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Non-Archimedean and Tropical Techniques in Arithmetic Geometry

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Doctor of Philosophy

Mathematics

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B.S., Emory University, 2014

M.Sc., Emory University, 2016

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Abstract

Non-Archimedean and Tropical Techniques in Arithmetic Geometry

By Jackson Salvatore Morrow

Let K be a number field, and let C/K be a curve of genus $g \geq 2$. In 1983, Faltings famously proved that the set $C(K)$ of K-rational points is finite. Given this, several questions naturally arise:

1. How does this finite quantity $\#C(K)$ varies in families of curves?

2. What is the analogous result for degree $d > 1$ points on C?

3. What can be said about a higher dimensional variant of Faltings result?

In this thesis, we will prove several results related to the above questions.

In joint with with J. Gunther, we prove, under a technical assumption, that for each positive integer $d > 1$, there exists a number B_d such that for each $g > d$, a positive proportion of odd hyperelliptic curves of genus g over $\mathbb{Q}$ have at most B_d "unexpected" points of degree d. Furthermore, we may take $B_2 = 24$ and $B_3 = 114$.

Our other results concern the strong Green–Griffiths–Lang–Vojta conjecture, which is the higher dimensional version of Faltings theorem (neé the Mordell conjecture). More precisely, we prove the strong non-Archimedean Green–Griffiths–Lang–Vojta conjecture for closed subvarieties of semi-abelian varieties and for projective surfaces admitting a dominant morphism to an elliptic curve.

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Contents

Chapter 1

Introduction

Let K be a number field, and let C/K be a curve of genus $g \geq 2$. In 1983, Faltings [Fal83] famously proved that the set $C(K)$ of K-rational points is finite. Given this, several questions naturally arise:

1. How does this finite quantity $\#C(K)$ varies in families of curves?
2. What is the analogous result for degree $d > 1$ points on C?
3. What can be said about a higher dimensional variant of Faltings result?

In this thesis, we will prove several results related to the above questions on rational points on varieties.

In joint with with J. Gunther, we prove, under a technical assumption, that for each positive integer $d > 1$, there exists a number B_d such that for each $g > d$, a positive proportion of odd hyperelliptic curves of genus g over $\mathbb{Q}$ have at most B_d "unexpected" points of degree d. More precisely, our result reads as follows.

Theorem A ([GM19]). *Let $d > 1$ be a positive integer and suppose Assumption 3.3.7 holds.*

1. *If d is odd, there exists a number B_d such that for each $g > d$, a positive*

proportion of genus g hyperelliptic curves over $\mathbb{Q}$ with a rational Weierstrass point have at most B_d points of degree d.

2. *If d is even, there exists a number B_d such that for each $g > d$, a positive proportion of genus g hyperelliptic curves over $\mathbb{Q}$ with a rational Weierstrass point have at most B_d points of degree d not obtained by pulling back degree $\frac{d}{2}$ points of $\mathbb{P}^1$.*

3. *We may take $B_2 = 24$ and $B_3 = 114$.*

Our other results concern the strong Green–Griffiths–Lang–Vojta conjecture, which is the higher dimensional version of Faltings theorem (neé the Mordell conjecture). First, we prove the strong non-Archimedean Green–Griffiths–Lang–Vojta conjecture for closed subvarieties of semi-abelian varieties.

Theorem B ([Mor19]). *Let K be an algebraically closed complete non-Archimedean valued field of characteristic zero. Let X be a closed subvariety of a semi-abelian variety G over K. Let $Sp(X)$ be the union of the subvarieties of X which are translates of positive-dimensional closed subgroups of G. Then, X is groupless modulo* $\mathrm{Sp}(X)$ *if and only if X is K-analytically Brody hyperbolic modulo* $\mathrm{Sp}(X)$.

Our second result concerns the algebraic degeneracy of non-Archimedean entire curves in projective varieties of general type admitting a dominant morphism to an elliptic curve.

Theorem C ([Mor20]). *Let K be an algebraically closed, complete, non-Archimedean valued field of characteristic zero. Let X/K be a projective variety of general type dominating an elliptic curve. Then, any non-Archimedean entire curve $\varphi\colon \mathbb{G}_{m,K}^{\mathrm{an}} \to X^{\mathrm{an}}$ is algebraically degenerate.*

Using the above result, we prove the strong non-Archimedean Green–Griffiths–Lang conjecture for projective surfaces admitting a dominant morphism to an elliptic curve.

Theorem D ([Mor20])**.** *Let K be an algebraically closed, complete, non-Archimedean valued field of characteristic zero, and let S/K be a projective surface admitting a dominant morphism to an elliptic curve. Then, S is pseudo-groupless if and only if S is pseudo-K-analytically Brody hyperbolic.*

1.1 Organization

In Chaper 2, we recall some background on the Chabauty–Coleman method and its generalization to dth symmetric products of curves. In Chapter 3, we describe some results on the arithmetic and geometry of hyperelliptic Jacobians and prove Theorem A. In Chapter 4, we give a summary of the various notions of hyperbolicity in the complex analytic, algebraic, and non-Archimedean analytic setting and discuss the Green–Griffiths–Lang–Vojta conjectures which relate these various notions of hyperbolicity. In Chapter 5, we state our results on the non-Archimedean Green–Griffiths–Lang–Vojta conjectures as well as some consequences of our results. Finally, in Chapter 6, we prove Theorem B, and in Chapter 7, we prove Theorems C and D.

Chapter 2

Background — Rational points on curves

Let K be a number field, and let C/K be a curve of genus $g \geq 2$. In 1983, Faltings [Fal83] proved that the set $C(K)$ of K-rational points is finite. Almost forty years earlier, Chabauty [Cha41] proved the finiteness of $C(K)$ under the additional assumption that the Mordell–Weil rank of the Jacobian of C is less than the genus of C. In [Col85], Coleman used the theory of p-adic integration to make Chabauty's proof effective, and to this day, the Chabauty–Coleman method is one of the most powerful tools for determining rational points on curves.

In this background section, we will recall some results on p-adic integration and the Chabauty–Coleman method; we refer the reader to [MP12, Sik09, Par16] for a fuller account of these techniques. We also prove some auxiliary lemmas, which are needed in Chapter 3.

2.1 The method of Chabauty–Coleman

Fix $C/\mathbb{Q}$ a curve of genus $g \geq 2$, and p a prime number. We make use of p-adic integration on the Jacobian variety J of our curve. Let $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$. After we base change from $\mathbb{Q}$ to $\mathbb{Q}_p$, we have an integration pairing

$$\begin{aligned} H^0(C_{\mathbb{Q}_p}, \Omega^1) \times J(\mathbb{C}_p) &\to \mathbb{C}_p \\ (\omega, D) &\mapsto \int_0^D \omega \end{aligned}$$

that is $\mathbb{Q}_p$-linear in the left factor, and a group homomorphism in the right. The kernel on the left is trivial, and on the right is the torsion subgroup $J(\mathbb{C}_p)_{\text{tors}}$.

Let r be the rank of $J(\mathbb{Q})$ as a finitely generated abelian group (for the rest of the paper, r will denote this rank for whatever curve is under consideration). We identify $J(\mathbb{Q})$ with its image in $J(\mathbb{Q}_p)$ and $J(\mathbb{C}_p)$. Within the former, its p-adic closure $\overline{J(\mathbb{Q})} \subset J(\mathbb{Q}_p)$ will be a finitely generated $\mathbb{Z}_p$-module of rank at most r. Define

$$\Lambda_C := \left\{ \omega \in H^0(C_{\mathbb{Q}_p}, \Omega^1) \;\middle|\; \int_0^D \omega = 0 \text{ for all } D \in J(\mathbb{Q}) \right\}.$$

This is a $\mathbb{Q}_p$-vector space of dimension at least $g - r$.

Suppose further that p is a prime of good reduction for our curve C. For a point $P \in C(\mathbb{C}_p)$, let $\overline{P} \in C_{\mathbb{F}_p}(\overline{\mathbb{F}_p})$ denote its reduction at p. Then given a nonzero form $\omega \in H^0(C_{\mathbb{Q}_p}, \Omega^1)$, we can scale it by an element of $\mathbb{Q}_p^\times$ to give a *normalized* form, which we take to mean it reduces to a nonzero element $\overline{\omega} \in H^0(C_{\mathbb{F}_p}, \Omega^1)$. For a normalized form ω, and a point $\overline{P} \in C_{\mathbb{F}_p}(\overline{\mathbb{F}_p})$, we define $n(\omega, \overline{P})$ to be the order of vanishing of $\overline{\omega}$ at $\overline{P}$. As long as $\Lambda_C \neq \{0\}$, we then define

$$n(\Lambda_C, \overline{P}) = \min_{\text{normalized } \omega \in \Lambda_C} n(\omega, \overline{P}).$$

By [Sto06, Theorem 6.4], the lower the rank is, the better we can control these minimal orders of vanishing.

Theorem 2.1.1 (Stoll)**.** *Let $C/\mathbb{Q}$ be a curve of genus $g \geq 2$, with rank $r \leq g - 1$,* *and let p be a prime of good reduction. Then* $\sum_{\overline{P}\in C_{\mathbb{F}_p}(\overline{\mathbb{F}_p})} n(\Lambda_C, \overline{P}) \leq 2r$.

For our purposes, we need forms that achieve these minima at different points simultaneously.

Lemma 2.1.2. *Let $C/\mathbb{Q}$ be a curve of genus $g \geq 2$, and let p be a prime of good reduction. Let $\overline{P_1}, \ldots, \overline{P_d} \in C_{\mathbb{F}_p}(\overline{\mathbb{F}_p})$. Suppose $r \leq g-1$ and $p \geq d$. Then there exists a normalized $\omega \in \Lambda_C$ such that $n(\Lambda_C, \overline{P_i}) = n(\omega, \overline{P_i})$ for $i = 1, \ldots, d$.*

Proof. We proceed by induction on d. The base case $d = 1$ is immediate from the definition of $n(\Lambda_C, \overline{P_1})$. Suppose there is a normalized form ω' such that $n(\Lambda_C, \overline{P_i}) = n(\omega', \overline{P_i})$ for $i = 1, \ldots, d-1$. If $n(\Lambda_C, \overline{P_d}) = n(\omega', \overline{P_d})$, we may take $\omega = \omega'$. Otherwise, choose a normalized ω'' such that $n(\Lambda_C, \overline{P_d}) = n(\omega'', \overline{P_d})$. If $n(\Lambda_C, \overline{P_i}) = n(\omega'', \overline{P_i})$ for $i = 1, \ldots, d-1$, we may take $\omega = \omega''$.

So suppose without loss of generality that $n(\omega'', \overline{P_1}) > n(\Lambda_C, \overline{P_1})$. Let $t_2, \ldots, t_{d-1}$ be uniformizers at $\overline{P_2}, \ldots, \overline{P_{d-1}}$, respectively. Write both $\overline{\omega'}$ and $\overline{\omega''}$ with respect to each uniformizer:

$$\overline{\omega'} = a_i t_i^{n_i} dt_i,$$

$$\overline{\omega''} = b_i t_i^{n_i} dt_i, \text{ for } i = 2, \ldots, d-1,$$

where for each $a_i, b_i \in \overline{\mathbb{F}_p}(C_{\mathbb{F}_p})$, the geometric function field of the reduction, we have $0 = v_{\overline{P_i}}(a_i) \leq v_{\overline{P_i}}(b_i)$. Since $p \geq d$, there exists $0 \neq \alpha \in \mathbb{F}_p$ such that $\alpha \cdot b_i(\overline{P_i}) \neq -a_i(\overline{P_i})$ for $i = 2, \ldots, d-1$. Choosing any $\tilde{\alpha} \in \mathbb{Z}_p$ whose reduction mod p is α, we may take $\omega = \omega' + \tilde{\alpha}\omega''$. $\square$

Lemma 2.1.3. *Let $C/\mathbb{Q}$ be a curve of genus $g \geq 2$, and let p be a prime of good reduction. Let $\overline{P_1}, \ldots, \overline{P_d} \in C_{\mathbb{F}_p}(\overline{\mathbb{F}_p})$, and suppose $r \leq g-d$. Then there exist linearly independent, normalized $\omega_1, \ldots, \omega_d \in \Lambda_C$ such that $n(\Lambda_C, \overline{P_i}) = n(\omega_j, \overline{P_i})$ for all $i, j = 1, \ldots, d$.*

Proof. Take ω_1 to be ω as given by Lemma 2.1.2. By the rank condition, we can choose $\omega_2', \ldots, \omega_d' \in \Lambda_C$ to be normalized forms such that $\omega_1, \omega_2', \ldots, \omega_d'$ are linearly independent. Then each reduction $\overline{\omega_1 + p\omega_j'} = \overline{\omega_1}$, so we can take $\omega_j = \omega_1 + p\omega_j'$ for $j = 2, \ldots, d$. $\square$

2.2 Symmetric power Chabauty–Coleman

Now let $C/\mathbb{Q}$ be a genus g curve with a marked rational point, which we denote by ∞. For any $\omega \in H^0(C_{\mathbb{Q}_p}, \Omega^1)$, we define (locally analytic) functions

$$\begin{aligned} f_\omega \colon C(\mathbb{C}_p) &\longrightarrow \mathbb{C}_p \\ P &\longmapsto \int_0^{[P-\infty]} \omega, \end{aligned}$$

and more generally for d a positive integer,

$$\begin{aligned} F_\omega^d \colon C(\mathbb{C}_p) \times \cdots \times C(\mathbb{C}_p) &\longrightarrow \mathbb{C}_p \\ (P_1, \ldots, P_d) &\longmapsto f_\omega(P_1) + \cdots + f_\omega(P_d) = \int_0^{[P_1 + \cdots + P_d - d\infty]} \omega. \end{aligned}$$

The starting point of the Chabauty–Coleman method for examining rational points is that if $\omega \in \Lambda_C$, then for any $P \in C(\mathbb{Q})$, we have $f_\omega(P) = 0$, because $[P - \infty] \in J(\mathbb{Q})$. The starting point for our method, following [Sik09, Par16], is that for $\omega \in \Lambda_C$ and $(P_1, \ldots, P_d)$ a d-tuple of conjugate degree d points on C, we have $F_\omega^d(P_1, \ldots, P_d) = 0$, since $[P_1 + \cdots + P_d - d\infty] \in J(\mathbb{Q})$.

We wish to control these zeros. For $\overline{P} \in C_{\mathbb{F}_p}(\overline{\mathbb{F}_p})$, define the *residue disk*

$$D_{\overline{P}} = \left\{ Q \in C(\mathbb{C}_p) \;\middle|\; \overline{Q} = \overline{P} \right\}.$$

Let $D \subset \mathbb{C}_p$ be the open unit disk, i.e. elements with absolute value strictly less than 1. For $P \in C(\overline{\mathbb{Q}_p})$, [Sik09, Lemma 2.3] asserts that we can always choose a *well-behaved uniformizer* z_P at P, which has the following key properties. First, the function $z_P\colon D_{\overline{P}} \to D$ is a diffeomorphism, with $z_P(P) = 0$. Furthermore, for a finite extension $L/\mathbb{Q}_p(P)$, with uniformizing element π, we have that z_P defines a bijection between $C(L) \cap D_{\overline{P}}$ and the π-adic disc $\pi\mathcal{O}_L$, given by $Q \mapsto z_P(Q)$.

Remark 2.2.1. Let v be the valuation on $\mathbb{C}_p$, normalized so that $v(p) = 1$. For $P, Q \in C(\overline{\mathbb{Q}_p})$ such that $\overline{P} = \overline{Q}$, let e be the ramification degree of $\mathbb{Q}_p(P, Q)$. The above implies that $v(z_P(Q)) \geq \frac{1}{e}$.

We can formally expand a normalized form ω with respect to the uniformizer z_P, as

$$\left(\sum_{i=0}^{\infty} a_i z_P^i\right) dz_P,$$

where the coefficients live in $\mathbb{Q}_p(P)$, and are integral ($v(a_i) \geq 0$ for all i). We record a few important facts from [Sik09, Section 2] about this expansion. First, the power series $\sum_{i=0}^{\infty} a_i t^i$ is convergent on D. Second, there is a connection to orders of vanishing: the smallest index i for which we have $v(a_i) = 0$ is given by $i = n(\omega, \overline{P})$. Lastly, for $Q \in D_{\overline{P}}$, the restriction of f_ω to $D_{\overline{P}}$ is given by

$$f_\omega(Q) = \int_0^{[P-\infty]} \omega + \sum_{i=0}^{\infty} \frac{a_i}{i+1} z_P(Q)^{i+1}.$$

Similarly, for $P_1, P_2 \in C(\overline{\mathbb{Q}_p})$, the restriction of F_ω^2 to $D_{\overline{P_1}} \times D_{\overline{P_2}}$ is given by

$$F_\omega^2(Q_1, Q_2) = \int_0^{[P_1+P_2-2\infty]} \omega + \sum_{i=0}^{\infty} \frac{a_i}{i+1} z_{P_1}(Q_1)^{i+1} + \sum_{i=0}^{\infty} \frac{b_i}{i+1} z_{P_2}(Q_2)^{i+1}.$$

Analogous expansions of course hold for F_ω^d, for arbitrary d.

Chapter 3

Irrational points on hyperelliptic curves

In this chapter, we describe joint work [GM19] with J. Gunther, where we prove, under a technical assumption, that for each positive integer $d > 1$, there exists a number B_d such that for each $g > d$, a positive proportion of odd hyperelliptic curves of genus g over $\mathbb{Q}$ have at most B_d "unexpected" points of degree d. Furthermore, we may take $B_2 = 24$ and $B_3 = 114$.

