Distribution Agreement

Signature:

______________________________ ______________

Morad Ihab Hassan Date

Mathieu Moonshine: Mock Modular Lifts

By

Morad Ihab Hassan
Master of Science

Mathematics

Dr. John Duncan
Advisor

Dr. David Zureick-Brown
Committee Member

Dr. Gregory J. Martin
Committee Member

Accepted:

Lisa A. Tedesco, Ph.D.
Dean of the James T. Laney School of Graduate Studies

Date

Mathieu Moonshine: Mock Modular Lifts

By

Morad Ihab Hassan

Advisor: John Duncan, PhD

An abstract of
A thesis submitted to the Faculty of the
James T. Laney School of Graduate Studies of Emory University
in partial fulfillment of the requirements for the degree of
Master of Science
in Mathematics
2018

Abstract

Mathieu Moonshine: Mock Modular Lifts
By Morad Ihab Hassan

Classical Moonshine describes the remarkable phenomenon that the coefficients of Hauptmoduln are graded traces for the action of the Monster group, the largest of the 26 sporadic groups, on a graded infinite-dimensional module. A similar phenomena has been shown to hold for other sporadic groups, particularly the Mathieu group M_{24} where instead of Hauptmoduln, the graded traces are shown to be coefficients of mock modular forms. In a recent paper, Ono, Rolen, and Trebat-Leder relate a monstrous moonshine function to one of the Mathieu moonshine functions by constructing products of rational functions of the monstrous Hauptmodul via *generalized Borcherds lifts* on mock modular forms. The lifting procedure is that introduced in a paper by Bruinier-Ono. We conjecture a generalization of this lift for all the mock modular forms of Mathieu moonshine. In particular, our generalization relates Mathieu moonshine mock modular forms to the modular forms of Conway moonshine by evaluating Heegner points corresponding to various congruence subgroups of the modular group. We present data that supports our conjecture.

Mathieu Moonshine: Mock Modular Lifts

By

Morad Ihab Hassan

Advisor: John Duncan, PhD

A thesis submitted to the Faculty of the
James T. Laney School of Graduate Studies of Emory University
in partial fulfillment of the requirements for the degree of
Master of Science
in Mathematics
2018

Acknowledgement

I would like to thank my advisor Professor John Duncan for his support in my producing this thesis. Without his help and guidance, I would have never been able to accomplish what I have. I am also grateful to Professors David Zureick-Brown and Shanshuang Yang who have been there to help me shape my thoughts, both mathematical and otherwise. Finally, thank you Emory University, and specifically, the Math and Computer Science Department for providing me the opportunity to grow in the ways I have and for allowing me to indulge in this incredible pursuit.

Contents

List of Tables

1 Introduction

1.1 Monstrous Moonshine

Monstrous Moonshine refers to the surprising connection between the representation theory of the largest sporadic simple group G, known as the *monster group*, and functions which are invariant under the action of certain subgroups of $SL_2(\mathbb{R})$ known as *modular functions.* For example, taking the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}$ and $\gamma \in \Gamma_0(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : N|c\}$, we define the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \frac{a\tau + b}{c\tau + d}$$

Then, a meromorphic function $f : \mathbb{H} \mapsto \mathbb{C}$ is a modular function for $\Gamma_0(N)$ if $f(\tau) = f(\gamma\tau)$ where $\gamma \in \Gamma_0(N)$. We refer to N as the level of the modular subgroup.

This connection began between the irreducible representations of the monster G and the normalized modular Klein j-function:

$$J(\tau) := j(\tau) - 744 = \frac{(1 + 240\sum_{n=1}^{\infty}\sum_{d|n} d^3 q^n)^3}{q\prod_{n=1}^{\infty}(1-q^n)^{24}} - 744 = \frac{1}{q} + 196884q + 21493760q^2 + \dots \quad (1)$$

where John McKay saw the first coefficient of the Klein j-function's Fourier expansion could be written as a linear combination of the dimensions of the irreducible representations of the G. Then Thompson noticed the same was true with other coefficients as well such as

$$1 + 196883 + 21296876 = 21493760.$$

In Thompson [1979a], Thompson conjectured the existence of a special infinite-dimensional graded representation $V^\natural$ of G for which the dimension of the grade n part was the n-th Fourier coefficient of $J(\tau)$.

This conjecture was further expanded by Thompson, Conway, and Norton in Conway [1979/80], Thompson [1979b] who considered this connection for arbitrary $g \in G$ and conjectured that there existed a representation $V^\natural$ for whose McKay-Thompson series, written,

$$\tilde{T}_g(\tau) := \sum_{n=-1}^{\infty} tr(g|V_n^\natural)q^n, \tag{2}$$

were principal moduli, or Hauptmoduln, for certain genus zero groups commensurable with $SL_2(\mathbb{Z})$. Conway and Norton's conjectures were proved in full by Richard Borcherds Borcherds [1998] in 1992 through developing the theory of vertex operator algebras Borcherds [1986].

1.2 Mathieu Moonshine

A similar phenomena occurs between certain mock modular forms and the Mathieu group M_{24}. The observation made in Eguchi et al. [2011] was given the q-series

$$H(\tau) = 2q^{\frac{-1}{8}}(-1 + 45q + 231q^2 + \ldots) = q^{\frac{-1}{8}}\left(-2 + \sum_{n=1}^{\infty} t_n q^n\right),$$

the first few coefficients $45, 231, 770, \ldots$ are in fact the dimensions of irreducible representations of M_{24}.

We say a holomorphic function $h(\tau)$ on $\mathbb{H}$ is a *weakly holomorphic mock modular form of weight* w for a discrete subgroup Γ of $SL_2(\mathbb{Z})$ if it has at most exponential growth as $\tau \to \alpha$ for any $\alpha \in \mathbb{Q}$, and if there exists a homolorphic modular form $f(\tau)$ of weight $2 - w$ on Γ such that the *completion* $\hat{h}(\tau)$, given by

$$\hat{h}(\tau) = h(\tau) + (4i)^{w-1}\int_{-\overline{\tau}}^{\infty}(z+\tau)^{-w}\overline{f(-\overline{z})}dz, \tag{3}$$

is a (non-holomorphic) modular form of weight w for Γ for some multiplier system ψ. In

Table 1: The cycle shapes and attached *eta-functions* of the 26 conjugacy classes of the sporadic group M_{24}. The naming of conjugacy classes follows the ATLAS convention Conway et al. [1985]

[g]	cycle shape	$\eta_g(\tau)$
1A	1^{24}	$\eta(\tau)^{24}$
2A	$1^8 2^8$	$\eta(\tau)^8\eta(2\tau)^8$
2B	2^{12}	$\eta(2\tau)^{12}$
3A	$1^6 3^6$	$\eta(\tau)^6\eta(3\tau)^6$
3B	3^8	$\eta(3\tau)^8$
4A	$2^4 4^4$	$\eta(2\tau)^4\eta(4\tau)^4$
4B	$1^4 2^2 4^4$	$\eta(\tau)^4\eta(2\tau)^2\eta(4\tau)^4$
4C	4^6	$\eta(4\tau)^6$
5A	$1^4 5^4$	$\eta(\tau)^4\eta(5\tau)^4$
6A	$1^2 2^2 3^2 6^2$	$\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$
6B	6^4	$\eta(6\tau)^4$
7AB	$1^3 7^3$	$\eta(\tau)^3\eta(7\tau)^3$
8A	$1^2 2^1 4^1 8^2$	$\eta(\tau)^2\eta(2\tau)\eta(4\tau)\eta(8\tau)^2$
10A	$2^2 10^2$	$\eta(2\tau)^2\eta(10\tau)^2$
11A	$1^2 11^2$	$\eta(\tau)^2\eta(11\tau)^2$
12A	$2^1 4^1 6^1 12^1$	$\eta(2\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)$
12B	12^2	$\eta(12\tau)^2$
14AB	$1^1 2^1 7^1 14^1$	$\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)$
15AB	$1^1 3^1 5^1 15^1$	$\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)$
21AB	$3^1 21^1$	$\eta(3\tau)\eta(21\tau)$
23AB	$1^1 23^1$	$\eta(\tau)\eta(23\tau)$

this case, the function f is called the *shadow* of the mock modular form h and ψ is called the multiplier system of h. For more detail, see Cheng and Duncan [2012].

