Some Ramsey-type Theorems

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Abstract

Some Ramsey-type Theorems
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We consider three Ramsey-type problems.

Extending the concept of the Ramsey numbers, Erdős and Rogers introduced the function

$$f_{s,t}(n) = \min\{\max\{|W| : W \subseteq V(G) \text{ and } G[W] \text{ contains no } K_s\}\},$$

where the minimum is taken over all K_t-free graphs G of order n. We establish that for every $s \geqslant 3$ there exist constants c_1 and c_2 such that $f_{s,s+1}(n) \leqslant c_1(\log n)^{c_2}\sqrt{n}$. We also prove that for all $t - 2 \geqslant s \geqslant 4$, there exists a constant c_3 such that $f_{s,t}(n) \leqslant c_3\sqrt{n}$. In doing so, we give a partial answer to a question of Erdős.

To state our second problem, we introduce some notation. For a graph S, the h-subdivision $S^{(h)}$ is obtained by replacing each edge with a path of length $h+1$. For any graph S of maximum degree d on $s \geqslant s_0(h,d,\ell)$ vertices, we show there exists a graph G with $(\log s)^{20h}s^{1+1/(h+1)}$ edges having the following *Ramsey* property: any coloring of the edges of G with ℓ colors yields a monochromatic copy of the subdivided graph $S^{(h)}$. This result complements work of Pak regarding 'long' subdivisions of bounded degree.

Another question of Erdős, answered by Rődl and Ruciński, asks if for every pair of positive integers ℓ and k, there exist a graph H having $\text{girth}(H) = k$ and the property that every ℓ-coloring of the edges of H yields a monochromatic cycle C_k. Here, we establish that such a graph exists with at most $r^{O(k^2)}k^{O(k^3)}$ vertices, where $r = r_\ell(C_k)$ is the ℓ color Ramsey number for the cycle C_k. We also consider two closely related problems.

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Contents

Chapter 1

Introduction

1.1 Graphs

In combinatorics, a *graph* is a type of mathematical model consisting of two parts: a *vertex set* V and an *edge set* E of related pairs of vertices. Fundamental to computer science, network theory, and discrete mathematics, graphs also model problems in chemistry, physics, sociology, biology, epidemiology, and linguistics. For example, one may consider a social network graph in which the vertices represent people and the edges represent pairs of acquainted individuals. Graphs can be visualized by imagining the vertices as points in the plane and edges as line segments between related pairs.

1.2 Probabilistic Combinatorics

Over the last two-thirds of a century, probabilistic reasoning has played an active and influential role in the development of graph theory. This area of mathematics, known as *probabilistic combinatorics*, includes the study of random graphs and algorithms. Random graphs may be equated with average case analysis in the sense that a particular instance of a problem may often be thought of as being generated by some underlying random process. Along with the related study of random algorithms, this has played a significant role in the theory of algorithmic design. Probabilistic tools, however, are not only limited to the study of random structures and algorithms, but also have applications to problems that themselves do not involve randomness. Examples of major mathematical theorems in which ideas in probabilistic combinatorics have played an important role include Szemerédi's Theorem (every set of integers with positive density contains arbitrarily long arithmetic progressions), the related Green-Tao Theorem (the prime numbers contain arbitrarily long arithmetic progressions), and a recent theorem of Keevash on the existence of designs (resolving a famous question of Steiner from 1853). As this relatively new area of mathematics matures, additional connections to other areas of mathematics are likely to emerge.

1.3 Ramsey Numbers

A red/blue coloring of a graph is a partition of the edges into two classes. For graphs H and G, we write $H \to G$ if every red/blue coloring of the edges of H yields G as a monochromatic subgraph. The *Ramsey number* $r(G)$ is the minimum number of vertices in a graph H with the property $H \to G$. That is,

$$r(G) := \min\Big\{|V(H)| : H \to G\Big\},$$

where without loss of generality H can be assumed to be a complete graph K_n on n vertices in which all $\binom{n}{2}$ possible edges are present. In 1930, economist and mathematician Frank Ramsey established that $r(G) \leqslant 4^{v(G)}$ where $v(G)$ denotes the number of vertices in G.

One of the first applications of the probabilistic method was a proof of Erdős [22] from 1947 that established a lower bound for $r(K_s)$. The current best known bounds, due respectively to Spencer [71] and Conlon [15], are:

$$(1 - o(1))e^{-1}\sqrt{2}s \cdot 2^{s/2} \leqslant r(K_s) \leqslant s^{c \log s/\log\log s} \cdot 2^{2s},$$

where c is a constant that does not depend on s and $o(1) \to 0$ as $s \to \infty$. This problem has attracted a great deal of attention, although the asymptotic behavior of $r(K_s)$ is still unknown, as the exponents in the lower and upper bounds stated above differ by a factor of four. In particular, W.T. Gowers [40] writes, 'I consider this to be one of the major problems in combinatorics and have devoted many months of my life unsuccessfully trying to solve it'.

Over the last half century, many variations of the Ramsey number problem have been considered. The development of probabilistic combinatorics had an intimate relationship with this study, and new powerful probabilistic methods have been developed as a result of this inquiry. Moreover, many of the open problems in this area encapsulate fundamental gaps in our existing knowledge of graphs.

1.4 Overview of Results

This thesis focuses on three distinct Ramsey-type problems addressed in three corresponding chapters. Each of these chapters is self contained. In the overview below, we provide a very brief and (relatively) nontechnical description of the problems addressed and the significance of our results. We defer the formal statements of our theorems and the discussion of the relevant historical background to the introductions provided in the following chapters.

Our next chapter concerns a generalization of the Ramsey numbers due to Erdős and Rogers [24], which concerns the size of the largest K_s-free set necessarily present in every K_k-free graph on n vertices. In contrast, the standard Ramsey number problem can be phrased as asking for the size of the largest independent set necessarily present in every K_k-free graph on n vertices. This generalization of Erdős and Rogers has received considerable attention over the last 50 years, having been addressed by Bollobás and

Hind [11], Krivelevich [50,51], Alon and Krivelevich [3], Dudek and Rödl [21], Dudek and Mubayi [18], and most recently Wolfovitz [75]. Here, we improve upon the best known bounds for many values of s and t. To do so, we provide a random three stage probabilistic construction and make use of the Local Lemma and theory of projective planes. This chapter is based upon joint work with Andrzej Dudek and Vojtěch Rödl [19].

Our second chapter concerns the size-Ramsey numbers of short subdivisions. Whereas the Ramsey number problem asks for the minimum number of vertices in a graph H with the property $H \to G$, the size-Ramsey number problem asks for the minimum number of edges in a graph H with the property $H \to G$. This is one of the most basic extensions of the traditional Ramsey problem, and has been the topic of much research. After being introduced by Erdős, Faudree, Rousseau, and Schelp [30] in 1978, it was subsequently studied by Beck [6], Haxell, Kohayakawa, and Łuczak [44], Friedman and Pippenger [37], and Dellamonica [17]. See also [35, 43, 46, 58–60], or the more general recent survey on graph Ramsey theory [16]. One of the most significant open problems in this area is to determine the size-Ramsey number of graphs of bounded degree. Although some progress on the bounded-degree problem has been made by Rödl and Szemerédi [64] and Kohayakawa, Rödl, Schacht, and Szemerédi [49], a rather large gap between the best known upper and lower bounds still remains. Here, we will investigate the size-Ramsey numbers of bounded degree graphs that have the additional property that the set of vertices of degree greater than two induces an independent set. To do so, we consider 'short' subdivisions of graphs, which are obtained by replacing edges in a graph by paths of some fixed 'short' length. Pak [57] in 2002 considered the closely related problem for 'long' subdivisions of bounded degree, where the length of the subdivisions is logarithmic in terms of the number of vertices. We make use of the sparse regularity lemma, ideas from a paper of Gerke, Kohayakawa, Rödl, and Steger [38], a hypergraph version of Hall's

Theorem due to Aharoni and Haxel [2], and a new embedding lemma. This chapter is based upon joint work with Yoshiharu Kohayakawa and Vojtěch Rödl [76].

Our third result has its roots in a problem suggested by Paul Erdős [27], which asks if for every positive integer k, there exists a graph H having $\mathrm{girth}(H) = k$ and the Ramsey property $H \to C_k$. The existence of such graphs was first established by Rödl and Ruciński in [62]. We raise the question of determining the least number of vertices such a graph may have. Whereas the bounds implicitly following from the known constructions are rather large, we provide a new random construction that yields much improved bounds. This construction is analyzed by way of the Container Lemma of Saxton and Thomason [67]. In this section, we also apply our technique to two other problems. The former concerns monochromatic arithmetic progression in an arbitrarily colored set of integers and the latter hypergraphs with larger chromatic number and girth. This chapter is based upon joint work with Hiệp Hàn, Vojtěch Rödl, and Mathias Schacht [42].