3.1 Introduction

Let K be a number field, and let C/K be a curve of genus $g \geq 2$. In 1983, Faltings [Fal83] proved that the set $C(K)$ of K-rational points is finite. Given this, one can ask how the finite quantity $\#C(K)$ varies in families of curves. Recently, multiple works have considered this question, for the family of all hyperelliptic curves over $\mathbb{Q}$ with a rational Weierstrass point [PS14, RT18], the family with a rational non-Weierstrass point [SW18], and the entire family of hyperelliptic curves over $\mathbb{Q}$ [Bha13].

In this paper, for a hyperelliptic curve $C/\mathbb{Q}$, instead of rational points we consider the *degree* d points of C, which we take to mean the set

$$\left\{P \in C(\overline{\mathbb{Q}}) \mid [\mathbb{Q}(P) : \mathbb{Q}] = d\right\}.$$

Since C is defined over $\mathbb{Q}$, this set is partitioned into d-tuples of Galois-conjugate points.

Before stating our main theorems, we consider an extended example: quadratic points, i.e. $d = 2$. Because we allow the quadratic extension to vary, there are infinitely many such points: for almost any point of $\mathbb{P}^1(\mathbb{Q})$, its pre-image under the hyperelliptic map will be a pair of conjugate quadratic points on C. We will call these *expected* quadratic points. More simply, for a hyperelliptic equation $y^2 = f(x)$, these are the quadratic points given by plugging in a rational number for x, and then solving for y.

But there can also be *unexpected* quadratic points, whose x-coordinate is irrational but whose y-coordinate is contained in the same quadratic field. For example, the genus 4 curve defined by $y^2 = x^9 + x^3 - 1$ contains infinitely many expected points, such as $(0, \pm i), (-1, \pm\sqrt{-3})$, and $(2, \pm\sqrt{519})$, but also contains unexpected points like $(\pm i, \pm i)$, $(\zeta_3, \pm 1)$, and $(-\zeta_3, \pm\sqrt{-3})$, where ζ_3 is a primitive third root of unity. In general, it is no small feat to compute these points explicitly for a given curve.

Example 3.1.1. Let $C/\mathbb{Q}$ be the hyperelliptic curve with affine model given by

$$C\colon y^2 = f(x) = x(x^2 + 2)(x^2 + 43)(x^2 + 8x - 6).$$

In [Sik09, Section 6.1], Siksek determined the set of quadratic points on C. Besides the infinitely many expected quadratic points of the form $(x, \pm\sqrt{f(x)})$, for $x \in \mathbb{Q}$, there are exactly 9 pairs of unexpected quadratic points on C. For example, there are the three pairs below:

$$\mathcal{Q}_1 = \left\{(\sqrt{6}, 56\sqrt{6}), (-\sqrt{6}, -56\sqrt{6})\right\}, \mathcal{Q}_2 = \left\{(\sqrt{-2}, 0), (-\sqrt{-2}, 0)\right\},$$
$$\mathcal{Q}_3 = \left\{\left(\frac{-164 + \sqrt{22094}}{49}, \frac{257704352 - 1648200\sqrt{22094}}{823543}\right), \text{ conjugate}\right\}.$$

Example 3.1.2. The modular curve $X_0(29)$ is a genus 2 curve with affine model

$$y^2 + (-x^3 - 1)y = -x^5 - 3x^4 + 2x^2 + 2x - 2.$$

In [BN15, Table 5], Bruin and Najman determined that there are exactly 4 pairs of unexpected quadratic points on $X_0(29)$:

$$P_1 = \left(\sqrt{-1} - 1, 2\sqrt{-1} + 4\right), \qquad P_2 = \left(\sqrt{-1} - 1, \sqrt{-1} - 1\right),$$
$$P_3 = \left(\frac{1}{4}(\sqrt{-7} + 1), \frac{1}{16}(-11\sqrt{-7} - 7)\right), \quad P_4 = \left(\frac{1}{4}(\sqrt{-7} + 1), \frac{1}{8}(5\sqrt{-7} + 9)\right),$$

along with their respective images under the hyperelliptic involution.

Addtional works of Ozman–Siksek [OS18] and Box [Box19] have classified the unexpected quadratic points on the modular curves $X_0(N)$ for various values of N.

The examples above demonstrate a general phenomenon: by further work of Faltings [Fal91, p. 550], we know that for any hyperelliptic curve of genus $g \geq 4$, there are only finitely many of these *unexpected* quadratic points. Thus, one can ask how many there are on a typical hyperelliptic curve.

To make that question rigorous, we need a way of ordering curves. A genus g hyperelliptic curve C over $\mathbb{Q}$ has a marked rational Weierstrass point ∞ if and only if it can be given an affine model of the form

$$y^2 = f(x) = x^{2g+1} + a_2x^{2g-1} + a_3x^{2g-2} + \cdots + a_{2g+1}, \tag{3.1.2.1}$$

with $f(x) \in \mathbb{Z}[x]$ separable, such that ∞ is not contained in this affine patch. Furthermore, C has a unique such *minimal* equation, for which there is no prime p such that $p^{2i} \mid a_i$ for each $i \geq 2$. Define the *height* of C to be

$$H(C) \coloneqq \max\left\{|a_i|^{1/i}\right\},$$

where the a_i's are coefficients for the minimal equation of C.

To study this question, we use and refine work of Park [Par16], which uses tropical intersection theory. However, the techniques of that pre-print seem to be missing a technical hypothesis, along the lines of (†) in our Assumption 3.3.7, in order to be valid; see Section 7.6 and Remark 3.3.6 for an explanation. This is the source of the conditional nature of some of our results.

We can now state our first conditional theorem.

Theorem 3.1.3. *Suppose Assumption 3.3.7 holds. Then for each $g > 2$, a positive proportion of genus g hyperelliptic curves over $\mathbb{Q}$ with a rational Weierstrass point, when ordered by height, have at most 24 quadratic points not obtained by pulling back points of* $\mathbb{P}^1(\mathbb{Q})$.

More precisely, let $\mathcal{F}_g$ denote the set of $\mathbb{Q}$-isomorphism classes of genus g hyperelliptic curves defined over $\mathbb{Q}$, with a marked rational Weierstrass point. The above says that if $\mathcal{F}'_g \subset \mathcal{F}_g$ corresponds to those curves satisfying the conditions of Theorem 3.1.3, then

$$\liminf_{X\to\infty} \frac{\#\{C \in \mathcal{F}'_g \mid H(C) < X\}}{\#\{C \in \mathcal{F}_g \mid H(C) < X\}} > 0.$$

The bound of Theorem 3.1.3 does not hold for all hyperelliptic curves, as shown by the following example, told to us by Michael Stoll.

Example 3.1.4. Let $f_1(x), \ldots, f_9(x)$ be distinct irreducible monic quadratic polynomials with rational coefficients. Write their product as the square of a degree 9 polynomial, plus a remainder polynomial $r(x)$ of degree at most 8. Then $y^2 = -r(x)$ will usually have at least 36 "unexpected" quadratic solutions.

Next we consider cubic points, i.e. degree 3 points. Unlike the case of $d = 2$, where the geometry — in this case, the existence of a 2:1 map to $\mathbb{P}^1$ — imposes infinitely many quadratic points, there need not be any cubic points. We prove the following theorem on their sparsity.

Theorem 3.1.5. *Suppose Assumption 3.3.7 holds. Then for each $g > 3$, a positive proportion of genus g hyperelliptic curves over $\mathbb{Q}$ with a rational Weierstrass point, when ordered by height, have at most 114 cubic points.*

The $d = 2$ and $d = 3$ cases turn out to be prototypical, and we can now state our main theorem concerning points of arbitrary degree d.

Theorem 3.1.6 (Theorem A)**.** *Let $d > 1$ be a positive integer and suppose Assumption 3.3.7 holds.*

1. *If d is odd, there exists a number B_d such that for each $g > d$, a positive proportion of genus g hyperelliptic curves over $\mathbb{Q}$ with a rational Weierstrass point have at most B_d points of degree d.*

2. *If d is even, there exists a number B_d such that for each $g > d$, a positive proportion of genus g hyperelliptic curves over $\mathbb{Q}$ with a rational Weierstrass point have at most B_d points of degree d not obtained by pulling back degree $\frac{d}{2}$ points of $\mathbb{P}^1$.*

3. *We may take $B_2 = 24$ and $B_3 = 114$.*

Remark 3.1.7.

- If $d = 1$, we may unconditionally take $B_1 = 1$, by [PS14, Theorem 10.3] (in the case $g > 2$) and [RT18, Theorem 1.2] ($g = 2$).

- For arbitrary d, we show that $B_d = (3d \cdot (2(2d^2+3)^d+1))^d$.

- The hypothesis $g > d$ is natural: that is exactly when the symmetric product $C^{(d)}$ is of general type, for a curve of genus $g \geq 2$. Since a degree d point of C gives a rational point of $C^{(d)}$, this is when one would expect them to be rare.

Terminology

While we do not use the term in theorem statements, we will call a point of even degree d on a hyperelliptic curve *unexpected* if it does not map to a degree $d/2$ point on $\mathbb{P}^1$. If d is odd, we call any point of degree d *unexpected*.

Throughout the paper, we use "asymptotically" when considering hyperelliptic curves of fixed genus with a marked rational Weierstrass point, with increasing height and "congruence conditions" when considering the coefficients of the minimal equation in (3.1.2.1).

3.2 Arithmetic and geometry of hyperelliptic Jacobians

First, we recall results of Bhargava and Gross on the average size of 2-Selmer groups of Jacobians of hyperelliptic curves.

Theorem 3.2.1 ([BG13, Theorem 1.1])**.** *When all hyperelliptic curves of fixed genus $g \geq 1$ over $\mathbb{Q}$ having a rational Weierstrass point are ordered by height, the average size of the 2-Selmer groups of their Jacobians is at most 3.*

Furthermore, the same result holds if one averages within a family defined by a finite set of congruence conditions.

This gives immediate corollaries concerning the average rank of $\mathrm{J}(\mathbb{Q})$, where we write J for the Jacobian of a curve C.

Corollary 3.2.2 ([BG13, Corollary 1.2]). *When all hyperelliptic curves of fixed genus $g \geq 1$ over $\mathbb{Q}$ having a rational Weierstrass point are ordered by height, the average rank of the Mordell–Weil groups of their Jacobians is at most $\frac{3}{2}$.*

Furthermore, the same result holds if one averages within a family defined by a finite set of congruence conditions.

Corollary 3.2.3. *When all hyperelliptic curves of fixed genus $g \geq 1$ over $\mathbb{Q}$ having a rational Weierstrass point are ordered by height, at least 25% have* $\operatorname{rank} \mathrm{J}(\mathbb{Q}) = 0$ *or* 1*. The same holds if one averages within a congruence family.*

Furthermore, either a positive proportion have rank 0, or at least 50% have rank 1.

Next, we recall a deep theorem of Faltings about rational points on subvarieties of abelian varieties.

Theorem 3.2.4 ([Fal94, p. 175]). *Let X be a closed subvariety of an abelian variety A, with both defined over a number field K. Then the set $X(K)$ equals a finite union $\cup B_i(K)$, where each B_i is a translated abelian subvariety of A contained in X.*

To conclude this section, we prove that asymptotically, 100% of hyperelliptic curves with a rational Weierstrass point over $\mathbb{Q}$ have finitely many *unexpected* degree d points, as described in Theorem A.

Proposition 3.2.5. *Fix $g \geq 2$. Asymptotically, 100% of genus g hyperelliptic curves over $\mathbb{Q}$ with a rational Weierstrass point have geometrically simple Jacobian.*

Proof. Let $t_2, \ldots, t_{2g+1}$ be indeterminates. The polynomial

$$F(x, t_2, \ldots, t_{2g+1}) = x^{2g+1} + t_2 x^{2g-1} + \cdots + t_{2g+1}$$

has Galois group S_{2g+1} over the field $\mathbb{Q}(t_2, \ldots, t_{2g+1})$. One way to see this is to note that its specialization at $t_2 = \cdots = t_{2g-1} = 0$ and $t_{2g} = t_{2g+1} = -1$ gives $x^{2g+1} - x - 1$, which has Galois group S_{2g+1}, by [Osa87, Corollary 3] or [NV79].

By a criterion of Zarhin [Zar74, Theorem 1], the Jacobian of the hyperelliptic curve given by $y^2 = f(x)$ is geometrically simple whenever $f(x)$ has Galois group $S_{\deg f}$.

Let $\mathcal{E} = \mathcal{E}_g$ be the complement in $\mathbb{A}^{2g}$ of the discriminant locus for equations of the form $y^2 = x^{2g+1} + a_2 x^{2g-1} + \cdots + a_{2g+1}$. We apply a version of Hilbert irreducibility (our height weights coordinates unequally, so some care must be taken); see [Coh81, Theorem 2.1], adapted as in [Coh81, Section 5, Notes (iii)]. It implies that asymptotically 100% of all the integer points in $\mathcal{E}$, when ordered by height, give hyperelliptic curves whose Jacobians are geometrically simple. A sieving argument shows that a positive proportion of the integer points of $\mathcal{E}$ give minimal equations, so asymptotically 100% of minimal equations give curves with geometrically simple Jacobians. $\square$

For a curve C and a positive integer d, let $C^{(d)}$ denote its d-th symmetric product, the points of which correspond to effective degree d divisors on C. Note that a conjugate d-tuple of points on C gives a rational point of the symmetric product.

Proposition 3.2.6. *Let d be a positive integer, and let $g > d$ be an integer.*

1. *If d is odd, then asymptotically, 100% of genus g hyperelliptic curves over $\mathbb{Q}$ with a rational Weierstrass point have finitely many degree d points.*

2. *If d is even, then asymptotically, 100% of genus g hyperelliptic curves over $\mathbb{Q}$ with a rational Weierstrass point have finitely many degree d points not obtained by pulling back degree $\frac{d}{2}$ points of $\mathbb{P}^1$.*

Proof. First, note that since the map from a hyperelliptic curve C to $\mathbb{P}^1$ has degree two, the image of a d-tuple of conjugate points on C is either a d-tuple of conjugate points on $\mathbb{P}^1$, or possibly a $\frac{d}{2}$-tuple of conjugate points on $\mathbb{P}^1$ if d is even.

We may assume C has geometrically simple Jacobian J, by Proposition 3.2.5. Map the symmetric product $C^{(d)}$ to J via the Abel–Jacobi map given by the rational Weierstrass point, i.e. $P_1 + \cdots + P_d \mapsto [P_1 + \cdots + P_d - d \cdot \infty]$.

Since $d < g$, the image W_d is a proper closed subvariety of J. Since J is geometrically simple, W_d contains no translate of a positive-dimensional abelian subvariety of J. By Theorem 3.2.4, $W_d(\mathbb{Q})$ is finite.

Lastly, we can ignore the positive-dimensional fibers of $C^{(d)} \to J$, which correspond to effective degree d divisors D on C such that D has positive rank. On a hyperelliptic curve, any such divisor D must contain a subdivisor of the form $P + \iota(P)$, where P is some point of C and ι is the hyperelliptic involution, which switches points within the same fiber [ACGH85, p. 13].

But by the first paragraph, if d is odd, no d-tuple of conjugate points can make up such a D. If d is even, it is only possible if

$$D = P_1 + \iota(P_1) + \cdots + P_{\frac{d}{2}} + \iota(P_{\frac{d}{2}}),$$

which will map to a $\frac{d}{2}$-tuple of conjugate points on $\mathbb{P}^1$. Thus in either case, the set we wish to show is finite injects into the finite set $W_d(\mathbb{Q})$. □

Besides the correction (see p. 2 and Section 7.6), Park's [Par16] most general results require a second technical hypothesis (*loc. cit.* Assumption 1.3) involving excess analytic intersection of the zero loci of the F^d_ω's for $\omega \in \Lambda_C$. In this section, we prove an assertion of Park (*loc. cit.* p. 2) that this assumption always holds when $r \leq 1$; to ease notation and terminology, we restrict to the hyperelliptic setting.

Fix a hyperelliptic curve $C/\mathbb{Q}$ of genus $g \geq 3$, with a rational Weierstrass point ∞, and embed C in its Jacobian J via the Abel–Jacobi map $C \to J$ given by ∞. Let p be a prime of good reduction for C. Let $W_d \coloneqq C + \cdots + C \subset J$ be the image of all degree d effective divisors, and let Λ_C be as in Section 2.1. Define J^{Λ_C} to be the kernel of pairing with elements of Λ_C, i.e.

$$J^{\Lambda_C} := \left\{ D \in J(\mathbb{C}_p) \;\middle|\; \int_0^D \omega = 0,\ \forall\, \omega \in \Lambda_C \right\}.$$

Note that J^{Λ_C} is also a $\mathbb{C}_p$-analytic manifold (in the sense of Bourbaki and Serre [Ser06, Chapter III]), and in fact a p-adic Lie group. If we assume that $J(\mathbb{Q})$ has rank ≤ 1, then J^{Λ_C} has dimension 0 or 1 (as a manifold). The next two lemmas use the topology on $J(\mathbb{C}_p)$ and $C(\mathbb{C}_p) \times \cdots \times C(\mathbb{C}_p)$ given by their structures as $\mathbb{C}_p$-analytic manifolds, unless otherwise noted.

