the refractor $R$ evaluated on the set $\omega$. Similarly, if we write $G_{convex}(R; \omega)$ for some Borel set $\omega \subseteq T$ and some refractor $R \in \mathcal{R}^{\epsilon'}_{convex}(T)$, this is specifically the energy function for the refractor $R$ evaluated on the set $\omega$. We will be using this when we are talking about multiple refractors and we need to specify the energy function for each refractor.

Here we consider the case of the refractor problem (3.13) where the set $T$

is finite. We prescribe the measure $F$ in (3.13) as a Dirac measure concentrated at points in $T$. We now introduce notation for refractors in the discrete case. Let $T = \{x_1, x_2, \ldots, x_k\}$. We set $H_i = H(x_i)$ and the eccentricity of $H_i$ we denote by $\epsilon_i$. For the hyperboloids $H_1, \ldots, H_k$ define the refractor

$$R = \partial\left(\bigcap_{i=1}^{k} \tilde{H}_i\right) \in \mathcal{R}^{\epsilon'}_{convex}(T). \tag{3.14}$$

Since each hyperboloid $H_i$ is uniquely defined by its eccentricity $\epsilon_i$, the refractor $R$ can be identified with the point with coordinates $(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ in the region

$$\epsilon_1 \geq \epsilon', \epsilon_2 \geq \epsilon', \ldots, \epsilon_k \geq \epsilon' \tag{3.15}$$

in $k-$dimensional euclidean space. Thus we can write a refractor $R \in \mathcal{R}^{\epsilon'}_{convex}(T)$ as $(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$. We start with a uniqueness theorem for the discrete case.

**Theorem 3.4.1.** *Let $T = \{x_1, \ldots, x_k\}$ be a collection of $k$ distinct points that satisfy Hypothesis H1. Suppose we are given positive real numbers $\gamma$ and $\epsilon'$ such that $\epsilon' \geq \epsilon_0 + \gamma$ where $\epsilon_0$ is defined by (3.5). Assume we are given a nonnegative $g \in L^1(\mathbb{S}^2)$ such that $g > 0$ inside $D_T^{\delta_\gamma}$ and $g \equiv 0$ outside $D_T^{\delta_\gamma}$ where $\delta_\gamma = \frac{1}{\epsilon_0+\gamma}$. Let $f_1, \ldots, f_k$ be a collection of positive real numbers such that*

$$\sum_{i=1}^{k} f_i = \mu_g(D_T^{\delta_\gamma}). \tag{3.16}$$

*Let $\overline{R} = (\overline{\epsilon_1}, \ldots, \overline{\epsilon_k})$ and $\tilde{R} = (\tilde{\epsilon_1}, \ldots, \tilde{\epsilon_k})$ be refractors in $\mathcal{R}^{\epsilon'}_{convex}(T)$ such that $G_{convex}(\tilde{R}; x_i) = G_{convex}(\overline{R}; x_i) = f_i$ for all $i \in [k]$.*

*Then the inequality $\tilde{\epsilon_j} \geq \overline{\epsilon_j}$ for some $j$ implies that $\tilde{\epsilon_i} \geq \overline{\epsilon_i}$ for all $i \in [k]$. Furthermore, the equality $\tilde{\epsilon_j} = \overline{\epsilon_j}$ for some $j$ implies that $\tilde{\epsilon_i} = \overline{\epsilon_i}$ for all $i \in [k]$.*

*Proof.* Let $J$ be a nonempty subset of $[k]$ such that for any $i \in J$, $\tilde{\epsilon_i} > \overline{\epsilon_i}$, and for any $i \in [k] \setminus J$, $\tilde{\epsilon_i} \leq \overline{\epsilon_i}$. Note that $m \in V_{convex}(\tilde{R}; \{x_i | i \in J\})$ if and only if there exists some $i \in J$ such that $\hbar_{x_i,\tilde{\epsilon_i}}(m) \geq \hbar_{x_\ell,\tilde{\epsilon_\ell}}(m)$ for all $\ell \in [k] \setminus J$. For this $m$: since $\hbar_{x_i,\overline{\epsilon_i}}(m) > \hbar_{x_i,\tilde{\epsilon_i}}(m)$ for all $i \in J$ and $\hbar_{x_\ell,\overline{\epsilon_\ell}}(m) \leq \hbar_{x_\ell,\tilde{\epsilon_\ell}}(m)$ for all $\ell \in [k] \setminus J$, then there exists some $i \in J$ such that $\hbar_{x_i,\overline{\epsilon_i}}(m) > \hbar_{x_\ell,\overline{\epsilon_\ell}}(m)$ for all $\ell \in [k] \setminus J$. Thus, any $m \in V_{convex}(\tilde{R}; \{x_i | i \in J\})$ is an interior point of $V_{convex}(\overline{R}; \{x_i | i \in J\})$; in other words, $V_{convex}(\tilde{R}; \{x_i | i \in J\}) \subseteq \mathrm{Int}(V_{convex}(\overline{R}; \{x_i | i \in J\}))$. Recall that, by Lemma 3.4.1, $V_{convex}(\tilde{R}; \{x_i | i \in J\})$ is closed and, since $f_1, \ldots, f_k$ are positive, $V_{convex}(\overline{R}; \{x_i | i \in J\})$ is nonempty. Then $\mathrm{Int}(V_{convex}(\overline{R}; \{x_i | i \in J\})) \setminus V_{convex}(\tilde{R}; \{x_i | i \in J\})$ is open and nonempty. So $\mu_g(V_{convex}(\overline{R}; \{x_i | i \in J\}) \setminus V_{convex}(\tilde{R}; \{x_i | i \in J\})) > 0$. Therefore we must have

$$\sum_{i \in J} f_i = G_{convex}(\tilde{R}; \{x_i | i \in J\}) < G_{convex}(\overline{R}; \{x_i | i \in J\}) = \sum_{i \in J} f_i \tag{3.17}$$

which is a contradiction because $G_{convex}(\tilde{R}; x_i) = G_{convex}(\overline{R}; x_i) = f_i$ for all $i \in [k]$. The theorem is proved. $\square$

Observe that for all refractors $R \in \mathcal{R}^{\epsilon'}_{convex}(T)$, there exists a function $K : T \to [\epsilon', \infty)$ such that $R = \partial(\bigcap_{x \in T} \tilde{H}_{K(x)}(x))$. Since each hyperboloid $\tilde{H}_{K(x)}(x)$ is uniquely determined by $K$, the refactor $R$ can be identified with the function $K : T \to [\epsilon', \infty)$. Thus we can write a refractor $R \in \mathcal{R}^{\epsilon'}_{convex}(T)$ as $[K]$ where $K : T \to [\epsilon', \infty)$; note that $[K] = \left\{ m \sup_{x \in T} \hbar_{x,K(x)}(m) \,\middle|\, m \in \mathrm{Int}\left( \bigcap_{x \in T} A_x^{\frac{1}{K(x)}} \right) \right\}$. Given a refractor $R \in \mathcal{R}^{\epsilon'}_{convex}(T)$, we call $K : T \to [\epsilon', \infty)$ *the maximal function of* $R$ if $R = \left\{ m \max_{x \in T} \hbar_{x,K(x)}(m) \,\middle|\, m \in \mathrm{Int}\left( \bigcap_{x \in T} A_x^{\frac{1}{K(x)}} \right) \right\}$. We proceed with the following lemma.

**Lemma 3.4.2.** *Let $R \in \mathcal{R}^{\epsilon'}_{convex}(T)$ be a refractor such that for all $x \in T$, $V_{convex}(\{x\})$ is nonempty. Then there exists a unique $K : T \to [\epsilon', \infty)$ that is the maximal function* *of R.*

*Proof.* By Lemma 2 in [13], $T \subset \bigcap_{x\in T} \tilde{H}_{\epsilon_x}(x)$. Therefore, by Definition 3.3.1, that $m \in V_{convex}(\{x\})$ if and only if there exists a corresponding $\epsilon'_x \geq \epsilon'$ such that $\hbar_{x,\epsilon'_x}(m) = \sup_{x\in T} \hbar_{x,\epsilon_x}(m)$. Define $K(x) = \epsilon'_x$ for all $x \in T$. Then $R = \left\{ m \max_{x\in T} \hbar_{x,K(x)}(m) \,\middle|\, m \in \operatorname{Int}\left(\bigcap_{x\in T} A_x^{\frac{1}{K(x)}}\right)\right\}$. □

We now conclude with a uniqueness theorem for more general measures and target sets.

**Theorem 3.4.2.** *Let $T$ be a target set that satisfies Hypothesis H1. Let $F$ be a nonnegative, finite, Borel measure on Borel subsets of $T$. Suppose we are given positive real numbers $\gamma$ and $\epsilon'$ such that $\epsilon' \geq \epsilon_0 + \gamma$ where $\epsilon_0$ is defined by (3.5). Assume we are given a nonnegative $g \in L^1(\mathbb{S}^2)$ such that $g > 0$ inside $D_T^{\delta_\gamma}$ and $g \equiv 0$ outside $D_T^{\delta_\gamma}$ where $\delta_\gamma = \frac{1}{\epsilon_0+\gamma}$ such that*

$$F(T) = \mu_g(D_T^{\delta_\gamma}). \tag{3.18}$$

*Let $\overline{R}$ and $\tilde{R}$ be refractors in $\mathcal{R}^{\epsilon'}_{convex}(T)$ such that for all nonempty Borel $\omega \subseteq T$: $F(\omega) = G_{convex}(\tilde{R};\omega) = G_{convex}(\overline{R};\omega)$, $V_{convex}(\tilde{R};\omega) \neq \varnothing$, and $V_{convex}(\overline{R};\omega) \neq \varnothing$.*

*Assume that $J$ is closed subset of $T$, then there exists unique functions $\overline{K} : T \to [\epsilon', \infty)$ and $\tilde{K} : T \to [\epsilon', \infty)$ that are, respectively, maximal functions of $\overline{R}$ and $\tilde{R}$ such that the inequality $\tilde{K}(x) \geq \overline{K}(x)$ for all $x \in J$ implies that $\tilde{K}(y) \geq \overline{K}(y)$ for all $y \in T$. Furthermore, the equality $\tilde{K}(x) = \overline{K}(x)$ for all $x \in J$ implies that $\tilde{K}(y) = \overline{K}(y)$ for all $y \in T$.*

*Proof.* By Lemma 3.4.2, there exists unique functions $\overline{K} : T \to [\epsilon', \infty)$ and $\tilde{K} : T \to [\epsilon', \infty)$ that are maximal functions of $\overline{R}$ and $\tilde{R}$ respectively. Note that $\overline{R} = [\overline{K}]$ and $\tilde{R} = [\tilde{K}]$.

Let $J$ be a nonempty closed subset of $T$ such that for any $x \in J$, $\tilde{K}(x) > \overline{K}(x)$, and for any $x \in T \setminus J$, $\tilde{K}(x) \leq \overline{K}(x)$. Note that $m \in V_{convex}(\tilde{R}; J)$ if and only if there exists some $z \in J$ such that $\mathscr{h}_{z,\tilde{K}(z)}(m) \geq \mathscr{h}_{z',\tilde{K}(z')}(m)$ for all $z' \in T \setminus J$. For this $m$: since $\mathscr{h}_{z,\overline{K}(z)}(m) > \mathscr{h}_{z,\tilde{K}(z)}(m)$ for all $z \in J$ and $\mathscr{h}_{z',\overline{K}(z')}(m) \leq \mathscr{h}_{z',\tilde{K}(z')}(m)$ for all $z' \in T \setminus J$, then there exists some $z \in J$ such that $\mathscr{h}_{z,\overline{K}(z)}(m) > \mathscr{h}_{z',\overline{K}(z')}(m)$ for all $z' \in T \setminus J$. Thus, any $m \in V_{convex}(\tilde{R}; J)$ is an interior point of $V_{convex}(\overline{R}; J)$; in other words, $V_{convex}(\tilde{R}; J) \subseteq \mathrm{Int}(V_{convex}(\overline{R}; J))$. Recall that, by Lemma 3.4.1, $V_{convex}(\tilde{R}; J)$ is closed. Then $\mathrm{Int}(V_{convex}(\overline{R}; J)) \setminus V_{convex}(\tilde{R}; J)$ is open and nonempty. So $\mu_g(V_{convex}(\overline{R}; J) \setminus V_{convex}(\tilde{R}; J)) > 0$. Therefore we must have

$$F(J) = G_{convex}(\tilde{R}; J) < G_{convex}(\overline{R}; J) = F(J) \tag{3.19}$$

which is a contradiction because $F(\omega) = G_{convex}(\tilde{R}; \omega) = G_{convex}(\overline{R}; \omega)$ for all Borel $\omega \subseteq T$. The theorem is proved. □

## 3.5 Weak Solutions in the Discrete Case

Here we consider the case of the refractor problem (3.13) where the set $T$ is finite. We prescribe the measure $F$ in (3.13) as a Dirac measure concentrated at points in $T$. Recall the notation for refractors in the discrete case that was introduced in the previous section before Theorem 3.4.1. We now recall Theorem 9 from [13].

**Theorem 3.5.1** (Theorem 9 in [13]). *Let $T = \{x_1, \dots, x_k\}$ be a collection of $k$ distinct points in $\mathbb{R}^3 \setminus \{\mathcal{O}\}$, $k > 2$. Assume that $T$ satisfies Hypothesis H1. Let $\gamma$, $\epsilon_M$, $\epsilon_{min}$, and $\epsilon_{max}$ be positive real numbers such that $\epsilon_0 + \gamma < \epsilon_M \leq \epsilon_{min} \leq \epsilon_{max} < \infty$, where $\epsilon_0$ is defined by (3.5). Assume we are given a nonnegative $g \in L^1(\mathbb{S}^2)$ such that $g \equiv 0$ outside $D_T^{\delta_\gamma}$ where $\delta_\gamma = \frac{1}{\epsilon_0 + \gamma}$. Let $f_1, \dots, f_k$ be nonnegative real numbers such that*

$$\sum_{i=1}^{k} f_i = \mu_g(D_T^{\delta_\gamma}). \tag{3.20}$$

*Suppose that there also exists some $\ell \in [k]$ such that for all $i \in [k]$, $i \neq \ell$,*

$$G_{convex}(R_\ell; x_i) \leq f_i \tag{3.21}$$

*where $R_\ell = (\epsilon_1 = \epsilon_{max}, ..., \epsilon_{\ell-1} = \epsilon_{max}, \epsilon_\ell = \epsilon_{min}, \epsilon_{\ell+1} = \epsilon_{max}, ..., \epsilon_k = \epsilon_{max})$, and*

$$G_{convex}(R_{\ell i}; x_\ell) < f_\ell \tag{3.22}$$

*where $R_{\ell i} = (\epsilon_1 = \epsilon_{max}, \ldots, \epsilon_{i-1} = \epsilon_{max}, \epsilon_i = \epsilon_M, \epsilon_{i+1} = \epsilon_{max}, \ldots, \epsilon_{\ell-1} = \epsilon_{max}, \epsilon_\ell = \epsilon_{min}, \epsilon_{\ell+1} = \epsilon_{max}, ..., \epsilon_k = \epsilon_{max})$. Then there exists a refractor $R = (\epsilon_1, \ldots, \epsilon_k) \in \mathcal{R}^{\epsilon_M}_{convex}(T)$ such that*

$$G_{convex}(R; x_i) = f_i \text{ for all } i \in [k]. \tag{3.23}$$

We will now use the above theorem to prove the following proposition.