Chapter 2

A Function of Erdős and Rogers

2.1 Introduction

In a graph G, a set $S \subseteq V(G)$ is *independent* if $G[S]$ does not contain a copy of K_2. More generally for any integer s, a set $S \subseteq V(G)$ can be called *s-independent* if $G[S]$ does not contain a copy of K_s. With this in mind, define the *s-independence number* of G, denoted by $\alpha_s(G)$, to be the size of the largest s-independent set in G. The classical Ramsey number $R(t,u)$ can be defined in this language as the least integer n such that every graph of order n contains either a copy of K_t or a 2-independent set of size u. In other words, $R(t,u)$ is the least integer n such that

$$u \leqslant \min\{\alpha_2(G) : G \text{ is a } K_t\text{-free graph of order } n\}.$$

Observe that the problem of determining the right hand side of the above inequality, which is a function of n and t, is equivalent to determining the classical Ramsey numbers.

A more general problem results by replacing the standard independence number by the s-independence number for some $2 \leqslant s < t$. Following this approach, in 1962 Erdős and Rogers [24] introduced the function

$$f_{s,t}(n) = \min\{\alpha_s(G) : G \text{ is a } K_t\text{-free graph of order } n\}.$$

The lower bound $k \leqslant f_{s,t}(n)$ means that every K_t-free graph of order n contains a subset of k vertices with no copy of K_s. The upper bound $f_{s,t}(n) < \ell$ means that there exists a K_t-free graph of order n such that every subset of ℓ vertices contains a copy of K_s.

The case $t = s + 1$ has received considerable attention over the last 50 years, in part due to the fact that it creates a general upper bound in the sense that for $t' > t$, we clearly have $f_{s,t'}(n) \leqslant f_{s,t}(n)$. The first nontrivial upper bound for $f_{s,s+1}(n)$ was established by Erdős and Rogers [24]. This problem of determining a better upper bound for $f_{s,s+1}(n)$ was subsequently addressed by Bollobás and Hind [11], Krivelevich [50, 51], Alon and Krivelevich [3], Dudek and Rödl [21], and most recently Wolfovitz [75]. The first nontrivial lower bound established by Bollobás and Hind [11] was later slightly improved by Krivelevich [50]. The most recent general bounds for $s \geqslant 3$ were of the form:

$$\Omega\left(\sqrt{\frac{n \log n}{\log \log n}}\right) = f_{s,s+1}(n) = O\big(n^{\frac{2}{3}}\big). \tag{2.1}$$

The lower bound of (2.1) was first explicitly stated by Dudek and Mubayi [18], and was based upon their observation that the result of Krivelevich [50] could be slightly strengthened by incorporating a result of Shearer [69]. The upper bound of (2.1) appears in [21], where it was also conjectured that for all sufficiently large s the upper bound could be improved to show that

$$f_{s,s+1}(n) = n^{\frac{1}{2}+o(1)}. \tag{2.2}$$

Recently, Wolfovitz [75] showed that (2.2) holds when $s = 3$. In this chapter, (2.2) is proved for every $s \geqslant 3$, establishing an upper bound that is tight up to a polylogarithmic factor. Our proof builds upon the ideas in [75], [21], [51], and [50].

Theorem 2.1. *For every $s \geqslant 3$ there is a constant $c = c(s)$ such that*

$$f_{s,s+1}(n) \leqslant c(\log n)^{4s^2}\sqrt{n}.$$

For the case $t = s+2$, it follows from a result of Sudakov [73] (see also [21] for a simplified formula) that $f_{s,s+2}(n) = \Omega(n^{a_2})$, where $\frac{1}{a_2} = 2 + \frac{2}{3s-4}$. On the other hand, clearly $f_{s,s+2}(n) \leqslant f_{s,s+1}(n)$. When $s \geqslant 4$, we establish an improved upper bound that omits the logarithmic factor.

Theorem 2.2. *For every $s \geqslant 4$ there is a constant $c = c(s)$ such that*

$$f_{s,s+2}(n) \leqslant c\sqrt{n}.$$

This establishes the following corollary which provides the best known bounds on $f_{s,t}(n)$ for $t < 2s$.

Corollary 2.3. *For every $6 \leqslant s + 2 \leqslant t$ there is a constant $c = c(s)$ such that*

$$f_{s,t}(n) \leqslant f_{s,s+2}(n) \leqslant c\sqrt{n}.$$

When $t \geqslant 2s$, the upper bound $c(\log n)^{1/(s-1)}n^{s/(t+1)}$ of Krivelevich [51] remains best. For all values of $t > s + 1$, the best lower bounds follow from a recursive formula defined by Sudakov [73, 74]. We will return to the results concerning the general case at the end of this chapter in Section 2.5. More related results are summarized in the survey [20].

We now turn our attention towards an old question of Erdős [23], asking if for fixed integers $s + 1 < t$,

$$\lim_{n\to\infty} \frac{f_{s+1,t}(n)}{f_{s,t}(n)} = \infty. \tag{2.3}$$

This central conjecture in the area is still wide open and asks for a rather precise estimation of $f_{s,t}(n)$. By a result of Sudakov [74], (2.3) holds for

$$(s,t) \in \{(2,4),(2,5),(2,6),(2,7),(2,8),(3,6)\}.$$

Observe that Theorem 2.2 together with the lower bound of [50] (and [21]) implies that for $s \geqslant 4$,

$$\frac{f_{s+1,s+2}(n)}{f_{s,s+2}(n)} \geqslant \frac{\Omega\left(\sqrt{\frac{n\log n}{\log\log n}}\right)}{O(\sqrt{n})} = \Omega\left(\sqrt{\frac{\log n}{\log\log n}}\right) \xrightarrow[n\to\infty]{} \infty.$$

That is, (2.3) holds for all pairs $(s,t) \in \{(4,6),(5,7),(6,8),\dots\}$.

In what follows, consider s to be an arbitrary fixed integer and n sufficiently large, i.e. $n \geqslant n_0(s)$. We will show that there exists a K_{s+1}-free graph of order n such that every subset of $c(\log n)^{4s^2}\sqrt{n}$ vertices contains a copy of K_s and that there exists a K_{s+2}-free graph of order n such that every subset of $c\sqrt{n}$ vertices contains a copy of K_s. Indeed, this establishes Theorems 2.1 and 2.2 as stated (for all n), since the constant factors can subsequently be inflated to accommodate the finitely many cases where $n \leqslant n_0$. For simplicity, we do not round numbers that are supposed to be integers either up or down; this is justified since these rounding errors are negligible to the asymptomatic calculations we will make.

In Section 2.2, we begin our construction by considering the random hypergraph $\mathbb{H}$ which is essentially the random hypergraph obtained from the affine plane by taking each hyperedge (line) with some uniform probability. We then use $\mathbb{H}$ in Section 2.3 to construct a random graph $\mathbb{G}$ by replacing each hyperedge by a complete s-partite graph. In Section 2.4, the proof of Theorem 2.2 considers an induced subgraph of $\mathbb{G}$ whereas the proof of Theorem 2.1 considers yet another random subgraph of $\mathbb{G}$ which is analyzed by way of the Local Lemma.

Below we will use the standard notation to denote the neighborhood and degree of $v \in G$ by $N_G(v)$ and $d_G(v)$, respectively.

2.2 The Hypergraph H

The *affine plane* of order q is an incidence structure on a set of q^2 points and a set of q^2+q lines such that: any two points lie on a unique line; every line contains q points; and every point lies on $q+1$ lines. It is well known that affine planes exist for all prime power orders. Clearly, an incidence structure can be viewed as a hypergraph with points corresponding to vertices and lines corresponding to hyperedges; we will use this terminology interchangeably.

In the affine plane, call lines L and L' *parallel* if $L \cap L' = \varnothing$. In the affine plane there exist $q+1$ sets of q pairwise parallel lines. (For more details see, e.g., [14].) Let $(V, \mathcal{L})$ be the hypergraph obtained by removing a parallel class of q lines from the affine plane of order q. The following lemma establishes some properties of this graph.

Lemma 2.4. *For q prime, the q-uniform, q-regular hypergraph $(V, \mathcal{L})$ of order q^2 satisfies:*

(P1) Any two vertices are contained in at most one hyperedge;

(P2) For every $A \in \binom{V}{q}$, $|\{L \in \mathcal{L} : L \cap A \neq \varnothing\}| \geqslant \frac{q^2}{2}$.