Lemma 3.2.7. *Let $0 < d < g$ be integers. Assume that J is geometrically simple and that $J(\mathbb{Q})$ has rank ≤ 1. Then $J^{\Lambda_C} \cap W_d$ consists of isolated points.*

Proof. If J^{Λ_C} is 0-dimensional, then we are done since J^{Λ_C} is a closed subset of J.

If J^{Λ_C} is 1-dimensional, let $P \in J^{\Lambda_C} \cap W_d$. We can choose a closed neighborhood V of P such that $V \cap J^{\Lambda_C}$ is diffeomorphic to a closed disk in $\mathbb{C}_p$ via [Ser06, Chapter III, Section 3]. Since W_d is a closed set, $V \cap J^{\Lambda_C} \cap W_d$ is given by the vanishing of a convergent 1-variable power series on this disk. Thus, $V \cap J^{\Lambda_C} \cap W_d$ is either a finite set of points or all of $V \cap J^{\Lambda_C}$.

But in the latter case, note that $V \cap J^{\Lambda_C} \cap W_d$ is a translate of a closed disk centered at the origin, which makes it a translate of an infinite subgroup of J^{Λ_C}. Its Zariski closure would then be a translate of a positive-dimensional abelian subvariety of J contained in W_d, but this contradicts our initial assumption that J is geometrically simple. Therefore, $V \cap J^{\Lambda_C} \cap W_d$ is a finite set of points, so P is isolated. $\square$

Let

$$(C^d)^{\Lambda_C} := \left\{ (P_1, \ldots, P_d) \in C(\mathbb{C}_p) \times \cdots \times C(\mathbb{C}_p) \;\middle|\; F_\omega^d(P_1, \ldots, P_d) = 0,\, \forall\, \omega \in \Lambda_C \right\},$$

the inverse image of J^{Λ_C} in C^d. For any subset $S \subset \Lambda_C$, let $(C^d)^S$ similarly denote pairs satisfying the condition for all $\omega \in S$.

Lemma 3.2.8. *Let $0 < d < g$ be integers. Assume that J is geometrically simple and that $J(\mathbb{Q})$ has rank ≤ 1. Let $P = (P_1, \dots, P_d)$ be a point of $C^d(\mathbb{C}_p)$ such that the divisor $P_1 + \cdots + P_d$ is non-special. Then P is an isolated point of $(C^d)^{\Lambda_C}$.*

Proof. The set of d-tuples in $C^d(\mathbb{C}_p)$ which give special divisors is a closed subset. Away from this subset, the fibers of the map $C^d \to W_d$ are finite. The result then follows from Lemma 3.2.7 and the fact that $C \times \cdots \times C$ is Hausdorff in its topology as a $\mathbb{C}_p$-analytic manifold. □

To conclude this section, we consider the locally affinoid structure of $(C^d)^{\Lambda_C}$.

Definition 3.2.9. For $P \in C(\mathbb{C}_p)$, z_P a well-behaved uniformizer at P, and $m > 0$, let

$$B_m(P, z_P) := \{Q \in C(\mathbb{C}_p) \mid v(z_P(Q)) \geq m\}.$$

Since F_ω^d has a convergent power series expression on the entire open polydisk $D_{\overline{P_1}} \times \cdots \times D_{\overline{P_d}}$, on any closed sub-polydisk it will actually give an element of that sub-polydisk's affinoid coordinate ring, which is a Tate algebra [BGR84, Section 7.1.1]. For any choices of $P_1, \dots, P_d$ and $z_{P_1}, \dots, z_{P_d}$ and $m > 0$, the set

$$(C^d)^{\Lambda_C} \cap (B_m(P_1, z_{P_1}) \times \cdots \times B_m(P_d, z_{P_d}))$$

will have finitely many irreducible components as an affinoid subset of $B_m(P_1, z_{P_1}) \times \cdots \times B_m(P_d, z_{P_d})$, by [BGR84, Sect. 7.1.1, Cor. 8]. These components can be zero-dimensional or positive-dimensional.

Lemma 3.2.10. *Let $C/\mathbb{Q}$ be a curve of genus $g \geq 2$, let d be a positive integer, and let p be a prime of good reduction for C. Suppose C has rank $r \leq g - d$. Let $P_1, \dots, P_d \in C(\mathbb{C}_p)$, let z_{P_i} be a well-behaved uniformizer at P_i for $i = 1, \dots, d$, and let*

$m > 0$. Then we can choose $\omega_1, \ldots, \omega_d \in \Lambda_C$ as in Lemma 2.1.3 such that furthermore the zero set

$$(C^d)^{\{\omega_1,\ldots,\omega_d\}} \cap (B_m(P_1, z_{P_1}) \times \cdots \times B_m(P_d, z_{P_d}))$$

has the same positive-dimensional components as

$$(C^d)^{\Lambda_C} \cap (B_m(P_1, z_{P_1}) \times \cdots \times B_m(P_d, z_{P_d})).$$

Proof. The proof is similar to [Par16, Proposition 5.7], and proceeds by induction.

Claim: For $k = 1, \ldots, d$, we can choose $\omega_1, \ldots, \omega_k$ as in Lemma 2.1.3 such that the set of components of codimension at most $k-1$ for

$$(C^d)^{\{\omega_1,\ldots,\omega_k\}} \cap (B_m(P_1, z_{P_1}) \times \cdots \times B_m(P_d, z_{P_d}))$$

and

$$(C^d)^{\Lambda_C} \cap (B_m(P_1, z_{P_1}) \times \cdots \times B_m(P_d, z_{P_d}))$$

are the same.

Proof of Claim. The base case $k = 1$ is trivial. For the induction step, suppose it holds for a given value of k, for forms $\omega_1, \ldots, \omega_k$. Choose ω_{k+1} linearly independent from $\omega_1, \ldots, \omega_k$, and as in Lemma 2.1.2, such that

$$(C^d)^{\{\omega_1,\ldots,\omega_{k+1}\}} \cap (B_m(P_1, z_{P_1}) \times \cdots \times B_m(P_d, z_{P_d}))$$

has the minimal number of codimension k components $V_1, \ldots, V_s$ for any such choice of ω_{k+1}.

Suppose the conclusion of the claim is false for $k+1$. Then without loss of generality, we may assume that V_s is not a component of $(C^d)^{\Lambda_C} \cap (B_m(P_1, z_{P_1}) \times \cdots \times B_m(P_d, z_{P_d}))$. Let $V_{s+1}, \ldots, V_t$ be any codimension k components of $(C^d)^{\{\omega_1, \ldots, \omega_k\}} \cap (B_m(P_1, z_{P_1}) \times \cdots \times B_m(P_d, z_{P_d}))$ that are distinct from each $V_1, \ldots, V_s$.

Since V_s is not a component of $(C^d)^{\Lambda_C} \cap (B_m(P_1, z_{P_1}) \times \cdots \times B_m(P_d, z_{P_d}))$, we may choose a normalized form $\omega' \in \Lambda_C$ such that $F_{\omega'}$ does not vanish at some point R_s of V_s. Then for any integer n, we have that $F^d_{\omega_{k+1}+p^n\omega'}$ does not vanish identically on V_s, since $F^d_{\omega_{k+1}}$ does but $F^d_{p^n\omega'} = p^n F^d_{\omega'}$ does not.

For each $i = s+1, \ldots, t$, we can choose a point R_i on V_i at which $F^d_{\omega_{k+1}}$ does not vanish. Choose n to be a sufficiently large positive integer such that $v(F_{p^n\omega'}(R_i)) = n + v(F_{\omega'}(R_i)) > v(F_{\omega_{k+1}}(R_i))$ for each $i = s+1, \ldots, t$. Then $F_{\omega_{k+1}+p^n\omega'}$ does not vanish identically on any of $V_{s+1}, \ldots, V_t$.

Note that $\omega_{k+1} + p^n\omega'$ has the same reduction as ω_{k+1}, and thus the conclusion of Lemma 2.1.3 still holds. Also, it is linearly independent from $\omega_1, \ldots, \omega_k$, since $F^d_{\omega_1}, \ldots, F^d_{\omega_k}$ vanish at R_s and our integration pairing is $\mathbb{Q}_p$-linear. But by construction, the codimension k components of $(C^d)^{\{\omega_1, \ldots, \omega_d, \omega_{k+1}+p^n\omega'\}} \cap (B_m(P_1, z_{P_1}) \times \cdots \times B_m(P_d, z_{P_d}))$ are contained in $\{V_1, \ldots, V_{s-1}\}$, which contradicts the minimality of ω_{k+1}. $\square$

The lemma is the $k = d$ case of the claim. $\square$

3.3 Bounding the number of unexpected degree d points

In this section, we prove the first two statements of Theorem A. We refer the reader to Subsection 3.1 for the definition of unexpected degree d points.

Let $(P_1, \ldots, P_d)$ be a conjugate d-tuple of degree d points.

Lemma 3.3.1. *Let $d > 1$ be a positive integer, let $C/\mathbb{Q}$ be a hyperelliptic curve of* *genus $g > d$, with a rational Weierstrass point, geometrically simple Jacobian with $r \leq 1$, and let p be an odd prime of good reduction for C. Let $P_1, \cdots, P_d \in C(\overline{\mathbb{Q}})$ be a conjugate d-tuple with well-behaved uniformizers $z_{P_1}, \ldots, z_{P_d}$. Let $(Q_1, \ldots, Q_d)$ be a d-tuple of unexpected conjugate degree d points with the same reduction as $(P_1, \ldots, P_d)$ modulo p.*

Then $\{(Q_1, \ldots, Q_d)\}$ is a zero-dimensional component of

$$(C^d)^{\Lambda_C} \cap (B_{\frac{1}{d^2}}(P_1, z_{P_1}) \times \cdots \times B_{\frac{1}{d^2}}(P_d, z_{P_d})).$$

Proof. By Remark 2.2.1, $\{(Q_1, \ldots, Q_d)\}$ is contained in

$$B_{\frac{1}{d^2}}(P_1, z_{P_1}) \times \cdots \times B_{\frac{1}{d^2}}(P_d, z_{P_d}).$$

By Lemma 3.2.8, we have that $\{(Q_1, \ldots, Q_d)\}$ is a zero-dimensional component of

$$(C^d)^{\Lambda_C} \cap (B_{\frac{1}{d^2}}(P_1, z_{P_1}) \times \cdots \times B_{\frac{1}{d^2}}(P_d, z_{P_d})).$$

□

In light of Lemma 3.3.1, we work to bound the number of these zero-dimensional components. We use the convention that $v(0) = \infty$. For a positive rational number m, let $D_m = \{\alpha \in \mathbb{C}_p \mid v(\alpha) \geq m\}$, let D_m^d denote its d-fold product, and let $\mathbb{C}_p\langle D_m^d \rangle$ denote the Tate algebra of functions in its affinoid coordinate ring. The key tool we use to control zero-dimensional components comes from Park [Par16] and builds off results of Rabinoff [Rab12] in tropical deformation and tropical intersection theory.

Before we can state the theorem, we recall some definitions.

Definition 3.3.2. For $F(t_1, \ldots, t_d) = \sum_{u \in \mathbb{Z}^d_{\geq 0}} a_u t^u \in \mathbb{C}_p\langle D_m^d \rangle$, we define the *tropi-*

calization of F as

$$\mathrm{Trop}(F) := \overline{\{(v(z_1), \ldots, v(z_d)) : F(z_1, \ldots, z_d) = 0, (z_1, \ldots, z_d) \in D_m^d)\}},$$

where we take the topological closure in $\mathbb{R}^d$.

Next, for a set $S \subset \mathbb{R}^d$, let $\mathrm{conv}(S)$ denote its convex hull. A necessary condition for a series to sum to 0 in a non-Archimedean field is that it has a term of minimal valuation, and that this term is not unique. In the below definition, the relevant $(w_1, \ldots, w_d)$ can thus be thought of as candidates for the coordinate-wise valuations of zeros for the power series F, where we only look for zeros whose coordinates have valuation at least m, and the $(u_1, \ldots, u_d)$ are the multi-indices of terms that could have minimal valuation after plugging in such a zero.

Definition 3.3.3. Let m be a positive rational number, and let $F(t_1, \ldots, t_d) = \sum_{u \in \mathbb{Z}_{\geq 0}^d} a_u t^u \in \mathbb{C}_p\langle D_m^d \rangle$. We define the *Newton polygon* of F (with respect to m) to be the set $\mathrm{New}_m(F) \subset \mathbb{R}^d$ given by

$$\begin{aligned}\mathrm{New}_m(F) := \mathrm{conv}(\{u = (u_1, \ldots, u_d) \in \mathbb{Z}_{\geq 0}^d \mid \exists (w_1, \ldots, w_d) \in \mathbb{Q}^d \text{ with each } w_i \geq m \text{ s.t.}\\ \exists\, u' \in \mathbb{Z}_{\geq 0}^d \text{ with } u \neq u' \text{ and } v(a_u) + \sum_i w_i u_i = v(a_{u'}) + \sum_i w_i u_i',\\ \text{and } \forall\, u'' \in \mathbb{Z}_{\geq 0}^d, v(a_u) + \sum_i w_i u_i \leq v(a_{u''}) + \sum_i w_i u_i''\}).\end{aligned}$$

A *polytope* is the convex hull of finitely many points of a Euclidean space. We need to define the mixed volume of a collection of polytopes in a given Euclidean space. For polytopes $Z_1, \ldots, Z_d \subset \mathbb{R}^d$ and positive real numbers $\lambda_1, \ldots, \lambda_d$, the volume of the scaled Minkowski sum $\lambda_1 Z_1 + \cdots + \lambda_d Z_d = \{\lambda_1 z_1 + \cdots + \lambda_d z_d \mid z_i \in Z_i\}$ is known to be given by a homogeneous polynomial of degree d in the coefficients $\lambda_1, \ldots, \lambda_d$. The *mixed volume* of the polytopes, denoted $\mathrm{MV}(Z_1, \ldots, Z_d)$, is defined to be the coefficient of $\lambda_1 \lambda_2 \cdots \lambda_d$ in that polynomial.

Example 3.3.4. For $a \geq 1$ and d a positive integer, let

$$Z_1 = \cdots = Z_d = \operatorname{conv}(\{e_1, \ldots, e_d, ae_1, \ldots, ae_d\}) \subset \mathbb{R}^d,$$

where e_i is the vector with a 1 in the i-th place and 0's elsewhere. Then

$$\lambda_1 Z_1 + \cdots + \lambda_d Z_d = \operatorname{conv}((\lambda_1 + \cdots + \lambda_d)\{e_1, \ldots, e_d, ae_1, \ldots, ae_d\}),$$

and a quick calculation gives $\operatorname{MV}(Z_1, \ldots, Z_d) = a^d - 1$.

Theorem 3.3.5 ([Par16, Theorem 4.18 & Proposition 5.7])**.** *Let m be a positive rational number, and let $F_1, \ldots, F_d \in \mathbb{C}_p\langle D_m^d \rangle$ be power series such that for any $i, j \in \{1, \ldots, d\}$, F_i has a term of the form ct_j^N, with $c \neq 0$ and $N > 0$.*

Suppose that the tropicalization of an isolated point in the intersection of the zero loci of the F_i is isolated in the intersection of the tropicalizations $\operatorname{Trop}(F_i)$. *Then the number of zero-dimensional components of the zero locus $V(F_1, \ldots, F_d)$ in $D_m^d \cap (\mathbb{C}_p^\times)^d$ is at most* $\operatorname{MV}(\operatorname{New}_m(F_1), \ldots, \operatorname{New}_m(F_d))$.

Remark 3.3.6. The condition that the tropicalization of an isolated point in the intersection of the zero loci of the F_i is isolated in the intersection of the tropicalizations does not appear in [Par16], but does appear in the previous theorem from [Rab12] that Park uses. It seems to be necessary if one wishes to use the mixed volume $\operatorname{MV}(\operatorname{New}_m(F_1), \ldots, \operatorname{New}_m(F_d))$ to bound the number of zero-dimensional components of $V(F_1, \ldots, F_d)$ in $D_m^d \cap (\mathbb{C}_p^\times)^d$.

If the tropicalization of an isolated point in the intersection of the zero loci of the F_i lands in a positive-dimensional component of the intersection of their tropicalizations, then one could hope to use work of Osserman–Rabinoff [OR11] to bound the number of zero-dimensional components in terms of the *stable tropical intersection*; see [JY16] for definitions and results concerning stable tropical intersections. However, one cannot use mixed volumes of Newton polygons to compute these stable intersection numbers.

Since we use Theorem 3.3.5 to explicitly bound the number zero-dimensional components from Lemma 3.3.1, we include an assumption.

Assumption 3.3.7. *For each $g \geq 2$, more than 75% of hyperelliptic curves over $\mathbb{Q}$ of genus g satisfy the following condition:*

for each positive rational number m and each residue disk D^d_m, the tropicalization of an isolated point in the intersection of $F^d_{\omega_i}$, where $\{\omega_1, \dots, \omega_d\}$ are d linearly independent 1-forms from Lemma 3.2.10, is isolated in the intersection of the tropicalizations $\mathrm{Trop}(F^d_{\omega_i})$. (†)

Furthermore, the same holds if one averages within a family defined by a finite set of congruence conditions.

Returning to the proof of the first two statements of Theorem A, we use Theorem 3.3.5 to bound the maximal number of zero-dimensional components mentioned in Lemma 3.3.1.