**Proposition 3.5.1.** *Let $T = \{x_1, \ldots, x_k\}$ be a collection of $k$ distinct points in $\mathbb{R}^3 \setminus \{\mathcal{O}\}$ such that $T$ satisfies Hypothesis H1 and $k_x = k_y$ for all $x, y \in T$. Suppose that we are given $\gamma > 0$ such that $\epsilon_0 + \gamma < \lim_{t \to K^+} \frac{1}{t-1}$ for $K = \frac{\max_{x \in T} |x|}{\min_{x \in T} |x|}$ and $\epsilon_0$ is defined by (3.5). Assume we are given a nonnegative $g \in L^1(\mathbb{S}^2)$ such that $g \equiv 0$ outside $D_T^{\delta_\gamma}$ where $\delta_\gamma = \frac{1}{\epsilon_0 + \gamma}$. Let $f_1, \ldots, f_k$ be nonnegative real numbers such that*

$$\sum_{i=1}^{k} f_i = \mu_g(D_T^{\delta_\gamma}) \tag{3.24}$$

*and for the $\ell \in [k]$ where $|x_\ell| = \max_{y \in T} |y|$, $f_\ell > 0$.*

*Then there exists an* $\epsilon_M \in (\epsilon_0 + \gamma, \lim_{t\to K^+} \frac{1}{t-1})$ *such that we can construct a convex, rotationally symmetric refractor* $R = (\epsilon_1, \ldots, \epsilon_k) \in \mathcal{R}^{\epsilon_M}_{convex}(T)$ *where*

$$G_{convex}(R; x_i) = f_i \text{ for all } i \in [k]. \tag{3.25}$$

*Proof.* Note that $k_x = k_y$ for all $x, y \in T$, implies that $\epsilon_0 = 1$ as defined by (3.5). The case where $k = 1$ is trivial; let $k \geq 2$. Assume that $|x_i| \geq |x_{i+1}|$ for all $i \in [k-1]$ and thus $f_1 > 0$. Recall that for $x \in T$ by Proposition 3.2.2, $\mathscr{k}_{\epsilon,x}(m) \to \frac{|x|}{2\langle m,k_x\rangle}$ as $\epsilon \to \infty$ for $m \in D_T^{\delta_\gamma}$.

Observe that

$$\frac{|x_1|}{2\langle m, k_x\rangle} < \frac{|x_k|(1-\epsilon_M^2)}{2\epsilon_M(1-\epsilon_M\langle m, k_x\rangle)} \text{ for } m \in D_T^{\delta_\gamma}, \tag{3.26}$$

if and only if

$$\frac{|x_1|}{2} < \frac{|x_k|(1-\epsilon_M^2)}{2\epsilon_M(1-\epsilon_M)}. \tag{3.27}$$

Thus we have that $\epsilon_M < \frac{1}{K-1}$ where $K = \frac{|x_1|}{|x_k|}$. Note that by Hypothesis H1 and the fact that $k \geq 2$, we have $1 < K < 2$ and $1 < \frac{1}{K-1} < \infty$. Thus we can have that $1 = \epsilon_0 < \epsilon_0 + \gamma < \epsilon_M < \frac{1}{K-1}$. If $k = 2$, by the continuity implied by Lemma 8 in [13], there exists a refractor $R = (\epsilon_1, \epsilon_2) \in \mathcal{R}^{\epsilon_M}_{convex}(T)$ such that $G_{convex}(R; x_i) = f_i$ for all $i \in [k]$.

If $k > 2$, we borrow language and notation from Theorem 3.5.1. By continuity, if $\epsilon_{min} = \epsilon_{max}$ is sufficiently large such that $\frac{1}{K-1} < \epsilon_{min} = \epsilon_{max}$, then, assuming that $\epsilon_M < \frac{1}{K-1}$, $G_{convex}(R_1; x_i) = 0$ and $G_{convex}(R_{1i}; x_1) = 0$ for all $i \in [k]$ such that $i \neq 1$. Therefore by Theorem 3.5.1, there exists a refractor $R = (\epsilon_1, \ldots, \epsilon_k) \in \mathcal{R}^{\epsilon_M}_{convex}(T)$ such that $G_{convex}(R; x_i) = f_i$ for all $i \in [k]$.

$\square$

The above proposition motivates our main result. Consider the following definition.

**Definition 3.5.1.** *Assume that we are given some $w, W \in (0,\infty)$ where $w > \frac{W}{2}$. Given some $m_* \in \mathbb{S}^2$, let*

$$S(m_*,\xi) = \{x \in \mathbb{R}^3 | w \leq |x| \leq W, \langle k_x, m_* \rangle \geq 1 - \xi\} \tag{3.28}$$

*where $1 - \cos\left(\frac{1}{2}\arccos\left(\frac{W}{2w}\right)\right) > \xi > 0$; note that $S(m_*,\xi)$ satisfies Hypothesis H1. $S(m_*,\xi)$ describes a conical cylinder of height $W - w$ with a bottom circle radius of $w\tan(\arccos(1-\xi))$ and a top circle radius of $W\tan(\arccos(1-\xi))$.*

**Theorem 3.5.2.** *Assume that we are given some $w, W \in (0,\infty)$ where $w > \frac{W}{2}$ and some $m_* \in \mathbb{S}^2$. Recall that $\delta_\gamma = \frac{1}{\epsilon_0+\gamma}$ where $\gamma > 0$ and $\epsilon_0$ is defined by (3.5).*

*Then there exists positive $\xi$ and $\gamma$ such that,*

1. *for any collection of $k$ distinct points $T = \{x_1, x_2, \cdots, x_k\} \subset S(m_*,\xi)$,*
2. *for any nonnegative $g \in L^1(\mathbb{S}^2)$ such that $g \equiv 0$ outside $D_T^{\delta_\gamma}$,*
3. *any collection $f_1, \ldots, f_k$ of nonnegative real numbers where $\sum_{i=1}^k f_i = \mu_g(D_T^{\delta_\gamma})$ and $f_\ell > 0$ for the $\ell \in [k]$ where $|x_\ell| = \max_{y\in T}|y|$,*

*there exists an $\epsilon_M > \epsilon_0 + \gamma$ such that we can construct a refractor $R = (\epsilon_1, \ldots, \epsilon_k) \in \mathcal{R}^{\epsilon_M}_{convex}(T)$ where*

$$G_{convex}(R; x_i) = f_i \text{ for all } i \in [k]. \tag{3.29}$$

*Proof.* Note that $k_x = k_y$ for all $x, y \in T$ if and only if $\min_{x,y\in T}\langle k_x, k_y\rangle = 1$. Observe that $\xi \to 0$ implies that $\min_{x,y\in T}\langle k_x, k_y\rangle \to 1$. Assume that $|x_\ell| = \max_{y\in T}|y|$. Let $\hbar_{Tmax,\epsilon}(m) = \max_{x\in T}\hbar_{x,\epsilon}(m)$ and $P_{Tmax}(m) = \max_{x\in T}\frac{|x|}{2\langle m,k_x\rangle}$ where $m \in D_T^{\delta_\gamma}$. Note that $\hbar_{Tmax,\epsilon}(m) \to \hbar_{x_\ell,\epsilon}(m)$ and $P_{Tmax}(m) \to \frac{|x_\ell|}{2\langle m,k_{x_\ell}\rangle}$ as $\min_{y\in T}\langle k_{x_\ell}, k_y\rangle \to 1$.

We borrow language and notation from Theorem 3.5.1. Choose an $\epsilon_{min}$; then, by continuity, there exists a $\xi > 0$ such that if $\min_{x,y \in T}\langle k_x, k_y\rangle \geq 1-\xi$, then we have $P_{Tmax}(m) < \hslash_{x_\ell,\epsilon_{min}}(m)$ for all $m \in D_T^{\delta_\gamma}$. Therefore by continuity there exists an $\epsilon_{max}$ such that $P_{Tmax}(m) < \hslash_{Tmax,\epsilon_{max}}(m) < \hslash_{x_\ell,\epsilon_{min}}(m)$ for all $m \in D_T^{\delta_\gamma}$.

Observe that $\epsilon_0 \to 1$ as $\min_{x,y\in T}\langle k_x, k_y\rangle \to 1$ and recall that a degenerate hyperboloid has eccentricity 1. Then, for sufficiently small $\gamma > 0$, by continuity we can choose $\epsilon_0 + \gamma < \epsilon_M < \epsilon_{min} < \epsilon_{max}$, such that $P_{Tmax}(m) < \hslash_{x_\ell,\epsilon_{max}}(m) < \hslash_{Tmax,\epsilon_{min}}(m) < \hslash_{x_\ell,\epsilon_M}(m)$ for all $m \in D_T^{\delta_\gamma}$.

Then $G_{convex}(R_\ell; x_i) = 0$ and $G_{convex}(R_{\ell i}; x_\ell) = 0$ for all $i \in [k]$ such that $i \neq \ell$. Then by Theorem 3.5.1, there exists a refractor $R = (\epsilon_1, \ldots, \epsilon_k) \in \mathcal{R}^{\epsilon_M}_{convex}(T)$ such that $G_{convex}(R; x_i) = f_i$ for all $i \in [k]$.

$\square$

## 3.6 Weak Solutions in the General Case

In this section, we extend the results of Theorems 3.5.2 and 3.4.1 to the case of more general sets $T$ and energy distributions $F$. We consider the case where prescribe the measure $F$ as a Lebesgue measure over $T$, specifically

$$F(\omega) = \int_\omega f(x)d\lambda(x) \text{ for any Borel set } \omega \subseteq T \tag{3.30}$$

for some given nonnegative function $f \in L^1(T)$; here $\lambda$ is the Lebesgue measure on $T$.

**Theorem 3.6.1.** *Assume that we are given some $w, W \in (0,\infty)$ where $w > \frac{W}{2}$ and some $m_* \in \mathbb{S}^2$. Recall that $\delta_\gamma = \frac{1}{\epsilon_0+\gamma}$ where $\gamma > 0$ and $\epsilon_0$ is defined by (3.5).*

*Then there exists positive $\xi$ and $\gamma$ such that,*

1. *for any closed subset $T \subseteq S(m_*, \xi)$,*

2. *for any nonnegative $g \in L^1(\mathbb{S}^2)$ such that $g \equiv 0$ outside $D_T^{\delta_\gamma}$,*

3. *for any nonnegative $f \in L^1(T)$ with a measure $F$ defined by (3.30) where $F(T) = \mu_g(D_T^{\delta_\gamma})$,*

*there exists an $\epsilon_M > \epsilon_0 + \gamma$ such that we can construct a convex refractor $R \in \mathcal{R}^{\epsilon_M}_{convex}(T)$ where $R$ that is a convex weak solution to the refractor problem (3.13).*

The following proof is similar to arguments made in [13], [3], and [11]. Specifically, the argument below follows the proof of Theorem 13 in [13] very closely with some adjustments to fit into this new context. Even though Theorem 13 in [13] is incorrect (see Appendix B), the type of argument presented in its proof is broadly applicable.

*Proof.* Recall that our definition of the energy function can also be considered as a measure of $T$.

If $\mu_g(D_T^{\delta_\gamma}) = 0$, then any refractor $R \in \mathcal{R}^{\epsilon_M}_{convex}(T)$ will do. Assume that $\mu_g(D_T^{\delta_\gamma}) > 0$.

Since $T$ is bounded, for any $\delta > 0$ there exists an $N \in \mathbb{N}$ such that for each $k \geq N$ there exists a partition of $T$ into $k$ Borel sets $\omega_1^k, \ldots, \omega_k^k$ such that

$$\operatorname{diam}(\omega_i^k) \leq \delta \text{ for any } k \geq N, i \in [k]. \tag{3.31}$$

For each $k \in \mathbb{N}$, we choose an $x_i^k \in \omega_i^k$ for $i \in [k]$, and put

$$F_i^k = F(\omega_i^k). \tag{3.32}$$

Define a measure $F^k$ on $T$ by

$$F^k(\omega) = \sum_{x_i^k \in \omega} F_i^k \text{ for any Borel set } \omega \subseteq T. \tag{3.33}$$

Note that $F^k$ converges weakly to $F$ as $k \to \infty$. For each $k$, there exists a unique nonempty $S_k = \{i \in [k] | F_i^k > 0\} \subseteq [k]$. Since $\{x_i^k\}_{i \in S_k}$ and $\{F_i^k\}_{i \in S_k}$ satisfies the assumptions of Theorem 3.5.2, there exists a convex refractor $R^k \in \mathcal{R}_{convex}^{\epsilon_M}(\{x_i^k | i \in S_k\}) \subseteq \mathcal{R}_{convex}^{\epsilon_M}(\{x_i^k | i \in [k]\}) \subseteq \mathcal{R}_{convex}^{\epsilon_M}(T)$ defined by hyperboloids with an eccentricity greater than or equal to some $\epsilon_M > \epsilon_0 + \gamma$ such that

$$G_{convex}(R^k; x_i^k) = F_i^k \text{ for } i \in [k]. \tag{3.34}$$

Let $G^k$ be the measure on $T$ defined by

$$G^k(\omega) = \sum_{x_i^k \in \omega} G_{convex}(R^k; x_i^k) \tag{3.35}$$

then obviously $F^k \equiv G^k$ for all $k \in \mathbb{N}$ and consequently, $G^k \to F$. To finish the proof we need to construct a refractor $R$ whose energy function, $G$, would be the limit of measures $G^k$. This refractor is constructed in the following manner as a limit of refractors $R^k$.

First, we note that since $g \equiv 0$ outside $D_T^{\delta_\gamma}$, we only need to consider the part of the refractor $R^k \cap C$ where $C$ is the cone created by the union of all rays from $\mathcal{O}$

that intersect $D_T^{\delta_\gamma}$. Also for some $\epsilon_M > \epsilon_0 + \gamma$ one can show that for any $R \in \mathcal{R}^{\epsilon_M}(T)$

$$(R \cap C) \subseteq \mathrm{B}(\mathcal{O}, b) \text{ for some } b > 0 \tag{3.36}$$

where $\mathrm{B}(\mathcal{O}, b)$ is the open ball centered at the origin $\mathcal{O}$ of radius $b$. Let us prove this statement. We define

$$b = \max_{x \in T, m \in \overline{D_T}^{\delta_\gamma}} \hslash_{x,\epsilon_M}(m). \tag{3.37}$$

Since $\hslash_{x,\epsilon_M}$ is a continuous function and $\overline{D_T}^{\delta_\gamma}$ is compact, this definition is correct and $b < \infty$. Thus for any $\epsilon > \epsilon_M$, $\hslash_{x,\epsilon}(m) \leq b$ and (3.36) is proved.

For each of the refractors $R^k$ we consider a bounded convex body

$$h_b^k = h^k \cap C_{\mathcal{O}, D_T^{\delta_\gamma}, \infty} \cap \mathrm{B}(\mathcal{O}, b) \tag{3.38}$$

where for each $k \in \mathbb{N}$ the set $h^k$ is defined by (3.6). By Blaschke's selection theorem (see [19] or Appendix C), there exists a subsequence of $\{h_b^k\}$ which we again denote by $\{h_b^k\}$, which converges to some convex body $h_b$.

We show now that for each point $r \in [\partial(h_b) \cap C_{\mathcal{O}, D_T^{\delta_\gamma}, \infty}] \setminus \partial(\mathrm{B}(\mathcal{O}, b))$ there exists a hyperboloid $H^r(x)$ which is supporting to $h^b$ at point $r$.

Let $r \in [\partial(h_b) \cap C_{\mathcal{O}, D_T^{\delta_\gamma}, \infty}] \setminus \partial(\mathrm{B}(\mathcal{O}, b))$. Then there exists a sequence $\{r_k\}$ that converges to $r$ where each $r_k \in R^k$. Let $H(x_k)$ be a supporting hyperboloid to $R^k$ at $r_k$. Since $T$ is compact, $\{x_k\}$ contains a subsequence, which we will denote by $\{x_k^*\}$, converging to some $x \in T$. The convex body $\tilde{H}(x_k^*)$ bounded by $H(x_k^*)$ contains the body $h_b^k$. The corresponding sequence $\{\tilde{H}(x_k^*)\}$ converges to the body $\tilde{H}^r(x)$ containing $h_b$ and $h_b^k$ converges to $h_b$. Therefore $\tilde{H}^r(x)$ contains $\partial(h_b)$. It follows that $H^r(x)$ is supporting to $h_b$ at $r$.