This essentially states that a mock modular form can be adjusted by its *shadow* and multiplier system to obtain a modular form but loses its holomorphicity in the process.

It has been shown that the function $H(\tau)$ is a *weakly holomorphic mock modular form of weight* $1/2$ on $SL_2(\mathbb{Z})$ with *shadow* $24\eta(\tau)^3$ Eguchi and Hikami [2009], Dabholkar et al. [2012]. This is very much like the case of Monstrous Moonshine. This occurrence of information being encoded in a function which expressed modular(-like) properties led to a similar exploration of moonshine with the M_{24} group and conjectures for the existence of certain infinite dimensional representations of M_{24} that will behave interestingly for all $g \in M_{24}$.

The McKay-Thompson series for each $g \in M_{24}$ were proposed and are listed in Section 4.1. We arrive to a conjecture similar to that of Conway and Norton: does there exist a $K = \oplus_{n=0}^{\infty} K_n$ module of M_{24} such that the McKay-Thomspon series, given by

$$H_g(\tau) = q^{\frac{-1}{8}} \left(-2 + \sum_{n=1}^{\infty} tr(g|K_{8n-1})q^n \right), \tag{4}$$

also display special mock modular properties? Terry Gannon confirms the existence of such a module in Gannon [2016].

We conclude this subsection by recalling properties of M_{24} which are relevant for our results. The group M_{24} may be characterized as the automorphism group of the unique doubly even self-dual binary code of length 24 with no words of weight 4, also known as the *(extended) binary Golay Code* (see Cheng and Duncan [2012] for more information). This perspectives permits M_{24} a natural permutation representation of degree 24 which we denote R. Via R, we may assign cycle shapes to each of its elements. Any cycle shape arising from an element of M_{24} is of the form $i_1^{l_1} i_2^{l_2} \cdots i_r^{l_r}$ where

$$\sum_{s=1}^{r} l_s i_s = 24,$$

for some $l_s \in \mathbb{N}$ and $1 \leq i_1 < \ldots < i_r \leq 23$ with $r \geq 1$. To each of the elements $g \in M_{24}$, we can attach an *eta-product*, to be denoted η_g, which is the function on the upper half-plane given by

$$\eta_g(\tau) := \prod_s \eta(i_s \tau)^{l_s} \tag{5}$$

where $\prod_s i_s^{l_s}$ is the cycle shape attached to g, and $\eta(\tau)$ is the Dedekind eta function defined as $\eta(\tau) := q^{1/24} \prod_{n>0}(1 - q^n)$ for $q(\tau) := e(\tau)$ where $e(x) := e^{2\pi i x}$.

1.3 Conway Moonshine

We direct the reader's attention to one more significant case of moonshine, Conway Moonshine. Similar to Mosntrous Moonshine, in Conway [1979/80] Conway and Norton also describe an assignment of genus zero groups $\Gamma_g < SL_2(\mathbb{R})$, to elements g of the Conway group, C_{O_0}. In addition to this assignment, several important results display the qualities of Conway moonshine that are analogues to Monstrous moonshine. The first is recalled from Conway [1979/80], Queen [1981].

Theorem 1.1. *For any $g \in C_{O_0}$, regarded as a subgroup of $SO(\alpha)$, the function*

$$t_g := \frac{\eta_g(\tau)}{\eta_g(2\tau)} \tag{6}$$

is a principal modulus for a genus zero group $\Gamma_g < SL_2(\mathbb{R})$ containing some $\Gamma_0(N)$.

Here each η_g is an attachment similar to one described earlier in Mathieu moonshine. In fact, we will soon see that our choice to attach η_g products to M_{24} was partly inspired by Conway moonshine. We note C_{O_0} is not representable as a permutations on 24 elements, so the choice of $\eta_g(\tau)$ is determined by what is referred to in Duncan and Mack-Crane [2015] as the *Frame shape* of g.

Remark 1.2. It is a fact M_{24} is a subgroup of C_{O_0} (which can be remembered by reading the exposition in Duncan and Mack-Crane [2015]). Interestingly, our assignment η_g for some element $g \in M_{24}$ is equal to the assignment of η_g to this very same g as an element of C_{O_0}.

Let T_g^s be defined as

$$T_g^s := t_g(\tau/2) + C_g, \tag{7}$$

where C_g is constant added to normalize the function, or remove the non-vanishing constant term. Then, Duncan and Mack-Crane show in Duncan and Mack-Crane [2015] that a $\frac{1}{2}\mathbb{Z}$-graded infinite-dimensional C_{O_0} module, $V^{s\natural} = \bigoplus_{n\geq 0} V^{s\natural}_{n/2}$, can be constructed which obtains the coefficients of these T_g^s's through the module's McKay-Thompson series. Furthermore, each of these T_g^s's are special, in that $T_g^s(2\tau)$ is the unique normalized principal modulus attached to the genus zero group Γ_g. As we will see, these T_g^s's will be directly involved in our research.

2 Our Conjecture

We propose generalizations of the generalized Borcherds lifts introduced in Bruinier and Ono [2010] and used in Ono et al. [2015] to construct meromorphic modular functions on $\Gamma_0(N_g)$ from certain mock modular forms arising from umbral moonshine. We conjecture an extension of this lift to the special case of Mathieu moonshine mock modulars, and in doing so, we establish a connection between the Mathieu group's mock-modular graded twists and rational products of modular functions arising from Conway moonshine.

To precisely state our generalization, we set up the following notation. Let Δ be a negative *fundamental discriminant* and $r^2 = \Delta \pmod 8$. Let the *order* of g be denoted by $ord(g)$. Note that a *fundamental discriminant* Δ is an integer such that $\Delta = 1 \pmod 4$ and is square-free or $D = 4m$, where $m = 2$ or $3 \pmod 4$ and m is square-free. Then $\Psi_{\Delta,r}$ is a function which takes H_g and produces a modular form, referred to as the *Lift* of H_g and defined in the following way,

$$\Psi_{\Delta,r}(\tau, H_g) := \prod_{n=0}^{\infty} P_{\Delta,r}(q^n, g), \tag{8}$$

for

$$P_{\Delta,r}(x, g) := \prod_{b \in \mathbb{Z}/|\Delta|\mathbb{Z}} \exp\left(\ell_{\Delta,r}\left(e(b/\Delta)x, g\right)\left(\tfrac{\Delta}{b}\right)\right), \tag{9}$$

where

$$\ell_{\Delta,r}(y,g) := \sum_{k=1}^{\infty}(-1)^k \frac{y^k}{k} c_{g^k}(|\Delta|n^2, rn), \tag{10}$$

and $c_g(|\Delta|n^2, rn)$ is defined by

$$c_g(|\Delta|n^2, rn) := \begin{cases} tr(g|K_{|\Delta|n^2}) & \text{if } rn \equiv 1 \bmod 4, \\ -tr(g|K_{|\Delta|n^2}) & \text{if } rn \equiv -1 \bmod 4, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

This notation is lengthy, yet as we will see, is rather simple to follow in practice. Before continuing, We provide an example calculation for $\Psi_{\Delta,r}$ for an arbitrary (Δ, r).