Lemma 3.3.8. *Let $d > 1$ be a positive integer and let $C/\mathbb{Q}$ be a hyperelliptic curve of genus $g > d$, with a rational Weierstrass point, geometrically simple Jacobian with $r \leq 1$, good reduction at a prime $p > d^2 + 3$, and which satisfies condition (†).*

Let $P_1, \cdots, P_d \in C^d(\overline{\mathbb{Q}})$ be a conjugate d-tuple with well-behaved uniformizers $z_{P_1}, \dots, z_{P_d}$. Then there are at most 3^d ordered tuples of unexpected conjugate degree d points in $D_{\overline{P_1}} \times \cdots \times D_{\overline{P_d}}$, i.e. with the same reduction as $(P_1, \dots, P_d)$ modulo p.

Proof. Using Lemmas 2.1.3 and 3.2.10, we can choose linearly independent, normalized forms $\omega_1, \dots, \omega_d \in \Lambda_C$ such that $n(\omega_i, \overline{P_j}) = n(\Lambda_C, \overline{P_j})$ for $i, j \in \{1, \dots, d\}$. By Theorem 2.1.1, these numbers are at most 2.

Viewed as a function on D^d via $z_{P_1}, \dots, z_{P_d}$, $F^d_{\omega_1}$ is given by

$$\sum_{i=0}^{\infty} \frac{a_{1,i}}{i+1} t_1^{i+1} + \cdots + \sum_{i=0}^{\infty} \frac{a_{d,i}}{i+1} t_d^{i+1}.$$

For each $i \in \{1, \ldots, d\}$, we have that $v(a_{i,j}) = 0$ for some $j \leq 2$. Similar statements holds for $F^d_{\omega_2}, \ldots, F^d_{\omega_d}$.

By a standard Newton polygon argument (cf. proof of [Sto06, Lemma 6.1 & Proposition 6.3] using our assumption that $p > d^2 + 3$) applied to each of these sums, we see that $\mathrm{New}_{1/d^2}(F^d_{\omega_1}), \mathrm{New}_{1/d^2}(F^d_{\omega_2}), \ldots,$ and $\mathrm{New}_{1/d^2}(F^d_{\omega_d})$ are contained in the set

$$\mathrm{conv}(\{(1, 0, \ldots, 0), (3, 0, \ldots, 0), (0, 1, \ldots, 0), (0, 3, \ldots, 0), \ldots, (0, 0, \ldots, 1), (0, 0, \ldots, 3)\}) \subset \mathbb{R}^d.$$

By Theorem 3.3.5 and Example 3.3.4, there are at most $3^d - 1$ zero-dimensional components of interest away from $(P_1, \ldots, P_d)$. Since $(P_1, \ldots, P_d)$ could be a d-tuple of unexpected conjugate degree d points, we have at most 3^d zero-dimensional components in D^d. We are now done by Lemmas 3.2.10 and 3.3.1. □

Proof of Theorem A. Choose a prime $p > d^2 + 3$. Among all hyperelliptic curves of genus g with a rational Weierstrass point, those with good reduction at p are defined by finitely many congruence conditions on their minimal equations, and thus constitute a positive proportion of all such curves. Proposition 3.2.5 tells us that asymptotically, 100% of curves in this subfamily have geometrically simple Jacobian, and by Corollary 3.2.3, at least 25% of these curves also have Jacobian rank $r \leq 1$. Furthermore, under Assumption 3.3.7, a positive proportion of these curves satisfy condition (†).

Let C be such a curve of genus g. Given a d-tuple $(Q_1, \ldots, Q_d)$ of conjugate degree d points, the reduction of each Q_i is certainly contained in $C_{\mathbb{F}_p}(\mathbb{F}_{p^{m_i}})$ for some $1 \leq m_i \leq d$. Note that since C is an odd hyperelliptic curve with good reduction at p, the size of $C_{\mathbb{F}_p}(\mathbb{F}_{p^{m_i}})$ is less than or equal to $2p^{m_i} + 1$. Crudely, there are at most $(d \cdot (2p^d + 1))^d$ possible reductions for $(Q_1, \ldots, Q_d)$ modulo p. By Lemma 3.3.8, there are at most 3^d ordered tuples of unexpected conjugate degree d points with each reduction. Thus, we may take

$$B_d = (3d \cdot (2p^d + 1))^d \leq (3d \cdot (2(2d^2 + 3)^d + 1))^d,$$

by Bertrand's postulate [Tch62, Tome I, p. 64]. $\square$

3.4 Explicit bounds on the number of unexpected quadratic points

In this section, we prove Theorem 3.1.3.

Lemma 3.4.1. *Let $C/\mathbb{Q}$ be a hyperelliptic curve of genus $g \geq 3$, with a rational Weierstrass point, geometrically simple Jacobian with $r \leq 1$, and let p be an odd prime of good reduction for C.*

Let $P_1, P_2 \in C(\overline{\mathbb{Q}})$ be either two rational points or a pair of conjugate quadratic points, with well-behaved uniformizers z_{P_1}, z_{P_2}. Let (Q_1, Q_2) be a pair of unexpected conjugate quadratic points with the same reduction as (P_1, P_2). Then $\{(Q_1, Q_2)\}$ is a zero-dimensional component of $(C^2)^{\Lambda_C} \cap (B_{\frac{1}{2}}(P_1, z_{P_1}) \times B_{\frac{1}{2}}(P_2, z_{P_2}))$.

Proof. The field $\mathbb{Q}_p(P_1, P_2, Q_1, Q_2)$ is certainly contained in the compositum of the three quadratic extensions of $\mathbb{Q}_p$. Since p is odd, that compositum has ramification degree $e = 2$ over $\mathbb{Q}_p$. The result now follows from Lemma 3.3.1. $\square$

Lemma 3.4.2. *Suppose $p \neq 2$ or 5, and $\sum_{i=0}^{\infty} a_i t^i \in \mathbb{C}_p[\![t]\!]$ is a power series with integral coefficients. If $v(a_0) = 0$, then for $f(t) = \sum_{i=0}^{\infty} \frac{a_i}{i+1} t^{i+1}$, we have $\mathrm{New}_1(f) = \emptyset$ and $\mathrm{New}_{1/2}(f) \subset [1, 3]$. If $v(a_1) = 0$ or $v(a_2) = 0$, then $\mathrm{New}_1(f), \mathrm{New}_{1/2}(f) \subset [1, 3]$.*

Proof. We begin with the case where $v(a_0) = 0$. Then for any $w \geq 1$ and $i > 0$, we have $v(\frac{a_0}{1}) + 1 \cdot w = w < v(\frac{a_i}{i+1}) + (i+1) \cdot w$. To see this, note that the right-hand side is no smaller than $-v(i+1)+(i+1)w$, and $v(i+1)$ is strictly less than i for $i>0$ and $p>2$.

For $w \geq \frac{1}{2}$, things can be different: for example, suppose $p=3$, $v(a_2)=0$, and $w=\frac{1}{2}$. Then $\frac{1}{2}=v(\frac{a_2}{3})+3\cdot w=v(\frac{a_0}{1})+1\cdot w$. But past $i=2$, strict inequality holds, so we have $\mathrm{New}_{1/2}(f)\subset[1,3]$. The rest of the cases proceed similarly. $\square$

Lemma 3.4.3. *Let $C/\mathbb{Q}$ be a hyperelliptic curve of genus $g\geq 3$, with a rational Weierstrass point, geometrically simple Jacobian with $r\leq 1$, good reduction at 3, which satisfies condition (†).*

Let $P_1, P_2 \in C(\overline{\mathbb{Q}})$ be either two rational points or a pair of conjugate quadratic points, with $\overline{P_1}, \overline{P_2} \in C_{\mathbb{F}_3}(\mathbb{F}_3)$. Then there are at most 8 ordered pairs (Q_1, Q_2) of unexpected conjugate quadratic points in $D_{\overline{P_1}} \times D_{\overline{P_2}}$ that are not equal to (P_1, P_2).

Proof. Using Lemmas 2.1.3 and 3.2.10, choose linearly independent, normalized forms $\omega_1, \omega_2 \in \Lambda_C$ such that $n(\omega_i, \overline{P_j}) = n(\Lambda_C, \overline{P_j})$ for $i, j \in \{1,2\}$. By Theorem 2.1.1, these numbers are at most 2. Viewed as a function on D^2 via z_{P_1} and z_{P_2}, $F^2_{\omega_1}$ is given by

$$\sum_{i=0}^{\infty} \frac{a_i}{i+1} t_1^{i+1} + \sum_{i=0}^{\infty} \frac{b_i}{i+1} t_2^{i+1}.$$

A similar statement holds for $F^2_{\omega_2}$.

By construction of ω_1 (and similarly for ω_2), we have $v(a_i)=0$ for some $i\leq 2$, and $v(b_j)=0$ for some $j\leq 2$. Now by Lemma 3.4.2 applied to each of these two sums, we see that both $\mathrm{New}_{1/2}(F^2_{\omega_1})$ and $\mathrm{New}_{1/2}(F^2_{\omega_2})$ are contained in the set

$$\mathrm{conv}(\{(1,0),(3,0),(0,1),(0,3)\}) \subset \mathbb{R}^2.$$

By Theorem 3.3.5 and Example 3.3.4, there are at most $3^2-1=8$ zero-dimensional components of interest away from (P_1, P_2). We are done by Lemmas 3.2.10 and 3.4.1. $\square$

Lemma 3.4.4. *Let $C/\mathbb{Q}$ be a hyperelliptic curve of genus $g \geq 3$, with a rational Weierstrass point, geometrically simple Jacobian with $r \leq 1$, good reduction at 3, which satisfies condition (†).*

Let $P_1, P_2 \in C(\overline{\mathbb{Q}})$ be a pair of conjugate quadratic points, with $\overline{P_1}, \overline{P_2} \in C_{\mathbb{F}_3}(\mathbb{F}_9) \setminus C_{\mathbb{F}_3}(\mathbb{F}_3)$. If $n(\Lambda_C, \overline{P_1}) = 1$, there are at most 8 pairs (Q_1, Q_2) of unexpected conjugate quadratic points in $D_{\overline{P_1}} \times D_{\overline{P_2}}$ that are not equal to (P_1, P_2). If $n(\Lambda_C, \overline{P_1}) = 0$, there are no such pairs.

Proof. The proof is similar to that of Lemma 3.4.3, with two changes. First, one can consider New_1 instead of $\mathrm{New}_{1/2}$, using Remark 2.2.1 and the fact that $\mathbb{Q}_3(P_1, Q_1)$ is unramified. Second, note that since P_1 and P_2 reduce to (necessarily distinct) points outside of $C_{\mathbb{F}_3}(\mathbb{F}_3)$, we know that P_1 and P_2 remain conjugate over $\mathbb{Q}_3$. Thus their reductions are conjugate over $\mathbb{F}_3$, so we have $n(\Lambda_C, \overline{P_1}) = n(\Lambda_C, \overline{P_2})$. By Theorem 2.1.1, this common value can only be 0 or 1. □

Lemma 3.4.5. *For each $g \geq 3$, there exists a congruence family of genus g hyperelliptic curves with a rational Weierstrass point, such that any curve C in the family has good reduction at 3, and satisfies $C_{\mathbb{F}_3}(\mathbb{F}_3) = \{\overline{\infty}\}$ and $C_{\mathbb{F}_3}(\mathbb{F}_9) = \{\overline{\infty}, (0, \pm\alpha), (1, \pm\alpha), (2, \pm\alpha)\}$, with $\alpha \in (\mathbb{F}_9 \setminus \mathbb{F}_3)$.*

Proof. For each g, consider the families of hyperelliptic curves whose reduction mod 3 is given by:

$$y^2 = f_g(x) = x^{2g+1} + 2x^9 + 2 \qquad \text{for } g \equiv 1 \mod 4,$$

$$y^2 = f_g(x) = x^{2g+1} + 2x^{15} + 2 \qquad \text{for } g \equiv 2 \mod 4,$$

$$y^2 = f_g(x) = x^{2g+1} + 2x^5 + 2 \qquad \text{for } g \equiv 3 \mod 4,$$

$$y^2 = f_g(x) = x^{2g+1} + x^3 + x + 2 \qquad \text{for } g \equiv 0 \mod 4 \text{ and } g \equiv 0, 1 \mod 3,$$

$$y^2 = f_g(x) = x^{2g+1} + x^9 + x^3 + 2 \qquad \text{for } g \equiv 0 \mod 4 \text{ and } g \equiv 2 \mod 3.$$

For $g = 3$, one checks directly that $f_g(x)$ represents as few squares as is possible: $f(\mathbb{F}_3) = \{2\}$, and $f(\mathbb{F}_9 \setminus \mathbb{F}_3) \subset (\mathbb{F}_9 \setminus \mathbb{F}_9^2)$. This is equivalent to the lemma's condition on $\mathbb{F}_3$- and $\mathbb{F}_9$-points. In general, for $g \equiv 3 \mod 4$, note that $f_g(x) = x^{2g+1} + 2x^5 + 2$ defines the exact same function as $x^7 + 2x^5 + 2$ on $\mathbb{F}_9$, as x^k and x^{k+8r} define the same function on $\mathbb{F}_9$ for natural numbers k, r. The same argument works for the other values of g.

We conclude by showing that these polynomials are square-free over $\mathbb{F}_3$. Over any field, the discriminant of a trinomial $x^n + ax^k + b$ is

$$(-1)^{\frac{n(n-1)}{2}} b^{k-1}[n^{n_1} b^{n_1-k_1} + (-1)^{n_1+1}(n-k)^{n_1-k_1} k^{k_1} a^{n_1}]^d,$$

where $d = (n, k)$ and $n = n_1 d, k = k_1 d$ [Swa62, Theorem 2]. Thus for $g \equiv 1, 2, 3 \mod 4$, the discriminant of f_g in $\mathbb{F}_3$ is

$$\pm[(2g+1) + (2g+1-k)^2 k \cdot 2],$$

with $k = 1, 7$, or 5, respectively. This is non-zero in $\mathbb{F}_3$ for any value of g.

For $g \equiv 0 \mod 4$, we use a different method. It suffices to show $f_g(x)$ and $f_g'(x)$ have no common factors. If $g \equiv 1 \mod 3$, then $f_g'(x) = 1$, so this is clear. If $g \equiv 2 \mod 3$, then $f_g'(x) = -x^{2g}$, and so this also is clear. Lastly if $g \equiv 0 \mod 3$, then $f_g'(x) = x^{2g}+1$, so any common factor would divide $f_g(x) - x f_g'(x) = x^3 + 2 = (x+2)^3$. Thus we see f_g and f_g' are coprime. □

Proof of Theorem 3.1.3. For a given value of g, the family given by Proposition 3.2.5 and Lemma 3.4.5 comprises a positive proportion of all hyperelliptic curves with a rational Weierstrass point, since the latter is defined by finitely many congruence conditions. By Corollary 3.2.3, at least 25% of the curves in this family have rank $r \leq 1$; this is still a positive proportion of all the curves. Furthermore, under Assumption 3.3.7, a positive proportion of these curves satisfy condition (†).

Let C be such a curve. Any pair of conjugate quadratic points on C that reduce to $\mathbb{F}_3$-points will have to lie in $D_{\overline{\infty}} \times D_{\overline{\infty}}$. We may choose $P_1 = P_2 = \infty$ and apply Lemma 3.4.3 to conclude there are at most eight ordered pairs (Q_1, Q_2) of unexpected conjugate quadratic points in this residue class. But since $\overline{Q_1} = \overline{Q_2}$, each *unordered* pair is counted twice, so there are at most four unordered pairs of such quadratic points.

If the minimal Weierstrass model of C is $y^2 = f(x)$, note that (for some choice of square root) the pair of *expected* quadratic points $(i, \pm\sqrt{f(i)})$ reduces to $(\bar{i}, \pm\alpha)$ for each $\bar{i} = 0, 1, 2$. In $D_{(\bar{i},\alpha)} \times D_{(\bar{i},-\alpha)}$, we may choose $P_1 = (i, \sqrt{f(i)}), P_2 = (i, -\sqrt{f(i)})$, and apply Lemma 3.4.4. If $n(\Lambda_C, (\bar{i}, \alpha)) = 0$, we conclude there are no unexpected pairs in this residue class. If $n(\Lambda_C, (\bar{i}, \alpha)) = 1$, there are at most eight.

By Theorem 2.1.1, at most one value of $\bar{i} = 0, 1, 2$ will have $n(\Lambda_C, (\bar{i}, \alpha)) = n(\Lambda_C, (\bar{i}, -\alpha)) = 1$, and all the others will be 0. Thus, there are at most $4 + 8 = 12$ unordered pairs of unexpected conjugate quadratic points. $\square$

3.5 Explicit bounds on the number of cubic points

In this section, we prove Theorem 3.1.5.

Lemma 3.5.1. *Suppose* $\sum_{i=0}^{\infty} a_i t^i \in \mathbb{C}_3[\![t]\!]$ *is a power series with integral coefficients. If* $v(a_0) = 0$, $v(a_1) = 0$, *or* $v(a_2) = 0$, *then for* $f(t) = \sum_{i=0}^{\infty} \frac{a_i}{i+1} t^{i+1}$, *we have* $\mathrm{New}_{1/3}(f) \subset [1, 3]$.