We now define the refractor $R = \partial(\bigcap_{x \in T} \tilde{H}^r(x))$ and show that the sequence of measures $G^k$, that is equivalent to the energy functions corresponding to the refractors $R^k$, converges weakly to the measure $G$, which is the energy function of the refractor $R$.

Let $\alpha^k_{convex}$ and $\alpha_{convex}$ be the refractor maps corresponding to $R^k$ and $R$ respectively. By Theorem D.1 (see [19], Appendix D), the refractor maps $\alpha^k_{convex}$ for $k \in \mathbb{N}$ and $\alpha_{convex}$ are single-valued functions almost everywhere. Furthermore, for almost all $m \in D_T^{\delta_\gamma}$ the hyperboloids $H_k$ supporting to $R^k$ at points $r_k(m)$ converge to the hyperboloid $H$ supporting to $R$ at the point $r(m)$. Thus, $\alpha^k_{convex}(m)$ converges to $\alpha_{convex}(m)$ almost everywhere.

If given a set of cardinality one, $\{z\}$, let $\mathrm{Ele}(\{z\}) = z$. Let $Y^k(m) = \{x \in \alpha^k_{convex}(m) | x \in \{x_i^k\}_{i \in [k]}\}$ and let $J^k(m) \subseteq [k]$ be the set of indices such that $\{x_i^k | i \in J^k(m)\} = Y^k(m)$. Let $z \in T$,

$$K^k(m) = x^k_{\min J^k(m)}, \tag{3.39}$$

and

$$K(m) = \begin{cases} \mathrm{Ele}(\alpha_{convex}(m)) & \text{if } |\alpha_{convex}(m)| = 1 \\ z & \text{if } |\alpha_{convex}(m)| > 1 \end{cases}. \tag{3.40}$$

Then for any continuous function $u$ on $T$ we have

$$\int_T u dG^k = \int_{D_T^{\delta_\gamma}} u(K^k(m)) d\mu_g(m) \longrightarrow \int_{D_T^{\delta_\gamma}} u(K(m)) d\mu_g(m) = \int_T u dG \tag{3.41}$$

as $k \to \infty$, that is, the measures $\{G^k\}$ converge weakly to $G$. $\square$

## 3.7 Rotationally Symmetric Convex Refractors on the Surface of a Right Circular Cone

We now use the results we obtained earlier in this chapter to find concrete results for the rotationally symmetric case. We focus on the case where the target set is on the surface of a right circular cone. Both Theorems 3.6.1 and 3.5.1 imply this next result.

**Corollary 3.7.1.** *Let $T$ be a closed set that satisfies Hypothesis H1 and $k_x = k_y$ for all $x, y \in T$. Assume that we are given $\gamma > 0$ such that $\epsilon_0 + \gamma < \lim_{t \to K^+} \frac{1}{t-1}$ for $K = \frac{\max_{x \in T} |x|}{\min_{x \in T} |x|}$ and $\epsilon_0$ is defined by (3.5). Also, assume we are given a nonnegative $g \in L^1(\mathbb{S}^2)$ such that $g \equiv 0$ outside $D_T^{\delta_\gamma}$ where $\delta_\gamma = \frac{1}{\epsilon_0 + \gamma}$. Suppose we are given a nonnegative $f \in L^1(T)$ and a measure $F$ defined by (3.30) such that*

$$F(T) = \mu_g(D_T^{\delta_\gamma}). \tag{3.42}$$

*Then there exists an $\epsilon_M \in (\epsilon_0 + \gamma, \lim_{t \to K^+} \frac{1}{t-1})$ such that we can construct a convex rotationally symmetric refractor $R \in \mathcal{R}_{convex}^{\epsilon_M}(T)$ where $R$ that is a convex weak solution to the refractor problem (3.13).*

We will use the above corollary to create rotationally symmetric refractors with the target set on a right circular cone.

**Definition 3.7.1.** *Let $k \geq 2$, $d > 0$, $1 > \xi > 0$, and $m_*, m' \in \mathbb{S}^2$ such that $\langle m_*, m' \rangle = 0$. We create a Cartesian coordinate system centered at $\mathcal{O}$ where $m_*$ is the direction of our $z$-axis, $m'$ is the direction of our $x$-axis, and $m_* \times m'$ is the direction of our $y$-axis. Let $(x, y, z)'$ represent a point in this system.*

*Recall that, given a point $(x, y, z)' \in \mathbb{R}^3$, there exists $r \in [0, \infty)$, $\phi \in [0, \pi]$,*

$\theta \in [0, 2\pi)$, *such that*

$$x = r\cos\theta\sin\phi \tag{3.43}$$

$$y = r\sin\theta\sin\phi \tag{3.44}$$

$$z = r\cos\phi. \tag{3.45}$$

*Let $Q$ be a closed subset of the interval $(0, \infty)$. Define the set of points $T^{\xi}_{k,Q}(m_*, m')$ as*

$$\left\{\left(d\cos\left(\frac{2\pi j}{k}\right)\sin\left(\arccos(\xi)\right), d\sin\left(\frac{2\pi j}{k}\right)\sin\left(\arccos(\xi)\right), d\xi\right)' | j \in I, d \in Q\right\} \tag{3.46}$$

*where $I = \{0, 1, \ldots, k-1\}$.*

*Define the set $T^{\xi}_{\infty,Q}(m_*, m')$ as*

$$\left\{\left(d\cos(\theta)\sin\left(\arccos(\xi)\right), d\sin(\theta)\sin\left(\arccos(\xi)\right), d\xi\right)' | \theta \in [0, 2\pi), d \in Q\right\}. \tag{3.47}$$

**Proposition 3.7.1.** *Let $m_*, m' \in \mathbb{S}^2$ such that $\langle m_*, m'\rangle = 0$. Let $Q$ be a closed subset of the interval $(0, \infty)$, $k \geq 2$, and $1 > \xi > 0$ such that $T = T^{\xi}_{k,Q}(m_*, m')$ satisfies Hypothesis H1 such that $\epsilon_0 < \lim_{t\to K^+} \frac{1}{t-1}$ for $K = \frac{\max_{x\in T}|x|}{\min_{x\in T}|x|}$ where $\epsilon_0$ is defined by (3.5). Assume that we are given $\gamma > 0$ such that $\epsilon_0 + \gamma < \lim_{t\to K^+}\frac{1}{t-1}$. Also, assume we are given a nonnegative $g_* \in L^1(\mathbb{S}^2)$ that is rotationally symmetric about the axis defined by the ray of direction $m_*$ originating at $\mathcal{O}$. Let the function $g \in L^1(\mathbb{S}^2)$ be defined as $g \equiv g_*$ inside $D^{\delta_\gamma}_T$ and $g \equiv 0$ outside $D^{\delta_\gamma}_T$ where $\delta_\gamma = \frac{1}{\epsilon_0+\gamma}$.*

*Suppose we are given a nonnegative $f \in L^1(T)$ such that for every $d \in Q$:*

$$f\left(\left(d\cos\left(\frac{2\pi j}{k}\right)\sin\left(\arccos(\xi)\right), d\sin\left(\frac{2\pi j}{k}\right)\sin\left(\arccos(\xi)\right), d\xi\right)'\right) \tag{3.48}$$

*is constant for all* $j \in \{0, 1, \ldots, k-1\}$*. Let* $F$ *be the measure defined by (3.30) and*

$$F(T) = \mu_g(D_T^{\delta_\gamma}). \tag{3.49}$$

*Then there exists an* $\epsilon_M \in (\epsilon_0 + \gamma, \lim_{t \to K^+} \frac{1}{t-1})$ *such that we can construct a convex refractor* $R \in \mathcal{R}_{convex}^{\epsilon_M}(T)$ *where* $R$ *that is a convex weak solution to the refractor problem (3.13).*

*Proof.* We create a Cartesian coordinate system centered at $\mathcal{O}$ where $m_*$ is the direction of our $z$-axis, $m'$ is the direction of our $x$-axis, and $m_* \times m'$ is the direction of our $y$-axis. Let $(x, y, z)'$ represent a point in this system.

Recall that, given a point $(x, y, z)' \in \mathbb{R}^3$, there exists $r \in [0, \infty)$, $\phi \in [0, \pi]$, $\theta \in [0, 2\pi)$, such that

$$x = r \cos\theta \sin\phi \tag{3.50}$$

$$y = r \sin\theta \sin\phi \tag{3.51}$$

$$z = r \cos\phi. \tag{3.52}$$

Let $T_j = \left\{ \left(d \cos\left(\frac{2\pi j}{k}\right) \sin\left(\arccos(\xi)\right), d \sin\left(\frac{2\pi j}{k}\right) \sin\left(\arccos(\xi)\right), d\xi\right)' | d \in Q \right\}$. Note that $T = \bigcup_{i \in \{0,1,\ldots,k-1\}} T_i$.

Let $m_1, m_2 \in \mathbb{S}^2$ such that $\langle m_1, m_2 \rangle > -1$ and $\mathscr{L}(m_1, m_2)$ be the shortest arc on $\mathbb{S}^2$ between the points $m_1$ and $m_2$. For all $j \in \{0, 1, \ldots, k-1\}$, let $B_j \subset \overline{D_T^{\delta_\gamma}}$ be the set $\{t \in \mathscr{L}(m_*, y) | y \in \partial(D_T^{\delta_\gamma}) \cap \partial(A_x^{\delta_\gamma})\}$ where $x \in T_j$. For $d \in Q$, since $T_{k,\{d\}}^{\xi}(m_*, m')$ defines the points of a regular $k$-gon centered at the axis defined by the ray of direction $m_*$ originating at $\mathcal{O}$, then $\mu_g(B_j) = \frac{\mu_g(D_T^{\delta_\gamma})}{k}$ for all $j$. For all $j \in \{0, 1, \ldots, k-1\}$, let $g_j$ be a function over $\mathbb{S}^2$ such that $g_j \equiv g$ inside $\mathrm{Int}(B_j)$ and

$g_j \equiv 0$ outside $\mathrm{Int}(B_j)$. Note that $\mu_g(\mathrm{Int}(B_j)) = \mu_{g_j}(D_{T_j}^{\delta_\gamma})$ for all $j$.

Let $m^j \in \mathbb{S}^2$ be the unit vector such that a ray originating from $\mathcal{O}$ of direction $m^j$ intersects every point in $T_j$. Also, let $f_j$ be the restriction of the function $f$ to the set $T_j$ and $F_j$ be the corresponding measure as defined by (3.30). By Proposition 3.7.1, for all $j \in \{0, 1, \ldots, k-1\}$, there exists a refractor $R_j = \partial \left( \bigcap_{d \in Q} \tilde{H}_{\epsilon_d}(dm^j) \right) \in \mathcal{R}^{\epsilon_M}_{convex}(T_j)$, that is rotationally symmetric about the axis defined by the ray starting at $\mathcal{O}$ with direction $m^j$, such that $R_j$ that is a convex weak solution to the refractor problem (3.13) where $F_j(\omega) = \mu_{g_j}(V(R_j; \omega))$ for all Borel $\omega \subseteq T$.

Since for $x \in R_j$, we have that $|x|$ increases as $\langle k_x, m^j \rangle$ decreases for all $j \in \{0, 1, \ldots, k-1\}$, then for $j \in \{0, 1, \ldots, k-1\}$, defining $r_j(m)$ as the point of intersection between $R_j$ and the ray originating from $\mathcal{O}$ in direction $m$, we have $|r_j(m)| = \max_{i \in \{0,1,\ldots,k-1\}} |r_i(m)|$ for all $m \in B_j$.

Thus

$$\partial \left( \bigcap_{j \in \{0, \ldots, k-1\}} \left( \bigcap_{d \in Q} \tilde{H}_{\epsilon_d}(dm^j) \right) \right) \in \mathcal{R}^{\epsilon_M}_{convex}(T) \tag{3.53}$$

is our refractor. $\square$

With an argument similar to that we use in the proof of Theorem 3.6.1, we obtain the following result from Proposition 3.7.1.

**Theorem 3.7.1.** *Let $m_*, m' \in \mathbb{S}^2$ such that $\langle m_*, m' \rangle = 0$. Let $1 > \xi > 0$ and $Q$ be a closed subset of the interval $(0, \infty)$ such that $T = T^\xi_{\infty,Q}(m_*, m')$ satisfies Hypothesis H1 where $\epsilon_0 < \lim_{t \to K^+} \frac{1}{t-1}$ for $K = \frac{\max_{x \in T}|x|}{\min_{x \in T}|x|}$ and $\epsilon_0$ is defined by (3.5). Assume that we are given $\gamma > 0$ such that $\epsilon_0 + \gamma < \lim_{t \to K^+} \frac{1}{t-1}$. Also, assume we are given a nonnegative $g \in L^1(\mathbb{S}^2)$ that is rotationally symmetric about the axis defined by the ray of direction $m_*$ originating at $\mathcal{O}$ such that $g \equiv 0$ outside $D_T^{\delta_\gamma}$ where $\delta_\gamma = \frac{1}{\epsilon_0 + \gamma}$.*

*Assume that we have a nonnegative $f \in L^1(T)$ such that for every $d \in Q$:*

$$f\left((d\cos(\theta)\sin(\arccos(\xi)), d\sin(\theta)\sin(\arccos(\xi)), d\xi)'\right) \tag{3.54}$$

*is constant for all $\theta \in [0, 2\pi)$. Let $F$ be the measure defined by (3.30) and*

$$F(T) = \mu_g(D_T^{\delta_\gamma}). \tag{3.55}$$

*Then there exists an $\epsilon_M \in (\epsilon_0 + \gamma, \lim_{t\to K^+} \frac{1}{t-1})$ such that we can construct a convex rotationally symmetric refractor $R \in \mathcal{R}^{\epsilon_M}_{convex}(T)$ where $R$ is a convex weak solution to the refractor problem (3.13).*

## 3.8 Discussion

In this section, we make progress on the original formulation of the refractor problem introduced in [13]. We proved an existence theorem for the case where the points in the target set are sufficiently close to each other, Theorem 3.6.1. We also proved a uniqueness theorem for the case when the target set is finite, Theorem 3.4.1, and for the general case, Theorem 3.4.2. Also, we proved an existence theorem for the rotationally symmetric case, Theorem 3.7.1. Rotationally symmetric cases are not only practically useful because this case provides a model situation [16], but also because rotationally symmetric solutions can be used to recover nonrotationally symmetric solutions from irradiance distributions without special symmetry assumptions [15].

For Theorem 3.6.1, a possible avenue for further research would be to find precise values for $\xi$ and $\gamma$ such that the theorem holds. Another potential avenue for research is to find an explicit algorithm to find the appropriate hyperboloids for the discrete case. In addition, Theorems 3.6.1 and 3.7.1 provide motivation for the following conjecture.

**Conjecture 3.8.1.** *Let $T$ be a target set that satisfies Hypothesis H1 and $\epsilon_0 < \lim_{t\to K^+} \frac{1}{t-1}$ for $K = \frac{\max_{x\in T}|x|}{\min_{x\in T}|x|}$ when $\epsilon_0$ is defined by (3.5). Assume that we are given a $\gamma > 0$ such that $\epsilon_0 + \gamma < \lim_{t\to K^+} \frac{1}{t-1}$. Also, assume that we have a nonnegative $g \in L^1(\mathbb{S}^2)$ where $g \equiv 0$ outside $D_T^{\delta_\gamma}$ where $\delta_\gamma = \frac{1}{\epsilon_0+\gamma}$. Assume that we are given a nonnegative $f \in L^1(T)$ and $F$ is the measure defined by (3.30) such that*

$$F(T) = \mu_g(D_T^{\delta_\gamma}). \tag{3.56}$$

*Then there exists an $\epsilon_M \in (\epsilon_0 + \gamma, \lim_{t\to K^+} \frac{1}{t-1})$ such that there exists a convex refractor $R \in \mathcal{R}^{\epsilon_M}_{convex}(T)$ where $R$ is a convex weak solution to the refractor problem (3.13).*

# Chapter 4

# Generalized Weak Solutions to the Virtual Source Reflector Problem

## 4.1 Introduction

This chapter deals with the virtual source reflector problem introduced in the previous chapter and the introduction. We reuse some definitions and terminology from the previous chapter; specifically, we again make use of the 'refractor' terminology. We also reference results that were found in the previous chapter. Thus it might be prudent to review the previous chapter with special focus on Chapters 3.1-3.3. We introduce the following concept.