Proof. By construction, $(V, \mathcal{L})$ is q-uniform, q-regular, and satisfies (P1). Consider $A = \{v_1, v_2, \ldots, v_q\}$. Define $d_+(v_i) = |\{L \in \mathcal{L} : L \cap \{v_1, v_2, \ldots, v_i\} = \{v_i\}\}|$. Hence by property (P1), $d_+(v_i) \geqslant q - i + 1$. We now compute

$$|\{L \in \mathcal{L} : L \cap A \neq \varnothing\}| \geqslant \sum_{i=1}^{q} d_+(v_i) \geqslant \binom{q+1}{2} \geqslant \frac{q^2}{2}.$$

$\square$

The objective of this section is to establish the existence of a certain hypergraph $(V, \mathcal{L}') \subseteq (V, \mathcal{L})$ by considering a random sub-hypergraph of $(V, \mathcal{L})$. Preceding this, we introduce some terminology. For $B \subseteq V$, define

$$\mathcal{L}'_A = \{L \in \mathcal{L}' : L \cap A \neq \varnothing\}, \qquad \text{and} \qquad \mathcal{L}'_{B,\gamma} = \{L \in \mathcal{L}' : |L \cap B| \geqslant \gamma\}.$$

Call $S \subseteq V$ $\mathcal{L}'$-*complete* if every pair of points in S is contained in some common line in $\mathcal{L}'$. Let $L(x, y)$ denote the unique line in $\mathcal{L}$ containing x and y, provided such a line exists.

We will now distinguish 3 types of $\mathcal{L}'$-*dangerous* subsets as depicted in Figure 2.1. The first two types have 5 vertices $\{v_1, v_2, v_3, v_4, x\}$ and third type has 6 vertices $\{v_1, v_2, v_3, v_4, y, z\}$. All 3 types of dangerous sets must be $\mathcal{L}'$-complete and have 4 points $\{v_1, v_2, v_3, v_4\}$ in general position. Additionally we specify:

Type 1 $\mathcal{L}'$-dangerous

The points $\{v_1, v_2, v_3, v_4, x\}$ are in general position.

Type 2 $\mathcal{L}'$-dangerous

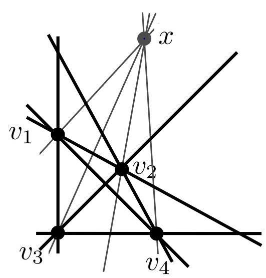
The point x is contained in precisely one of the 6 lines $L(v_i, v_j)$ for $1 \leqslant i < j \leqslant 4$. Up to relabeling, say $x \in L(v_2, v_3)$.

Type 3 $\mathcal{L}'$-dangerous

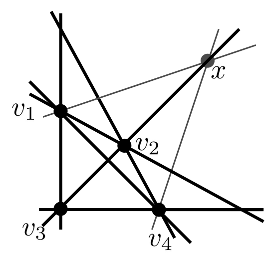
The points y and z are each contained in exactly two of the lines $L(v_i, v_j)$ for $1 \leqslant i < j \leqslant 4$. Up to relabeling, say $y \in L(v_1, v_3) \cap L(v_2, v_4)$ and $z \in L(v_1, v_2) \cap L(v_3, v_4)$.

All concepts above were defined relative to the subset $\mathcal{L}' \subseteq \mathcal{L}$. Obviously we can define the concepts $\mathcal{L}$-complete, $\mathcal{L}$-dangerous, $\mathcal{L}_A$, and $\mathcal{L}_{B,\gamma}$ related to the set $\mathcal{L}$ analogously.

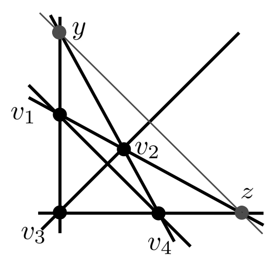
We are now ready to state the main result of this section.



(a) Type 1



(b) Type 2



(c) Type 3

Figure 2.1: Types of dangerous sets.

Lemma 2.5. *Let q be a sufficiently large prime and $\alpha = (\log q)^2$. Then, there exists a q-uniform hypergraph $H = (V, \mathcal{L}')$ of order q^2 such that:*

(H1) *Any two vertices are contained in at most one hyperedge;*

(H2) *For every $v \in V$, $d_H(v) \leqslant 2\alpha$;*

(H3) *$|\mathcal{D}| \leqslant 2\alpha^8 q$, where $\mathcal{D}$ is the set of $\mathcal{L}'$-dangerous subsets;*

(H4) *For every $A \in \binom{V}{q}$, $|\mathcal{L}'_A| \geqslant \alpha q/4$;*

(H5) *For every integer $1 \leqslant \gamma \leqslant q/16$ and every $B \in \binom{V}{16\gamma q}$, $|\mathcal{L}'_{B,\gamma}| \geqslant \alpha q/8$.*

Before proving the above lemma, we state a basic form of the Chernoff bound (as appearing in Corollary 2.3 of [45]) and state the union bound and Markov Inequality. We let $\mathrm{Bi}(n,p)$ denote a binomial random variable bas n events that each occur with probability p.

Chernoff Bound. *If $X \sim Bi(n,p)$ and $0 < \varepsilon \leqslant \frac{3}{2}$, then*

$$\Pr\Big(|X - E(X)| \geqslant \varepsilon \cdot E(X)\Big) \leqslant 2\exp\left\{-\frac{E(X)\varepsilon^2}{3}\right\}.$$

Union Bound. *If E_i are events, then*

$$\Pr\left(\bigcup_{i=1}^{k} E_i\right) \leqslant k \cdot \max\{\Pr(E_i) : i \in [k]\}.$$

Markov Inequality. *If X is a nonnegative random variable and $a > 0$, then*

$$\Pr\Big(X \geqslant a\Big) \leqslant \frac{E(X)}{a}.$$

Proof of Lemma 2.5. Take $(V, \mathcal{L})$ to be a hypergraph established by Lemma 2.4. Let $\mathbb{H} = (V, \mathcal{L}')$ be a random sub-hypergraph of $(V, \mathcal{L})$ where every line in $\mathcal{L}$ is taken independently with probability

$$\frac{\alpha}{q} = \frac{(\log q)^2}{q}.$$

Since $\mathbb{H}$ is a subgraph of $(V, \mathcal{L})$ any two vertices are in at most one line, so $\mathbb{H}$ always satisfies (H1). We will show $\mathbb{H}$ fails to satisfy (H2) and (H4) with probability at most $o(1)$ and that $\mathbb{H}$ fails to satisfy (H3) with probability at most $\frac{1}{2}$. Together this implies $\mathbb{H}$ satisfies (H1)-(H4) with probability at least $1 - \frac{1}{2} - o(1)$, establishing the existence of a hypergraph H that satisfies (H1)-(H4). Finally, we use a counting argument to show that any such H necessarily satisfies (H5).

(H2): We first show that the probability that there exists a vertex of degree greater than 2α is $o(1)$. Observe for fixed $v \in \mathbb{H}$, $d_{\mathbb{H}}(v) \sim Bi(q, \frac{\alpha}{q})$ and $E(d_{\mathbb{H}}(v)) = \alpha$. So by the Chernoff bound with $\varepsilon = 1$,

$$\Pr\Big(d_{\mathbb{H}}(v) \geqslant 2\alpha\Big) \leqslant \Pr\Big(|d_{\mathbb{H}}(v) - \alpha| \geqslant \alpha\Big) \leqslant 2\exp\left\{-\frac{\alpha}{3}\right\}.$$

Thus by the union bound the probability that there exists some $v \in V$ with $d_{\mathbb{H}}(v) \geqslant 2\alpha$ is at most

$$q^2 \cdot 2\exp\left\{-\frac{\alpha}{3}\right\} = 2\exp\left\{2\log q - \frac{(\log q)^2}{3}\right\} = o(1).$$

(H3): In order to show $|\mathcal{D}| > 4\alpha^8 q$ with probability at most $\frac{1}{2}$, we begin by counting the number of $\mathcal{L}$-dangerous subsets of each type. Clearly the number of Type 1 $\mathcal{L}$-dangerous subsets is at most $\binom{q^2}{5}$. To count the number of Type 2 $\mathcal{L}$-dangerous subsets, first choose $\{v_1, v_2, v_3, v_4\}$ then x, observing x must lie on one of the 6 lines which each have at most q vertices. Thus there are at most $\binom{q^2}{4}(6q)$ configurations of this type. To count the number of Type 3 $\mathcal{L}$-dangerous subsets, observe the lines $L(v_i, v_j)$ for $1 \leqslant i < j \leqslant 6$

intersect at at most 3 points other than v_1, v_2, v_3, v_4. Hence there are at most $\binom{q^2}{4}\binom{3}{2}$ subsets of this type in $\mathcal{L}$.