Example 2.1. Let $[g] = 1A$, then we have $ord(g) = 1$. We have

$$\begin{aligned} \ell_{\Delta,r}(y,e) :=& \sum_{k=1}^{\infty}(-1)^k \frac{y^k}{k} c_{g^k}(|\Delta|n^2, rn) = \sum_{k=1}^{\infty}(-1)^k \frac{y^k}{k} c_e(|\Delta|n^2, rn) \\ =& c_e(|\Delta|n^2, rn)\log(1-y), \end{aligned}$$

and

$$\begin{aligned} P_{\Delta,r}(x,e) :=& \prod_{b \in \mathbb{Z}/|\Delta|\mathbb{Z}} \exp\left(\ell_{\Delta,r}\left(e(b/\Delta)x, e\right)\left(\tfrac{\Delta}{b}\right)\right) \\ =& \prod_{b \in \mathbb{Z}/|\Delta|\mathbb{Z}} (1 - e(b/\Delta)x)^{c_e(|\Delta|n^2, rn)\left(\frac{\Delta}{b}\right)}. \end{aligned}$$

Thus,

$$\Psi_{\Delta,r}(\tau, H_e) := \prod_{n=0}^{\infty} P_{\Delta,r}(q^n, e)$$

$$= \prod_{n=0}^{\infty} \prod_{b \in \mathbb{Z}/|\Delta|\mathbb{Z}} (1 - e(b/\Delta)q^n)^{c_e(|\Delta|n^2, rn)\left(\frac{\Delta}{b}\right)}$$

Now given a conjugacy class $[g]$, we define T_g in the following way,

$$T_g := \frac{\eta_g(\tau)}{\eta_g(2\tau)} + \chi(g), \tag{12}$$

where $\chi(g)$ is a constant added to normalize the rational form which can be found in Table 5. By defining T_g in this way, it is a meromorphic modular form on some discrete subgroup commensurable with $SL_2(\mathbb{Z})$ Duncan and Mack-Crane [2015]. We call a *Heegner point* a complex number the form $\alpha = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$ with $(a, b, c) = 1$, $N_g|a$ for $\Gamma_0(N_g)$, and $\Delta = b^2 - 4ac$ where Δ is a fundamental discriminant. Finally, we call a pair (Δ, r) *admissible* for g if Δ is a negative fundamental discriminant and $r^2 = \Delta \pmod{4N_g}$. Now, we can present our conjecture:

Conjecture 2.2. *Given an admissible pair (Δ, r) for conjugacy class $[g]$, then $\Psi_{\Delta,r}(\tau, H_g)$ satisfies the following:*

$$\Psi_{\Delta,r}(\tau, H_g) = \prod_{i=1}^{m} \frac{(T_g(\tau) - T_g(\alpha_i))^{\gamma_i}}{(T_g(\tau) - \overline{T_g(\alpha_i)})^{\gamma_i}}, \tag{13}$$

where each α_i is a satisfactory Heegner Point.

As a result of the conjecture, we have the following corollary.

Corollary 2.3. *The twisted graded trace satisfies the equality*

$$\frac{1}{\epsilon_\Delta} \sum_{i=1}^{m} \gamma_i (T_g(\alpha_i) - \overline{T_g(\alpha_i)}) = tr(g|K_\Delta), \tag{14}$$

with

$$\epsilon_\Delta = \sum_{b \in \mathbb{Z}/|\Delta|\mathbb{Z}} e(b/\Delta) \cdot \left(\frac{\Delta}{b}\right) \tag{15}$$

where α_i *is the same Heegner point and* γ_i *is some integer.*

Remark 2.4. This conjecture is inspired by an example calculation in Ono et al. [2015] for $[g] = 1A$ which utilizes the generalized Borcherds products formed in Bruinier and Ono [2010] to lift a weak harmonic Maass form of weight $1/2$ to a meromorphic modular functions. We present an example calculation for $[g] = 2A$.

Example 2.5. Let $[g] =$ 2A. Then the corresponding modular function is

$$T_g = \frac{\eta(\tau)^8}{\eta(4\tau)^8} + 8.$$

Picking admissible pair $(\Delta, r) = (-7, 1)$, then

$$\begin{aligned}\Psi_{-7,1}(\tau, H_g) =& \frac{(T_g(\tau) - T_g(\alpha))^2}{(T_g(\tau) - \overline{T_g(\alpha)})^2} \\ =& 1 + 6\sqrt{-7}q + (126 + 45\sqrt{-7})q^2 + \cdots ,\end{aligned}$$

where $\alpha = \frac{3+\sqrt{-7}}{8}$.

We check Corollary 2.3,

$$\epsilon_\Delta = \sum_{b \in \mathbb{Z}/|-7|\mathbb{Z}} e(-b/7) \left(\frac{-7}{b}\right) = -\sqrt{-7}$$

and

$$\frac{1}{\epsilon_\Delta} \sum_i \gamma_i T_g(\alpha_i) = 6 = c_g(7, 1) = tr(g|K_{7/8}).$$

We provide another example here with a different fundamental discriminant $\Delta = -15$.

The case when $[g] = 1\text{A}$ and $(\Delta, r) = (-15, 1)$ was already done in Ono et al. [2015], so again we proceed to the next case where $[g] = 2\text{A}$.

Example 2.6. Let $[g] = 2\text{A}$ and $(\Delta, r) = (-15, 1)$. Then

$$\Psi_{-15,1}(\tau, H_g) = \frac{(T_g(\tau) - T_g(\alpha_1))^2 (T_g(\tau) - T_g(\alpha_2))^2}{(T_g(\tau) - \overline{T_g(\alpha_1))^2} (T_g(\tau) - \overline{T_g(\alpha_2))^2}}$$

$$= 1 + 14\sqrt{-15}q + (-1470 + 231\sqrt{-15})q^2 + \cdots,$$

where $\alpha_1 = \frac{-1+\sqrt{15}}{8}$ and $\alpha_2 = \frac{7+\sqrt{-15}}{16}$.

We get

$$\epsilon_\Delta = \sqrt{-15},$$

and

$$\frac{1}{\epsilon_\Delta} \sum_i \gamma_i T_g(\alpha_i) = 14 = c_g\,(15, 1) = tr(g|K_{1,15/8}).$$

Remark 2.7. Our ability to follow through with these computations relies on computing satisfactory Heegner points. Thus, given $[g]$, our conjecture only concerns pairs (Δ, r) for which Heegner points exist. For example, no Heegner points exist for $15A$ given the admissible pair $(-7, 1)$, yet they do exist for this conjugacy class when we set the admissible pair to $(-15, 1)$ (see Tables 3 and 4).

Remark 2.8. There does not seem to be any conjugacy class for which there does not exist Heegner points for all admissible pairs. Given the data presented in Tables 3 and 4, the most likely candidate for such a choice is $21A$. But even that class will admit a quadratic representative for the pair $(-47, 1)$ and, as a result, admit some collection of Heegner points.

2.1 Calculating Heegner Points

Using Theorem 2.9 and given an admissible pair, we can calculate the quadratic forms which contain Heegner points as its roots.

Consider integral binary quadratic forms $[a, b, c](x, y) = ax^2 + bxy + cy^2$. Then $\Gamma_0(N)$ acts on such forms as follows:

$$[a, b, c] \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (x, y) = [a, b, c](\alpha x + \beta y, \gamma x + \delta y).$$

This action preserves the discriminant $\Delta = b^2 - 4ac$ and the greatest common divisor (a, b, c). We denote $\mathcal{Q}_\Delta$ and $\mathcal{Q}^0_\Delta$ to be the set of all quadratic forms of discriminant Δ and the subset of primitive forms, respectively. When defining a classification with respect to the subgroup $\Gamma_0(N)$, more invariants arise. Specifically, under the action by $\Gamma_0(N)$, the value of b modulo $2N$ which we shall denote as r remains unchanged and we always have $gcd(a, N) = N$. Then, for an integer r modulo $2N$ such that $\Delta = r^2 \pmod{4N}$, we set

$$\mathcal{Q}_{N,\Delta,r} = \{[a, b, c] \in \mathcal{Q}_\Delta : N|a, b = r \pmod{2N}\}$$

Then the subset of primitive forms, $\mathcal{Q}^0_{N,\Delta,r}$ is $\Gamma_0(N)$ invariant. We denote the quadratic forms from here on as $[aN, b, c]$.