**Definition 4.1.1.** *Assume that we are given an aperture $D \subseteq \mathbb{S}^2$. Let $U$ be an open subset of $\mathbb{S}^2$ such that $D \subseteq U$. Consider a not necessarily continuous function $\rho : U \to (0, \infty)$. If $m\rho(m)$ is a regular point for almost all $m \in D$, then the set $R = \{m\rho(m)|m \in U\} \subset \mathbb{R}^3$ is a* **generalized refractor***.*

In this chapter, we develop a novel notion of a weak solution to the PDE described by equation (4) in [13] (also see Appendix A), where we develop existence results. We first prove the existence of a generalized refractor under general target sets and measures (Theorem 4.3.1). We conclude with an existence result for the rotationally symmetric case (Theorem 4.4.1).

## 4.2 Generalized Weak Solutions Constructed from Hyperboloids

We use the same definitions and notation introduced in Chapter 3.2. Let $D_\omega^\delta = \operatorname{Int}\left(\bigcap_{x\in\omega} A_x^\delta\right)$ for $\omega \subseteq \mathbb{R}^3 \setminus \{\mathcal{O}\}$.

Let $\omega$ be a compact set in $\mathbb{R}^3 \setminus \mathcal{O}$, $K : \omega \to (1,\infty)$, and

$$\mathscr{D}_\omega^K = \operatorname{Int}\left(\bigcap_{x\in\omega} A_x^{\frac{1}{K(x)}}\right);$$

note that $D_\omega^\delta = \mathscr{D}_\omega^K$ if $K(x) = \frac{1}{\delta}$. Let $\tilde{H}_\epsilon(x)$ be the convex body bounded by $H_\epsilon(x)$. Then consider

$$X(\omega, K) = \partial h \text{ where } h = \bigcap_{x\in\omega} \tilde{H}_{K(x)}(x). \tag{4.1}$$

The polar radius of $X(\omega, K)$ relative to $\mathcal{O}$ can be represented as:

$$\mathcal{P}_{\omega,K}(m) = \sup_{x\in\omega} \mathscr{h}_{x,K(x)}(m),\ m \in \mathscr{D}_\omega^K. \tag{4.2}$$

We define $\mathcal{H}_{\omega,K}(m) = m\mathcal{P}_{\omega,K}(m)$. Given a compact set $\omega \subset \mathbb{R}^3 \setminus \mathcal{O}$, let $\mathscr{K}(\omega)$ be the set of all functions $\omega \to (1,\infty)$. We define

$$\mathcal{X}_\omega = \{\, X(\omega, K) |\, K \in \mathscr{K}(\omega)\}\,. \tag{4.3}$$

Consider a compact set $T \subseteq \mathbb{R}^3 \setminus \{\mathcal{O}\}$ and an aperture $D \subset \mathrm{Int}(\bigcup_{x\in T} A_x)$ that is an open set. Let $\Omega$ be a finite family of compact subsets of $T$ such that $\bigcup_{\omega\in\Omega} \omega = T$. Let $\mathscr{A}(T)$ be the set of all such families.

Given a $\Omega \in \mathscr{A}(T)$. Let $\sigma$ denote the standard measure on $\mathbb{S}^2$. Let $\mathscr{B}$ be a finite family of open subsets of $\mathbb{S}^2$ such that

1. for all $B \in \mathscr{B}$ there exists an $\omega \in \Omega$ such that $B \subseteq D_\omega^\delta$ for some $\delta \in (0,1)$,
2. $\sigma(\overline{D} \setminus \bigcup_{B\in\mathscr{B}} \overline{B}) = 0$,
3. $D \subseteq \bigcup_{B\in\mathscr{B}} \overline{B}$,
4. $\sigma(\overline{B} \cap \overline{B'}) = 0$ for all distinct $B, B' \in \mathscr{B}$.

Let the set $\mathcal{B}(D,\Omega)$ be the set of all such families.

Assume that we are given an $\Omega \in \mathscr{A}(T)$ and a $\mathscr{B} \in \mathcal{B}(D,\Omega)$. Let $\mathscr{U}_{\mathscr{B}}^{\Omega}$ be the set of all functions $u : \mathscr{B} \to \Omega$ such that for all $B \in \mathscr{B}$ we have $B \subseteq D_{u(B)}^\delta$ for some $\delta \in (0,1)$. Every element of $\mathcal{X}_\omega$ needs a well defined $K \in \mathscr{K}(\omega)$. So, given a $u \in \mathscr{U}_{\mathscr{B}}^{\Omega}$, let $\mathscr{V}_{u,\mathscr{B}}^{\Omega}$ be the set of all functions $v : \mathscr{B} \to \bigcup_{B\in\mathscr{B}} \mathscr{K}(u(B))$ such that for all $B \in \mathscr{B}$ we have $v(B) \in \mathscr{K}(u(B))$ and $B \subseteq \mathscr{D}_{u(B)}^{v(B)}$.

Thus we define a set

$$\mathcal{E}_T(D) = \left\{ \bigcup_{B\in\mathscr{B}} \mathcal{H}_{u(B),v(B)}[\overline{B}] \,\middle|\, \Omega \in \mathscr{A}(T),\ \mathscr{B} \in \mathcal{B}(D,\Omega),\ u \in \mathscr{U}_{\mathscr{B}}^{\Omega},\ v \in \mathscr{V}_{u,\mathscr{B}}^{\Omega} \right\}. \tag{4.4}$$

Given some $Z \in \mathcal{E}_T(D)$, let $y_Z^1(m) = Z \cap \{am | a \in [0,\infty)\}$ for $m \in D$ be the points of intersection $Z$ with a ray of direction $m$ originating from $\mathcal{O}$, and

$$\rho_Z(m) = \min_{x\in y_Z^1(m)} |x|,\ m \in D. \tag{4.5}$$

Observe that the function $\rho_Z$ is positive and not necessarily continuous. Let $W(\rho_Z) = \{m\rho_Z(m)|m \in D\}$ and observe that $m\rho_Z(m)$ is a regular point for almost all $m \in D$. Therefore $W(\rho_Z)$ is a generalized refractor per Definition 4.1.1. We now describe a set of generalized refractors

$$\mathcal{R}_D(T) = \{W(\rho_Z)|\, Z \in \mathcal{E}_T(D)\}\,. \tag{4.6}$$

We can also describe a set of refractors

$$\mathcal{R}_D^{cont}(T) = \{W(\rho_Z)|\, Z \in \mathcal{E}_T(D) \text{ and } \rho_Z \text{ is continuous}\}\,. \tag{4.7}$$

Clearly, $\mathcal{R}_D^{cont}(T) \subseteq \mathcal{R}_D(T)$.

For every generalized refractor $R \in \mathcal{R}_D(T)$, there exists an $\Omega_R \in \mathscr{A}(T)$ and a corresponding $\mathscr{B}_R \in \mathcal{B}(D, \Omega_R)$ such that there exists a $u_R \in \mathscr{U}_{\mathscr{B}_R}^{\Omega_R}$ and a $v_R \in \mathscr{V}_{u_R,\mathscr{B}_R}^{\Omega_R}$ such that $R = W\left(\rho_Z\right)$ where $Z = \bigcup_{B\in\mathscr{B}_R} \mathcal{H}_{u_R(B),v_R(B)}[\overline{B}]$.

As before, we denote by $g \in L^1(\mathbb{S}^2)$ the energy density of the source $\mathcal{O}$. Let us define for all measurable $X \subseteq \mathbb{S}^2$

$$\mu_g(X) = \int_X g(m)d\sigma(m) \tag{4.8}$$

where $\sigma$ denotes the standard measure on $\mathbb{S}^2$. Assume that $g \in L^1(\mathbb{S}^2)$ is a nonnegative function where $g \equiv 0$ outside of $D$. In order to formulate and solve the generalized refractor problem (in the framework of weak solutions to be defined below), we need to define a measure representing the energy generated by $g$ and redistributed by a generalized refractor $R \in \mathcal{R}_D(T)$.

Assume that we are given

1. an $R \in \mathcal{R}_D(T)$,

2. an $\Omega_R \in \mathscr{A}(T)$ and a corresponding $\mathscr{B}_R \in \mathcal{B}(D, \Omega_R)$,

3. a $u_R \in \mathscr{U}^{\Omega_R}_{\mathscr{B}_R}$ and a $v_R \in \mathscr{V}^{\Omega_R}_{u_R,\mathscr{B}_R}$ such that $R = W(\rho_Z)$ where

$$Z = \bigcup_{B \in \mathscr{B}_R} \mathcal{H}_{u_R(B), v_R(B)}[\overline{B}]. \tag{4.9}$$

Recall Definition 3.3.1. Given a $B \in \mathscr{B}_R$, a subset $S \subseteq u_R(B)$, and a $X_B = X(u_R(B), v_R(B))$ put

$$Q_B(S) = \{z \in X_B | \text{ there exists } x \in S \text{ such that } H(x) \text{ is supporting to } X_B \text{ at } z\}. \tag{4.10}$$

The intersection of $\overline{B}$ with the image of the set $Q_B(S)$ under radial projection on $\mathbb{S}^2$ we call $J_B(S)$. By Lemma 4 of [13], this set $J_B(S)$ is measurable for all Borel sets $S \subseteq u_R(B)$.

Given a Borel set $\omega \subseteq T$, we define the *visibility set of* $\omega$ as

$$V(\omega) = \bigcup_{B \in \mathscr{B}_R} J_B(u_R(B) \cap \omega). \tag{4.11}$$

Clearly, $V(\omega)$ is measurable as it is a finite union of measurable sets. We define *energy function* as

$$G(\omega) = \mu_g(V(\omega)). \tag{4.12}$$

For $m \in D$, let $r(m)$ be the set of the points of intersection between the generalized refractor $R \in \mathcal{R}_D(T)$ and the ray of direction $m$ originating at $\mathcal{O}$. The potentially multi-valued map $\alpha : D \to T$,

$$\alpha(m) = \{x \in u_R(B) | m\rho_Z(m) = \mathcal{H}_{u_R(B),v_R(B)}(m), \text{ and there exists } H(x) \text{ supporting to } X_B \text{ at } r(m)\} \quad (4.13)$$

is called the *generalized refractor map.*

Let $F$ be a nonnegative, finite, Borel measure on Borel subsets of the set $T$. We say that a generalized refractor $R \in \mathcal{R}_D(T)$ is a *generalized weak solution of the generalized refractor problem* if there exists

1. an $\Omega_R \in \mathscr{A}(T)$ and a corresponding $\mathscr{B}_R \in \mathcal{B}(D, \Omega_R)$,
2. a $u_R \in \mathscr{U}_{\mathscr{B}_R}^{\Omega_R}$ and a corresponding $v_R \in \mathscr{V}_{u_R,\mathscr{B}_R}^{\Omega_R}$ such that $R = W(\rho_Z)$ where

$$Z = \bigcup_{B \in \mathscr{B}_R} \mathcal{H}_{u_R(B),v_R(B)}[\overline{B}]$$

such that

$$F(\omega) = G(\omega) \text{ for any Borel set } \omega \subseteq T \quad (4.14)$$

and the refractor map $\alpha$ is such that $\alpha(m) \subseteq T$ for all $m \in D$.

Throughout this chapter, we concern ourselves with the case where $F$ is a discrete measure. We also consider the case where prescribe the measure $F$ as a Lebesgue measure over $T$, specifically

$$F(\omega) = \int_\omega f(x) d\lambda(x) \text{ for any Borel set } \omega \subseteq T \quad (4.15)$$

for some given nonnegative function $f \in L^1(T)$; here $\lambda$ is the Lebesgue measure on $T$.

## 4.3 Generalized Refractors

In this section, we show that using generalized refractors allows us to prove relatively broad existence theorems. We first proceed with the following lemma.

**Lemma 4.3.1.** *Let $T \subset \mathbb{R}^3 \setminus \{\mathcal{O}\}$ and $0 < \delta < 1$ such that $D_T^\delta \neq \varnothing$. Assume we are given a nonnegative $g \in L^1(\mathbb{S}^2)$ where $g \equiv 0$ outside $D_T^\delta$. Suppose that we are given a set $S$ that is open in $\mathbb{S}^2$. Let $f_1, f_2, \ldots, f_k$ be nonnegative real numbers such that*

$$\sum_{i=1}^{k} f_i = \mu_g(D_T^\delta \cap S). \tag{4.16}$$

*Then there exists disjoint open sets $\{B_i\}_{i=1}^k$ such that $\mu_g(B_i) = f_i$ for all $i$, $B_i \subseteq D_T^\delta \cap S$, and $\bigcup_{i=1}^k \overline{B_i} = \overline{D_T^\delta \cap S}$.*

*Specifically, there exists $\xi_0, \xi_1, \ldots, \xi_k$ where $\delta = \xi_0 \leq \xi_1 \leq \cdots \leq \xi_k = 1$ such that*

$$f_i = \mu_g\left([D_T^{\xi_{i-1}} \setminus D_T^{\xi_i}] \cap S\right) \tag{4.17}$$

*for all $i \in [k]$. So we have that*

$$B_i = \text{Int}([D_T^{\xi_{i-1}} \setminus D_T^{\xi_i}] \cap S) \tag{4.18}$$

*for all $i \in [k]$.*

*Proof.* We will first construct disjoint open sets $\{B_i\}_{i=1}^k$ such that $\mu_g(B_i) = f_i$ for all $i$, $B_i \subseteq D_T^\delta \cap S$, and $\bigcup_{i=1}^k \overline{B_i} = \overline{D_T^\delta \cap S}$.

Let $F_m = \sum_{i=1}^m f_i$ for all $m \in [k]$. Consider the following equation with respect to $\xi$:

$$F_m = \mu_g\left([D_T^\delta \setminus D_T^\xi] \cap S\right). \tag{4.19}$$

The set being measured in the above equation is the part of the border of the set $D_T^\delta$ that intersects with $S$; the thickness of the border being determined by $\xi$. We now show that the above equation has a solution using the intermediate value theorem.

Let $Q(\xi) = [D_T^\delta \setminus D_T^\xi] \cap S$. We can now rewrite the aforementioned equation as $F_m = \mu_g(Q(\xi))$. Observe that we can assume $\xi \in \mathbb{R}$ because $Q(\xi) = \varnothing$ when $\xi < \delta$ and $Q(\xi) = D_T^\delta \cap S$ when $\delta \geq 1$. Consider $\lim_{\xi \to \xi_*} |\mu_g(Q(\xi)) - \mu_g(Q(\xi_*))|$ where $\xi_* \in \mathbb{R}$.

Let $M_g = \operatorname{ess\,sup}\{g(m) | m \in \mathbb{S}^2\}$. Then we can use the squeeze theorem to evaluate the right-hand limit. It would be advantageous to observe that that $Q(\xi_1) \subseteq Q(\xi_2)$ when $\xi_1 \leq \xi_2$. Recall that we denote $\sigma$ as the standard measure on $\mathbb{S}^2$. Observe that

$$0 \leq \lim_{\xi \to \xi_*^+} |\mu_g(Q(\xi)) - \mu_g(Q(\xi_*))| = \lim_{\xi \to \xi_*^+} \mu_g\left(Q(\xi_*) \setminus Q(\xi)\right) \tag{4.20}$$

$$\leq M_g \lim_{\xi \to \xi_*^+} \sigma(Q(\xi_*) \setminus Q(\xi)) = 0. \tag{4.21}$$

Then, by the squeeze theorem, $\lim_{\xi \to \xi_*^+} |\mu_g(Q(\xi)) - \mu_g(Q(\xi_*))| = 0$.