Proof. We proceed as in Lemma 3.4.2. If $v(a_0) = 0$, then for $w \geq 1/3$ and $i > 2$, we have that $v(\frac{a_0}{1}) + 1 \cdot w = w < v(\frac{a_i}{i+1}) + (i+1) \cdot w$. Recall that the right-hand side is no smaller than $-v(i+1) + (i+1)w$, and so it suffices to prove that for $i > 2$, $w < -v(i+1) + (i+1)w$. A short induction argument on i using the non-Archimedean properties of v and taking $w = 1/3$ yields the desired result. The remaining cases follow in a similar fashion. $\square$

Lemma 3.5.2. *Let $C/\mathbb{Q}$ be a hyperelliptic curve of genus $g \geq 4$, with a rational Weierstrass point, geometrically simple Jacobian with $r \leq 1$, good reduction at 3, which satisfies condition (†).*

Let $P_1, P_2, P_3 \in C(\mathbb{Q})$ be three rational points. Then there are at most 26 ordered triples (Q_1, Q_2, Q_3) of conjugate cubic points in $D_{\overline{P_1}} \times D_{\overline{P_2}} \times D_{\overline{P_3}}$.

Proof. As in Lemma 3.4.3, we use Lemmas 2.1.3 and 3.2.10 to choose linearly independent, normalized forms $\omega_1, \omega_2, \omega_3 \in \Lambda_C$ such that $n(\omega_i, \overline{P_j}) = n(\Lambda_C, \overline{P_j})$ for $i, j \in \{1, 2, 3\}$. By Theorem 2.1.1, these numbers are at most 2. Viewed as a function on D^3 via z_{P_1}, z_{P_2}, and z_{P_3}, $F^3_{\omega_1}$ is given by

$$\sum_{i=0}^{\infty} \frac{a_i}{i+1} t_1^{i+1} + \sum_{i=0}^{\infty} \frac{b_i}{i+1} t_2^{i+1} + \sum_{i=0}^{\infty} \frac{c_i}{i+1} t_3^{i+1}.$$

A similar statement holds for $F^3_{\omega_2}$ and $F^3_{\omega_3}$.

By construction of ω_1 (and similarly for ω_2 and ω_3), we have $v(a_i) = 0$ for some $i \leq 2$, $v(b_j) = 0$ for some $j \leq 2$, and $v(c_k) = 0$ for some $k \leq 2$. Now by Lemma 3.5.1 applied to each of these three sums, we see that $\mathrm{New}_{1/3}(F^3_{\omega_1})$, $\mathrm{New}_{1/3}(F^3_{\omega_2})$, and $\mathrm{New}_{1/3}(F^3_{\omega_3})$ are contained in the set

$$\mathrm{conv}(\{(1,0,0), (3,0,0), (0,1,0,), (0,3,0), (0,0,1), (0,0,3)\}) \subset \mathbb{R}^3.$$

By Theorem 3.3.5 and Example 3.3.4, there are at most $3^3 - 1 = 26$ zero-dimensional components of interest. We are done by Lemmas 3.2.10 and 3.3.1. □

Lemma 3.5.3. *Let $C/\mathbb{Q}$ be a hyperelliptic curve of genus $g \geq 4$, with a rational Weierstrass point, geometrically simple Jacobian with $r \leq 1$, good reduction at 3, which satisfies condition (†).*

Let $P_1, P_2 \in C(\overline{\mathbb{Q}})$ be conjugate quadratic points, with $\overline{P_1}, \overline{P_2} \in C_{\mathbb{F}_3}(\mathbb{F}_9) \setminus C_{\mathbb{F}_3}(\mathbb{F}_3)$, and $P_3 \in C(\mathbb{Q})$ a rational point. If $n(\Lambda_C, \overline{P_1}) = 1$, there are at most 26 ordered triples

(Q_1, Q_2, Q_3) of conjugate cubic points in $D_{\overline{P_1}} \times D_{\overline{P_2}} \times D_{\overline{P_3}}$. If $n(\Lambda_C, \overline{P_1}) = 0$, there are no such triples.

Proof. In the first two coordinates, we can consider New_1 using Remark 2.2.1 and the fact that $\mathbb{Q}_3(P_1, Q_1)$ is unramified, whereas we have to consider $\mathrm{New}_{1/3}$ in the last coordinate. We also have that $n(\Lambda_C, \overline{P_1}) = n(\Lambda_C, \overline{P_2})$. Theorem 2.1.1 asserts that $n(\Lambda_C, \overline{P_1}) + n(\Lambda_C, \overline{P_2}) + n(\Lambda_C, \overline{P_3}) \leq 2$, so we have two cases. If $n(\Lambda_C, \overline{P_1}) = 0$, then there are no ordered triples by first statement of Lemma 3.4.2. If $n(\Lambda_C, \overline{P_1}) = 1$, then the result follows from the second statement of Lemma 3.4.2, Lemma 3.5.1, and the same computation as in Lemma 3.5.2. □

Lemma 3.5.4. *Let $C/\mathbb{Q}$ be a hyperelliptic curve of genus $g \geq 4$, with a rational Weierstrass point, geometrically simple Jacobian with $r \leq 1$, good reduction at 3, which satisfies condition (†).*

Let $P_1, P_2, P_3 \in C(\overline{\mathbb{Q}})$ be three conjugate cubic points, with $\overline{P_1}, \overline{P_2}, \overline{P_3} \in C_{\mathbb{F}_3}(\mathbb{F}_{27}) \setminus C_{\mathbb{F}_3}(\mathbb{F}_3)$. Then there are no ordered triples (Q_1, Q_2, Q_3) of conjugate cubic points in $D_{\overline{P_1}} \times D_{\overline{P_2}} \times D_{\overline{P_3}}$ not equal to (P_1, P_2, P_3).

Proof. The proof is similar to that of Lemma 3.5.2 with two changes. First, one can consider New_1 instead of $\mathrm{New}_{1/3}$ in all the coordinates, again using Remark 2.2.1 and the fact that $Q_3(P_1, Q_1)$ is unramified. Second, note that since P_1, P_2, and P_3 reduce to (necessarily distinct) points outside of $C_{\mathbb{F}_3}(\mathbb{F}_3)$, we know that P_1, P_2, and P_3 remain conjugate over $\mathbb{Q}_3$. Thus their reductions are conjugate over $\mathbb{F}_3$, so we have $n(\Lambda_C, \overline{P_1}) = n(\Lambda_C, \overline{P_2}) = n(\Lambda_C, \overline{P_3})$. By Theorem 2.1.1, this common value can only be 0, and then the result follows from Lemma 3.4.2. □

Proof of Theorem 3.1.5. For a given value of g, the family given by Proposition 3.2.5 and Lemma 3.4.5 comprises a positive proportion of all hyperelliptic curves with a rational Weierstrass point, since the latter is defined by finitely many congruence conditions. By Corollary 3.2.3, at least 25% of the curves in this family have rank $r \leq$

1; this is still a positive proportion of all the curves. Furthermore, under Assumption 3.3.7, a positive proportion of these curves satisfy condition (†).

Let C be such a curve. Any triple of conjugate cubic points on C that reduce to $\mathbb{F}_3$-points will have to lie in $D_{\overline{\infty}} \times D_{\overline{\infty}} \times D_{\overline{\infty}}$. We may choose $P_1 = P_2 = P_3 = \infty$ and apply Lemma 3.5.2 to conclude there are at most 26 ordered triples (Q_1, Q_2, Q_3) of conjugate cubic points in this residue class. But since $\overline{Q_1} = \overline{Q_2} = \overline{Q_3}$, each *unordered* triple is overcounted by a factor of 6, so there are at most $\lfloor 26/6 \rfloor = 4$ unordered triples of such cubic points.

If the minimal Weierstrass model of C is $y^2 = f(x)$, note that (for some choice of square root) the pair of quadratic points $(i, \pm\sqrt{f(i)})$ reduces to $(\bar{i}, \pm\alpha)$ for each $\bar{i} = 0, 1, 2$. Any triple of conjugate cubic points on C where two of the points reduce to $(\mathbb{F}_9 \setminus \mathbb{F}_3)$-points will have to lie in $D_{(\bar{i},\alpha)} \times D_{(\bar{i},-\alpha)} \times D_{\overline{\infty}}$ for some $\bar{i}$. In $D_{(\bar{i},\alpha)} \times D_{(\bar{i},-\alpha)} \times D_{\overline{\infty}}$, we may choose $P_1 = (i, \sqrt{f(i)})$, $P_2 = (i, -\sqrt{f(i)})$, and $P_3 = \infty$ and apply Lemma 3.5.3 with the values $n(\Lambda_C, (\bar{i}, \alpha))$, $n(\Lambda_C, (\bar{i}, -\alpha))$, and $n(\Lambda_C, \overline{\infty})$ to count ordered triples.

The last case is when a triple of conjugate cubic points on C reduces to (necessarily distinct) $(\mathbb{F}_{27} \setminus \mathbb{F}_3)$-points. In this setting, Lemma 3.5.4 asserts that there are no triples in this residue class away from their centers. Since C is hyperelliptic and has good reduction at 3, any *unordered* triple of conjugate cubic points (over $\mathbb{F}_3$) will have to lie over an unordered triple of cubic points of $\mathbb{P}^1_{\mathbb{F}_3}$, of which there are only $((3^3 + 1) - (3 + 1))/3 = 8$.

Using Theorem 2.1.1, we get the worst bound on the number of 0-dimensional components in all residue classes by assuming that for some $\bar{i} = 0, 1, 2$, $n(\Lambda_C, (\bar{i}, \alpha)) = n(\Lambda_C, (\bar{i}, -\alpha)) = 1$, and all the others will be 0. To conclude, there are at most

$$4 + 26 + 8 = 38$$

unordered triples of conjugate cubic points. $\square$

Chapter 4

Background — Hyperbolicity

In this chapter, we recall definitions and results on varieties of general type and hyperbolicity in the complex analytic, algebraic, and non-Archimedean analytic setting, and we conclude by stating the conjectures of Green–Griffiths–Lang–Vojta.

For a more detailed account of these definitions and conjectures, we highly recommend Javanpeykar's survey [Jav20].

4.1 Varieties of general type

To begin, we recall some basic notions of varieties of general type.

A variety X over K is *of general type* if the dimension of X equals the Kodaira dimension of X (or equivalently, the dimension of X equals the transcendence degree of the fraction field of the canonical ring minus one). It is well-known that being of general type descends along finite surjective morphism and that a variety of general type cannot be a group variety. We summarize these properties in the following lemma, which will be used later.

Lemma 4.1.1. *Let X be a projective variety of general type over K, and let $Y \to X$ denote a finite surjective morphism. Then Y is not a group variety.*

4.2 Hyperbolicity in complex analytic setting

We start with the classical notion of Brody hyperbolicity for complex varieties.

Definition 4.2.1. A complex-analytic space X is *Brody hyperbolic* if every holomorphic map $\mathbb{C} \to X$ is constant (i.e., if X does not admit an entire curve). A variety X over $\mathbb{C}$ is *Brody hyperbolic* if the complex analytification $X(\mathbb{C})$ is Brody hyperbolic.

Classical results in complex analysis lead to the following classification of Brody hyperbolic projective curves.

Theorem 4.2.2 (Liouville, Riemann, Schwarz, Picard)**.** *Let $X/\mathbb{C}$ be a smooth projective connected curve. Then X is Brody hyperbolic if and only if the genus of X is greater than 1.*

Remark 4.2.3. It is implicit in Theorem 4.2.2 that an elliptic curve is not Brody hyperbolic. More generally, an abelian variety of dimension g over $\mathbb{C}$ is not Brody hyperbolic as its associated complex-analytic space is uniformized by $\mathbb{C}^g$ by Riemann's uniformization theorem. We note that A even admits a dense entire curve, so it should be considered to be as far as possible from being Brody hyperbolic.

We now turn to the notion of Kobayashi hyperbolicity for complex varieties. In [**?**], Kobayashi defined an intrinsic pseudo-metric on a connected complex manifold X; a pseudo-metric d is a non-negative real valued function d satisfying the conditions of a metric except that $d(x, y)$ can equal zero for $x \neq y$.

Definition 4.2.4. Given two points $x, y \in X$, *a chain of holomorphic disks connecting x to y* consists of the following data: points $x = x_0, x_1, \ldots, x_{k-1}, x_k = y$ of X, points $a_1, \ldots, a_k$, $b_1, \ldots, b_k$ of the open unit disk D, and holomorphic mappings $f_1, \ldots, f_k$ of D into X such that $f_i(a_i) = x_{i=1}$ and $f_i(b_i) = x_i$ for $i = 1, \ldots, k$. The *length* of a chain of holomorphic disks connecting x to y is

$$\sum_{i=1}^{k} \rho(a_i, b_i),$$

where $\rho(\cdot, \cdot)$ is the Poincaré metric on D.

Definition 4.2.5. For a connected complex manifold X, the *Kobayashi pseudo-metric* $d_X(x, y)$ is the infimum of the length over all chains of holomorphic disks connecting x to y. If d_X is a metric on X, we say that X is *Kobayashi hyperbolic.*

Example 4.2.6. The Kobayashi pseudo-metric on $\mathbb{C}$ is identically zero. It suffices to show that $d_{\mathbb{C}}(0, 1) = 0$. To do so, consider the function given by multiplication by r on the unit disc. As $r \to \infty$, the hyperbolic distance between 0 and $1/r$ tends to 0. The intuitive reason why the Kobayashi pseudo-metric on $\mathbb{C}$ is identically zero is because we can make arbitrarily small dilations of the unit disk in $\mathbb{C}$.

The Kobayashi pseudo-metric plays an important role in characterizing complex manifolds as illustrated by the two theorems.

Theorem 4.2.7 ([Bar72])**.** *If X is Kobayashi hyperbolic, then d_X defines the complex topology of X.*

Theorem 4.2.8 ([Bro78])**.** *If X is compact, then X is Kobayashi hyperbolic if and only if X is Brody hyperbolic.*

We now introduce the "pseudo-fications" of these notions.

Definition 4.2.9. Let $X/\mathbb{C}$ be a variety and let Δ be a closed subset of X. We say that X is *Brody hyperbolic modulo* Δ if every holomorphic non-constant morphism $\mathbb{C} \to X(\mathbb{C})$ factors over $\Delta(\mathbb{C})$.

Definition 4.2.10. A variety X over $\mathbb{C}$ is *pseudo-Brody hyperbolic* if there is a proper closed subset $\Delta \subsetneq X$ such that X is Brody hyperbolic modulo Δ.

Remark 4.2.11.

- The notion of pseudo-Brody hyperbolicity is a birational invariant.
- A variety is Brody hyperbolic if and only if it is Brody hyperbolic modulo $\emptyset$.

Definition 4.2.12. Let $X/\mathbb{C}$ be a variety and let Δ be a closed subset of X. We say that X is *Kobayashi hyperbolic modulo* Δ if, for every $x \neq y \in X(\mathbb{C}) \setminus \Delta(\mathbb{C})$, the Kobayashi pseud-metric $d_X(x, y)$ is positive.

Definition 4.2.13. A variety X over $\mathbb{C}$ is *pseudo-Kobayashi hyperbolic* if there is a proper closed subset $\Delta \subsetneq X$ such that X is Kobayashi hyperbolic modulo Δ.

Remark 4.2.14.

- The notion of pseudo-Kobayashi hyperbolicity is a birational invariant.
- The notion of pseudo-Kobayashi hyperbolicity remains quite mysterious. Indeed, it is not known whether a pseudo-Brody hyperbolic projective variety $X/\mathbb{C}$ is pseudo-Kobayashi hyperbolic.

4.3 Hyperbolicity in the algebraic setting

Now, we recall definitions and results concerning notions of hyperbolicity in the algebraic setting. We will follow [JK19, JX19].

Definition 4.3.1 ([JK19, Definition 2.1])**.** A finite type scheme X over K is *groupless (over K)* if, for every finite type connected group scheme G over K, every morphism of K-schemes $G \to X$ is constant.

When X/K is proper, one can use Chevalley's theorem [Con02] on finite type connected algebraic groups over K to show that being groupless can be tested on abelian varieties.

Lemma 4.3.2 ([JK19, Lemma 2.4 & Lemma 2.5]). *Let X/K be a proper variety. Then X is groupless over K if and only if for every abelian variety A/K, every morphism $A \to X$ is constant.*

We conclude our discussion on hyperbolicity in the algebraic setting by recalling a more general notion of groupless introduced by Javanpeykar–Xie [JX19] following ideas of Vojta [Voj15].

Definition 4.3.3 ([JX19, Definition 3.1]). Let X/K be a variety and let $\Delta \subset X$ be a closed subscheme. X is *groupless modulo* Δ *(over K)* if, for every finite type connected group scheme G/K and every dense open subscheme $U \subset G$ with $\operatorname{codim}(G \setminus U) \geq 2$, every non-constant morphism $U \to X$ factors over Δ.

Definition 4.3.4 ([JX19, Definition 3.2]). We say that X is *pseudo-groupless (over K)* if there exists a proper closed subscheme $\Delta \subset X$ such that X is groupless modulo Δ.

Remark 4.3.5. The reader might wonder why one considers "big" open subschemes $U \subset G$ in Definition 4.3.3. The reason is that Vojta [Voj15, Section 4] proved that for $A/\mathbb{C}$ an abelian variety and $U \subset A$ a dense open subscheme with $\operatorname{codim}(A \setminus U) \geq 2$, U is not Brody hyperbolic, and in fact, U admits a Zariski dense holomorphic curve. As such, one should think of "big" open subschemes of group schemes as being as far from hyperbolic as possible.

Remark 4.3.6 ([JX19, Remark 3.3]). Let X be a proper variety over K and let $\Delta \subset X$ be a closed subscheme. By the valuative criterion for properness, X is groupless modulo Δ if and only if for every finite type, connected group scheme G over K and every dense open subscheme $U \subset G$, every non-constant morphism $U \to X$ factors over Δ.