We define $m := \left(N, r, \frac{r^2-\Delta}{4N}\right)$. For $[aN, b, c] \in \mathcal{Q}^0_{N,\Delta,r}$, define $m_1 := (N, b, a)$, $m_2 := (N, b, c)$, which are coprime and have product m. Then, the following is true Gross et al. [1987]:

Theorem 2.9. *There is a 1:1 correspondence between $\Gamma_0(N)$ equivalence classes of the forms $[aN, b, c] \in \mathcal{Q}^0_{N,\Delta,r}$ satisfying $(N, b, c) = m_1$, $(N, a, b) = m_2$ and the $SL_2(\mathbb{Z})$ equivalence classes of the forms in $\mathcal{Q}^0_{\Delta,r}$ given by $\mathcal{Q} = [aN, b, c] \mapsto \mathcal{Q}' = [aN_1, b, cN_2]$, where N_1N_2 is any decomposition of N into coprime positive factors satisfying $(m_1, N_2) = (m_2, N_1) = 1$. In*

$[g]_{M_{24}}$	Γ_g	$[g]_{C_{O_0}}$	Frame Shape
1A	$2-$	1A	1^{24}
2A	$4-$	2B	$1^8 2^8$
2B	$4\|2-$	2D	2^{12}
3A	$6+3$	3B	$1^6 3^6$
3B	$6\|3$	3D	3^8
4A	$8\|2-$	4G	$2^4 4^4$
4B	$8-$	4E	$1^4 2^2 4^4$
4C	$8\|4-$	4H	4^6
5A	$10+5$	5B	$1^4 5^4$
6A	$12+3$	6K	$1^2 2^2 3^2 6^2$
6B	$12\|6-$	6P	6^4
7AB	$14+7$	7B	$1^3 7^3$
8A	$16-$	8G	$1^2 2^1 4^1 8^2$
10A	$20\|2+5$	10J	$2^2 10^2$
11A	$22+11$	11A	$1^2 11^2$
12A	$24\|2+3$	12P	$2^1 4^1 6^1 12^1$
12B	$24\|12-$	12S	12^2
14AB	$28+7$	14C	$1^1 2^1 7^1 14^1$
15AB	$30+5,5,15$	15D	$1^1 3^1 5^1 15^1$
21AB	$42\|3+7$	21C	$3^1 21^1$
23AB	$46+23$	23AB	$1^1 23^1$

Table 2: Shows the modular subgroups corresponding to each $g \in M_{24}$. Note that since $M_{24} < C_{O_0}$, then we are able to look at Duncan and Mack-Crane [2015] to determine the appropriate modular subgroup. The modular subgroups are meant to be read $\Gamma_g(N_g)$ where the column below Γ_g lists the level N_g. Definitions for each of the levels can be found in Duncan and Mack-Crane [2015].

particular, $|\mathcal{Q}^0_{N,\Delta,r}/\Gamma_0(N)| = 2^v|\mathcal{Q}^0_{N,\Delta,r}/SL_2(\mathbb{Z})|$, *where* v *is the number of prime factors of* m.

Note that $2^v|\mathcal{Q}^0_{N,\Delta,r}/SL_2(\mathbb{Z})| = 2h(\Delta)$ for $\Delta < 0$, where $h(\Delta)$ is the class number of $\mathbb{Q}(\sqrt{\Delta})$ the factor 2 arises because $\mathcal{Q}^0_\Delta$ also contains negative semi-definite forms.

Thus, we obtain a simple method for constructing the quadratic forms that hold the desired Heegner points. Given Δ, we solve for r satisfying the condition $\Delta \equiv_{4N} r^2$. Then we calculate b such that $b \equiv_{2N} r$, and all $(Na \cdot c)$ such that $\frac{b^2-\Delta}{4} = aN \cdot c$. To reconstruct a polynomial given $[g], \Delta$, we can look at the tables below. We can select any pair, but we'd need select a, c such that the conditions defined above are satisfied.

Remark 2.10. By Duncan and Mack-Crane [2015], there is a Frame shape for each $g \in C_{O_0}$ which can be expressed in a fashion similar to the cycle shapes from the permutation representation of elements in M_{24}. For each frame shape, the same T_g function is defined and attached to each $g \in C_{O_0}$. Since M_{24} is a subgroup of C_{O_0}, we can naturally associate each frame shape to each $g \in M_{24}$ which in turn enables us to attach a T_g to each $g \in M_{24}$. It is proven in Duncan and Mack-Crane [2015] that each of these T_g are principal modulus for a genus zero group $\Gamma_g < SL_2(\mathbb{Z})$. For this reason, we can tell the level of each of these subgroups by referring to Table 2.

Example 2.11. Let $[g] = 2A$ where $\Delta = -7$, we look to Table 3 below for viable representations on $\Gamma_0(4)$: $Q_{\pm 1} = 4x^2 \pm 3x + 1, Q_{\pm 2} = 8x^2 \pm 5x + 1, Q_{\pm 3} = 4x^2 \pm 5x + 2$. There are, however, only 2 quadratic forms for $[2A]$ up to equivalence since $m = 1$ and so $|\mathcal{Q}^0_{4,-7,1}/\Gamma_0(4)| = 2$. Extracting the roots with positive imaginary components from these polynomials and inputting them into $T_{2A}(\tau)$ determines the uniqueness of the polynomial. That is, only inequivalent representatives will provide roots that produce distinctive outputs. A quick calculation will show that $Q_{\pm 1}$ may serve as our polynomial representatives for [2A] and provides us with Heegner points $\alpha_{\pm} = \frac{\pm 3 + \sqrt{-7}}{8}$ which we used in our earlier example.

Remark 2.12. Not all conjugacy classes will necessarily have a quadratic representatives given Δ and N. If there is no such r, then $\mathcal{Q}_{N,\Delta,r}$ is empty. For example, when $[g] = 3A$ and $(\Delta, r) = (-7, 1)$, there are no quadratic representatives.

Remark 2.13. We only calculate the Heegner points for conjugacy classes whose cycle shapes fixed one point. That is, whenever the cycle shape contains a $(1)^l$ for some positive integer l. We do this because the modular subgroups are considerably simpler as can be seen above in Table 2.

Remark 2.14. To verify the choice of polynomial, it has always been fruitful to check Corollary 3.3 with the provided Heegner points.