The argument for the left-hand limit is very similar:

$$0 \leq \lim_{\xi \to \xi_*^-} |\mu_g(Q(\xi)) - \mu_g(Q(\xi_*))| = \lim_{\xi \to \xi_*^-} \mu_g\left(Q(\xi) \setminus Q(\xi_*)\right) \tag{4.22}$$

$$\leq M_g \lim_{\xi \to \xi_*^-} \sigma(Q(\xi) \setminus Q(\xi_*)) = 0. \tag{4.23}$$

Then, by the squeeze theorem, $\lim_{\xi \to \xi_*^-} |\mu_g(Q(\xi)) - \mu_g(Q(\xi_*))| = 0$. We therefore have $\lim_{\xi \to \xi_*} |\mu_g(Q(\xi)) - \mu_g(Q(\xi_*))| = 0$ which implies that
$\lim_{\xi \to \xi_*} \mu_g(Q(\xi)) = \mu_g(Q(\xi_*))$. So the function $\mu_g(Q(\xi))$ is continuous on $\mathbb{R}$.

Since $\mu_g(Q(1)) = \mu_g\left(D_T^\delta \cap S\right)$ and $\mu_g(Q(0)) = 0$, then by the intermediate value theorem, for all $n \in [k]$ there exists a $0 < \xi_n < 1$ such that $F_n = \mu_g(Q(\xi_n))$. Since the function $\mu_g(Q(\xi))$ is, by design, monotonically non-decreasing and $F_i \leq F_{i+1}$, we have that $\xi_i \leq \xi_{i+1}$. Since $F_k = \mu_g(D_T^\delta \cap S)$, we can set $\xi_k = 1$. Therefore, if we set $\xi_0 = \delta$,

$$
\begin{aligned}
f_i &= F_{i-1} - F_i && (4.24)\\
&= \mu_g(Q(\xi_{i-1}) \setminus Q(\xi_i)) && (4.25)\\
&= \mu_g\left([D_T^{\xi_{i-1}} \setminus D_T^{\xi_i}] \cap S\right) && (4.26)
\end{aligned}
$$

for all $i \in [k]$. So we have that

$$B_i = \mathrm{Int}([D_T^{\xi_{i-1}} \setminus D_T^{\xi_i}] \cap S) \tag{4.27}$$

for all $i \in [k]$. □

We can now investigate the case where $T$ is any finite collection of points. Consider the following lemma that is proven in detail in Appendix E.

**Lemma 4.3.2.** *Let $(\Omega, S, \nu)$ be a finite nonatomic measure and $A$ a measurable set in $S$ with $\nu(A) > 0$ and subsets $A_1, A_2, \ldots, A_n$ and let $m_1, m_2, \ldots, m_n > 0$. Then there are disjoint subsets $B_k \subseteq A_k$ with $\nu(B_k) = m_k$ for all $k \in [n]$ if and only if*

$$\nu\left(\bigcup_{i \in I} A_i\right) \geq \sum_{i \in I} m_i \tag{4.28}$$

*for all $I \subseteq [n]$.*

For a set $S \subseteq \mathbb{R}^3 \setminus \{\mathcal{O}\}$ and a $1 > \delta > 0$, let $D_S^{\delta,\cup} = \mathrm{Int}\left(\bigcup_{x \in S} A_x^\delta\right)$. We now can prove the following theorem.

**Proposition 4.3.1.** *Let $T = \{x_1, x_2, \ldots, x_k\} \subset \mathbb{R}^3 \setminus \{\mathcal{O}\}$ where $k \in \mathbb{N}$ and let*

$0 < \delta < 1$. *Assume we are given a nonnegative* $g \in L^1(\mathbb{S}^2)$ *where* $g \equiv 0$ *outside* $D_T^{\delta,\cup}$. *Let* $f_1, f_2, \ldots, f_k$ *be nonnegative real numbers such that*

$$\sum_{i\in Q} f_i \le \mu_g \left( D^{\delta,\cup}_{\{x_i \in T \mid i \in Q\}} \right) \tag{4.29}$$

*for all* $Q \subsetneq [k]$ *and*

$$\sum_{i=1}^{k} f_i = \mu_g \left( D_T^{\delta,\cup} \right). \tag{4.30}$$

*Let* $F$ *be a discrete measure over* $T$ *such that* $F(\{x_i\}) = f_i$.

*Then there exists a generalized refractor* $R \in \mathcal{R}_{D_T^{\delta,\cup}}(T)$ *that is a generalized weak solution to the generalized refractor problem (4.14).*

*Proof.* The measure $\mu_g$ is clearly nonatomic; see Appendix E. By Lemma 4.3.2 there exists $k$ disjoint subsets $\{B_i\}$ such that $B_i \subseteq \mathrm{Int}(A^\delta_{x_i})$ and $\mu_g(B_i) = f_i$. Let $S^\delta_Q = \bigcap_{i\in Q} \mathrm{Int}(A^\delta_{x_i}) \setminus \bigcup_{i\notin Q} \mathrm{Int}(A^\delta_{x_i})$ for a nonempty $Q \subseteq [k]$. Then given some nonempty $Q \in [k]$, let $f_{i,Q} = \mu_g(B_i \cap S^\delta_Q)$. It is clear that $f_i = \sum_{Q\in\{P\in 2^{[k]} \mid i\in P\}} f_{i,Q}$, $S^\delta_Q = \bigcup_{i=1}^{k} B_i \cap S^\delta_Q$, and $B_i = \bigcup_{Q\in\{P\in 2^{[k]} \mid i\in P\}} B_i \cap S^\delta_Q$.

By Lemma 4.3.1, there exists disjoint open subsets $B_{i,Q} \subset \mathrm{Int}(S^\delta_Q)$ such that $\mu_g(B_{i,Q}) = f_{i,Q}$. Then we define $B^*_i = \bigcup_{Q\in\{P\in 2^{[k]} \mid i\in P\}} B_{i,Q}$.

For $\epsilon > \frac{1}{\delta}$ and

$$Z = \bigcup_{i=1}^{k} h_{x_i,\epsilon}[\overline{B^*_i}],$$

the refractor $W(\rho_Z) \in \mathcal{R}_{D_T^{\delta,\cup}}(T)$ is a generalized weak solution to the generalized refractor problem (4.14). $\square$

The following is a corollary of Theorem 3.6.1 and will be used in the proofs of the main result.

**Corollary 4.3.1.** *Assume that we are given some $w, W \in (0,\infty)$ where $w > \frac{W}{2}$ and some $m_* \in \mathbb{S}^2$. Recall that $\delta_\gamma = \frac{1}{\epsilon_0+\gamma}$ where $\gamma > 0$ and $\epsilon_0$ is defined by (3.5).*

*Then there exists positive $\xi$ and $\gamma$ such that,*

1. *for any closed subset $T \subseteq S(m_*, \xi)$,*
2. *for any nonnegative $g \in L^1(\mathbb{S}^2)$ such that $g \equiv 0$ outside $D_T^{\delta_\gamma}$,*
3. *for any nonnegative $f \in L^1(T)$ with a measure $F$ defined by (3.30) where $F(T) = \mu_g(D_T^{\delta_\gamma})$,*

*there exists an $\epsilon_M > \epsilon_0 + \gamma$ such that we can construct a convex refractor $\mathcal{H}_{T,K}[D_T^{\delta_\gamma}] \in \mathcal{R}^{cont}_{D_T^{\delta_\gamma}}(T)$ that is a generalized weak solution to the generalized refractor problem (4.14) such that $K : T \to [\epsilon_M, \infty)$.*

We will now use Proposition 4.3.1 to motivate the next result. First, we define relevant terms.

**Definition 4.3.1.** *Given some $w, W \in (0,\infty)$ where $w > \frac{W}{2}$, some $m_* \in \mathbb{S}^2$, and a $\xi \in \left(0, 1 - \cos\left[\frac{1}{2}\arccos\left(\frac{W}{2w}\right)\right]\right)$, let*

$$S(m_*, \xi, w, W) = \{x \in \mathbb{R}^3 | w \le |x| \le W, \langle k_x, m_* \rangle \ge 1 - \xi\}. \tag{4.31}$$

*Recall that, when given a target set $T$ that satisfies Hypotheses H1, $\delta_\gamma = \frac{1}{\epsilon_0+\gamma}$ where $\gamma > 0$ and $\epsilon_0$ is defined by (3.5). Given a set $\omega \subset \mathbb{R}^3 \setminus \{\mathcal{O}\}$ that satisfies Hypothesis H1, we say that $\omega$* **satisfies Corollary 4.3.1** *if there exists $w, W \in (0,\infty)$, and $m_* \in \mathbb{S}^2$, such that there exists $\xi \in \left(0, 1 - \cos\left[\frac{1}{2}\arccos\left(\frac{W}{2w}\right)\right]\right)$ and $\gamma \in (0,\infty)$ where*

1. *$\omega \subseteq S(m_*, \xi, w, W)$,*

2. *for any closed* $T \subseteq S(m_*, \xi, w, W)$*, any nonnegative* $g \in L^1(\mathbb{S}^2)$ *where* $g \equiv 0$ *outside* $D_T^{\delta_\gamma}$*, and for any positive* $f \in L^1(T)$ *with a measure* $F$ *defined by (4.15) such that* $F(T) = \mu_g(D_T^{\delta_\gamma})$*, there exists* $\epsilon_M > \epsilon_0 + \gamma$ *such that we can construct a convex refractor* $\mathcal{H}_{T,K}[D_T^{\delta_\gamma}] \in \mathcal{R}^{cont}_{D_T^{\delta_\gamma}}(T)$ *that is a generalized weak solution to the generalized refractor problem (4.14) such that* $K : T \to [\epsilon_M, \infty)$*.*

*On a related note, given an* $\omega$ *that satisfies Corollary 4.3.1, we say that* $\gamma$ **is a value such that** $\omega$ **satisfies Corollary 4.3.1** *if there exists* $w, W \in (0, \infty)$*,* $\xi \in \left(0, 1 - \cos\left[\frac{1}{2}\arccos\left(\frac{W}{2w}\right)\right]\right)$*, and* $m_* \in \mathbb{S}^2$*, where conditions 1. and 2. are satisfied.*

**Theorem 4.3.1.** *Let* $T$ *be a compact target set in* $\mathbb{R}^3 \setminus \{\mathcal{O}\}$*and let* $\lambda$ *be the Lebesgue measure of* $T$*. Let* $\{\omega_i\}_{i=1}^n$ *be a collection of closed subsets of* $T$ *such that:*

1. $\lambda(\omega_i) > 0$ *for all* $i \in [n]$*,*
2. $\bigcup_{i=1}^n \omega_i = T$*,*
3. $\omega_i$ *satisfies Hypothesis H1 for all* $i \in [n]$*,*
4. $\omega_i$ *satisfies Corollary 4.3.1 for all* $i \in [n]$*.*

*For every* $i \in [n]$*, suppose* $\gamma_i$ *is a value such that* $\omega_i$ *satisfies Corollary 4.3.1. Let* $\delta_i = \frac{1}{\epsilon_{0,i} + \gamma_i}$ *where* $\epsilon_{0,i}$ *is defined by (3.5) with respect to the set* $\omega_i$*. Assume we are given a nonnegative* $g \in L^1(\mathbb{S}^2)$ *such that* $g \equiv 0$ *outside* $\bigcup_{i=1}^n D_{\omega_i}^{\delta_i}$*. Assume we have a nonnegative* $f \in L^1(T)$ *and a measure* $F$ *defined by (4.15) such that for all* $S \subsetneq [n]$

$$F\left(\bigcup_{i \in S} \omega_i\right) \leq \mu_g\left(\bigcup_{i \in S} D_{\omega_i}^{\delta_i}\right) \tag{4.32}$$

*and*

$$F(T) = \mu_g\left(\bigcup_{i \in [n]} D_{\omega_i}^{\delta_i}\right). \tag{4.33}$$

*Then we can construct a generalized refractor* $R \in \mathcal{R}_{\bigcup_{i=1}^n D_{\omega_i}^{\delta_i}}(T)$ *that is a generalized weak solution to the generalized refractor problem (4.14).*

*Proof.* Let $f_i \in L^1(T)$ be a nonnegative function such that $f_i \equiv 0$ outside $\omega_i$ and $f = \sum_{i=1}^n f_i$. Also, let

$$F_i(S) = \int_S f_i(x) d\lambda(x) \text{ for any Borel set } S \subseteq T. \tag{4.34}$$

Note that $F(S) = \sum_{i=1}^n F_i(S)$ for any Borel set $S \subseteq T$. Let $D_i = D_{\omega_i}^{\delta_i}$. Observe that for all nonempty $S \subsetneq [n]$

$$\sum_{i \in S} F_i\left(\bigcup_{i \in S} \omega_i\right) \leq F\left(\bigcup_{i \in [n]} \omega_i\right) \leq \mu_g\left(\bigcup_{i \in S} D_i\right) \tag{4.35}$$

and

$$\sum_{i \in [n]} F_i(T) = F(T) = \mu_g\left(\bigcup_{i \in [n]} D_i\right). \tag{4.36}$$

By Lemma 4.3.2 there exists $k$ disjoint subsets $\{B_i\}$ such that $B_i \subseteq \mathrm{Int}(D_i)$ and $\mu_g(B_i) = F_i(\omega_i)$. Let $S_Q = \bigcap_{i \in Q} \mathrm{Int}(D_i) \setminus \bigcup_{i \notin Q} \mathrm{Int}(D_i)$ for a nonempty $Q \subseteq [n]$. Then given some nonempty $Q \in [n]$, let $f_{i,Q} = \mu_g(B_i \cap S_Q)$. It is clear that $F_i(\omega_i) = \sum_{Q \in \{P \in 2^{[k]} | i \in P\}} f_{i,Q}$, $S_Q = \bigcup_{i=1}^k B_i \cap S_Q$, and $B_i = \bigcup_{Q \in \{P \in 2^{[k]} | i \in P\}} B_i \cap S_Q$.

By Lemma 4.3.1, there exists disjoint open subsets $B_{i,Q} \subset \mathrm{Int}(S_Q)$ such that $\mu_g(B_{i,Q}) = f_{i,Q}$. Then we define $B_i^* = \bigcup_{Q \in \{P \in 2^{[k]} | i \in P\}} B_{i,Q}$. Let

$$g_i(m) = \begin{cases} g(m) & m \in B_i^* \\ 0 & m \notin B_i^* \end{cases}. \tag{4.37}$$

Then, by Corollary 4.3.1, since

$$F_i(\omega_i) = \mu_{g_i}(D_i), \tag{4.38}$$

there exists $\epsilon_{M,i} > \epsilon_{M,i} > \epsilon_{0,i} + \gamma_i$ such that we can construct a convex refractor $\mathcal{H}_{\omega_i,K_i}[D_i] \in \mathcal{R}^{cont}_{D_i}(\omega_i)$ that is a generalized weak solution to the generalized refractor problem (4.14) such that $K_i : \omega_i \to [\epsilon_{M,i}, \infty)$.

Therefore, the generalized refractor $W(\rho_Z) \in \mathcal{R}_{\bigcup_{i\in[n]} D_i}(T)$ where

$$Z = \bigcup_{i\in[n]} \mathcal{H}_{\omega_i,K_i}[\overline{B_i^*}] \tag{4.39}$$

is a generalized weak solution to the generalized refractor problem (4.14). □

## 4.4 The Rotationally Symmetric Case

In Chapter 3.6, we addressed the case of rotationally symmetric refractors when the target set is on a right circular cone. Here, we address the rotationally symmetric case with target sets on a broader category of rotationally symmetric surfaces. We begin with the discrete case on a ring. The following lemma can be proved identically to Proposition 3.7.1. Recall Definition 3.7.1, as we make some use of the notation.