Remark 4.3.7 ([JX19, Remark 3.4]). Let X be a proper variety over K which does not contain any rational curves and let $\Delta \subset X$ be a closed subscheme. Then X is groupless modulo Δ if and only if for every finite type connected group scheme G/K, every non-constant morphism factors over Δ. This follows because every finite type connected group scheme G over a field of characteristic zero is smooth [Sta15, Lemma 047N] and since any morphism from a "big" open $U \subset G$ to X will uniquely extend to a morphism $G \to X$ by Remark 4.3.6 and [Deb01, Corollary 1.44].

Furthermore, a proper variety is groupless if and only if it is groupless modulo the emptyset.

Conjecture 4.5.2 posits that being pseudo-groupless is equivalent to being of general type. Since being of general type is a birational invariant and descends along finite étale morhpisms, we need to know that being pseudo-groupless also satisfies these properties for this conjecture to have any hope of being true.

Lemma 4.3.8 ([JX19, Lemmas 3.8 & Lemma 3.9])**.** *Let X and Y be proper varieties, and assume that X is birational to Y. Then, X is pseudo-groupless over K if and only if Y is pseduo-groupless over K.*

Remark 4.3.9 ([Jav20, Remark 6.7])**.** The notion of groupless is not a birational invariant. Indeed, for C/K a projective curve of genus ≥ 2, the surface $C \times C$ is groupless, but the blow-up S of $C \times C$ at a point is not because it contains a rational curve in the exceptional locus of the blow-up $S \to C \times C$. Thus, S is groupless modulo this exceptional locus, and hence S is pseudo-groupless.

Lemma 4.3.10 ([Jav20, Lemma 6.5])**.** *Let $f\colon X \to Y$ be a finite étale morphism of proper varieties over K. Then X is pseudo-groupless over K if and only if Y is pseudo-groupless over K.*

Another very important property of pseudo-grouopless is that for a proper variety, one can test for it on "big" open subsets of abelian varieties.

Proposition 4.3.11 ([JX19, Corollary 3.17])**.** *If X/K is a proper variety and Δ is a closed subscheme of X, then X is groupless modulo Δ if and only if for every abelian* *variety A/K and every dense open subscheme $U \subset A$ with* $\operatorname{codim}(A \setminus U) \geq 2$*, every non-constant morphism of varieties $U \to X$ factors over Δ.*

4.4 Hyperbolicity in the non-Archimedean analytic setting

We now discuss non-Archimedean notions of hyperbolicity following [JV18, Mor19].

Definition 4.4.1 ([JV18, Definition 2.3, Lemma 2.14, Lemma 2.15])**.** A variety X over K is *K-analytically Brody hyperbolic* if

- every analytic morphism $\mathbb{G}_{m,K}^{\mathrm{an}} \to X^{\mathrm{an}}$ is constant, and
- for every abelian variety A over K, every morphism $A \to X$ is constant.

In [Mor19], we provided a definition of pseudo-K-analytically Brody hyperbolic.

Definition 4.4.2 ([Mor19, Definition 2.2])**.** Let X be a variety over K and let $\Delta \subset X$ be a proper closed subscheme. Then X is *K-analytically Brody hyperbolic modulo* Δ if

- every non-constant analytic morphism $\mathbb{G}_{m,K}^{\mathrm{an}} \to X^{\mathrm{an}}$ factors over Δ^{an}, and
- for every abelian variety A over K and every dense open subset $U \subset A$ with $\operatorname{codim}(A \setminus U) \geq 2$, every non-constant morphism $U \to X$ of schemes factors over Δ.

Definition 4.4.3 ([Mor19, Definition 2.5])**.** We say that X is *pseudo-K-analytically Brody hyperbolic* if there exists some proper closed subscheme $\Delta \subset X$ such that X is K-analytically Brody hyperbolic modulo Δ.

Remark 4.4.4. It is not hard to see that a proper scheme X over K is K-analytically Brody hyperbolic if and only if X is K-analytically Brody hyperbolic modulo the emptyset. Indeed, if X is K-analytically Brody hyperbolic and proper, then X does not contain rational curves.

Remark 4.4.5. By Proposition 4.3.11, we have that a proper variety X/K is K-analytically Brody hyperbolic modulo Δ if and only if every non-constant analytic morphism $\mathbb{G}_{m,K}^{\mathrm{an}} \to X^{\mathrm{an}}$ factors over Δ^{an} and X is groupless modulo Δ over K.

Remark 4.4.6. Let A/K be an abelian variety and let G/K be a semi-abelian variety. By [Moc12, Lemma A.2], for every dense open subset $U \subset A$ with $\mathrm{codim}(A \setminus U) \geq 2$, we have that any morphism $U \to G$ extends uniquely to a morphism $A \to G$. Using this result, we immediately have that a closed subscheme X of a semi-abelian variety G is K-analytically Brody hyperbolic if and only if X is K-analytically Brody modulo $\emptyset$.

As with pseudo-groupless, we can prove that pseudo-K-analytically Brody hyperbolicity is a birational invariant and descends along finite étale morphisms.

Lemma 4.4.7. *Let X and Y be proper varieties, and assume that X is birational to Y. Then, X is pseudo-K-analytically Brody hyperbolic if and only if Y is pseduo-K-analytically Brody hyperbolic.*

Proof. The proof follows *mutatis mutandis* from [JX19, Lemmas 3.8 & 3.9]. □

Lemma 4.4.8. *Let $f\colon X \to Y$ be a finite étale morphism of proper varieties over K. Then X is pseudo-K-analytically Brody hyperbolic over K if and only if Y is pseudo-K-analytically Brody hyperbolic.*

Proof. Using Remark 4.4.5 and Lemma 4.3.10, it suffices to consider analytic morphisms of tori. In this setting, the result follows from the arguments of [JV18, Proposition 2.13]. □

We also recall that Conjecture 4.5.2 is true for projective curves.

Theorem 4.4.9 ([Che94, Theorem 3.6]). *Let C/K be a connected, projective curve. Then, the following are equivalent:*

- *C is of general type,*
- *C is pseudo-groupless,*
- *C is pseudo-K-analytically Brody hyperbolic.*

We now introduce the notions of non-Archimedean entire curves and algebraic degeneracy.

Definition 4.4.10. For a K-analytic space $\mathscr{X}$, an analytic morphism

$$\varphi\colon \mathbb{G}_{m,K}^{\mathrm{an}} \to \mathscr{X}$$

is called *non-Archimedean entire curve* in $\mathscr{X}$.

Remark 4.4.11. In the complex analytic setting, the exponential and logarithm maps provide an isomorphism between $\mathbb{C}$ and $\mathbb{C}^{\times}$, and so Brody hyperbolicity could equivalently be defined by the non-existence of non-constant morphisms from $\mathbb{C}^{\times}$ into a complex analytic manifold. However, in the non-Archimedean setting, the exponential map is not convergent everywhere, and so we do not have such an isomorphism; in fact, by [Che94, Proposition 3.3], every analytic map from $\mathbb{A}_K^{1,\mathrm{an}} \to \mathbb{G}_{m,K}^{\mathrm{an}}$ is constant. As a result, testing hyperbolicity on analytic morphisms from $\mathbb{A}_K^{1,\mathrm{an}}$ or $\mathbb{G}_{m,K}^{\mathrm{an}}$ can yield different results. For example, a result of Cherry (*loc. cit.* Theorem 3.5) states that for an abelian variety A/K, every analytic map $\mathbb{A}_K^{1,\mathrm{an}} \to A^{\mathrm{an}}$ is constant.

However, there can exist non-constant analytic morphisms $\mathbb{G}_{m,K}^{\mathrm{an}} \to A^{\mathrm{an}}$ if A does not have good reduction over $\mathcal{O}_K$. The reason for this is that analytic tori appear in the non-Archimedean uniformization of abelian varieties [BL84, Theorem 8.8]. Moreover, Definition 4.4.10 appears to be the "correct" one as it aligns with our desideratum and gives insight into the K-analytic Brody hyperbolicity of a K-analytic space.

Definition 4.4.12. A variety X/K is *algebraically degenerate* if, for every non-Archimedean entire curve $\varphi\colon \mathbb{G}_{m,K}^{\mathrm{an}} \to X^{\mathrm{an}}$, there exists a proper closed subscheme $Y_\varphi \subset X$, which depends on φ, such that $\varphi(\mathbb{G}_{m,K}^{\mathrm{an}}) \subset Y_\varphi$.

Remark 4.4.13. When X is proper, being algebraically degenerate is equivalent to the non-existence of a Zariski dense non-Archimedean entire curve. Indeed, if a non-Archimedean entire curve is not Zariski dense, then its Zariski closure is a proper closed subscheme of X by Berkovich analytic GAGA [Ber90, Corollary 3.4.13].

4.5 The conjectures of Green–Griffiths–Lang–Vojta

To conclude this section, we state the weak and strong forms of the Green–Griffiths–Lang–Vojta conjecture for varieties as well as their non-Archimedean versions [Jav20].

First, we state the weak Green–Griffiths–Lang–Vojta conjecture

Conjecture 4.5.1 (Weak Green–Griffiths–Lang)**.** *Let K be an algebraically closed field of characteristic zero, and let X/K be a quasi-projective variety. Then the following statements are equivalent:*

1. *X is groupless over K,*
2. *Every integral subvariety of X is of log-general type,*
3. *If $K = \mathbb{C}$, then X is Brody hyperbolic,*
4. *If K is a complete, non-Archimedean valued field, then X is K-analytically Brody hyperbolic.*

Conjecture 4.5.2 (Strong Green–Griffiths–Lang–Vojta). *Let K be an algebraically closed field of characteristic zero, and let X/K be a quasi-projective variety. Then the following statements are equivalent:*

1. *X is pseudo-groupless over K,*

2. *X is of log-general type;*

3. *If $K = \mathbb{C}$, then X is pseudo-Brody hyperbolic,*

4. *If K is a complete, non-Archimedean valued field, then X is pseudo-K-analytically Brody hyperbolic.*

Remark 4.5.3. We show that the equivalence (1)$\Leftrightarrow$(4) in Conjecture 4.5.2 implies (1)$\Leftrightarrow$(4) in Conjecture 4.5.1. Assume that X is groupless, so in particular, X is pseudo-groupless. By Conjecture 4.5.2, we have that X is K-analytically Brody hyperbolic modulo some proper closed subset $\Delta \subset X$. Now, since X is groupless, it follows that Δ is groupless. Repeating the above argument, we see that Δ is K-analytically Brody hyperbolic, and hence we can conclude that X is K-analytically Brody hyperbolic.

Chapter 5

Statement of results on the strong non-Archimedean Green–Griffiths–Lang–Vojta conjecture

In this short chapter, we will state our results concerning the non-Archimedean Green–Griffiths–Lang–Vojta conjecture.

Our first result is the non-archimedean analogue of results of Abramovich, Faltings, Kawamata, Noguchi, Ueno, Vojta [Abr94, Fal91, Fal94, Kaw80, Nog98, Uen73, Voj96].

Theorem 5.0.1 (Theorem B). *Let K be an algebraically closed complete non-Archimedean valued field of characteristic zero. Let X be a closed subvariety of a semi-abelian variety G over K. Let $Sp(X)$ be the union of the subvarieties of X which are translates of positive-dimensional closed subgroups of G. Then, X is groupless modulo* $\mathrm{Sp}(X)$ *if and only if X is K-analytically Brody hyperbolic modulo* $\mathrm{Sp}(X)$*.*

A direct consequence of Theorem B is the following characterization of groupless closed subvarieties of a semi-abelian variety.

Corollary 5.0.2 ([Mor19, Corollary B]). *Let K be an algebraically closed complete non-archimedean valued field of characteristic zero, and let X be a closed subvariety of a semi-abelian variety G over K. Then, X is groupless if and only if X is K-analytically Brody hyperbolic.*

Our second result concerns the algebraic degeneracy of non-Archimedean entire curves in projective varieties of general type admitting a dominant morphism to an elliptic curve.

Theorem 5.0.3 (Theorem C). *Let K be an algebraically closed, complete, non-Archimedean valued field of characteristic zero. Let X/K be a projective variety of general type dominating an elliptic curve. Then, any non-Archimedean entire curve $\varphi\colon \mathbb{G}_{m,K}^{\mathrm{an}} \to X^{\mathrm{an}}$ is algebraically degenerate (Definition 4.4.12).*

Using Theorem C, we prove (2)$\Leftrightarrow$(3) in Conjecture 4.5.2 for projective surfaces admitting a dominant morphism to an elliptic curve.

Theorem 5.0.4 (Theorem D). *Let K be an algebraically closed, complete, non-Archimedean valued field of characteristic zero, and let S/K be a projective surface admitting a dominant morphism to an elliptic curve. Then, S is pseudo-groupless if and only if S is pseudo-K-analytically Brody hyperbolic.*

As an immediate corollary of Theorem D, we characterize projective groupless surfaces admitting a dominant morphism to an elliptic curve, and hence prove Conjecture 4.5.1 for such projective surfaces.

Corollary 5.0.5 ([Mor20, Corollary C]). *Let K be an algebraically closed, complete, non-Archimedean valued field of characteristic zero, and let S/K be a projective surface admitting a dominant morphism to an elliptic curve. Then, S is groupless over K if and only if S is K-analytically Brody hyperbolic.*

Chapter 6

The non-Archimedean Green–Griffiths–Lang–Vojta for closed subvarieties of a semi-abelian variety

In this chapter, we describe our results from [Mor19] on the-Archimedean Green–Griffiths–Lang–Vojta for closed subvarieties of a semi-abelian variety.

6.1 Introduction

Our starting point is the following theorem, which is the culmination of results in [Fal91, Fal94, Abr94, Voj96, Nog98]. The definitions of the notions appearing in the following theorem are stated in [Lan87, p. 78] and [Jav20, Definitions 7.1, 8.1].

Theorem 6.1.1 (Abramovich, Faltings, Kawamata, Noguchi, Ueno, Vojta)**.** *Let X be a closed subvariety of a semi-abelian variety G over $\mathbb{C}$. Let $\mathrm{Sp}(X)$ be the union of the subvarieties of X which are translates of positive-dimensional closed subgroups of*

G. Then the following statements hold.

1. *The subset* $\mathrm{Sp}(X)$ *is Zariski closed in* X.

2. *The variety* X *is of log-general type if and only if* $\mathrm{Sp}(X) \neq X$.

3. *The variety* X *is arithmetically hyperbolic modulo* $\mathrm{Sp}(X)$.

4. *The variety* X *is Brody hyperbolic modulo* $\mathrm{Sp}(X)$.

Our main result is the non-Archimedean analogue of the statements (2), (3), and (4) in Theorem 6.1.1.

Theorem 6.1.2 (Theorem B)**.** *Let K be an algebraically closed complete non-Archimedean valued field of characteristic zero. Let X be a closed subvariety of a semi-abelian variety G over K. Let $Sp(X)$ be the union of the subvarieties of X which are translates of positive-dimensional closed subgroups of G. Then, X is groupless modulo* $\mathrm{Sp}(X)$ *if and only if X is K-analytically Brody hyperbolic modulo* $\mathrm{Sp}(X)$.

A direct consequence of Theorem B is the following characterization of groupless closed subvarieties of a semi-abelian variety.

Corollary 6.1.3. *Let K be an algebraically closed complete non-Archimedean valued field of characteristic zero, and let X be a closed subvariety of a semi-abelian variety G over K. Then, X is groupless if and only if X is K-analytically Brody hyperbolic.*

6.2 Non-Archimedean entire curves in semi-abelian varieties

Let G be a semi-abelian variety over K. Since G is semi-abelian, there is a split torus $T_1 \subset G$, an abelian variety A over K, and a short exact sequence of commutative group schemes

$$0 \to T_1 \to G \to A \to 0.$$

Our goal is to prove that, if $\phi\colon \mathbb{G}_m^{\mathrm{an}} \to G^{\mathrm{an}}$ is a morphism, then the Zariski closure of its image is the translate of the analytification of an *algebraic* subgroup of G; see Proposition 6.2.5 for a precise statement.

We start by recalling that line bundles on analytifications of tori are trivial.

Lemma 6.2.1. *Let X be a separated, good, strictly K-analytic space, and let X_0 denote the associated rigid analytic space. Then* $\mathrm{Pic}(X) \cong \mathrm{Pic}(X_0)$.

Proof. This follows from [Ber93, Corollary 1.3.5] and the bottom of *loc. cit.* p. 37. □

Lemma 6.2.2. *If L is a line bundle on a split torus $\mathbb{G}_{m,K}^{r,\mathrm{an}}$, then L is trivial.*

Proof. Since the Berkovich analytification of $\mathbb{G}_{m,K}^r$ is a separated, good, strictly K-analytic space, our result follows from [FvdP04, Theorem 6.3.3.(2)] and Lemma 6.2.1. □

Lemma 6.2.3. *Let $\phi\colon \mathbb{G}_m^{\mathrm{an}} \to G^{\mathrm{an}}$ be a morphism, and let $\widetilde{\phi}\colon \mathbb{G}_m^{\mathrm{an}} \to \widetilde{G}$ be a lift of this morphism to the universal cover $\widetilde{G}$ of G^{an}. Then, the image $\widetilde{\phi}(\mathbb{G}_m^{\mathrm{an}})$ is contained inside a split torus T^{an} of $\widetilde{G}$.*

Proof. Let $\widetilde{A}$ be the universal covering of A^{an}. By [BL84, Uniformization Theorem 8.8], there is a semi-abelian variety H over K with $\widetilde{A} \cong H^{\mathrm{an}}$, an abelian variety B over K with good reduction over $\mathcal{O}_K$, a split torus $T_2 \subset H$, and a short exact sequence of commutative group schemes

$$0 \to T_2 \to H \to B \to 0.$$

Let $\widetilde{G}$ be the universal covering space of G^{an}. Note that there is a structure of a commutative Berkovich analytic group on $\widetilde{G}$ which makes $\widetilde{G} \to G^{\mathrm{an}}$ into a homomorphism. By the universal property of universal covering spaces, the surjective homomorphism $G^{\mathrm{an}} \to A^{\mathrm{an}}$ lifts uniquely to a homomorphism $\widetilde{G} \to H^{\mathrm{an}}$.