Table 3: Quadratic representatives given $\Delta = -7$

$[g]_{M_{24}}$	$[r_{-7}]$	$\{(b, aN \cdot c)_{-7}\}$
1A	$[1,3,5,7]$	$\{(1,2),(3,4)\}$
2A	$[3,5,11,13]$	$\{(3,4),(5,8)\}$
2B		
3A	-	-
3B	-	-
4A	$[5,11,21,27]$	$\{(5,8),(11,32)\}$
4B		
4C		
5A	-	-
6A	-	-
6B	-	-
7AB	$[7,21,35,49]$	$\{(7,14),(21,112)\}$
8A	$[11,21,43,53]$	$\{(11,32),(21,112)\}$
10A	-	-
11A	$[9,13,31,35,53,57,75,79]$	$\{(9,22),(13,44),(31,242),(35,308)\}$
12A	-	-
12B	-	-
14AB	$[21,35,77,91]$	$\{(21,112),(35,308)\}$
15AB	-	-
21AB	-	-
23AB	$[19,27,65,73,111,119,157,165]$	$\{(19,92),(27,184),(65,1058),(73,1334)\}$

Table 4: Quadratic representatives given $\Delta = -15$

$[g]_{M_{24}}$	$[r_{-15}]$	$\{(b, aN \cdot c)\}$
1A	$[1,3,5,7]$	$\{(1,4),(3,6)\}$
2A	$[1,7,9,15]$	$\{(1,4),(7,16)\}$
2B		
3A	$[3,9,15,21]$	$\{(3,6),(9,24)\}$
3B		
4A	$[7,9,23,25]$	$\{(7,16),(9,24)\}$
4B		
4C		
5A	$[5,15,25,35]$	$\{(5,10),(15,60)\}$
6A	[9,15,33,39]	$\{(9,24),(15,60)\}$
6B		
7AB	-	-
8A	$[7,25,39,57]$	$\{(7,16),(25,160)\}$
10A	-	-
11A	-	-
12A	$[9,39,57,87]$	$\{(9,24),(39,384)\}$
12B		
14AB	-	-
15AB	$[15,45,75,105]$	$\{(15,60),(45,510)\}$
21AB	-	-
23AB	$[13,33,59,79,105,125,151,171]$	$\{(13,46),(33,276),(59,874),(79,1564)\}$

3 Generalized Borcherds Products

Borcherds Products or *Borcherds Lifts* is a method to construct meromorphic modular forms on the special orthogonal group, $SO^+(2, n^-)$ from weakly holomorphic modular forms on $SL_2(\mathbb{Z})$. That is, given a weakly holomorphic modular form f, there is an associated Ψ_f which is a convergent infinite product of complex valued functions which display modular properties and are holormorphic except for poles. A succinct, yet revealing account of Borcherds Products can be found in Hill [2012].

The *Generalized Borcherds Products* are defined in terms of vector valued weak harmonic Maass forms in Bruinier and Ono [2010] and are used to to prove a special case of our conjecture for $[g] = 1A$ in Ono et al. [2015]. Its purpose is to take a *harmonic weak maass forms* of weight 1/2 and produce a meromorphic modular form. Following the examples of the papers previously mentioned, we will also provide the background on vector-valued modular forms and follow closely the results presented in Ono et al. [2015], Bruinier and Ono [2010], but in less generality as to only satisfy the lift for H_e. See Bruinier and Ono [2010] for more general levels and functional equations.

3.1 A Lattice Related to $\Gamma_0(2)$

We define a lattice L and a dual lattice L' related to $\Gamma_0(2)$ such that the components of our vector-valued modular forms will be labeled by the elements of L'/L Ono et al. [2015], Bruinier and Ono [2010]. This lattice L should be even. We consider the quadratic space

$$V := \{X \in Mat_2(\mathbb{Q}) : tr(X) = 0\}$$

with the quadratic form $P(X) := 2det(X)$. The, the corresponding bilinear form is $(X, Y) := -2tr(XY)$. Let L be the lattice

$$L := \left\{ \begin{pmatrix} b & -a/2 \\ c & -b \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

The dual lattice is then given by

$$L := \left\{ \begin{pmatrix} b/4 & -a/2 \\ c & -b/4 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

It will be helpful to view elements of L' as both matrices and as quadratic forms, with the matrix

$$X = \begin{pmatrix} b/4 & -a/2 \\ c & -b/4 \end{pmatrix}$$

corresponding to the integral binary quadratic form

$$Q = [2c, b, a] = 2cx^2 + bxy + ay^2.$$

Note that $P(X) = -Disc(Q)/8 = -(b^2 - 8ac)/8$. We refer to the finite abelian group L'/L as a *discriminant group* and as the *discriminant form* of the lattice L when L'/L has a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form induced by P. We can identify L'/L with $\frac{1}{4}\mathbb{Z}/\mathbb{Z}$ by defining the isomorphism $\phi : L'/L \mapsto \frac{1}{4}\mathbb{Z}/\mathbb{Z}$ where

$$\phi\left(\begin{pmatrix} b/4 & -a/2 \\ c & -b/4 \end{pmatrix} + L\right) = b/4 + \mathbb{Z}.$$

The quadratic form P with the quadratic form $\frac{h}{4} \to \frac{-h^2}{8}$ on $\mathbb{Q}/\mathbb{Z}$. We will also occasionally identify $\frac{h}{4} \in \mathbb{Q}/\mathbb{Z}$ with $h \in \mathbb{Z}/4\mathbb{Z}$. For a fundamental discriminant D and $r/4 \in L'/L$ with

$r^2 = D$ $(mod\ 8)$, let

$$Q_{D,r} := \{Q = [2c, b, a] : a, b, c \in \mathbb{Z}, Disc(Q) = D, b = r \pmod 4\}.$$

The action of $\Gamma_0(2)$ on this set is given by the usual action of congruence subgroups on binary quadratic forms: Write integral binary quadratic forms $[a, b, c](x, y) := ax^2+bxy+cy^2$. Then $\Gamma_0(N)$ acts on such forms as follows

$$[a, b, c] \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (x, y) = [a, b, c](\alpha x + \beta y, \gamma x + \delta y).$$

3.2 Weil Representation

We write $Mp_2(\mathbb{R})$ for the metaplectic two-fold cover of $SL_2(\mathbb{R})$. The elements of this group are pairs $(M, \phi(\tau))$, where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $\phi : \mathbb{H} \mapsto \mathbb{C}$ is a holomorphic function such that $\phi(\tau)^2 = c\tau + d$. The multiplication is defined by

$$(M, \phi(\tau))(M', \phi'(\tau)) = (MM', \phi(M'\tau)\phi'(\tau)).$$

We denote the integral metaplectic group, the inverse image of $\Gamma : SL_2(\mathbb{Z})$ under the covering map, by $\widetilde{\Gamma} := Mp_2(\mathbb{Z})$.

The generators of $\widetilde{\Gamma}$ are $T := ((\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}), 1)$ and $S := ((\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}), \sqrt{\tau})$. Here, $\sqrt{\tau} = \sqrt{|\tau|}e^{\arg(\tau)i}$ for $-\pi < \arg(\tau) < \pi$ where $|\cdot|$ is the complex modulus.

The Weil Representation associated with a discriminant form L'/L is a representation of $\widetilde{\Gamma}$ on the group algebra $\mathbb{C}[L'/L]$. We denote the standard basis elements of $\mathbb{C}[L'/L]$ by ν_h, $h \in L'/L$, and write $\langle \cdot, \cdot \rangle$ for the standard scalar product such that $\langle \nu_h, \nu_{h'} \rangle = \delta_{h,h'}$. The Weil representation ρ_L associated with the discriminant form L'/L is the unitary representation of $\widetilde{\Gamma}$ on $\mathbb{C}[L'/L]$ defined by

$$\rho_L(T)(\nu_h) := e(h^2/2)\nu_h$$

and

$$\rho_L(S)(\nu_h) := \frac{e((b^- - b^+)/8)}{\sqrt{|L'/L|}} \sum_{h' \in L'/L} e(-(h,h'))\nu_{h'},$$

where (b^+, b^-) is the signature of the vector space V defined earlier. That is, b^+ is the dimension of a maximal positive definite subspace of V and b^- is the dimension of a maximal negative definite subspace Here $e(x) := e^{2\pi i x}$.

3.3 Vector Valued Modular Forms

If $f : \mathbb{H} \mapsto \mathbb{C}[L'/L]$ is a function, then we can write $f = \sum_{h \in L'/L} f_h \nu_h$ for its decomposition into components with respect to the standard basis of $\mathbb{C}[L'/L]$. Let $k \in \frac{1}{2}\mathbb{Z}$, and let $M^!_{k,\rho_L}$ denote the space of $\mathbb{C}[L'/L]$-valued weakly holomorphic modular forms. Modular forms are denoted by M_{k,ρ_L} and cusp forms are denoted by S_{k,ρ_L}. Cusp forms are modular forms which vanish at cusps.