Since $\mathcal{L}$-dangerous subsets of Type 1, Type 2, and Type 3 have 10, 8, and 7 lines respectively, an $\mathcal{L}$-dangerous subset of each type will be $\mathcal{L}'$-dangerous with respective probabilities $\left(\frac{\alpha}{q}\right)^{10}$, $\left(\frac{\alpha}{q}\right)^{8}$, and $\left(\frac{\alpha}{q}\right)^{7}$. By the linearity of expectation, we now compute

$$\begin{aligned}
E(|\mathcal{D}|) &\leqslant \binom{q^2}{5}\cdot\left(\frac{\alpha}{q}\right)^{10} + \binom{q^2}{4}(6q)\cdot\left(\frac{\alpha}{q}\right)^{8} + \binom{q^2}{4}\binom{3}{2}\cdot\left(\frac{\alpha}{q}\right)^{7} \\
&\leqslant \alpha^{10} + \frac{q\alpha^8}{4} + \frac{q\alpha^7}{8} \\
&\leqslant q\alpha^8.
\end{aligned}$$

Thus, the Markov inequality yields,

$$\Pr\left(|\mathcal{D}| \geqslant 2\alpha^8 q\right) \leqslant \Pr\left(|\mathcal{D}| \geqslant 2E(|\mathcal{D}|)\right) \leqslant \frac{1}{2}.$$

(H4): We will now prove that the probability that there exists $A \in \binom{V}{q}$ such that $|\mathcal{L}'_A| < \frac{\alpha q}{4}$ is $o(1)$. Begin by considering any fixed $A \in \binom{V}{q}$. Then by Lemma 2.4, $|\mathcal{L}_A| \geqslant \frac{q^2}{2}$, so we may fix $X \subseteq \mathcal{L}_A$ with $|X| = \frac{q^2}{2}$. Let $X' = X \cap \mathcal{L}'$. Since each line in X appears in X' independently with probability $\frac{\alpha}{q}$, $|X'| \sim Bi(\frac{q^2}{2}, \frac{\alpha}{q})$ and $E(|X'|) = \frac{\alpha q}{2}$. Hence by the Chernoff bound with $\varepsilon = \frac{1}{2}$,

$$\Pr\left(|\mathcal{L}'_A| < \frac{\alpha q}{4}\right) \leqslant \Pr\left(|X'| < \frac{\alpha q}{4}\right) \leqslant \Pr\left(\left||X'| - \frac{\alpha q}{2}\right| \geqslant \frac{\alpha q}{4}\right) \leqslant 2\exp\left\{-\frac{\alpha q}{24}\right\}.$$

Consequently by the union bound, the probability that there exits some $A \subseteq V$, $|A| = q$, with $|\mathcal{L}'_A| < \frac{\alpha q}{4}$ is at most

$$\binom{q^2}{q} \cdot 2\exp\left\{-\frac{\alpha q}{24}\right\} \leqslant q^{2q} \cdot 2\exp\left\{-\frac{(\log q)^2 q}{24}\right\}$$
$$= 2\exp\left\{2q\log q - \frac{q(\log q)^2}{24}\right\}$$
$$= o(1).$$

(H5): Finally, we will establish the following deterministic property: If H satisfies (H2) and (H4), then H also satisfies (H5).

Consider arbitrary fixed $0 \leqslant \gamma \leqslant \frac{q}{16}$ and $B \in \binom{V}{16\gamma q}$. Let $B = B_1 \cup B_2 \cup \cdots \cup B_{16\gamma}$ be a partition of B into 16γ sets of size q. Consider the auxiliary bipartite graph Aux with bipartition $\{B_1, B_2, \ldots, B_{16\gamma}\} \cup \mathcal{L}'$. Join B_i to $L \in \mathcal{L}'$ if $B_i \cap L \neq \varnothing$. By property (H4) $d_{Aux}(B_i) \geqslant \frac{\alpha q}{4}$ for all $i \in [16\gamma]$, and thus the number of edges in Aux satisfies

$$|e(Aux)| \geqslant \frac{\alpha q}{4}|\{B_1, B_2, \ldots, B_{16\gamma}\}| = 4\alpha q\gamma. \tag{2.4}$$

On the other hand, clearly $d_{Aux}(L) \leqslant |\{B_1, B_2, \ldots, B_{16\gamma}\}| = 16\gamma$ for all $L \in \mathcal{L}'$ and by definition $d_{Aux}(L') \leqslant \gamma$ for all $L' \in \{\mathcal{L}' \backslash \mathcal{L}'_{B,\gamma}\}$. Also keeping in mind that by (H2)
$|\mathcal{L}' \backslash \mathcal{L}'_{B,\gamma}| \leqslant |\mathcal{L}'| = \sum_{v \in V} \frac{d_H(v)}{q} \leqslant q^2 \frac{2\alpha}{q} = 2\alpha q$, we compute

$$|e(Aux)| \leqslant |\mathcal{L}'_{B,\gamma}| \cdot 16\gamma + |\{\mathcal{L}' \backslash \mathcal{L}'_{B,\gamma}\}| \cdot \gamma \leqslant |\mathcal{L}'_{B,\gamma}| \cdot 16\gamma + 2\alpha q \cdot \gamma. \tag{2.5}$$

Comparing (2.4) and (2.5), we obtain

$$4\alpha q\gamma \leqslant |e(Aux)| \leqslant |\mathcal{L}'_{B,\gamma}| \cdot 16\gamma + 2\alpha q\gamma,$$

which yields $|\mathcal{L}'_{B,\gamma}| \geqslant \frac{\alpha q}{8}$. $\square$

2.3 The Graph G

Based upon the hypergraph H established in the previous section, we will construct a graph G with the following properties.

Lemma 2.6. *Let q be a sufficiently large prime, $\alpha = (\log q)^2$, $\beta = (\log q)^{4s^2}$, and $s \geqslant 3$. Then, there exists a graph $G = (V, E)$ of order q^2 such that:*

(G1) *For every $C \in \binom{V}{16sq}$, $G[C]$ contains a copy of K_s;*

(G2) *For every $U \in \binom{V}{64s\beta q}$, $G[U]$ contains $\frac{\alpha\beta^2 q}{16}$ edge disjoint copies of K_s;*

(G3) *Every edge $xy \in E$ is in at most $6^s\alpha^{2s-2}$ copies of K_{s+1};*

(G4) *If $s \geqslant 4$, then G can be made K_{s+2}-free by removing $2\alpha^8 q$ vertices.*

Proof. Fix a hypergraph $H = (V, \mathcal{L}')$ as established by Lemma 2.5. Construct the random graph $\mathbb{G}$ as follows. For every $L \in \mathcal{L}'$, let $\chi_L : L \to [s]$ be a random partition of the vertices of L into s classes, where for every $v \in L$, a class $\chi_L(v) \in [s]$ is assigned uniformly and independently at random. Then, let $xy \in E$ if $\{x, y\} \subseteq L$ for some $L \in \mathcal{L}'$ and $\chi_L(x) \neq \chi_L(y)$. Thus for every $L \in \mathcal{L}'$, $\mathbb{G}[L]$ is a complete s-partite graph with vertex partition $L = \chi_L^{-1}(1) \cup \chi_L^{-1}(2) \cup \cdots \cup \chi_L^{-1}(s)$ (where the classes need not have the same size and the unlikely event that a class is empty is permitted). Observe that not only are $G_{H'}[L]$ and $G_{H'}[L']$ are edge disjoint for distinct $L, L' \in \mathcal{L}'$, but also that the partitions for L and L' were determined independently.

We will show $\mathbb{G}$ does not satisfy (G1) and (G2) with probability at most $o(1)$ and that $\mathbb{G}$ always satisfies (G3) and (G4). Hence the probability that $\mathbb{G}$ satisfies properties (G1)-(G4) is at least $1 - o(1)$, implying the existence of a graph G described in the lemma.