The image of $\mathbb{G}_m^{\mathrm{an}} \to \widetilde{G} \to H^{\mathrm{an}}$ is contained in T_2^{an}. Indeed, the morphism $\mathbb{G}_m^{\mathrm{an}} \to H^{\mathrm{an}} \to B^{\mathrm{an}}$ is constant, since B^{an} has good reduction [Che94, Theorem 3.2], and so the image of $\mathbb{G}_m^{\mathrm{an}}$ in H^{an} lands inside its torus T_2^{an} (up to translation).

Since T_1^{an} is simply-connected [Ber90, Section 6.3], the subgroup $T_1^{\mathrm{an}} \subset G^{\mathrm{an}}$ lifts to a subgroup $T_1^{\mathrm{an}} \subset \widetilde{G}$. Note that the homomorphism $T_1^{\mathrm{an}} \to H^{\mathrm{an}}$ factors over T_2^{an}, and that the morphism $T_1^{\mathrm{an}} \to T_2^{\mathrm{an}}$ is algebraic [Che94, Proposition 3.4], i.e., the analytification of some morphism $T_1 \to T_2$. Let T_3 be the image of this morphism, which is again a split torus.

Let F be the inverse image of $T_3^{\mathrm{an}} \subset H^{\mathrm{an}}$ in $\widetilde{G}$. Note that F is a closed subgroup of $\widetilde{G}$ and that the kernel of the homomorphism $F \to T_3^{\mathrm{an}}$ equals T_1^{an}. Thus, there is a short exact sequence of rigid analytic groups

$$0 \to T_1^{\mathrm{an}} \to F \to T_3^{\mathrm{an}} \to 0.$$

By Lemma 6.2.2, the above sequence splits, and so F is the analytification of a split torus T. This shows that the image $\widetilde{\phi}(\mathbb{G}_m^{\mathrm{an}})$ is contained inside the split torus T^{an}, as required. $\square$

Lemma 6.2.4. *Let $\phi\colon \mathbb{G}_m^{\mathrm{an}} \to G^{\mathrm{an}}$ be a morphism. Suppose that the image of $\mathbb{G}_m^{\mathrm{an}} \to G^{\mathrm{an}} \to A^{\mathrm{an}}$ is Zariski dense. Then, the image of ϕ is Zariski dense in G^{an}.*

Proof. Lemma 6.2.3 asserts that $\phi(\mathbb{G}_m^{\mathrm{an}})$ is an analytic subgroup F' of G^{an}, as it is the composition of group homomorphisms $\widetilde{\phi}$ and the uniformization map, which is an analytic group homomorphism. Since $\mathbb{G}_m^{\mathrm{an}} \to G^{\mathrm{an}} \to A^{\mathrm{an}}$ is Zariski dense, F' dominates A^{an}, and this analytic group homomorphism has kernel T_1^{an}. Moreover, we have the following morphism of short exact sequences of analytic groups:

$$\begin{array}{ccccccccc}
0 & \to & T_1^{\mathrm{an}} & \longrightarrow & F' & \longrightarrow & A^{\mathrm{an}} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f} & & \| & & \\
0 & \to & T_1^{\mathrm{an}} & \longrightarrow & G^{\mathrm{an}} & \longrightarrow & A^{\mathrm{an}} & \longrightarrow & 0.
\end{array}$$

By Berkovich analytic GAGA [Ber90, Corollary 3.4.10], $\mathrm{Pic}(A^{\mathrm{an}})$ is in bijective correspondence with $\mathrm{Pic}(A)$, which implies that F' is in fact an algebraic subgroup of G^{an}. Moreover, the short five lemma tells us that the morphism f must be an isomorphism. $\square$

Proposition 6.2.5. *Let $\phi\colon \mathbb{G}_m^{\mathrm{an}} \to G^{\mathrm{an}}$ be a morphism. Then, the Zariski closure of $\phi(\mathbb{G}_m^{\mathrm{an}})$ in G^{an} is the analytification of the translate of of an algebraic subgroup of G.*

Proof. Let $\psi\colon \mathbb{G}_m^{\mathrm{an}} \to A^{\mathrm{an}}$ be the composition of ϕ and the surjective homomorphism $G^{\mathrm{an}} \to A^{\mathrm{an}}$. By Lemma 6.2.3, the image $\phi(\mathbb{G}_m^{\mathrm{an}})$ is an analytic subgroup of G^{an}. Therefore, the image $\psi(\mathbb{G}_m^{\mathrm{an}})$ is an analytic subgroup of A^{an}. Thus, the Zariski closure of the image of ψ is an abelian subvariety E^{an} of A^{an} (see [Che94, Proof of Theorem 3.6]).

Now, let F be the preimage of E inside G, and note that F is a semi-abelian variety (as it is a closed subgroup of G). Clearly, the image of the morphism $\phi\colon \mathbb{G}_m^{\mathrm{an}} \to G^{\mathrm{an}}$ is contained in F^{an}. Now, by construction, the image of the composed morphism $\mathbb{G}_m^{\mathrm{an}} \to F^{\mathrm{an}} \to E^{\mathrm{an}}$ is Zariski dense in E. Therefore, by Lemma 6.2.4, the image of $\mathbb{G}_m^{\mathrm{an}}$ in F^{an} is the analytification of the translate of an algebraic subgroup of F. In particular, it is the analytification of the translate of an algebraic subgroup of G. $\square$

The following example shows that the image of an algebraic group under an analytic homomorphism is not necessarily an algebraic subgroup.

Example 6.2.6. Let E/K be an elliptic curve with multiplicative reduction and let $\phi\colon \mathbb{G}_{m,K}^{\mathrm{an}} \to E^{\mathrm{an}}$ be the universal covering of E^{an}. Consider the semi-abelian variety $G = \mathbb{G}_{m,K} \times E$. Let $\mathbb{G}_{m,K}^{\mathrm{an}} \to \mathbb{G}_{m,K}^{\mathrm{an}} \times E^{\mathrm{an}}$ be the morphism defined by $z \mapsto (z, \phi(z))$. The image of this morphism is *not* an algebraic subgroup of G^{an}. However, its Zariski closure equals G^{an}.

To end this section, we prove Theorem B.

Proof of Theorem B. Proposition 6.2.5 tells us that the Zariski closure of every analytic morphism $\mathbb{G}_m^{\mathrm{an}} \to X^{\mathrm{an}} \subset G^{\mathrm{an}}$ is contained in $\mathrm{Sp}(X)^{\mathrm{an}}$. To conclude the proof, it suffices to show that for every abelian variety A over K and every dense open subset $U \subset A$ with $\mathrm{codim}(A \setminus U) \geq 2$, we have that every non-constant morphism $U \to X$ of schemes factors over $\mathrm{Sp}(X)$. By Remark 4.4.6, every morphism $U \to X \subset G$ extends to a morphism $A \to X \subset G$. Now, by [Iit76, Theorem 2], any morphism between semi-abelian varieties is the composition of a group homomorphism and a translation, so that the image of $A \to X \subset G$ factors over $\mathrm{Sp}(X)$, as desired. $\square$

Chapter 7

The non-Archimedean Green–Griffiths–Lang for projective surfaces dominating an elliptic curve

In this chapter, we will prove our results from [Mor20]. and on the non-Archimedean Green–Griffiths–Lang for projective surfaces dominating an elliptic curve.

7.1 Introduction

The works of Dethloff–Yu [DL07], Noguchi–Winkelmann–Yamanoi [NWY07,NWY13], and Winkelmann [Win11] proved a weak variant of Green-Griffiths–Lang conjecture when the irregularity $q(X)$ of a complex variety X is equal to the dimension. In particular, they show that for a smooth projective variety $X/\mathbb{C}$ of general type with $q(X) = \dim X$, every complex entire curve $f\colon \mathbb{C} \to X(\mathbb{C})$ is algebraically degenerate. To the author's knowledge, there does not appear to be any literature on this conjecture when $q(X) < \dim X$.

Our first result concerns the algebraic degeneracy of non-Archimedean entire curves in projective varieties of general type admitting a dominant morphism to an elliptic curve, which are varieties with irregularity less than their dimension.

Theorem 7.1.1. *Let K be an algebraically closed, complete, non-Archimedean valued field of characteristic zero. Let X/K be a projective variety of general type dominating an elliptic curve. Then, any non-Archimedean entire curve $\varphi\colon \mathbb{G}_{m,K}^{\mathrm{an}} \to X^{\mathrm{an}}$ is algebraically degenerate.*

Using the above result, we prove the strong non-Archimedean Green–Griffiths–Lang conjecture for projective surfaces admitting a dominant morphism to an elliptic curve.

Theorem 7.1.2. *Let K be an algebraically closed, complete, non-Archimedean valued field of characteristic zero, and let S/K be a projective surface admitting a dominant morphism to an elliptic curve. Then, S is pseudo-groupless if and only if S is pseudo-K-analytically Brody hyperbolic.*

7.2 Related results

We now describe results related to the Green–Griffiths–Lang conjecture for surfaces.

In the complex analytic setting, the works of Bogomolov [Bog77] and McQuillan [McQ98] proved the classical Green–Griffiths–Lang conjecture for projective surfaces $S/\mathbb{C}$ of general type with enough 2-jets (e.g., when $13c_1^2(S) > 9c_2(S)$). As mentioned above, previous results on the classical Green–Griffiths–Lang conjecture focused on varieties where the irregularity was greater than or equal to the dimension. Moreover, our results are complementary to these as we focus on varieties with irregularity strictly less than the dimension.

In the algebraic setting, Javanpeykar–Xie [JX19, Lemma 3.23] proved that projective integral pseudo-groupless surfaces are of general type. Combining their result with Theorem D, the remaining case of Conjecture 4.5.2 for surfaces of irregularity one is (1) $\Rightarrow$ (2).

In the non-Archimedean analytic setting, Cherry [Che94, Theorem 3.6] proved Conjecture 4.5.2 for closed subvarieties of an abelian variety, and hence for proper surfaces of irregularity greater than two. His proof follows from the uniformization theorem of Bosch–Lütkebohmert [BL84, Theorem 8.8] and the study of analytic maps from tori into semi-abelian varieties. Combining Cherry's results with Theorem D, the remaining cases of (2)$\Leftrightarrow$(3) in Conjecture 4.5.2 for irregular surfaces are those surfaces with irregularity two.

7.3 Surfaces of general type of irregularity one

We now discuss the extensive work on determining projective surfaces of general type with irregularity one. Note that such surfaces satisfy the conditions of Theorem D.

Let X/K be a (minimal) surface of general type. If the geometric genus of X is zero, then we have that the irregularity of X is zero since the Euler characteristic $\chi(\mathcal{O}_X)$ is positive . Moreover, minimal surfaces of general type with $p_g(X) = q(X) = 1$ are the irregular surfaces with the lowest geometric genus. It is well-known that for any minimal, irregular surface X/K of general type, Debarre's inequality $K_X^2 \geq 2p_g(X)$ holds [Deb82], and on the other hand, the Miyaoka–Yau inequality [Miy77, Yau77, Yau78] yields $K_X^2 \leq 9\chi(\mathcal{O}_X)$. This tells us that if $p_g(X) = q(X) = 1$, we have $2 \leq K_X^2 \leq 9$.

Many authors have tried to classify the minimal surfaces of general type with $p_g(X) = q(X) = 1$ with prescribed K_X^2 as above. Surfaces with $p_g(X) = q(X) = 1$ and $K_X^2 = 2$ were classified by [Cat81], and the works of [CC91,CC93] dealt with the $K_X^2 =$

3 case. For higher values of K_X^2, there are some sporadic examples (cf. [Cat99, Pol05, Pol06, Pol08, CP09, Pol09, MP10]). The latter results of Polizzi focus on classifying minimal surfaces of general type which are isogenous to a product of curves. In another direction, Takahashi [Tak98] showed that for all values $p_g(X) \geq 2$, there exists a minimal algebraic surface of general type with $K_X^2 = 3p_g(X)$ and $q(X) = 1$.

7.4 Semi-coverings

To conclude this preliminary section, we recall the notion of semi-covering from work of Brazas [Bra12].

Definition 7.4.1 ([Bra12, Definition 3.1])**.** A *semi-covering* is a local homeomorphism with continuous lifting of paths and homotopies.

By [Bra12, Remark 3.6], one can check that a covering is in fact a semi-covering, so this notion generalizes the notion of covering. Unlike coverings, path-connected semi-coverings satisfy a very useful "two out of three" property.

Proposition 7.4.2 ([Bra12, Lemma 3.4 & Corollary 3.5])**.** *Let $p\colon X \to Y$, $q\colon Y \to Z$ and $r = q \circ p$ be maps of path-connected spaces. If two of p, q, r are semi-coverings, then so is the third.*

A covering space of a path-connected, locally path-connected topological group is a topological group, and the exact same argument proves the analogous result for semi-coverings.

Lemma 7.4.3. *Let G be a path-connected, locally path-connected topological group. Let $p\colon H \to G$ be a semi-covering. Then H is also a topological group and p is a group homomorphism.*

To conclude the preliminary section, we recall that the structure of a K-analytic space can be lifted along a semi-coverings, and more generally local homeomorphisms. The proof of Lemma 7.4.4 is identical to [For81, Chapter I, Theorem 4.6].

Lemma 7.4.4. *Let $\mathscr{X}$ be a K-analytic space, let $\mathscr{Y}$ be a Hausdorff topological space, and let $p\colon \mathscr{Y} \to \mathscr{X}$ be a local homeomorphism. Then there exists a unique K-analytic space structure on $\mathscr{Y}$ such that p is analytic.*

We combine Lemmas 7.4.3 and 7.4.4 into the following useful corollary.

Corollary 7.4.5. *Let $\mathscr{G}$ be a K-analytic group, let $\mathscr{H}$ be a Hausdorff topological space, and let $p\colon \mathscr{H} \to \mathscr{G}$ be a semi-covering. Then there exists a unique K-analytic group structure on $\mathscr{H}$ such that p is an analytic group homomorphism.*

7.5 Non-Archimedean entire curves in projective varieties of general type dominating an elliptic curve

In this section, we will prove Theorem C. Let X/K be a projective variety of general type, let E/K be an elliptic curve, let $a\colon X \to E$ be a dominant morphism, and let $\varphi\colon \mathbb{G}_{m,K}^{\mathrm{an}} \to X^{\mathrm{an}}$ be a non-Archimedean entire curve. By Remark 4.4.13, it suffices to prove that there does not exist a non-Archimedean entire curve, which is Zariski dense. We also note that we can and do assume that X has dimension ≥ 2, as Theorem C is true when X is a curve by Theorem 4.4.9.

If E has good reduction over $\mathcal{O}_K$, then our result follows immediately.

Lemma 7.5.1. *With the notation as above, suppose that E has good reduction over $\mathcal{O}_K$. Then φ cannot be Zariski dense.*

Proof. The composed morphism $a^{\mathrm{an}} \circ \varphi\colon \mathbb{G}_{m,K}^{\mathrm{an}} \to E^{\mathrm{an}}$ is constant by [Che94, Theorem 3.2], and hence the image of φ is contained in a fiber F of a, which has dimension

≤ 1. Furthermore, the Zariski closure of F will be of dimension ≤ 1, and hence our claim follows. $\square$

For the remainder of the section, we will assume that E has multiplicative reduction over $\mathcal{O}_K$ and suppose to the contrary that $\varphi\colon \mathbb{G}_{m,K}^{\mathrm{an}} \to X^{\mathrm{an}}$ is Zariski dense.

We will prove (Proposition 7.5.5) that such a φ will factor as $\mathbb{G}_{m,K}^{\mathrm{an}} \to G^{\mathrm{an}} \to X^{\mathrm{an}}$ where G^{an} is the analytification of a connected algebraic group and $G^{\mathrm{an}} \to X^{\mathrm{an}}$ is an algebraic, finite, surjective morphism. The result will then follow from Lemma 4.1.1.

The condition that E/K has multiplicative reduction allows us to import techniques from algebraic topology. In particular, E^{an} is topologically uniformized by $\mathbb{G}_{m,K}^{\mathrm{an}}$, and hence, as a K-analytic space, E^{an} is isomorphic to $\mathbb{G}_{m,K}^{\mathrm{an}}/q^{\mathbb{Z}}$ for some $q \in K$ with $0 < |q| < 1$. Since $\mathbb{G}_{m,K}^{\mathrm{an}}$ is simply connected [Ber90, Section 6.3], the morphisms φ and $a^{\mathrm{an}} \circ \varphi$ uniquely lift to the topological universal cover of X^{an} and of E^{an}. To summarize the situation, we have the following diagram:

$$\begin{tikzcd}
\mathbb{G}_{m,K}^{\mathrm{an}} \arrow[r, "\widetilde{\varphi}"] \arrow[dr, "\varphi"'] & \widetilde{X} \arrow[r, "\widetilde{a}"] \arrow[d, "\pi_X"] & \mathbb{G}_{m,K}^{\mathrm{an}} \arrow[d, "\pi_E"] \\
 & X^{\mathrm{an}} \arrow[r, "a^{\mathrm{an}}"] & E^{\mathrm{an}},
\end{tikzcd}$$

where $\pi_X\colon \widetilde{X} \to X^{\mathrm{an}}$ and $\pi_E\colon \mathbb{G}_{m,K}^{\mathrm{an}} \to E^{\mathrm{an}}$ are the universal covering morphisms.