**Lemma 4.4.1.** *Let $m_*, m' \in \mathbb{S}^2$ such that $\langle m_*, m'\rangle = 0$ and $1 > \xi > 0$. Let $T = T^{\xi}_{k,\{d\}}(m_*, m')$. Let $0 < \delta < 1$ and assume we are given a nonnegative $g_* \in L^1(\mathbb{S}^2)$ that is rotationally symmetric about the axis defined by the ray of direction $m_*$ originating at $\mathcal{O}$. Let the function $g \in L^1(\mathbb{S}^2)$ be defined as $g \equiv g_*$ inside $D^\delta_T$ and $g \equiv 0$ outside $D^\delta_T$. Let $F$ be a discrete measure over $T$ such that $F(\{x\}) = \frac{\mu_g(D^\delta_T)}{k}$ for all $x \in T$.*

*Then the convex refractor $\mathcal{H}_{T,K}[D^\delta_T] \in \mathcal{R}^{cont}_{D^\delta_T}(T)$, where $K : T \to (\frac{1}{\delta}, \infty)$ is a* *constant function, is a generalized weak solution to the generalized refractor problem (4.14).*

The following lemma is a consequence of applying the argument from the proof of Theorem 3.6.1 to Lemma 4.4.1.

**Lemma 4.4.2.** *Let $T = T^{\xi}_{\infty,\{d\}}(m_*, m')$. Let $0 < \delta < 1$ and assume we are given a nonnegative $g \in L^1(\mathbb{S}^2)$ that is rotationally symmetric about the axis defined by the ray of direction $m_*$ originating at $\mathcal{O}$ such that $g \equiv 0$ outside $D^{\delta}_T$.*

*Assume we have a positive $f \in L^1(T)$ such that $f(x)$ is constant for all $x \in T$. Let $F$ be the measure defined by (4.15) and*

$$F(T) = \mu_g(D^{\delta_\gamma}_T). \tag{4.40}$$

*Then the convex refractor $\mathcal{H}_{T,K}[D^{\delta}_T] \in \mathcal{R}^{cont}_{D^{\delta}_T}(T)$, where $K : T \to (\frac{1}{\delta}, \infty)$ is a constant function, is a generalized weak solution to the generalized refractor problem (4.14).*

We now prove the following result where the target set is a collection of rings. For convenience, we define $T^1_{\infty,d}(m_*, m') = \{dm_*\}$.

**Theorem 4.4.1.** *Let $T = \bigcup_{j=1}^{q} T^{\alpha_j}_{\infty,d_j}(m_*, m')$ where $0 < \alpha_j \leq 1$ for all $j \in [q]$. Let $0 < \delta < 1$ and assume we are given a nonnegative $g \in L^1(\mathbb{S}^2)$ that is rotationally symmetric about the axis defined by the ray of direction $m_*$ originating at $\mathcal{O}$ such that $g \equiv 0$ outside $D^{\delta}_T$.*

*Let $f \in L^1(T)$ be a nonnegative function such that $f(x) = c_j$ for a constant $c_j \geq 0$ when $x \in T^{\alpha_j}_{\infty,\{d_j\}}(m_*, m')$ for all $j \in [q]$, and $F$ the measure defined by (4.15).*

*Assume that*

$$F(T) = \mu_g(D_T^\delta). \tag{4.41}$$

*Let* $\mathcal{P}_j(m) = \max_{x \in T^{\alpha_j}_{\infty,d_j}(m_*,m')} \frac{|x|}{2\langle k_x, m\rangle}$ *for all* $m \in D_T^\delta$*. If* $\mathcal{P}_{j+1} \leq \mathcal{P}_j$ *for all* $m \in D_T^\delta$*, then there exists a rotationally symmetric refractor* $R \in \mathcal{R}^{cont}_{D_T^\delta}(T)$ *that is a generalized weak solution of the problem (4.14).*

*Proof.* By Lemma 4.3.1, there exists disjoint open subsets of $D_T^\delta$, $\{B_j\}_{j=1}^q$, defined by

$$B_j = \mathrm{Int}(D_T^{\xi_{j-1}} \setminus D_T^{\xi_j}) \tag{4.42}$$

for all $j \in [q]$ where $\xi_j \leq \xi_{j+1}$, $\xi_q = 1$, and $\xi_0 = \delta$, such that

$$\mu_g(B_j) = F(T^{\alpha_j}_{\infty,d_j}(m_*, m'))$$

and $\bigcup_{j\in[q]} \overline{B}_j = \overline{D_T^\delta}$. Let $T_j = T^{\alpha_j}_{\infty,d_j}(m_*, m')$ and consider the functions $\mathcal{P}_{T_j,K_j}(m)$ and $\mathcal{H}_{T_j,K_j}(m)$ where $m \in D_T^\delta$ for all $j \in [q]$ and $K_j(x) = \epsilon_j$ for all $x \in T_j$. Please note that since the set $T$ is rotationally symmetric, then the set $D_T^\delta$ is an open disk centered at $m_*$; thus for all $\xi_j$, there exists a $\beta_j$ such that $D_T^{\xi_j} = \{m \in \mathbb{S}^2 | \langle m_*, m\rangle > \beta_j\}$.

By Lemma 4.4.2, our problem is reduced to finding the set

$$Z = \bigcup_{j=1}^{q} \mathcal{H}_{T_j,K_j}[\overline{B_j}] \tag{4.43}$$

where $K_j(x) = \epsilon_j$ for all $x \in T_j$. As $W(\rho_Z) \in \mathcal{R}^{cont}_{D_T^\delta}(T)$ will be our refractor. By rotational symmetry, this can be done with clever choices of $\epsilon_j$.

Let $\epsilon_1 > \frac{1}{\delta}$. Given such an $\epsilon_1$, if there exists choices of $\epsilon_j$ such that

$$\mathcal{P}_{T_j,K_j}(m)(m) = \mathcal{P}_{T_{j+1},K_{j+1}}(m)$$

for $m \in \{m \in \mathbb{S}^2 | \langle m_*, m\rangle = \beta_j\}$ for all $j \in [q-1]$, then it is possible to construct a refractor $R$ that is simply connected and almost everywhere smooth.

Let $a \in [q-1]$. If $\mathcal{P}_{T_a,K_a}(m) = P$ for $m \in \{m \in \mathbb{S}^2 | \langle m_*, m\rangle = \beta_a\}$, then we will show it is possible to select an $\epsilon_{a+1}$ such that $\mathcal{P}_{T_{a+1},K_{a+1}}(m) = P$ for $m \in \{m \in \mathbb{S}^2 | \langle m_*, m\rangle = \beta_a\}$.

Observe that for $m \in \{m \in \mathbb{S}^2 | \langle m_*, m\rangle = \beta_a\}$,

$$\lim_{\epsilon_a \to \frac{1}{\xi_a}^+} \mathcal{P}_{T_a,K_a}(m)(m) = \infty \text{ and } \lim_{\epsilon_a \to \infty} \mathcal{P}_{T_a,K_a}(m) = \mathcal{P}_a(m). \tag{4.44}$$

Similarly,

$$\lim_{\epsilon_{a+1} \to \frac{1}{\xi_a}^+} \mathcal{P}_{T_{a+1},K_{a+1}}(m) = \infty \text{ and } \lim_{\epsilon_{a+1} \to \infty} \mathcal{P}_{T_{a+1},K_{a+1}}(m) = \mathcal{P}_{a+1}(m). \tag{4.45}$$

By the ordering of $T$, we have that $\infty > P > \mathcal{P}_a(m) \geq \mathcal{P}_{a+1}(m)$. So by the intermediate value theorem, there exists an $\epsilon_{a+1}$ such that $\mathcal{P}_{T_{a+1},K_{a+1}}(m) = P$ on $\{m \in \mathbb{S}^2 | \langle m_*, m\rangle = \beta_a\}$.

Since both $\mathcal{P}_j$ and $B_j$ are rotationally symmetric about the axis defined by the ray originating from $\mathcal{O}$ with the direction $m_*$, the refractor $R$ is also rotationally symmetric. □

The theorem above can create refractors with a variety of rotationally symmetric target sets and distributions. For example, take the following lemma.

**Lemma 4.4.3.** *Let* $0 < \xi \leq 1$, $0 < \delta < 1$, *and*

$$P^{\xi}_{\infty,d}(m; m_*, m') = \max_{x \in T^{\xi}_{\infty,d}(m_*,m')} \frac{|x|}{2\langle k_x, m\rangle} \tag{4.46}$$

*for all*

$$m \in D^{\delta}_{T^{\xi}_{\infty,d}(m_*,m')}. \tag{4.47}$$

*Then* $d_1 > d_2$ *implies that*

$$P^{\xi}_{\infty,d_1}(m; m_*, m') > P^{\xi}_{\infty,d_2}(m; m_*, m') \tag{4.48}$$

*for all*

$$m \in D^{\delta}_{T^{\xi}_{\infty,d_1}(m_*,m') \cup T^{\xi}_{\infty,d_2}(m_*,m')} \tag{4.49}$$

*and* $\xi_1 > \xi_2$ *implies that*

$$P^{\xi_2}_{\infty,d}(m; m_*, m') > P^{\xi_1}_{\infty,d}(m; m_*, m') \tag{4.50}$$

*for all*

$$m \in D^{\delta}_{T^{\xi_1}_{\infty,d}(m_*,m') \cup T^{\xi_2}_{\infty,d}(m_*,m')}. \tag{4.51}$$

*Proof.* Firstly, let

$$m \in D^{\delta}_{T^{\xi}_{\infty,d_1}(m_*,m') \cup T^{\xi}_{\infty,d_2}(m_*,m')} \tag{4.52}$$

and $d_1 > d_2$, then

$$\begin{aligned} P^{\xi}_{\infty,d_1}(m; m_*, m') &= \max_{x \in T^{\xi}_{\infty,d_1}(m_*,m')} \frac{d_1}{2\langle k_x, m\rangle} && (4.53)\\ &> \max_{x \in T^{\xi}_{\infty,d_2}(m_*,m')} \frac{d_2}{2\langle k_x, m\rangle} && (4.54)\\ &= P^{\xi}_{\infty,d_2}(m; m_*, m'). && (4.55) \end{aligned}$$

Finally, let

$$m \in D^{\delta}_{T^{\xi_1}_{\infty,d}(m_*,m') \cup T^{\xi_2}_{\infty,d}(m_*,m')} \tag{4.56}$$

and $\xi_1 > \xi_2$, then

$$P^{\xi_2}_{\infty,d}(m; m_*, m') = \max_{x \in T^{\xi_2}_{\infty,d}(m_*,m')} \frac{d}{2\langle k_x, m\rangle} \tag{4.57}$$

$$> \max_{x \in T^{\xi_1}_{\infty,d}(m_*,m')} \frac{d}{2\langle k_x, m\rangle} \tag{4.58}$$

$$= P^{\xi_1}_{\infty,d}(m; m_*, m'). \tag{4.59}$$

□

If the target distribution is on a rotationally symmetric surface whose polar radius is nondecreasing with respect to the angular distance from the direction of the axis of rotation, then both Theorem 4.4.1 and Lemma 4.4.3 imply the existence of a refractor. Specifically, we have the following corollary.

**Corollary 4.4.1.** *Consider a nonincreasing positive function $w$ on $(0, 1]$. Let $T = \bigcup_{j=1}^{q} T^{\alpha_j}_{\infty,w(\alpha_j)}(m_*, m') \in \mathbb{R}^3 \setminus \{\mathcal{O}\}$ where $0 < \alpha_j \leq 1$ for all $j \in [q]$. Let $0 < \delta < 1$ and assume we are given a nonnegative $g \in L^1(\mathbb{S}^2)$ that is rotationally symmetric about the axis defined by the ray of direction $m_*$ originating at $\mathcal{O}$ such that $g \equiv 0$ outside $D^\delta_T$.*

*Let $f \in L^1(T)$ be a nonnegative function such that $f(x) = c_j$ for a constant $c_j \geq 0$ when $x \in T^{\alpha_j}_{\infty,\{w(\alpha_j)\}}(m_*, m')$ for all $j \in [q]$, and $F$ the measure defined by (4.15). Assume that*

$$F(T) = \mu_g(D^\delta_T). \tag{4.60}$$

*Then there exists a rotationally symmetric refractor $R \in \mathcal{R}^{cont}_{D^\delta_T}(T)$ that is a generalized weak solution of the problem (4.14).*

## 4.5 Discussion

In this chapter, we proved a relatively broad existence theorem for generalized refractors. Generalized refractors might be hard to construct in a real-world setting and may cause unintentional distortion in the intended irradiance distribution when accounting for interference caused by light waves. Interference and distortions caused by waves are, in general, not accounted for in the assumptions of geometric optics. However, these issues are more likely to be avoided when dealing with refractors. So, generalized refractors can have value when motivating the construction of refractors or, at least, hinting at avenues for possible research; for example, Theorem 4.3.1 inspires this next conjecture.

**Conjecture 4.5.1.** *Let $T$ be a compact target set in $\mathbb{R}^3 \setminus \{\mathcal{O}\}$ and let $\lambda$ be the Lebesgue measure of $T$. Let $\{\omega_i\}_{i=1}^n$ be a collection of closed subsets of $T$ such that:*

1. *$\lambda(\omega_i) > 0$ for all $i \in [n]$,*
2. *$\bigcup_{i=1}^n \omega_i = T$,*
3. *$\omega_i$ satisfies Hypothesis H1 for all $i \in [n]$,*
4. *$\omega_i$ satisfies Corollary 4.3.1 for all $i \in [n]$.*

*For every $i \in [n]$, suppose $\gamma_i$ is a value such that $\omega_i$ satisfies Corollary 4.3.1. Let $\delta_i = \frac{1}{\epsilon_{0,i} + \gamma_i}$ where $\epsilon_{0,i}$ is defined by (3.5) with respect to the set $\omega_i$. Assume that we are given a nonnegative $g \in L^1(\mathbb{S}^2)$ such that $g \equiv 0$ outside $\bigcup_{i=1}^n D_{\omega_i}^{\delta_i}$. Assume that we have a nonnegative $f \in L^1(T)$and a measure $F$ defined by (4.15) such that for all $S \subsetneq [n]$*

$$F\left(\bigcup_{i \in S} \omega_i\right) \leq \mu_g\left(\bigcup_{i \in S} D_{\omega_i}^{\delta_i}\right) \tag{4.61}$$

*and*

$$F\left(T\right)=\mu_{g}\left(\bigcup_{i\in[n]}D_{\omega_{i}}^{\delta_{i}}\right). \tag{4.62}$$

*Then we can construct a refractor* $R \in \mathcal{R}^{cont}_{\bigcup_{i=1}^{n} D^{\delta_i}_{\omega_i}}(T)$ *that is a generalized weak solution to the generalized refractor problem (4.14).*

For the rotationally symmetric case, we observe that Theorem 4.4.1 proves the existence of refractors with rotationally symmetric distributions on rotationally symmetric surfaces with a target set made up of finitely many rings. A natural next step, if possible, would be to proceed with a limit or smoothing argument on the refractors described in Theorem 4.4.1 so that we can recover rotationally symmetric solutions when the target set is not a collection of rings. Another approach that would improve the work on the rotationally symmetric case would be to 'stack' more complex rotationally symmetric reflectors. Corollary 4.4.1 motivates the following conjecture.

**Conjecture 4.5.2.** *Consider a nonincreasing positive function* $w : Q \to (0,\infty)$ *where* $Q$ *is a closed subset of* $(0,1]$. *Let* $T = \bigcup_{\alpha\in Q} T^{\alpha}_{\infty,w(\alpha)}(m_*, m')$ *where* $\alpha \in Q$. *Let* $0 < \delta < 1$ *and assume we are given a nonnegative* $g \in L^1(\mathbb{S}^2)$ *that is rotationally symmetric about the axis defined by the ray of direction* $m_*$ *originating at* $\mathcal{O}$ *such that* $g \equiv 0$ *outside* $D^{\delta}_{T}$.