(G1): Consider any $C \in \binom{V}{16sq}$. We will bound the probability that $G[C] \not\supset K_s$. By (H5) with $\gamma = s$, the set of lines $\mathcal{L}'_{C,s}$ that intersect C in at least s vertices has cardinality $|\mathcal{L}'_{C,s}| \geqslant \frac{\alpha q}{8}$. For each $L \in \mathcal{L}'_{C,s}$, let X_L be the event $K_s \nsubseteq \mathbb{G}[L \cap C]$. Since $|L \cap C| \geqslant s$ for all $L \in \mathcal{L}'_{C,s}$, $\Pr(X_L) \leqslant 1 - \frac{s!}{s^s}$. By independence,

$$\Pr\Big(K_s \notin \mathbb{G}[C]\Big) \leqslant \Pr\Big(\bigcap_{L \in \mathcal{L}'_{C,s}} X_L\Big) \leqslant \left(1 - \frac{s!}{s^s}\right)^{|\mathcal{L}'_{C,s}|}$$
$$\leqslant \left(1 - \frac{s!}{s^s}\right)^{\frac{\alpha q}{8}} \leqslant \exp\left\{-\frac{s!}{s^s}\frac{\alpha q}{8}\right\}.$$

So by the union bound, the probability that there exists a subset of $16sq$ vertices in $\mathbb{G}$ that contains no K_s is at most

$$\binom{q^2}{16sq} \exp\left\{-\frac{s!}{s^s}\frac{\alpha q}{8}\right\} \leqslant q^{16sq} \exp\left\{-\frac{s!}{s^s}\frac{\alpha q}{8}\right\}$$
$$= \exp\left\{16sq\log q - \frac{s! q(\log q)^2}{8s^s}\right\} = o(1).$$

(G2): For arbitrary $U \in \binom{V}{64s\beta q}$, we will bound the probability that $G[U]$ does not contain $\frac{\alpha\beta^2 q}{16}$ edge disjoint copies of K_s. By (H5) with $\gamma = 4s\beta$, we may fix a subset $\mathcal{Z}_U \subseteq \mathcal{L}'_{U,4s\beta}$ of exactly $\frac{\alpha q}{8}$ lines with the property that each line has intersection at least $4s\beta$ with U. We will consider the lines in $\mathcal{Z}_U$ that contain the complete balanced s-partite graph on $2s\beta$ vertices, which we denote by $K_{2\beta,\dots,2\beta}$. Define $\mathcal{Z}'_U = \{L \in \mathcal{Z}_U : K_{2\beta,\dots,2\beta} \subseteq \mathbb{G}[L \cap U]\}$. The graph $K_{2\beta,\dots,2\beta}$ certainly contains at least β^2 edge disjoint K_s (Since we may choose a prime $\beta \leqslant p \leqslant 2\beta$ and it follows from [1] that we may then decompose $K_{p,\dots,p}$ into p^2 edge disjoint copies of K_s; this suffices for our purposes, but stronger results are know). Thus if we show $|\mathcal{Z}'_U| \geqslant \frac{\alpha q}{16}$ it will imply that $\mathbb{G}[U]$ contains at least $|\mathcal{Z}'_U| \cdot \beta^2 \geqslant \frac{\alpha\beta^2 q}{16}$ edge disjoint copies of K_s.

For $L \in \mathcal{Z}_U$, let Y_L be the event that $L \notin \mathcal{Z}'_U$ and fix $L_{4s\beta} \subseteq L \cap U$, $|L_{4s\beta}| = 4s\beta$. Now Y_L will occur only if $|\chi_L^{-1}(i) \cap L_{4s\beta}| < 2\beta$ for some $i \in [s]$. Defining

$X_i = |\chi_L^{-1}(i) \cap L_{4s\beta}|$, observe $X_i \sim Bi(4s\beta, \frac{1}{s})$ and $E(X_i) = 4\beta$. Chernoff's inequality reveals

$$\Pr\Big(X_i < 2\beta\Big) \leqslant \Pr\left(|X_i - E(X_i)| \geqslant \frac{E(X_i)}{2}\right) \leqslant 2\exp\left\{-\frac{4\beta}{12}\right\} = 2\exp\left\{-\frac{\beta}{3}\right\}.$$

By the union bound, $\Pr(Y_L) \leqslant \Pr\left(\bigcup_{i\in s}(X_i \leqslant 2\beta)\right) \leqslant s \cdot 2\exp\left\{-\frac{\beta}{3}\right\}$.

By independence, the probability that Y_L occurs for at least $\frac{\alpha q}{16} = \frac{|\mathcal{Z}_U|}{2}$ of the lines in $\mathcal{Z}_U$ is at most

$$\begin{aligned}\binom{|\mathcal{Z}_U|}{\frac{|\mathcal{Z}_U|}{2}}\left(2s\exp\left\{-\frac{\beta}{3}\right\}\right)^{|\mathcal{Z}_U|/2} &\leqslant 4^{|\mathcal{Z}_U|/2}\left(2s\exp\left\{-\frac{\beta}{3}\right\}\right)^{|\mathcal{Z}_U|/2} \\ &= \left(8s\exp\left\{-\frac{\beta}{3}\right\}\right)^{\frac{\alpha q}{16}}\end{aligned}$$

That is, we have shown $|\mathcal{Z}'_U| < \frac{\alpha q}{16}$ with probability at most $\left(8s\exp\left\{-\frac{\beta}{3}\right\}\right)^{\frac{\alpha q}{16}}$ for fixed U. Thus by the union bound, the probability that there exits some $U \subseteq V$ with $|U| = 64s\beta q$ such that $|\mathcal{Z}'_U| < \frac{\alpha q}{16}$ is at most

$$\begin{aligned}\binom{q^2}{64s\beta q}\left(8s\exp\left\{-\frac{\beta}{3}\right\}\right)^{\frac{\alpha q}{16}} &\leqslant q^{64s\beta q}(8s)^{\frac{\alpha q}{16}}\left(\exp\left\{-\frac{\beta}{3}\right\}\right)^{\frac{\alpha q}{16}} \\ &\leqslant \exp\left\{64s\beta q\log q + \frac{\alpha q}{16}\log(8s) - \frac{\alpha\beta q}{48}\right\} = o(1).\end{aligned}$$

(G3): For any $xy \in E$, we will show the number of copies of K_{s+1} that contain xy is at most $6^s\alpha^{2s-2}$. Let $L \in \mathcal{L}'$ be the unique line such that $\{x, y\} \subseteq L$ as depicted in Figure 2.2. Let $N = (N_H(x) \cap N_H(y))\backslash L$ be the set of all vertices not on L that are collinear with both x and y. Since $d_H(x), d_H(y) \leqslant 2\alpha$ by (H2), we infer that $|N| \leqslant 4\alpha^2$. Because $K_{s+1} \nsubseteq G[L]$, if a K_{s+1} is to contain x and y it must contain at least one vertex $v \in N$.

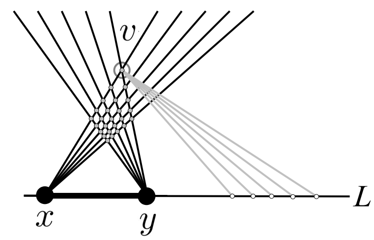


Figure 2.2: Counting K_{s+1} in $\mathbb{G}$ that contains a fixed edge xy by considering lines in H.

There are at most $|N| \leqslant 4\alpha^2$ choices for this vertex v. Once v has been chosen, each of the remaining $s-2$ vertices of the K_{s+1} must lie in N or in $L \cap N_H(v)$. Since $|N| + |L \cap N_H(v)| \leqslant 4\alpha^2 + 2\alpha$, the number of K_{s+1} containing the edge xy is at most $4\alpha^2(4\alpha^2 + 2\alpha)^{s-2} \leqslant 6^s\alpha^{2s-2}$.

(G4): We will finally show that if $s \geqslant 4$, G can be made K_{s+2} free be removing at most $2\alpha^8 q$ vertices. By (H3), all $\mathcal{L}'$-dangerous sets can be destroyed by removing $2\alpha^8 q$ vertices, so it suffices to shown that every K_{s+2} in $\mathbb{G}$ contains a $\mathcal{L}'$-dangerous subset.

Let K be any copy of K_{s+2} in $\mathbb{G}$. By assumption $s \geqslant 4$, so K must have at least 6 vertices, which clearly form a $\mathcal{L}'$-complete set.

We first show that K contains 4 vertices in general position. Suppose otherwise. Then there is some line $L \in \mathcal{L}'$ that contains 3 vertices $\{p_1, p_2, p_3\}$ of K. Since $K_{s+1} \nsubseteq \mathbb{G}[L]$, there must exist two vertices a and b in $K \backslash L$. Observe $\{a, b\}$ and any 2 vertices in $\{p_1, p_2, p_3\} \backslash L(a, b)$ are in general position.

Now fix 4 vertices $\{v_1, v_2, v_3, v_4\}$ of K that are in general position and let u_1, u_2 be any two other vertices of K. Three cases are now considered. If either u_1 or u_2 do not lie on any of the 6 lines $L(v_i, v_j)$ for $1 \leqslant i < j \leqslant 4$, then there is a $\mathcal{L}'$-dangerous subset of Type 1. If either u_1 or u_2 lie on exactly one line in $L(v_i, v_j)$ for $1 \leqslant i < j \leqslant 4$, then there is a $\mathcal{L}'$-dangerous subset of Type 2. In the remaining case where both u_1 and u_2 each lie on at least

2 lines in $L(v_i, v_j)$ for $1 \leqslant i < j \leqslant 4$, then there is a $\mathcal{L}'$-dangerous subset of Type 3. $\square$

2.4 Proof of Theorems 2.1 and 2.2

Consider any sufficiently large integer n and $s \geqslant 3$. By Bertrand's postulate, we can find a prime q such that $4n \leqslant q^2 \leqslant 16n$. Fix a graph G procured by Lemma 2.6 of order q^2 and as before take

$$\alpha = (\log q)^2 \qquad \text{and} \qquad \beta = (\log q)^{4s^2}.$$

Theorem 2.1 and Theorem 2.2 are now proved by considering different subgraphs of G of order n.