Assume$k \leq 1$, and a twice continuously differentiable function $f : \mathbb{H} \mapsto \mathbb{C}[L'/L]$ is called a *harmonic weak Maass form* (of weight k with respect to $\widetilde{\Gamma}$ and ρ_L) if it satisfies:

- $f(\gamma\tau) = \phi(\tau)^{2k}\rho_L(\gamma,\phi)f(\tau)$ for all $(\gamma,\phi) \in \widetilde{\Gamma}$,
- there is a $C > 0$ such that $f(\tau) = O(e^{Cw})$ as $w \mapsto \infty$,
- $\Delta_k f = 0$,

where

$$\Delta_k := -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial u}\right)$$

is the weight k hyperbolic Laplace operator.

We denote the vector space of these harmonic weak Maass forms by $\mathcal{H}_{k,\rho_L}$. Moreover, for $f \in \mathcal{H}_{k,\rho_L}$ we have the associated Fourier expansion

$$f(\tau) = P_f(\tau) = \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Z}+Q(h) \\ -\infty \ll n \leq 0}} c^+(n,h) e(n\tau) \nu_h$$

and some $\epsilon > 0$. Here, P_f is uniquely determined by f and is called the *principal part* of f. The Fourier expansion of any $f \in H_{k,\rho_L}$ gives a unique decomposition $f = f^+ + f^-$, where

$$f^+(\tau) = \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c^+(n,h) e(n\tau) \nu_h,$$

$$f^-(\tau) = \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n < 0}} c^-(n,h) W(2\pi n w)(e(n\tau) \nu_h,$$

and $W(x) = W_k(x) := \int_{-2x}^{\infty} e^{-t} t^{-k} dt = \Gamma(1-k, 2|x|)$ for $x < 0$.

Note that there is an antilinear differential operator defined by

$$\xi_k : H_{k,\rho_L} \mapsto S_{2-k,\overline{\rho_L}}, f(\tau) \mapsto \xi_k(f)(\tau) := 2iy^k \frac{\overline{\partial}}{\partial \overline{\tau}},$$

where $\overline{\rho_L}$ is the complex conjugate representation. This will be important in the following subsection. The Fourier expansion of $\xi_k(f)$ is given by

$$\xi_k(f) = -\sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n > 0}} (4\pi n)^{1-k} \overline{c^-(-n,h)} e(n\tau) \nu_h$$

The kernel of ξ_k is equal to $M^!_{k,\rho_L}$, and we have the following sequence:

$$0 \mapsto M^!_{k,\overline{\rho_L}} \mapsto H_{k,\overline{\rho_L}} \mapsto S_{2-k,\rho_L} \mapsto 0.$$

We call $\xi_k(f)$ the *shadow* of f. Note that $\xi_k(f)$ uniquely determines f^-, but f^+ is only determined up to addition of a weakly holomorphic modular form.

3.4 Lifting H_e

Here, we use all the earlier definitions and results to display the proof presented in Ono et al. [2015] which shows that H_e's generalized Borcherds Lift is a meromorphic modular form of the type stated earlier in the conjecture. We define a *mock modular form* of weight k to be the holomorphic part of some harmonic weak Maass form of weight k. The following will involve Jacobi forms and the reader may read 4.2 for review on Jacobi forms and definitions of $\varphi_1(\tau,z), \mu_{1,0}(\tau,z), \theta_r(\tau,z)$, and $R(u,\tau)$. We define the Jacobi form

$$\psi(\tau,z) := 2\varphi_1(\tau,z)\mu^g_{1,0}(\tau,z).$$

We can break up ψ into a finite part ψ_F and a polar part ψ_P. The polar part is given by

$$\psi_P(\tau,z) = 24\mu_{2,0}(\tau,z).$$

Then the mock modular form H_e is defined by $H_e := H_{e,1}$ where

$$\psi_F(\tau,z) = \psi(\tau,z) - \psi_P(\tau,z) = \sum_{h\in\mathbb{Z}/4\mathbb{Z}} H_{e,h}(\tau)\theta_{2,h}(\tau,z),$$

where

$$\theta_{2,h}(\tau,z) := \sum_{n=h \pmod 4} q^{n^2/8}y^k.$$

Note that ψ satisfies an optimal growth condition,

$$q^{1/8}H_{e,h}(\tau) = O(1)$$

as $\tau \to i\infty$ for all $h \in \mathbb{Z}/4\mathbb{Z}$.

We also define the shadow $S_e(\tau)$, the non-holomorphic part $F_{e,h}(\tau)$, and the harmonic weak Maass form $\hat{H}_e(\tau)$ corresponding to the mock modular form H_e via their components:

$$\begin{aligned}
S_{e,h}(\tau) :=& \sum_{n=h \pmod 4} nq^{n^2/8}, \\
F_{e,h}(\tau) :=& \int_{-\bar{\tau}}^{i\infty} \frac{S_h(z)}{\sqrt{-i(z+\tau)}} dz \\
=& -4q^{-(h-2)^2/8} R\left(\frac{h-2}{4}(4\tau) + \frac{1}{2}, 4\tau\right), \text{ and} \\
\hat{H}_{e,h}(\tau) :=& H_{e,h}(\tau) + F_{e,h}(\tau).
\end{aligned}$$

Note that by definition, $S_{e,h}(\tau) = S_{e,-h}(\tau)$. Therefore, $S_{e,0} = S_{e,2} = 0$. The same is true of $H_{e,h}$. We can write this in terms of Shimura's theta functions as $S_{e,h}(\tau) = \theta(\tau; h, 4, 4, x)$ by Shimura [1973]. Then using the transformation laws for his θ-functions, we get that $S_{e,h}$ transforms as follows:

$$\begin{aligned}
S_{e,h}(\tau)(\tau+1) =& e(h^2/8) S_{e,h}(\tau), \text{ and} \\
S_{e,h}(-1/\tau) =& \tau^{3/2} \frac{e(-1/8)}{\sqrt{4}} \sum_{k \pmod 4} e(kh/4) S_{e,k}(\tau).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
S_e(\tau+1) =& \rho_L(T) S_e(\tau), \text{ and} \\
S_e(-1/\tau) =& \tau^{3/2} \rho_L(S) S_e(\tau).
\end{aligned}$$

From these transformations, we see that $S_e(\tau) : \mathbb{H} \mapsto \mathbb{C}[L'/L]$ is a weight 3/2 vector-valued modular form transforming under the Weil representation ρ_L, i.e an element of the space $M_{3/2,\rho_L}$. From Cheng et al. [2014], we know that H_e is a mock modular form with shadow S_e. This gives the following theorem:

Theorem 3.1. *We have that $\hat{H}_e(\tau) : \mathbb{H} \mapsto \mathbb{C}[L'/L]$ is a weight $1/2$ vector valued harmonic weak Maass form transforming under the restriction of the Weil representation $\overline{\rho_L}$ to the preimage of $\Gamma_0(2)$ in $\tilde{\Gamma}$. Moreover, it has shadow $S_e(\tau)$, non-holomorphic part F_e, and principal part $P_{\hat{H}}(\tau) = -2q^{-1/8}(\nu_1 - \nu_3)$.*

3.5 Generalized Borcherds Products

Now that we are able to rewrite the mock modular form H_e as a harmonic weak Maass form, we can introduce the application of the generalized Borcherds Lifts defined in Bruinier and Ono [2010] to obtain a meromorphic modular form. Recalling the result from Ono et al. [2015], we call a pair (Δ, r) admissible if Δ is a negative fundamental discriminant and $r^2 = \Delta \pmod 8$. We also let $e(a) := e^{2\pi i a}$.