*Let* $f \in L^1(T)$ *be a nonnegative function such that, for all* $\alpha \in Q$, $f(x)$ *is constant for all* $x \in T^{\alpha}_{\infty,\{w(\alpha)\}}(m_*, m')$, *and* $F$ *the measure defined by (4.15). Assume that*

$$F(T) = \mu_g(D^{\delta}_{T}). \tag{4.63}$$

*Then there exists a rotationally symmetric refractor* $R \in \mathcal{R}^{cont}_{D^{\delta}_{T}}(T)$ *that is a generalized weak solution of the problem (4.14).*

# Appendix A

# Formulation of the PDE for the Virtual Source Reflector Problem

Here we present, along with necessary background, equation (4) from [13]; the following formulation is copied from [13]. The formulation of the PDE involves the energy conservation law and the Jacobian of the refractor map $m \to x$ where $m \in \mathbb{S}^2$ and $x \in T$, associated with a particular refractor $R$. An explicit expression for the Jacobian and the conservation law was derived in [14] in the case when $R$ is a graph over the input aperture $D \subset \mathbb{S}^2$. In this case, since $R$ is a graph over the input aperture $D \subset \mathbb{S}^2$, its position vector can be written as $m\rho(m)$ for $m \in D$, for some positive function $\rho$. Put $\partial_i \equiv \frac{\partial}{\partial u_i}, i = 1, 2$, where $u_1, u_2$ are some local coordinates on the unit sphere $\mathbb{S}^2$. Let $e_{ij} = \langle \partial_i m, \partial_j m \rangle$ be coefficients of the first fundamental form of $\mathbb{S}^2$, $g_{ij} = \partial_i \rho \partial_j \rho + \rho^2 e_{ij}$, $Q_{ij} = g_{ij} - \partial_i \rho \partial_j \rho$, $Q^{ij} = (Q_{ij})^{-1}$, $H_{ij} = \langle -\partial_i y(m), \partial_j (m\rho) \rangle$, $t(m) = |x(m) - m\rho(m)|$, $s_i = \partial_i t - \partial_i \rho$, $l = \rho + t$, and $l_i = \frac{\partial l}{\partial u_i}$.

With the introduced notation, the conservation law can be written as

$$g(m)\sqrt{\frac{\det\{[tH_{ik}+Q_{ik}]Q^{ks}[tH_{sj}+Q_{sj}]+l_i l_j\}}{\det e_{ij}}}=f(x(m)) \tag{A.1}$$

If one can find a function $\rho$ such that the map $x(m)=m\rho(m)-t(m)y(m):D\to T$, where $y$ is defined by (1.1) with the normal field $n$ on the refractor described by $\rho$ and the equation (A.1) is satisfied at each interior point of $D$, then the refractor given by $m\rho(m),\ m\in D$, is the desired refractor.

# Appendix B

# No Set of Points Satisfies Both Hypotheses H1 and H2

The paper of Kochengin et al. [13] specifies two assumptions with regard to the target set $T$ that they call Hypothesis H1 and Hypothesis H2. Lemma 11, Theorem 12, and Theorem 13 are proven with the assumption that both Hypotheses H1 and H2 hold for $T$. In this section, we will prove that Hypotheses H1 and H2 are contradictory. We start by restating some key definitions from [13] and proceed with the proof.

Let $k_x = \frac{x}{|x|}$. We are given a set of points $T \subset \mathbb{R}^3 \setminus \{\mathcal{O}\}$, where $c = \min_{x,y \in T} \langle k_x, k_y \rangle$, $\ell = \min_{x \in T} |x|$, and $L = \max_{x \in T} |x|$.

We also use the definition of $\epsilon_0$ provided in Lemma 2 of [13]:

$$\epsilon_0 = \frac{\ell + \sqrt{\ell^2 - 2L\ell c + L^2}}{2\ell c - L}. \tag{B.1}$$

In [13] we have Hypothesis H1 which is as follows.

**Hypothesis H1.** *$T$ is a compact subset of $\mathbb{R}^3$ contained in a half space of $\mathbb{R}^3$, $\ell > 0$, and $2\ell c > L$.*

Assume that $T$ satisfies Hypothesis H1. By definition $L \geq \ell > 0$, therefore we can write $L = \ell(1+\delta)$ where $\delta \geq 0$. We can also observe that $L > 0$ implies that $2\ell c > 0$. Thus $c > 0$. Observe that $c$ is the cosine of the largest angle between two points in $T$, then $1 \geq c$ since cosine is bounded above by 1.

Copying directly from [13] we present Hypothesis H2 as follows.

**Hypothesis H2.** *We say that $T$ satisfies Hypothesis H2 if*

1. *inequalities (22)-(24) in [13] hold for $T$,*
2. *for some number $\gamma' > 0$ condition (28) in [13] is satisfied*

As the title reveals, we prove that no set of points $T$ satisfies both Hypotheses H1 and H2. The inequalities I will focus on are (22) and (23) in [13], namely

$$2\ell - L\epsilon_0 > 0 \tag{22}$$

and

$$\epsilon_0 > \frac{\ell\epsilon_0 + \sqrt{\ell^2\epsilon_0^2 - 2\ell L\epsilon_0 + L^2\epsilon_0^2}}{2\ell - L\epsilon_0}. \tag{23}$$

The central idea of our main proof is showing that these two inequalities contradict each other when given Hypothesis H1.

However, we first need to prove the following claim.

**Claim B.1.** *Let a set of points $T$ satisfy Hypothesis H1, then $\epsilon_0 \geq 1$.*

*Proof.* Assume to the contrary that there exists a $\ell, L$ and $c$ such that $\epsilon_0 < 1$ and Hyopthesis H1 is also satisfied. Recall that we can write $L = \ell(1+\delta)$ where $\delta \geq 0$. Thus we can rewrite (B.1) as

$$\begin{aligned}
\epsilon_0 &= \frac{\ell + \sqrt{\ell^2 - 2L\ell c + L^2}}{2\ell c - L} && \text{(B.2)} \\
&= \frac{\ell + \sqrt{\ell^2 - 2\ell^2(1+\delta)c + \ell^2(1+\delta)^2}}{2\ell c - \ell(1+\delta)} && \text{(B.3)} \\
&= \frac{1 + \sqrt{1 - 2(1+\delta)c + (1+\delta)^2}}{2c - (1+\delta)}. && \text{(B.4)}
\end{aligned}$$

Since $\epsilon_0 < 1$, we have

$$1 > \frac{1 + \sqrt{1 - 2(1+\delta)c + (1+\delta)^2}}{2c - (1+\delta)}. \tag{B.5}$$

Hypotheses H1 tells us that $2\ell c - L > 0$, which implies that $2c - (1+\delta)$ is positive. We now have that

$$\begin{aligned}
2c - (1+\delta) &> 1 + \sqrt{1 - 2(1+\delta)c + (1+\delta)^2} && \text{(B.6)} \\
\Rightarrow 2(c-1) - \delta &> \sqrt{2(1-c) + 2\delta(1-c) + \delta^2}. && \text{(B.7)}
\end{aligned}$$

Since $1 \geq c > 0$, we have that $0 \geq c - 1$. Since $\delta \geq 0$, we have that the LHS of (B.7) is non-positive and the RHS is non-negative. A contradiction since $\epsilon_0 \geq 1$. □

Now for the main result.

**Theorem B.1.** *Let a set of points $T$ satisfy Hypothesis H1, then $T$ does not satisfy Hypothesis H2.*

*Proof.* Let us rewrite $L = \ell(1+\delta)$ when $\delta \geq 0$. Thus we can rewrite (22) as

$$2 - (1+\delta)\epsilon_0 > 0. \tag{B.8}$$

and we can rewrite (23) as

$$\epsilon_0 > \frac{\epsilon_0 + \sqrt{\epsilon_0^2 - 2(1+\delta)\epsilon_0 + (1+\delta)^2\epsilon_0^2}}{2 - (1+\delta)\epsilon_0}. \tag{B.9}$$

Assume to the contrary that there exists a set $T$ such that H2 can be satisfied. Then there exists an $\epsilon_0$ and a $\delta$ that satisfies both inequalities (22) and (23), or, equivalently, (B.8) and (B.9). If $T$ satisfies inequality (B.8), thus we can obtain the following inequality from (B.9):

$$\epsilon_0(2 - (1+\delta)\epsilon_0) - \epsilon_0 > \sqrt{\epsilon_0^2 - 2(1+\delta)\epsilon_0 + (1+\delta)^2\epsilon_0^2}. \tag{B.10}$$

With a little bit of algebra on each side, we obtain

$$-\delta\epsilon_0^2 - (\epsilon_0^2 - \epsilon_0) > \sqrt{(1+2\delta)(\epsilon_0^2 - \epsilon_0) + \epsilon_0^2\delta^2}. \tag{B.11}$$

By Claim B.1, $\epsilon_0 \geq 1$; thus $\epsilon_0^2 \geq \epsilon_0$. That combined with the fact that $\delta \geq 0$, we have that the LHS of (B.11) is non-positive and the RHS is non-negative. This would make inequality (B.11) incorrect. Thus when given H1, the inequality (B.9) is not valid when given inequality (B.8). Therefore the inequalities (22)-(24) in [13] for Hypothesis H2 are contradictory. So $T$ cannot satisfy Hypothesis H2. □

# Appendix C

# Blaschke's Selection Theorem

Here we state Blaschke's Selection theorem [19] and discuss its consequences in relation to our proof of Theorem 3.6.1.

**Theorem C.1.** *Every bounded sequence of convex bodies has a subsequence that converges to a convex body.*

Specifically, in the context of the proof of Theorem 3.6.1, since each $h_b^k$ is convex and bounded by $\mathrm{B}(\mathcal{O}, b)$, the sequence $\{h_b^k\}$ has a convergent subsequence.

# Appendix D

# Reidemeister's Theorem About Singular Points on Convex Sets

Here we state a theorem of Reidemeister about singular points on convex sets and discuss its consequences in relation to our proof of Theorem 3.6.1. We first start with the following definitions from [19].

**Definition D.1.** *Let $A \subset \mathbb{R}^n$ be a subset and $H \subset \mathbb{R}^n$ a hyperplane and let $H^+, H^-$ denote the two closed halfspaces bounded by $H$. We say that $H$* **supports** *$A$* **at** *$x$ if $x \in A \cap H$ and either $A \subset H^+$ or $A \subset H^-$. Further, $H$ is a* **support plane** *of $A$ or* **supports** *$A$ if $H$ supports $A$ at some point $x$, which is necessarily a boundary point of $A$.*

**Definition D.2.** *By $\mathcal{K}^n$ we denote the set of all convex bodies in $\mathbb{R}^n$ and by $\mathcal{K}^n_n$ the subset of convex bodies with interior points (thus, the lower index $n$ stands for the dimension of the bodies).*

**Definition D.3.** *If the supporting hyperplane to $K$ at $x$ is unique, then $x$ is called a* **regular** *or* **smooth** *point of $K$. Otherwise, $x$ is* **singular**.

We then state the theorem.

**Theorem D.1.** *That the set of singular boundary points of a convex body $K \in \mathcal{K}_n^n$ is of $(n-1)$-dimensional Hausdorff measure zero.*

In the context of the proof of Theorem 3.6.1, if a point $x$ on a refractor $R$ is regular, then the supporting hyperplane to $R$ at $x$ is unique. Since our refractors are constructed from hyperboloids: a unique supporting hyperplane of a refractor $R$ at some point $x$ implies the existence of a unique supporting hyperboloid to $R$ at $x$. Then, by the Chapter 3.3 definition of the refractor map, $\alpha^k_{convex}$ and $\alpha_{convex}$ are single-valued almost everywhere.

# Appendix E

# A Constructive Proof of Lemma 4.3.2

We seek to constructively prove the following theorem, which is the same as Lemma 4.3.2.

**Theorem E.1.** *Let $(\Omega, S, \nu)$ be a finite nonatomic measure and $A$ a measurable set in $S$ with $\nu(A) > 0$ and subsets $A_1, A_2, \ldots, A_n$ and let $m_1, m_2, \ldots, m_n > 0$.*

*Then there are disjoint subsets $B_k \subseteq A_k$ with $\nu(B_k) = m_k$ for all $k \in [n]$ if and only if*

$$\nu\left(\bigcup_{i\in I} A_i\right) \geq \sum_{i\in I} m_i \tag{E.1}$$

*for all $I \subseteq [n]$.*

## E.1 Nonatomic Measures

The following definition is taken from [10].

**Definition E.1.1.** *A measure $\nu$ on $(\Omega, S)$ is said to be* **nonatomic** *if $\nu(\{x\}) = 0$ for every $x \in \Omega$.*

An equivalent and potentially helpful definition from [6] can be seen below.

**Definition E.1.2.** *Let $(\Omega, S, \nu)$ be a $\sigma$-finite measure. Then an* **atom** *of $\nu$ is a set $A \in S$ with $\nu(A) > 0$ such that for all $C \in S$ with $C \subset A$, either $\nu(C) = 0$ or $\nu(C) = \nu(A)$. By $\sigma$-finiteness, we have $\nu(A) < \infty$. $(\Omega, S, \nu)$ or $\nu$ is called* **nonatomic** *if it has no atoms.*

Equivalently, $(\Omega, S, \nu)$ or $\nu$ is called *nonatomic* if for any measurable set $A$ with $\nu(A) > 0$ there exists a measurable subset $B$ of $A$ such that $\nu(A) > \nu(B) > 0$.

The following corollary is a consequence of Proposition A.1 in [6].

**Corollary E.1.1.** *Let $(\Omega, S, \nu)$ be a finite nonatomic measure with $\nu(\Omega) > 0$. Then if $A$ is a measurable set in $S$ with $\nu(A) > 0$, then for any real number $c$ with $\nu(A) \geq c \geq 0$ there exists a measurable subset $B$ of $A$ such that $\nu(B) = c$.*

We also have another corollary following from Proposition A.2 in [6].

**Corollary E.1.2.** *Let $(\Omega, S, \nu)$ be a finite nonatomic measure and $A$ be a measurable set in $S$ with $\nu(A) > 0$. Let $r_i$ for $i \in [n]$ be numbers with $r_i > 0$ and $\sum_{i=1}^{n} r_i = \nu(A)$. Then $A$ can be decomposed as a union of disjoint sets $R_i \in S$ with $\nu(R_i) = r_i$ for $i \in [n]$.*

## E.2 Generalization of Hall's Matching Theorem

In addition to Corollaries E.1.1 and E.1.2, we also need to use Hall's matching theorem; see ([8], Theorem 2.1.2 in [4]). Hall's theorem itself will not be covered in this write-up, as there is a lot of literature already in existence. It would also require me to write a few pages of elementary graph theory to cover the requisite background to make the statement of the theorem understandable.

The following corollary can be considered as a generalization of Hall's matching theorem. In fact, it is Exercise 2.9 in [4] and is proven with a clever application of Hall's matching theorem.

**Corollary E.2.1.** *Let $A$ be a finite set with subsets $A_1, \ldots, A_n$, and let $d_1, \ldots, d_n \in \mathbb{N}$. Then there are disjoint subsets $D_k \subseteq A_k$, with $|D_k| = d_k$ for all $k \in [n]$, if and only if*

$$\left| \bigcup_{i \in I} A_i \right| \geq \sum_{i \in I} d_i \tag{E.2}$$

*for all $I \subseteq [n]$.*

The above claim is a generalization of Hall's matching theorem because if we set $d_i = 1$ for all $i \in [n]$, then it would be an equivalent statement. If we consider the case where each point has a weight of $\xi > 0$, then we obtain the following corollary.