Proof of Theorem 2.2. Consider the case where $s \geqslant 4$. To prove the theorem, we will show there exists a K_{s+2}-free induced subgraph of G of order n with the property that every subset of order $64s\sqrt{n}$ contains a copy of K_s.

By (G1), every set of size $16sq$ in G contains K_s, so certainly every subset of size $64s\sqrt{n} \geqslant 16sq$ in any induced subgraph of G must also contain a copy of K_s. Thus it will suffice to show that there is a K_{s+2}-free subset of G of order n. But by (G4), we know that there is a set $R \subseteq V(G)$ of size $|R| = 2\alpha^8 q \leqslant n$ such that $G[V\backslash R]$ will be K_{s+2}-free. Finally since $|V\backslash R| \geqslant 4n - n \geqslant n$, the induced graph of G on any n vertices in $V\backslash R$ will have the desired properties. $\square$

Proof of Theorem 2.1. For $s \geqslant 3$, we will concentrate on constructing a K_{s+1}-free graph G' on q^2 vertices with the property that every subset of size $64s\beta q$ vertices contains a copy of K_s. Since $\log(4n) \leqslant 2\log n$,

$$64s\beta q = 64s(\log q)^{4s^2} q \leqslant 64s(\log 4n)^{4s^2} 4n \leqslant 2^{4s^2+8}(\log n)^{4s^2} n,$$

and so any induced subgraph of G' of order n will also be K_{s+1}-free and have the property that every set of order $2^{4s^2+8}(\log n)^{4s^2} n$ contains a copy of K_s, exactly as desired.

Let G' be a random subgraph of G where each edge is taken with probability

$$\frac{1}{\gamma}, \quad \text{where } \gamma = (\log q)^8.$$

For a set $S \in \binom{V(G)}{s+1}$ that spans a copy of K_{s+1} in G, let A_S to be the event that all the edges of S are in G'. Hence, $\bigcap \overline{A_S}$ means that $K_{s+1} \nsubseteq G'$. For a set $U \in \binom{V(G)}{64s\beta q}$ let $\mathcal{K}_U$ be a (fixed) set of

$$m = \frac{1}{16}\alpha\beta^2 q$$

edge disjoint copies K_s contained in U, which are known to exist by (G2). Define B_U to be the event that none of the m edge disjoint K_s appear in G'. Hence, $\bigcap \overline{B_U}$ implies that for every $U \in \binom{V(G)}{64s\beta q}$ one of the disjoint copies of K_s in $G[U]$ appears in G'. It will suffice to show that the probability that $\left(\bigcap \overline{A_S}\right) \cap \left(\bigcap \overline{B_U}\right)$ occurs is nonzero. In order to show this, we apply the Local Lemma (see, e.g., Lemma 5.1.1 in [4]).

Lovász Local Lemma. *Let $E_1, E_2, \ldots, E_k$ be events in an arbitrary probability space. A directed graph D on the set of vertices $\{1, 2, \ldots, k\}$ is called a* dependency digraph *for the events $E_1, E_2, \ldots, E_k$ if for each i, $1 \leqslant i \leqslant k$, the event E_i is mutually independent of all the events $\{E_j : (i,j) \notin D\}$. Suppose that D is a dependency digraph for the above events and suppose there are real numbers $z_1, \ldots, z_k$ such that $0 \leqslant z_i < 1$ and $Pr(E_i) \leqslant z_i \prod_{(i,j)\in D}(1 - z_j)$ for all $1 \leqslant i \leqslant k$. Then,* $\Pr\left(\bigcap_{i=1}^{k} \overline{E_i}\right) > 0$.

Let D be a dependency graph that corresponds to all events A_S and B_U. Observe that A_S depends only on the $\binom{s+1}{2}$ edges in S and B_U depends only on the $m\binom{s}{2}$ edges of the K_s in $\mathcal{K}_U$. Also, observe that the number of events of the type B_U is $\binom{q^2}{64s\beta q} \leqslant q^{64s\beta q}$. Thus by (G3), a fixed event A_S depends on at most

$$d_{AA} = \binom{s+1}{2} 6^s \alpha^{2s-2}$$

other events $A_{S'}$ and at most

$$d_{AB} = q^{64s\beta q}$$

events B_U. Similarly, a fixed event B_U depends on at most

$$d_{BA} = m\binom{s}{2} 6^s \alpha^{2s-2}$$

events A_S and at most

$$d_{BB} = q^{64s\beta q}$$

other events $B_{U'}$. Let

$$x = \frac{1}{\alpha^{2s^2}} \quad \text{and} \quad y = \frac{1}{(\log q)^{4s^2} q^{64s\beta q}}.$$

To finish the proof, due to the Local Lemma it suffices to show that

$$\left(\frac{1}{\gamma}\right)^{\binom{s+1}{2}} \leqslant x(1-x)^{d_{AA}}(1-y)^{d_{AB}}, \tag{2.6}$$

$$\left(1 - \left(\frac{1}{\gamma}\right)^{\binom{s}{2}}\right)^m \leqslant y(1-x)^{d_{BA}}(1-y)^{d_{BB}}. \tag{2.7}$$

First we show that (2.6) holds. Using the fact that $e^{-2x} \leqslant 1 - x$ for x sufficiently small (observe that $x \to 0$ with $q \to \infty$), a sufficient condition for (2.6) will be

$$\left(\frac{1}{\gamma}\right)^{\binom{s+1}{2}} \leqslant x\, e^{-2xd_{AA}}\, e^{-2yd_{AB}},$$

and equivalently,

$$\binom{s+1}{2}\log(\gamma) \geqslant \log\left(\frac{1}{x}\right) + 2xd_{AA} + 2yd_{AB}.$$

The latter immediately follows from the following three inequalities (which can be easily verified):

$$\begin{aligned}
\frac{2s^2}{2s^2+2s}\binom{s+1}{2}\log(\gamma) &\geqslant \log\left(\frac{1}{x}\right),\\
\frac{s}{2s^2+2s}\binom{s+1}{2}\log(\gamma) &\geqslant 2xd_{AA},\\
\frac{s}{2s^2+2s}\binom{s+1}{2}\log(\gamma) &\geqslant 2yd_{AB}.
\end{aligned}$$

Similarly, using the facts that $e^{-2y} \leqslant 1-y$ for y sufficiently small and that $1-\left(\frac{1}{\gamma}\right)^{\binom{s}{2}} \leqslant e^{-\left(\frac{1}{\gamma}\right)^{\binom{s}{2}}}$, (2.7) will be satisfied if

$$e^{-m\left(\frac{1}{\gamma}\right)^{\binom{s}{2}}} \leqslant y\, e^{-2xd_{BA}}\, e^{-2yd_{BB}},$$

and equivalently,

$$m\left(\frac{1}{\gamma}\right)^{\binom{s}{2}} \geqslant \log\left(\frac{1}{y}\right) + 2xd_{BA} + 2yd_{BB}.$$

As before the latter will follow from the following easy to check inequalities:

$$\frac{1}{3}m\left(\frac{1}{\gamma}\right)^{\binom{s}{2}} \geqslant \log\left(\frac{1}{y}\right),$$
$$\frac{1}{3}m\left(\frac{1}{\gamma}\right)^{\binom{s}{2}} \geqslant 2xd_{BA},$$
$$\frac{1}{3}m\left(\frac{1}{\gamma}\right)^{\binom{s}{2}} \geqslant 2yd_{BB}.$$

This completes the proof of Theorem 2.1. □

2.5 Concluding Remarks

We close this chapter by discussing how the asymptotic behavior of $f_{s,t}(n)$ changes for different values of $3 \leqslant s < t$.

If the difference between s and t is fixed, we make the following observation based upon the lower bound in Sudakov [73] (and Fact 3.5 in [21]) and Corollary 2.3.

Observation 2.7. *For any $\varepsilon > 0$ and an integer $k \geqslant 2$ there is a constant $s_0 = s_0(k, \varepsilon)$ such that for all $s \geqslant s_0$,*

$$\Omega\left(n^{\frac{1}{2}-\varepsilon}\right) = f_{s,s+k}(n) = O(\sqrt{n}).$$

In view of this observation and Theorem 2.2 we ask the following.