Theorem 3.2. *Let $c(n, r)$ be the n-th Fourier coefficient of the r-th component of H_e or $tr(e|K_{n,r})$. Let (Δ, r) be an admissible pair. Then the twisted generalized Borcherds product is*

$$\Psi_{\Delta,r}(\tau, H_e) := \prod_{n=1}^{\infty} P_\Delta(q^n)^{c_e\left(\Delta n^2, rn\right)}$$

where

$$P_\Delta(x) := \prod_{b \in \mathbb{Z}/|\Delta|\mathbb{Z}} \left[1 - e\left(b/\Delta\right) x\right]^{\left(\frac{\Delta}{b}\right)}$$

is a rational function in $T_e(\tau)$ with a discriminant Δ Heegner divisor.

Corollary 3.3. *By the preceding theorem,*

$$\Psi_{\Delta,r}(\tau, H_e) := \prod_i (T_e(\tau) - T_e(\alpha_i))^{\gamma_i}$$

for some discriminant Δ *and Heegner points* α_i*. Thus, we have*

$$c_e\left(\Delta n^2, rn\right) = \frac{1}{\epsilon_\Delta} \sum_i \gamma_i T_e(\alpha_i)$$

where

$$\epsilon_\Delta = \sum_{b \in \mathbb{Z}/|\Delta|\mathbb{Z}} e(b/\Delta) \left(\frac{\Delta}{b}\right)$$

Remark 3.4. *Generalized Borcherds Products* are the prime result used in this paper. In that, we propose a generalization of the Borcherds Lift applied here to the case of other H_g mock modular functions dependent on the conjugacy class of g.

4 Definitions Modular Forms, Jacobi Forms, Theta Functions, etc.

We conclude this paper with an appendix of the relevant functions involved in our computations. First beginning with the mock-modular McKay-Thompson series corresponding with Mathieu Moonshine. We define the following:

$$\Lambda_M(\tau) := Mq\frac{d}{dq}\left(log\frac{\eta(M\tau)}{\eta(\tau)}\right) = \frac{M(M-1)}{24} + M\sum_{k>0}\sum_{d|k} d\left(q^k - Mq^{Mk}\right) \tag{16}$$

which is a modular form of weight two for $\Gamma_0(N)$ if $M|N$.

[g]	$\chi(g)$	$\tilde{T}_g(\tau)$
$1A$	24	0
$2A$	8	$16\Lambda_2$
$2B$	0	$-24\Lambda_2 + 8\Lambda_4 = 2\eta(\tau)^8/\eta(2\tau)^4$
$3A$	6	$6\Lambda_3$
$3B$	0	$2\eta(\tau)^6/\eta(3\tau)^2$
$4A$	0	$4\Lambda_2 - 6\Lambda_4 + 2\Lambda_8 = 2\eta(2\tau)^8/\eta(4\tau)^4$
$4B$	4	$4(-\Lambda_2 + \Lambda_4)$
$4C$	0	$2\eta(\tau)^4\eta(2\tau)^2/\eta(4\tau)^2$
$5A$	4	$2\Lambda_5$
$6A$	2	$2(-\Lambda_2 - \Lambda_3 + \Lambda_6)$
$6B$	0	$2\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2/\eta(6\tau)^2$
$7AB$	3	Λ_7
$8A$	2	$-\Lambda_4 + \Lambda_8$
$10A$	0	$2\eta(\tau)^3\eta(2\tau)\eta(5\tau)/\eta(10\tau)$
$11A$	2	$2(\Lambda_{11} - 11\eta(\tau)^2\eta(11\tau)^2)/5$
$12A$	0	$2\eta(\tau)^3\eta(4\tau)^2\eta(6\tau)^3/\eta(2\tau)\eta(12\tau)$
$12B$	0	$2\eta(\tau)^4\eta(4\tau)^2\eta(6\tau)^3/\eta(2\tau)\eta(3\tau)\eta(12\tau)^2$
$14AB$	1	$(-\Lambda_2 - \Lambda_7 + \Lambda_{14} - 14\eta(\eta)\eta(2\tau)\eta(7\tau)\eta(14\tau))/3$
$15AB$	1	$(-\Lambda_3 - \Lambda_5 + \Lambda_{15} - 15\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau))/4$
$21AB$	0	$(7\eta(\tau)^3\eta(7\tau)^3/\eta(3\tau)\eta(21\tau) - \eta(\tau)^6/\eta(3\tau)^2)/3$
$23AB$	1	$(\Lambda_{23} - 23f_{23,1} + 23f_{23,3})/11$

Table 5: In this table, we list the relevant functions utilized in defining the various weight 1/2 mock modular forms of H_g described in Preposition 4.1. Here $f_{23,1}$ and $f_{23,3}$ are defined in Duncan et al. [2015]

4.1 Mathieu (Mock) Modular Functions

For M_{24}, we have the following candidates introduced mostly in Cheng [2010], Gaberdiel et al. [2010a,b], Eguchi and Hikami [2011] for the associated McKay-Thompson series H_g. From them, we have the following result:

Proposition 4.1. *Let $H : \mathbb{H} \mapsto \mathbb{C}$ given by*

$$H(\tau) = \frac{-2E_2(\tau) + 48F_2(\tau)}{\eta(\tau)^3} = 2q^{-\frac{1}{8}}(-1 + 45q + 231q^2 + \ldots) \tag{17}$$

where

$$F_2(\tau) = \sum_{\substack{r>s>0 \\ r-s=1 \ mod\ 2}} (-1)^r s q^{rs/2} = q + q^2 - q^3 + q^4 + \ldots. \tag{18}$$

and

$$E_2 = 1 - 24 \sum_{n>0} \sum_{k|n} q^n \tag{19}$$

Then for all $g \in M_{24}$, the function

$$H_g(\tau) = \frac{\chi(g)}{24} H(\tau) - \frac{\tilde{T}_g(\tau)}{\eta(\tau)^3} \tag{20}$$

is a mock modular form for $\Gamma_0(N_g)$ of weight $1/2$ with shadow $\chi(g)\eta(\tau)^3$. Moreover, we have

$$\hat{H}_g(\tau) = \psi(\gamma) jac(\gamma, \tau)^{1/4} \hat{H}_g(\gamma\tau),$$

for $\gamma \in \Gamma_0(n_g)$ where

$$\hat{H}_g(\tau) = H_g(\tau) + \chi(g)(4i)^{-1/2} \int_{-\overline{\tau}}^{\infty} (z+\tau)^{-1/2} \overline{\eta(-\overline{z})} dz,$$

and the multiplier system is given by $\psi(\gamma) = \epsilon(\gamma)^{-3} \rho_{n_g|h_g}(\gamma)$. Here

$$jac(\gamma, \tau) = (c\tau + d)^{-2}.$$

and

$$\rho_{n_g|h_g}(\gamma) = e(-cd/n_g h_g).$$

4.2 Theta Functions

We define the Jacobi theta functions $\theta_i(\tau, z)$ as follows for $q := e(\tau)$ and $y := e(z)$ for $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$ as in Ono et al. [2015]

$$\theta_2(\tau, z) := q^{1/8} y^{1/2} \prod_{n=1}^{\infty} (1 - q^n)(1 + yq^n)(1 + y^{-1} q^{n-1})$$

$$\theta_3(\tau, z) := \prod_{n=1}^{\infty}(1-q^n)(1+yq^{n-1/2}(1+y^{-1}q^{n-1/2}$$

$$\theta_4(\tau, z) := \prod_{n=1}^{\infty}(1-q^n)(1-yq^{n-1/2}(1-y^{-1}q^{n-1/2}$$

We use them to define weight zero index 1 Jacobi form φ_1.

$$\varphi := 4(f_2^2 + f_3^2 + f_4^2),$$

where $f_i(\tau, z) := \theta_i(\tau, z)/\theta_i(\tau, 0)$ for $i = 2, 3, 4$.