**Corollary E.2.2.** *Let $A$ be a finite set with subsets $A_1, \ldots, A_n$ and let $d_1, \ldots, d_n \in \mathbb{N}$. Given a real $\xi > 0$, let $(A, 2^A, \eta)$ be a discrete measure defined as $\eta(X) = \xi|X|$ for all $X \in 2^A$.*

*Then there are disjoint subsets $D_k \subseteq A_k$, with $\eta(D_k) = \xi d_k$ for all $k \in [n]$, if and only if*

$$\eta\left(\bigcup_{i \in I} A_i\right) = \xi \left| \bigcup_{i \in I} A_i \right| \geq \xi \sum_{i \in I} d_i \tag{E.3}$$

*for all $I \subseteq [n]$.*

If $\xi = 1$, then we will obtain Corollary E.2.1. Therefore, Corollary E.2.2 is a generalization of Corollary E.2.1.

If we replace discrete weighted points with disjoint subsets with the same measure and apply that to Corollary E.2.2, we obtain the next corollary.

**Corollary E.2.3.** *Given a real $\xi > 0$, let $(\Omega, S, \nu)$ be a finite measure and $A = \{A_1, A_2, \ldots, A_\ell\}$ be a finite collection of disjoint measurable sets in $S$ where $\nu(A_i) = \xi$. Also, let $d_1, \ldots, d_n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \subseteq A$.*

*Then there are disjoint subsets $D_k \subseteq \alpha_k$, with $\nu\left(\bigcup_{D \in D_k} D\right) = \xi d_k$ for all $k \in [n]$, if and only if*

$$\nu\left(\bigcup_{i \in I}\left(\bigcup_{B \in \alpha_i} B\right)\right) = \xi\left|\bigcup_{i \in I} \alpha_i\right| \geq \xi \sum_{i \in I} d_i \tag{E.4}$$

*for all $I \subseteq [n]$.*

*Proof.* Let $(A, 2^A, \eta)$ be a discrete measure defined as $\eta(X) = \xi|X|$ for all $X \in 2^A$. Then by Corollary E.2.2, there are disjoint subsets $D_k \subseteq \alpha_k$, with $\eta(D_k) = \xi d_k$ for all $k \in [n]$, if and only if

$$\eta\left(\bigcup_{i \in I} \alpha_i\right) = \xi\left|\bigcup_{i \in I} \alpha_i\right| \geq \xi \sum_{i \in I} d_i \tag{E.5}$$

for all $I \subseteq [n]$.

Since $\bigcup_{i \in I} \alpha_i$ is a finite collection of disjoint measurable sets, then

$$\eta\left(\bigcup_{i \in I} \alpha_i\right) = \eta\left(\bigcup_{B \in \bigcup_{i \in I} \alpha_i} \{B\}\right) \tag{E.6}$$

$$= \sum_{B \in \bigcup_{i \in I} \alpha_i} \eta(\{B\}) \tag{E.7}$$

$$= \sum_{B \in \bigcup_{i \in I} \alpha_i} \nu(B) \tag{E.8}$$

$$= \nu\left(\bigcup_{B\in\bigcup_{i\in I}\alpha_i} B\right) \tag{E.9}$$

$$= \nu\left(\bigcup_{i\in I}\left(\bigcup_{B\in\alpha_i} B\right)\right). \tag{E.10}$$

□

Observe that Corollary E.2.3 is a generalization of Corollary E.2.2. As Corollary E.2.2 can be considered as a special case of Corollary E.2.3 where each set is a single point.

## E.3 Proof of Theorem E.1

Now, we prove Theorem E.1.

*Proof.* If such $B_k$'s exist, then for all $I$, $\nu\left(\bigcup_{i\in I} A_i\right) \geq \sum_{i\in I} m_i$.

Conversely, letting $Q$ be a nonempty subset of $[n]$, then we can define

$$S_Q = \left(\bigcap_{i\in Q} A_i\right) \setminus \left(\bigcup_{i\notin Q} A_i\right). \tag{E.11}$$

Note that if $P$ is also a nonempty subset of $[n]$ and $Q \neq P$, then $S_Q \cap S_P = \varnothing$. It would also be helpful to note that

$$A_k = \bigcup_{Q\in\{P\in 2^{[n]} \mid k\in P\}} S_Q \tag{E.12}$$

and

$$\bigcup_{i\in[n]} A_i = \bigcup_{Q\in 2^{[n]}\setminus\{\varnothing\}} S_Q. \tag{E.13}$$

Assume $\xi > 0$. Then by Corollary E.1.1, for each nonempty $Q \subseteq [n]$ there exists a subset of $S_Q$ of measure $\xi\left\lfloor \frac{\nu(S_Q)}{\xi} \right\rfloor$; let this subset be denoted as $E_{Q,\xi}$. Then by Corollary E.1.2, $E_{Q,\xi}$ can be partitioned into $\left\lfloor \frac{\nu(S_Q)}{\xi} \right\rfloor$ subsets of measure $\xi$. We can now define a set

$$A_{k,\xi} = \bigcup_{Q\in\{P\in 2^{[n]}|k\in P\}} E_{Q,\xi}. \tag{E.14}$$

Recall the number $m_k$ from the theorem statement. Now consider the number $\left\lfloor \frac{m_k}{\xi} \right\rfloor$, which by Corollaries E.1.1 and E.1.2 is the maximum number of disjoint subsets of measure $\xi$ for a set of measure $m_k$, and define $d_{k,\xi} = \left\lfloor \frac{m_k}{\xi} \right\rfloor - 2^{n+1}$. We want $d_{k,\xi}$ to be positive integers: so note that if $0 < \xi \leq \frac{m_k}{2^{n+1}+1}$, then $d_{k,\xi} > 0$.

Observe that given a nonempty $Q \subseteq [n]$ we have that

$$\left|\nu\left(\bigcup_{i\in Q} A_i\right) - \nu\left(\bigcup_{i\in Q} A_{i,\xi}\right)\right| = \left|\nu\left(\bigcup_{P\in\{T\in 2^{[n]}|T\cap Q\neq\varnothing\}} S_P\right)\right. \tag{E.15}$$

$$\left. -\nu\left(\bigcup_{P\in\{T\in 2^{[n]}|T\cap Q\neq\varnothing\}} E_{P,\xi}\right)\right| \tag{E.16}$$

$$= \left|\sum_{P\in\{T\in 2^{[n]}|T\cap Q\neq\varnothing\}} \nu(S_P) - \sum_{P\in\{T\in 2^{[n]}|T\cap Q\neq\varnothing\}} \nu\left(E_{P,\xi}\right)\right| \tag{E.17}$$

$$= \left|\sum_{P\in\{T\in 2^{[n]}|T\cap Q\neq\varnothing\}} \left[\nu(S_P) - \nu\left(E_{P,\xi}\right)\right]\right| \tag{E.18}$$

$$< \left|\sum_{P\in\{T\in 2^{[n]}|T\cap Q\neq\varnothing\}} \xi\right| = \xi(2^n - 2^{n-|Q|}). \tag{E.19}$$

Thus, as $\xi \to 0$, we have $\nu\left(\bigcup_{i\in Q} A_{i,\xi}\right) \to \nu\left(\bigcup_{i\in Q} A_i\right)$.

Also, note that

$$\nu\left(\bigcup_{i\in Q} A_i\right) \geq \nu\left(\bigcup_{i\in Q} A_{i,\xi}\right). \tag{E.20}$$

We use $d_{k,\xi}$ because if we are given a nonempty $Q \subseteq [n]$ then, by the work we did previously, $\nu\left(\bigcup_{i\in Q} A_{i,\xi}\right) > \nu\left(\bigcup_{i\in Q} A_i\right) - \xi(2^n - 2^{n-|Q|})$.

Since the theorem hypothesis states that $\nu\left(\bigcup_{i\in Q} A_i\right) \geq \sum_{i\in Q} m_i$, then

$$\begin{aligned}
\nu\left(\bigcup_{i\in Q} A_i\right) - \xi(2^n - 2^{n-|Q|}) &\geq \left(\sum_{i\in Q} m_i\right) - \xi(2^n - 2^{n-|Q|}) && \text{(E.21)}\\
&\geq \xi\left(\sum_{i\in Q}\left\lfloor\frac{m_i}{\xi}\right\rfloor\right) - \xi(2^n - 2^{n-|Q|}) && \text{(E.22)}\\
&\geq \xi\left(\sum_{i\in Q}\left\lfloor\frac{m_i}{\xi}\right\rfloor\right) - 2^{n+1}\xi && \text{(E.23)}\\
&\geq \xi\sum_{i\in Q}\left(\left\lfloor\frac{m_i}{\xi}\right\rfloor - 2^{n+1}\right) = \xi\sum_{i\in Q} d_{i,\xi}. && \text{(E.24)}
\end{aligned}$$

In summary, for sufficiently small $\xi$,

$$\nu\left(\bigcup_{i\in Q} A_{i,\xi}\right) > \nu\left(\bigcup_{i\in Q} A_i\right) - \xi(2^n - 2^{n-|Q|}) \geq \xi\sum_{i\in Q} d_{i,\xi} > 0. \tag{E.25}$$

Therefore, for nonempty all $I \subseteq [n]$ and a sufficiently small $\xi$, we have

$$\nu\left(\bigcup_{i\in I} A_{i,\xi}\right) > \xi\sum_{i\in I} d_{i,\xi} > 0. \tag{E.26}$$

Thus, by Corollary E.2.3, for sufficiently small $\xi$ there are disjoint subsets $B_{k,\xi} \subseteq A_{k,\xi}$ such that $\nu(B_{k,\xi}) = \xi d_{k,\xi}$ for all $k \in [n]$.

Given a nonempty $Q \subseteq [n]$ we have that $\sum_{i\in Q} m_i \geq \xi\sum_{i\in Q} d_{i,\xi}$. Therefore,

$$\left|\sum_{i\in Q} m_i - \xi\sum_{i\in Q} d_{i,\xi}\right| = \left|\sum_{i\in Q}(m_i - \xi d_{i,\xi})\right| \tag{E.27}$$

$$= \left|\sum_{i\in Q}\left(m_i - \xi\left\lfloor\frac{m_i}{\xi}\right\rfloor + \xi 2^{n+1}\right)\right| \tag{E.28}$$

$$= 2^{n+1}\xi|Q| + \left|\sum_{i\in Q}\left(m_i - \xi\left\lfloor\frac{m_i}{\xi}\right\rfloor\right)\right|. \tag{E.29}$$

Keep in mind that $\lim_{\xi\to 0} m_i - |\xi| \leq \lim_{\xi\to 0}\xi\left\lfloor\frac{m_i}{\xi}\right\rfloor \leq \lim_{\xi\to 0} m_i + |\xi|$. Therefore $\lim_{\xi\to 0}\xi\left\lfloor\frac{m_i}{\xi}\right\rfloor = m_i$. Thus as $\xi \to 0$ we have $\nu(B_{k,\xi}) = \xi d_{k,\xi} \to m_k$ for all $k \in [n]$ and

$$\nu\left(\bigcup_{i\in I} A_{i,\xi}\right) > \xi\sum_{i\in I} d_{i,\xi} \longrightarrow \sum_{i\in I} m_i \leq \nu\left(\bigcup_{i\in I} A_i\right) \tag{E.30}$$

for all nonempty $I \subseteq [n]$.

Given a sufficiently small $\xi > 0$, let us define $\xi_i = \frac{\xi}{2^i}$ for $i \in \mathbb{N}\cup\{0\}$. So for each $k \in [n]$ we can now construct a sequence $\{\nu(B_{k,\xi_i})\}_{i=0}^{\infty}$. Unless there exists an $N > 0$ such that $\nu(B_{k,\xi_N}) = m_i$, the sequence $\{\nu(B_{k,\xi_i})\}_{i=0}^{\infty}$ has no greatest term. Therefore, there exists a monotonically nondecreasing subsequence $\{\nu(B_{k,\xi_{f(i)}})\}_{i=0}^{\infty}$ where $f : \mathbb{N}\cup\{0\} \to \mathbb{N}\cup\{0\}$ and $f(i+1) \geq f(i)$.

Recall that by construction $A_{k,\xi_{f(i)}} \subseteq A_{k,\xi_{f(i+1)}}$. Similarly, we can always construct the sets $B_{k,\xi_{f(i)}}$ and $B_{k,\xi_{f(i+1)}}$ from the preexisting partitions that we used to construct $A_{k,\xi_{f(i)}}$ and $A_{k,\xi_{f(i+1)}}$ such that $B_{k,\xi_{f(i)}} \subseteq B_{k,\xi_{f(i+1)}}$. Thus, since the $\{\nu(B_{k,\xi_{f(i)}})\}_{i=0}^{\infty}$ sequence is increasing monotonically and by Corollary E.1.1, for every $k \in [n]$ there exists a sequence of sets

$$B_{k,\xi_{f(0)}} \subseteq B_{k,\xi_{f(1)}} \subseteq B_{k,\xi_{f(2)}} \subseteq \cdots \tag{E.31}$$

such that for every $i \in \mathbb{N} \cup \{0\}$: $B_{k,\xi_{f(i)}} \subseteq A_k$, $\nu(B_{k,\xi_{f(i)}}) = \xi_i d_{k,\xi_{f(i)}}$, and $B_{k,\xi_{f(i)}} \cap B_{k',\xi_{f(i)}} = \varnothing$ when $k \neq k'$. Let us define

$$B_{k,0} = \bigcup_{i=0}^{\infty} B_{k,\xi_{f(i)}}. \tag{E.32}$$

By our previous work, we know that $m_k = \nu\left(B_{k,0}\right)$. We now need to prove two more things:

1. there exists a set $B_k \subseteq B_{k,0}$ such that $B_k \subseteq A_k$ and $\nu(B_k) = \nu(B_{k,0})$
2. there exists sets $B_{k_1} \subseteq B_{k_1,0}$ and $B_{k_2} \subseteq B_{k_2,0}$ where $B_{k_1} \cap B_{k_2} = \varnothing$, $\nu(B_{k_1}) = \nu(B_{k_1,0})$ and $\nu(B_{k_2}) = \nu(B_{k_2,0})$ when $k_1 \neq k_2$.

If $B_{k,0}$ is not a subset of $A_k$, then $B_{k,0} \cap A_k^c \neq \varnothing$. If $\nu(B_{k,0} \cap A_k^c) > 0$, then there is an $\alpha \in \mathbb{N} \cup \{0\}$ such that $\nu(B_{k,\xi_{f(\alpha)}} \cap A_k^c) > 0$, which contradicts $B_{k,\xi_{f(\alpha)}} \subseteq A_k$. Therefore $\nu(B_{k,0} \cap A_k^c) = 0$. Thus, there exists a set of measure zero $N_k$ such that $\nu(B_{k,0} \setminus N_k) = m_k$ and $B_{k,0} \setminus N_k \subseteq A_k$.

Similarly, if we are given distinct $k_1, k_2 \in [n]$ and $B_{k_1,0} \cap B_{k_2,0} \neq \varnothing$, then if $\nu(B_{k_1,0} \cap B_{k_2,0}) > 0$, then there exists an $\alpha \in \mathbb{N} \cup \{0\}$ such that $\nu(B_{k_1,\xi_{f(\alpha)}} \cap B_{k_2,\xi_{f(\alpha)}}) > 0$. This contradicts $B_{k_1,\xi_{f(\alpha)}} \cap B_{k_2,\xi_{f(\alpha)}} = \varnothing$, thus $\nu(B_{k_1,0} \cap B_{k_2,0}) = 0$. Thus there exists sets of measure zero $E_{k_1}$ and $E_{k_2}$ such that $(B_{k,0} \setminus E_{k_1}) \cap (B_{k',0} \setminus E_{k_2}) = \varnothing$ if $k_1 \neq k_2$.

Therefore, there are disjoint sets $B_k \subseteq A_k$ such that $\nu(B_k) = m_k$ for all $k \in [n]$. Our proof is complete.

$\square$

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