Question 2.8. *For any $s \geqslant 3$, is $f_{s,s+2}(n) = o(\sqrt{n})$?*

Another interesting question results from fixing the ratio between s and t. The following is based upon [73] and [51] respectively.

Observation 2.9. *For any $\varepsilon > 0$ and $\lambda \geqslant 2$ there is a constant $s_0 = s_0(\lambda, \varepsilon)$ such that for all $s \geqslant s_0$,*

$$\Omega\left(n^{\frac{1}{2\lambda}-\varepsilon}\right) = f_{s,\lfloor \lambda s \rfloor}(n) = O\left(n^{\frac{1}{\lambda}}\right).$$

In particular, when $\lambda = 3$, we see $\Omega(n^{1/6-\varepsilon}) = f_{s,\lfloor \lambda s \rfloor}(n) = O(n^{1/3})$.

Question 2.10. *What is the asymptotic behavior of $f_{s,\lfloor \lambda s \rfloor}(n)$?*

Recall that Erdős [23] asked if for fixed $s + 2 \leqslant t$, $\lim_{n\to\infty} \frac{f_{s+1,t}(n)}{f_{s,t}(n)} = \infty$. We ask a similar question, that if answered in the affirmative would imply an affirmative answer to the question of Erdős.

Question 2.11. *For all $t > s \geqslant 3$, is $\lim_{n\to\infty} \frac{f_{s+1,t+1}(n)}{f_{s,t}(n)} = \infty$?*

Chapter 3

Size-Ramsey Numbers of Short Subdivisions

3.1 Introduction

For graphs H and G and an integer ℓ, we write $H \to (G)_\ell$ if every coloring of the edges of H with ℓ colors contains a monochromatic copy of G. In the two color case ($\ell = 2$), we omit the subscript and simply write $H \to G$. For a graph G, the study of which graphs H have the property $H \to G$ is a major area of research in extremal combinatorics. One of the most well-known questions of this nature is to determine the *Ramsey number* $r(G)$, which is the minimum number of vertices in a graph H with the property $H \to G$. That is,

$$r(G) := \min\Big\{|V(H)| : H \to G\Big\},$$

where without loss of generality H can be assumed to be a complete graph. A variation of this problem, introduced by Erdős, Faudree, Rousseau, and Schelp [30] in 1978, asks for the minimum number of edges in a graph H with the property $H \to G$. This is the *size Ramsey number* of G and is often denoted by $\hat{r}(G)$. In other words,

$$\hat{r}(G) := \min\Big\{|E(H)| : H \to G\Big\}.$$

Trivially, $\widehat{r}(G) \leqslant \binom{r(G)}{2}$, and a simple argument, attributed to Chvátal in [30], shows that $\widehat{r}(K_n) = \binom{r(K_n)}{2}$ for the case when G is the complete graph. For many sparse graphs G, as we will see, the bound $\widehat{r}(G) \leqslant \binom{r(G)}{2}$ is far from optimal.

One of the first problems investigated regarding the size Ramsey number was to determine the behavior of the function $\widehat{r}(P_n)$, where P_n is the path on n vertices. Erdős asked the following version of this question in [29]: Is it true that

$$\widehat{r}(P_n)/n \to \infty \qquad \text{and} \qquad \widehat{r}(P_n)/n^2 \to 0?$$

This was answered in the negative by Beck [6], who, using probabilistic methods, proved that $\widehat{r}(P_n) \leqslant 900n$. This result was extended in [44], where it was established that cycles also have linear size Ramsey numbers (in fact, it was shown this even holds for the induced version of the size Ramsey number). Another extension by Friedman and Pippenger [37] established the linearity of the size Ramsey number for trees with bounded degree. More recently, Dellamonica [17] was able to determine asymptotically the size Ramsey number of general trees, confirming a conjecture of Beck. Other related results include [43, 46].

A significant open problem is to determine the size Ramsey number of graphs of bounded degree. Letting $\Delta(G)$ denote the maximum degree of G, we define this function of interest by

$$\widehat{r}(n,d) := \max\Big\{\widehat{r}(G) : |V(G)| = n,\ \Delta(G) \leqslant d\Big\}.$$

In [7], Beck asked if $\widehat{r}(n,d)$ is always linear in n for fixed d. This was settled in the negative by Rödl and Szemerédi [64], who established that

$$\widehat{r}(n,3) = \Omega(n(\log n)^{1/60}).$$

That is, they constructed graphs G of order n and maximum degree 3 and argued that if H is any graph with fewer than $cn(\log n)^{1/60}$ edges, then H does not have the property $H \to G$. In the same paper, it was conjectured that for all d there exists $\varepsilon_d > 0$ such that

$$n^{1+\varepsilon_d} \leqslant \widehat{r}(n,d) \leqslant n^{2-\varepsilon_d}. \tag{3.1}$$

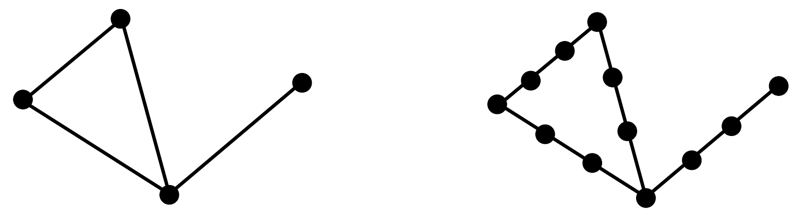
The upper bound in (3.1) was subsequently proved by Kohayakawa, Rödl, Schacht, and Szemerédi in [49]. The lower bound in (3.1), however, remains open and closing the rather large remaining gap between the upper and lower bounds for $\widehat{r}(n,d)$ is of considerable interest. For further results on size Ramsey numbers, see [35, 58–60], or the more general recent survey on graph Ramsey theory [16].

Subdivisions of Graphs

For a graph S and positive integer h, the *h-subdivision* of S, denoted $S^{(h)}$, is the graph obtained by replacing each edge of S with a path on h internal vertices as demonstrated in Figure 3.1 for the case $h = 2$. Having in mind that the size Ramsey numbers of trees are quite well-understood and that much regarding the size Ramsey numbers of bounded degree graphs remains open, we believe it is of interest to determine the size Ramsey numbers of subdivisions.

The size Ramsey number of 'long' subdivisions of bounded degree, which are subdivided graphs $S^{(h)}$ where $h > c\log|S^{(h)}|$ and the maximum degree of S is bounded, were studied by Pak [57] in 2002. Pak conjectured that $\widehat{r}(S^{(h)})$ is linear in terms of $|S^{(h)}|$ for such subdivisions and, by using results on mixing times of random walks on expanders, proved a weaker form of this conjecture up to a polylogarithmic factor.

Our main result relates to the size Ramsey number of 'short' subdivisions



(a) A graph S (b) The graph $S^{(2)}$

Figure 3.1: A graph and its subdivision

of bounded degree, which are subdivided graphs $S^{(h)}$ where h and the maximum degree of S are fixed and the number of vertices $|V(S)|$ is relatively large. To state a more general form of this result, we introduce the following definition.

Definition 3.1 (Universal Size Ramsey Number)**.** *For $h, d, \ell, s \in \mathbb{Z}^+$, define the universal size Ramsey number* $\mathrm{USR}(h, d, \ell, s)$ *to be the fewest number of edges in a graph H that has the following universal Ramsey property:*

$H \to (S^{(h)})_\ell$ *for every graph S on s vertices with maximum degree d.*

Theorem 3.2. *For any $h, d, \ell \in \mathbb{Z}^+$, there exists s_0 such that for all $s \geqslant s_0$,*

$$\mathrm{USR}(h, d, \ell, s) \leqslant (\log s)^{20h} s^{1+1/(h+1)}. \tag{3.2}$$

A corollary is that for any $h \geqslant 1$ and $d \geqslant 1$, there exists s_0 such that if S is any graph on $s \geqslant s_0$ vertices with maximum degree d,

$$\hat{r}(S^{(h)}) \leqslant (\log s)^{20h} s^{1+1/(h+1)}.$$

A short counting argument, which will be given in Section 3.6, yields the following lower bound.

Theorem 3.3. *For all $h, d, \ell, s \in \mathbb{Z}^+$ with $d \geqslant 3$,*

$$\mathrm{USR}(h, d, \ell, s) \geqslant \mathrm{USR}(h, d, 1, s) \geqslant s^{1+1/(h+1)-2/d(h+1)+o(1)}, \tag{3.3}$$

where $o(1) \to 0$ as $s \to \infty$.

That is, we obtain a bound for the number of edges in any graph H that contains $S^{(h)}$ as a subgraph for every graph S of maximum degree d on s vertices. Observe that for large d, the exponent in (3.2) is close to the exponent in (3.3).

We will also show that the proof of Theorem 3.2 can be extended to give the following more general theorem.