We define the Appell-Lerch sum as in Cheng et al. [2014] given by

$$\mu_{m,0}(\tau, z) := -\sum_{k\in\mathbb{Z}} q^{mk^2} y^{2mk} \frac{1+yq^k}{1-yq^k}.$$

It is the holomorphic part of a weight 1 index 2 *real-analytic Jacobi form.*

Finally,

$$R(\tau, z) := \sum_{\nu\in 1/2+\mathbb{Z}} \left\{ sgn(\nu) - E(\nu + a)\sqrt{2t} \right\} (-1)^{\nu-1/2} q^{-\nu^2/2} y^{-\nu},$$

where $t := \Im(\tau)$, $a := \frac{\Im(u)}{\Im(\tau)}$, and $E(z) := 2\int_0^z e^{-\pi u^2} du$.

References

Richard E. Borcherds. Vertex algebras, Kac-Moody algebras, and the Monster. *Proc. Nat. Acad. Sci. U.S.A.*, 83(10):3068–3071, 1986. ISSN 0027-8424. doi: 10.1073/pnas.83.10.3068.

URL https://doi.org/10.1073/pnas.83.10.3068.

Richard E. Borcherds. Automorphic forms with singularities on Grassmannians. *Invent. Math.*, 132(3):491–562, 1998. ISSN 0020-9910. doi: 10.1007/s002220050232. URL https://doi.org/10.1007/s002220050232.

Jan Bruinier and Ken Ono. Heegner divisors, L-functions and harmonic weak Maass forms. *Ann. of Math. (2)*, 172(3):2135–2181, 2010. ISSN 0003-486X. doi: 10.4007/annals.2010.172.2135. URL https://doi.org/10.4007/annals.2010.172.2135.

Miranda C. N. Cheng. $K3$ surfaces, $\mathcal{N} = 4$ dyons and the Mathieu group M_{24}. *Commun. Number Theory Phys.*, 4(4):623–657, 2010. ISSN 1931-4523. doi: 10.4310/CNTP.2010.v4.n4.a2. URL https://doi.org/10.4310/CNTP.2010.v4.n4.a2.

Miranda C. N. Cheng and John F. R. Duncan. The largest Mathieu group and (mock) automorphic forms. In *String-Math 2011*, volume 85 of *Proc. Sympos. Pure Math.*, pages 53–82. Amer. Math. Soc., Providence, RI, 2012. doi: 10.1090/pspum/085/1374. URL https://doi.org/10.1090/pspum/085/1374.

Miranda C. N. Cheng, John F. R. Duncan, and Jeffrey A. Harvey. Umbral moonshine and the Niemeier lattices. *Res. Math. Sci.*, 1:Art. 3, 81, 2014. ISSN 2522-0144. doi: 10.1186/2197-9847-1-3. URL https://doi.org/10.1186/2197-9847-1-3.

J. H. Conway. Monsters and moonshine. *Math. Intelligencer*, 2(4):165–171, 1979/80. ISSN 0343-6993. doi: 10.1007/BF03028594. URL https://doi.org/10.1007/BF03028594.

J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. *Atlas of finite groups*. Oxford University Press, Eynsham, 1985. ISBN 0-19-853199-0. Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray.

Atish Dabholkar, Sameer Murthy, and Don Zagier. Quantum Black Holes, Wall Crossing, and Mock Modular Forms. 2012.

John F. R. Duncan and Sander Mack-Crane. The moonshine module for Conway's group. *Forum Math. Sigma*, 3:e10, 52, 2015. ISSN 2050-5094. doi: 10.1017/fms.2015.7. URL `https://doi.org/10.1017/fms.2015.7`.

John F. R. Duncan, Michael J. Griffin, and Ken Ono. Proof of the umbral moonshine conjecture. *Res. Math. Sci.*, 2:Art. 26, 47, 2015. ISSN 2522-0144. doi: 10.1186/s40687-015-0044-7. URL `https://doi.org/10.1186/s40687-015-0044-7`.

Tohru Eguchi and Kazuhiro Hikami. Superconformal algebras and mock theta functions. II. Rademacher expansion for $K3$ surface. *Commun. Number Theory Phys.*, 3(3):531–554, 2009. ISSN 1931-4523. doi: 10.4310/CNTP.2009.v3.n3.a4. URL `https://doi.org/10.4310/CNTP.2009.v3.n3.a4`.

Tohru Eguchi and Kazuhiro Hikami. Note on twisted elliptic genus of $K3$ surface. *Phys. Lett. B*, 694(4-5):446–455, 2011. ISSN 0370-2693. doi: 10.1016/j.physletb.2010.10.017. URL `https://doi.org/10.1016/j.physletb.2010.10.017`.

Tohru Eguchi, Hirosi Ooguri, and Yuji Tachikawa. Notes on the $K3$ surface and the Mathieu group M_{24}. *Exp. Math.*, 20(1):91–96, 2011. ISSN 1058-6458. doi: 10.1080/10586458.2011.544585. URL `https://doi.org/10.1080/10586458.2011.544585`.

Matthias R. Gaberdiel, Daniel Grumiller, and Dmitri Vassilevich. Graviton 1-loop partition function for 3-dimensional massive gravity. *J. High Energy Phys.*, (11):094, 18, 2010a. ISSN 1126-6708. doi: 10.1007/JHEP11(2010)094. URL `https://doi.org/10.1007/JHEP11(2010)094`.

Matthias R. Gaberdiel, Stefan Hohenegger, and Roberto Volpato. Mathieu Moonshine in the elliptic genus of $K3$. *J. High Energy Phys.*, (10):062, 24, 2010b. ISSN 1126-6708. doi: 10.1007/JHEP10(2010)062. URL https://doi.org/10.1007/JHEP10(2010)062.

Terry Gannon. Much ado about Mathieu. *Adv. Math.*, 301:322–358, 2016. ISSN 0001-8708. doi: 10.1016/j.aim.2016.06.014. URL https://doi.org/10.1016/j.aim.2016.06.014.

B. Gross, W. Kohnen, and D. Zagier. Heegner points and derivatives of L-series. II. *Math. Ann.*, 278(1-4):497–562, 1987. ISSN 0025-5831. doi: 10.1007/BF01458081. URL https://doi.org/10.1007/BF01458081.

Richard Hill. Notes on Borcherds Products. 2012. URL http://www.math.unt.edu/~richter/afbridges/sites/default/files/lecture_notes_hill.pdf.

Ken Ono, Larry Rolen, and Sarah Trebat-Leder. Classical and umbral moonshine: connections and p-adic properties. *J. Ramanujan Math. Soc.*, 30(2):135–159, 2015. ISSN 0970-1249.

Larissa Queen. Modular functions arising from some finite groups. *Math. Comp.*, 37(156): 547–580, 1981. ISSN 0025-5718. doi: 10.2307/2007446. URL https://doi.org/10.2307/2007446.

Goro Shimura. Modular forms of half integral weight. pages 57–74. Lecture Notes in Math., Vol. 320, 1973.

J. G. Thompson. Some numerology between the Fischer-Griess Monster and the elliptic modular function. *Bull. London Math. Soc.*, 11(3):352–353, 1979a. ISSN 0024-6093. doi: 10.1112/blms/11.3.352. URL https://doi.org/10.1112/blms/11.3.352.

J. G. Thompson. Finite groups and modular functions. *Bull. London Math. Soc.*, 11(3): 347–351, 1979b. ISSN 0024-6093. doi: 10.1112/blms/11.3.347. URL https://doi.org/10.1112/blms/11.3.347.