A result of Cherry [Che94, Proposition 3.4] tells us that the morphism $\widetilde{a}\circ\widetilde{\varphi}\colon \mathbb{G}_{m,K}^{\mathrm{an}} \to \widetilde{X} \to \mathbb{G}_{m,K}^{\mathrm{an}}$ is *algebraic* (i.e., $\widetilde{a} \circ \widetilde{\varphi}\colon z \mapsto cz^d$ for some $c \in K$, $d \in \mathbb{Z}$). As a^{an} is dominant [Ber90, Proposition 3.4.7] and φ is assumed to be Zariski dense, we have that $a^{\mathrm{an}} \circ \varphi$ is Zariski dense, and hence we know that $c \neq 0$ and $d \neq 0$. Moreover, after translation and post-composition with the automorphism $z \mapsto z^{-1}$, we may and do assume that $\widetilde{a} \circ \widetilde{\varphi}\colon z \mapsto z^d$ where $d \in \mathbb{Z}_{>0}$.

We can further reduce to the case where $d = 1$ as follows. A result of Tate [Tat95, p. 325] states that the endomorphism $z \mapsto z^d$ on $\mathbb{G}_{m,K}^{\mathrm{an}}$ uniquely induces a morphism of smooth, proper, connected, commutative, 1-dimensional K-analytic groups $\psi\colon \mathbb{G}_{m,K}^{\mathrm{an}}/(q^{1/d})^{\mathbb{Z}} \to \mathbb{G}_{m,K}^{\mathrm{an}}/q^{\mathbb{Z}}$, which is in fact an isogeny [Tat95, p. 325, Theorem], whence a finite étale morphism. By Berkovich analytic GAGA [Ber90, Corollary 3.4.13], we have that $\mathbb{G}_{m,K}^{\mathrm{an}}/(q^{1/d})^{\mathbb{Z}}$ is algebraic (i.e., there exists an elliptic curve E'/K such that $E'^{\,\mathrm{an}} \cong \mathbb{G}_{m,K}^{\mathrm{an}}/(q^{1/d})^{\mathbb{Z}}$). Moreover, we can enhance the above diagram to

$$\begin{tikzcd}
\mathbb{G}_{m,K}^{\mathrm{an}} \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\pi_{E'}"'] \arrow[dr, "\varphi"'] & \widetilde{X} \arrow[r, "\widetilde{a}"] \arrow[d, "\pi_X"] & \mathbb{G}_{m,K}^{\mathrm{an}} \arrow[d, "\pi_E"] \\
E'^{\,\mathrm{an}} \arrow[rr, bend right, two heads, "\psi"'] & X^{\mathrm{an}} \arrow[r, "a^{\mathrm{an}}"] & E^{\mathrm{an}}
\end{tikzcd}$$

where $\psi\colon E'^{\,\mathrm{an}} \to E^{\mathrm{an}}$ is finite étale.

Consider the following fibered product $\mathscr{X}$, which exists as a K-analytic space.

$$\begin{tikzcd}
\mathscr{X} \arrow[r, two heads, "\psi'"] \arrow[d, "a'"] & X^{\mathrm{an}} \arrow[d, "a^{\mathrm{an}}"] \\
E'^{\,\mathrm{an}} \arrow[r, two heads, "\psi"] & E^{\mathrm{an}}
\end{tikzcd}$$

We claim that the K-analytic space $\mathscr{X}$ is algebraic. As finite étale morphisms are stable under base change [And03, Remarks 1.2.4(iv)], we have that the morphism $\psi'\colon \mathscr{X} \to X^{\mathrm{an}}$ is finite étale, and by Berkovich analytic GAGA [Ber90, Corollary 3.4.13], $\mathscr{X}$ is algebraic, so there exists a proper scheme of finite type X' such that $\mathscr{X} \cong X'^{\,\mathrm{an}}$.

Using this, we identify $\mathscr{X}$ with $X'^{\,\mathrm{an}}$. The commutativity of the above diagram says that there exists a unique morphism $\varphi'\colon \mathbb{G}_{m,K}^{\mathrm{an}} \to X'^{\,\mathrm{an}}$, and note that the morphism $\widetilde{a}' \circ \widetilde{\varphi}'\colon \mathbb{G}_{m,K}^{\mathrm{an}} \to \widetilde{X}' \to \mathbb{G}_{m,K}^{\mathrm{an}}$ is the identity map. Moreover, the morphism $\widetilde{\varphi}'$ is injective and $\widetilde{a}'$ is surjective. To summarize, we have reduced the proof of Theorem C to the setting in Figure 7.1.

In the next two lemmas, we show that the Zariski closure of $\varphi'(\mathbb{G}_{m,K}^{\mathrm{an}})$ is the

$$\begin{tikzcd}
\widetilde{X}' \arrow[r, two heads, shift left, "\widetilde{a}'"] \arrow[d, "\pi_{X'}"'] & \mathbb{G}_{m,K}^{\mathrm{an}} \arrow[l, hook', shift left, "\widetilde{\varphi}'"] \arrow[d, "\pi_{E'}"] \arrow[dl, "\varphi'"] \\
X'^{\,\mathrm{an}} \arrow[r, "a'"] & E'^{\,\mathrm{an}}
\end{tikzcd}$$

Figure 7.1: The case where $d = 1$

analytification of a closed, connected, algebraic group contained in $X'^{\,\mathrm{an}}$.

Lemma 7.5.2. *The morphism $\varphi' \colon \mathbb{G}_{m,K}^{\mathrm{an}} \to \varphi'(\mathbb{G}_{m,K}^{\mathrm{an}})$ is a semi-covering. Furthermore, $\varphi'(\mathbb{G}_{m,K}^{\mathrm{an}})$ is path-connected and locally path-connected.*

Proof. To begin, we claim that $\widetilde{\varphi}'(\mathbb{G}_{m,K}^{\mathrm{an}})$ is homeomorphic to $\mathbb{G}_{m,K}^{\mathrm{an}}$. Since $\mathbb{G}_{m,K}^{\mathrm{an}}$ is locally compact and $\widetilde{X}'$ is Hausdorff, the injection $\widetilde{\varphi}'$ is a local homeomorphism, and hence $\widetilde{\varphi}' \colon \mathbb{G}_{m,K}^{\mathrm{an}} \to \widetilde{\varphi}'(\mathbb{G}_{m,K}^{\mathrm{an}})$ is a bijective, local homeomorphism, which is a homeomorphism.

Note that $\pi_{X'}|_{\widetilde{\varphi}'(\mathbb{G}_{m,K}^{\mathrm{an}})} \colon \widetilde{\varphi}'(\mathbb{G}_{m,K}^{\mathrm{an}}) \to \varphi'(\mathbb{G}_{m,K}^{\mathrm{an}})$ is a covering map which fits into the following commutative diagram:

$$\begin{tikzcd}
\widetilde{\varphi}'(\mathbb{G}_{m,K}^{\mathrm{an}}) \arrow[r, equal] \arrow[d, "\pi_{X'}|_{\widetilde{\varphi}'(\mathbb{G}_{m,K}^{\mathrm{an}})}"'] & \mathbb{G}_{m,K}^{\mathrm{an}} \arrow[d, "\pi_{E'}"] \\
\varphi'(\mathbb{G}_{m,K}^{\mathrm{an}}) \arrow[r, "a'"] & E'^{\,\mathrm{an}}
\end{tikzcd}$$

Therefore, we have that $\varphi' \colon \mathbb{G}_{m,K}^{\mathrm{an}} \to \varphi'(\mathbb{G}_{m,K}^{\mathrm{an}})$ is the composition of a homeomorphism with a covering map, which is a semi-covering by Proposition 7.4.2.

To conclude, we have that $\varphi'(\mathbb{G}_{m,K}^{\mathrm{an}})$ is path-connected because the continuous image of a connected space is connected and a connected K-analytic space is path-connected [Ber90, Theorem 3.2.1]. Furthermore, since φ' is a local homeomorphism, we have that $\varphi'(\mathbb{G}_{m,K}^{\mathrm{an}})$ is locally path-connected. $\square$

Lemma 7.5.3. *The Zariski closure of the image $\varphi'(\mathbb{G}_{m,K}^{\mathrm{an}})$ is the analytification of a closed, connected algebraic group in X'.*

Proof. By Lemma 7.5.2 and Proposition 7.4.2, we have that $a' \colon \varphi'(\mathbb{G}_{m,K}^{\mathrm{an}}) \to E'^{\,\mathrm{an}}$ is a semi-covering because $\pi_{E'}$ is a covering map and $E'^{\,\mathrm{an}}$ is path-connected and locally path-connected. Moreover, we have that the image $\varphi'(\mathbb{G}_{m,K}^{\mathrm{an}})$ is a K-analytic group subvariety of $X'^{\,\mathrm{an}}$ by Corollary 7.4.5. Now, a lemma of Lang [Lan87, p. 84] tells us that the Zariski closure $\overline{\varphi'(\mathbb{G}_{m,K}^{\mathrm{an}})}$ of $\varphi'(\mathbb{G}_{m,K}^{\mathrm{an}})$ is a closed K-analytic group in $X'^{\,\mathrm{an}}$. Since $X'^{\,\mathrm{an}}$ is projective, Berkovich analytic GAGA [Ber90, Corollary 3.4.13] tells us that $\overline{\varphi'(\mathbb{G}_{m,K}^{\mathrm{an}})}$ is the analytification of a closed, connected algebraic group G of X'. □

Using Lemma 7.5.3, we identify the Zariski closure of the image $\varphi'(\mathbb{G}_{m,K}^{\mathrm{an}})$ in $X'^{\,\mathrm{an}}$ with G^{an}.

Lemma 7.5.4. *The composed morphism $G^{\mathrm{an}} \subset X'^{\,\mathrm{an}} \to X^{\mathrm{an}}$ is finite and surjective. Furthermore, the morphism $G^{\mathrm{an}} \to X^{\mathrm{an}}$ is algebraic (i.e., the morphism $G^{\mathrm{an}} \to X^{\mathrm{an}}$ is the analytification of a finite, surjective morphism $G \to X$ of K-schemes).*

Proof. By Lemma 7.5.3, the morphism $G^{\mathrm{an}} \subset X'^{\,\mathrm{an}}$ is a closed embedding, and hence the composed morphism $G^{\mathrm{an}} \subset X'^{\,\mathrm{an}} \to X^{\mathrm{an}}$ is finite. To prove surjectivity, note that the image of the composed morphism $G^{\mathrm{an}} \subset X'^{\,\mathrm{an}} \to X^{\mathrm{an}}$ is closed. Since the image of G^{an} in X^{an} contains the image $\varphi(\mathbb{G}_{m,K}^{\mathrm{an}})$ and φ is Zariski dense by assumption, we have that the composed morphism $G^{\mathrm{an}} \subset X'^{\,\mathrm{an}} \to X^{\mathrm{an}}$ is surjective. In the second statement, the algebraicity and finiteness of the morphism follows from Berkovich analytic GAGA [Ber90, Corollary 3.4.13] and the surjectivity follows from [Ber90, Proposition 3.4.6]. □

We can combine the above lemmas into the following result.

Proposition 7.5.5. *Let K be an algebraically closed, complete, non-Archimedean valued field of characteristic zero. Let X/K be a projective variety admitting a dominant morphism to an elliptic curve.*

A Zariski dense, analytic morphism $\mathbb{G}_{m,K}^{\mathrm{an}} \to X^{\mathrm{an}}$ factors as

$$\mathbb{G}_{m,K}^{\mathrm{an}} \to G^{\mathrm{an}} \to X^{\mathrm{an}}$$

where G^{an} is the analytification of a connected algebraic group and $G^{\mathrm{an}} \to X^{\mathrm{an}}$ is a finite surjective morphism. Moreover, the finite, surjective morphism $G^{\mathrm{an}} \to X^{\mathrm{an}}$ algebraizes (i.e., there exists a finite, surjective morphism of K-schemes $G \to X$).

Proof. This result is a combination of Lemmas 7.5.1, 7.5.2, 7.5.3, and 7.5.4. □

Proof of Theorem C. By Remark 4.4.5, it suffices to prove that there does not exist a Zariski dense, non-Archimedean entire curve in X^{an}. By Lemma 7.5.1, we may assume that E/K has multiplicative reduction over $\mathcal{O}_K$. Proposition 7.5.5 tells us that if there exists a Zariski dense, analytic morphism $\mathbb{G}_{m,K}^{\mathrm{an}} \to X^{\mathrm{an}}$, then X admits a finite cover by a connected algebraic group. However, this contradicts Lemma 4.1.1 as X is of general type. Therefore, we conclude that there cannot exist a Zariski dense, entire non-Archimedean curve in X^{an}. □

7.6 Non-Archimedean entire curves in projective surfaces dominating an elliptic curve

In this section, we will prove Theorem D. In particular, we will prove that for S/K a projective surface admitting a dominant morphism to an elliptic curve and for every proper closed $\Delta \subsetneq S$, S is groupless modulo Δ if and only if S is K-analytically Brody hyperbolic modulo Δ.

We will use the following results in our proof.

Proposition 7.6.1. *Let S/K be a projective surface admitting a domiant morphism to an elliptic curve E/K with good reduction over $\mathcal{O}_K$, and let $\Delta \subsetneq S$ be a proper* *closed subscheme. If S is groupless modulo Δ, then S is K-analytically Brody hyperbolic modulo Δ.*

Proof. To show that S is K-analytically Brody hyperbolic modulo Δ, it suffices to prove that the image of any non-constant analytic morphism $\varphi\colon \mathbb{G}_{m,K}^{\mathrm{an}} \to S^{\mathrm{an}}$ factors through Δ^{an} by Remark 4.4.5. Note that as E has good reduction over $\mathcal{O}_K$, the composed morphism $\mathbb{G}_{m,K}^{\mathrm{an}} \to E^{\mathrm{an}}$ is constant [Che94, Theorem 3.2]. Therefore, the morphism $\mathbb{G}_{m,K}^{\mathrm{an}} \to S^{\mathrm{an}}$ factors through fibre F^{an} of $S^{\mathrm{an}} \to E^{\mathrm{an}}$. Since F is a fibre of $X \to E$, it follows that $\dim F \leq 1$. As φ is non-constant and a has connected fibers [LP12, p. 352], we have that $\dim F \neq 0$.

Now, the Zariski closure $\overline{\varphi(\mathbb{G}_{m,K}^{\mathrm{an}})}$ of the image of φ is a connected, closed, 1-dimensional K-analytic subvariety of S^{an}, and by Berkovich analytic GAGA [Ber90, Corollary 3.4.13], we have that $\overline{\varphi(\mathbb{G}_{m,K}^{\mathrm{an}})}$ is a connected, projective curve (i.e., $\overline{\varphi(\mathbb{G}_{m,K}^{\mathrm{an}})}$ is isomorphic to Z^{an} for a connected, projective curve $Z \subset S$). Since the morphism $\varphi\colon \mathbb{G}_{m,K}^{\mathrm{an}} \to Z^{\mathrm{an}}$ is dominant, we see that Z is not birational to curve of general type by Theorem 4.4.9, and therefore, Z is a rational curve or birational to an elliptic curve. As S is groupless modulo Δ, this tells us that the image of $\varphi\colon \mathbb{G}_{m,K}^{\mathrm{an}} \to S^{\mathrm{an}}$ factors through Δ^{an}, as desired. □

Lemma 7.6.2. *Let S/K be a projective surface, and let $\Delta \subsetneq S$ be a proper closed subscheme. Suppose that S is groupless modulo Δ. If an analytic morphism $\varphi\colon \mathbb{G}_{m,K}^{\mathrm{an}} \to S^{\mathrm{an}}$ is not Zariski dense, then $\varphi(\mathbb{G}_{m,K}^{\mathrm{an}})$ is contained in Δ^{an}.*

Proof. This follows from the latter part of the proof of Proposition 7.6.1. □

We are now in a position to prove Theorem D and Corollary 5.0.5.

Proof of Theorem D.

($\Leftarrow$). This direction follows from Proposition 4.3.11 and Definition 4.4.2.

($\Rightarrow$). Let $\Delta \subsetneq S$ be a proper closed subscheme, and suppose that S is groupless modulo Δ. By Remark 4.4.5, we are reduced to show that any non-constant analytic morphism $\mathbb{G}_{m,K}^{\mathrm{an}} \to S^{\mathrm{an}}$ factors through Δ^{an}. By Proposition 7.6.1 and Lemma 7.6.2, we know this is true when E/K has good reduction over $\mathcal{O}_K$ or if the morphism φ is not Zariski dense, and so it suffices to prove that when E/K has multiplicative reduction over $\mathcal{O}_K$, the analytic morphism $\varphi\colon \mathbb{G}_{m,K}^{\mathrm{an}} \to X^{\mathrm{an}}$ cannot be Zariski dense. As a projective integral pseudo-groupless surface is of general type [JX19, Lemma 3.23], Theorem C asserts that φ cannot be Zariski dense, and therefore S is K-analytically Brody hyperbolic modulo Δ. $\square$

Proof of Corollary 5.0.5. This follows from Theorem D and Remarks 4.3.7, 4.4.4, and 4.4.5. $\square$

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