Theorem 3.4. *For any $h, d \in \mathbb{Z}^+$, there exists a constant $c_{h,d}$ such that the following holds. If Q is a graph with maximum degree at most d on q vertices with the property that every pair of vertices of degree greater than 2 are distance at least $h+1$ apart, then*

$$\hat{r}(G) \leqslant c_{h,d}(\log q)^{20h} q^{1+1/(h+1)}.$$

We believe that the exponent of the logarithm in both Theorems 3.2 and 3.4 could be substantially reduced, although our method does not allow for the dependency of the exponent of the logarithm on h to be removed. For the sake of clarity of presentation, we have opted not to make any attempt to optimize this power. We do believe, however, that removing the dependency on h or removing the logarithm entirely would be of interest. We also ask the following.

Question 3.5. *For every integer d, does there exist a constant c_d such that*

$$\hat{r}(S^{(h)}) \leqslant c_d h s^{1+1/(h+1)}$$

for every integer h and for every graph S on s vertices with maximum degree d?

Notation

We use fairly standard notation in this chapter, including the following. For a graph H and vertex subsets X_1 and X_2, we let $E_H(X_1, X_2)$ be the the set of edges between X_1 and X_2 and $e_H(X_1, X_2) = |E_H(X_1, X_2)|$. When unambiguous, we omit the subscript. Unless explicitly noted otherwise, a subgraph need not be induced. Also, as is standard, we omit floors and ceilings that do not affect the asymptotic nature of our calculations.

Organization

The rest of this chapter is organized as follows. Section 3.2 introduces an *Existence Lemma* (Lemma 3.12), a *Coloring Lemma* (Lemma 3.9), and an *Embedding Lemma* (Lemma 3.14), and then establishes Theorem 3.2 based upon these three lemmas. The proofs of these three lemmas are deferred to Sections 3.4, 3.3, and 3.5 respectively. Section 3.6 addresses Theorem 3.3. Section 3.7 addresses Theorem 3.4.

3.2 Proof of Theorem 3.2

The proof of Theorem 3.2 is based on an *Existence Lemma* (Lemma 3.12), a *Coloring Lemma* (Lemma 3.9), and an *Embedding Lemma* (Lemma 3.14). The Existence Lemma will establish the existence of a sparse graph G that has several properties including being a member of a class of graphs called $\mathcal{I}(N, p)$ (Definition 3.8). The Coloring Lemma will establish that, since $G \in \mathcal{I}(N, p)$, any ℓ-coloring of the edges of G yields a monochromatic subgraph H that is a member of a class of graphs called $\mathcal{H}(h, n, \varepsilon, q)$ (Definition 3.7). For appropriate parameters, we will have that the graph $H \in \mathcal{H}(h,n,\varepsilon,q)$ is also in a class of graphs called $\mathcal{J}(h,n,\delta)$ (Definition 3.13). For any graph S on s vertices that has maximum degree d, the Embedding Lemma will then establish that, since H is in $\mathcal{J}(h,n,\delta)$, the graph $S^{(h)}$ can be embedded into H. These lemmas together will be used to establish that $G \to (S^{(h)})_\ell$ for any graph S on s vertices with maximum degree d, as desired. The objective of this section is to introduce the terminology required to state these three lemmas and then to prove Theorem 3.2.

The following class describes graphs obtained from blowing up the cycle C_{h+1} by replacing each vertex by an independent set of size n and each edge by an arbitrary bipartite graph. In this definition and elsewhere, we say that H is a graph on $\bigsqcup_{i=1}^{h+1} X_i$ if $X_1, X_2, \ldots, X_{h+1}$ are pairwise disjoint sets and $V(H) = \bigcup_{i=1}^{h+1} X_i$. For notational convenience, we will index the sets X_i modulo $h+1$; in particular, we set $X_{h+2} := X_1$ and $X_0 := X_{h+1}$.

Definition 3.6. *Let $\mathcal{H}(h,n)$ bet the set of all graphs on $\bigsqcup_{i=1}^{h+1} X_i$ such that both the following hold:*

(i) $|X_i| = n$ *for all* $i \in [h+1]$.

(ii) $E(H) = \bigsqcup_{i=1}^{h+1} E_H(X_i, X_{i+1})$.

The following subclass of $\mathcal{H}(h,n)$ describes graphs where the bipartite graphs induced on (X_i, X_{i+1}) have density q and uniformly distributed edges.

Definition 3.7. *Let $\mathcal{H}(h,n,\varepsilon,q)$ be the set of all graphs H on $\bigsqcup_{i=1}^{h+1} X_i$ that are in $\mathcal{H}(h,n)$ and satisfy the following additional two properties:*

(iii) $e(X_i, X_{i+1}) = qn^2$ *for all* $i \in [h+1]$.

(iv) For any integer $i \in [h+1]$ and vertex subsets $\widehat{X}_i \subset X_i$ and $\widehat{X}_{i+1} \subset X_{i+1}$ each of size $|\widehat{X}_i|, |\widehat{X}_{i+1}| \geqslant \varepsilon n$,

$$(1-\varepsilon)q|\widehat{X}_i||\widehat{X}_{i+1}| \leqslant e(\widehat{X}_i, \widehat{X}_{i+1}) \leqslant (1+\varepsilon)q|\widehat{X}_i||\widehat{X}_{i+1}|.$$

In the context of the random graph $G(N, p)$, the next definition introduces a class of graphs having neither 'dense bipartite patches' nor 'large bipartite holes'.

Definition 3.8. *Let $\mathcal{I}(N, p)$ be the set of N-vertex graphs G that have both the following properties:*

(i) For all disjoint sets $V_1, V_2 \subset V(G)$ with $1 \leqslant |V_1| \leqslant |V_2| \leqslant pN|V_1|$,

$$e(V_1, V_2) \leqslant p|V_1||V_2| + e^2\sqrt{6} \cdot \sqrt{pN|V_1||V_2|}.$$

(ii) For all disjoint sets $V_1, V_2 \subset V(G)$ with $|V_1|, |V_2| \geqslant N(\log N)^{-1}$,

$$(1/2) \cdot p|V_1||V_2| \leqslant e(V_1, V_2) \leqslant 2 \cdot p|V_1||V_2|.$$

The following lemma is a deterministic statement about the previous two classes of graphs.

Lemma 3.9 (Coloring Lemma)**.** *For any $\varepsilon \in \mathbb{R}^+$ and $h, \ell \in \mathbb{Z}^+$, there exist $t, n_1 \in \mathbb{Z}^+$ such that, for all $n \geqslant n_1$,*

$$q := 4(\log n)^2 n^{-1+1/(h+1)}, \qquad N := tn, \qquad \textit{and} \qquad p := 4\ell q,$$

every graph $G \in \mathcal{I}(N, p)$ has the following property. Any ℓ-coloring of the edges of G yields disjoint vertex subsets $X_1, X_2, \ldots, X_{h+1} \subset V(G)$ and a monochromatic subgraph H on $\bigsqcup_{i=1}^{h+1} X_i$ such that $H \in \mathcal{H}(h, n, \varepsilon, q)$.

The Existence Lemma, which we state next, establishes that there exists a graph G on N vertices that exhibits several properties including being in $\mathcal{I}(N, p)$. Combined with the Coloring Lemma, this gives that, for appropriate parameters, any ℓ-coloring of such a graph G will not only contain a monochromatic copy of some $H \in \mathcal{H}(h, n, \varepsilon, q)$, but one that inherits certain additional desirable properties which will be used to embed $S^{(h)}$. We now describe these additional properties.

Definition 3.10 (Path Abundance)**.** *Let H be a graph on $\bigsqcup_{i=1}^{h+1} X_i$ with $H \in \mathcal{H}(h,n)$.*

- *For vertices $u, v \in X_1$, a* transversal path *between u and v is an (undirected) path with endpoints u and v that has exactly $h+2$ vertices and exactly one vertex from each X_i for all $i \in [h+1] \backslash \{1\}$.*
- *H is* $(1-\delta, \log n)$-path abundant *if for at least $(1-\delta)\binom{n}{2}$ pairs of vertices $\{u,v\} \in \binom{X_1}{2}$, there are at least $\log n$ transversal paths between u and v that are pairwise edge-disjoint.*

Definition 3.11 (Cluster-Free)**.** *Let F be a graph and $\mathcal{L} \subset \binom{V(F)}{2}$ be a set of pairs of vertices in F (that need not correspond to edges). Let $V(\mathcal{L}) = \bigcup_{\{u,v\} \in \mathcal{L}} \{u,v\}$ and $Z \subset V(F)$ be a subset of vertices with $Z \cap V(\mathcal{L}) = \varnothing$.*

- *An* $(\mathcal{L}, Z, h, \log n)$-cluster *is a set of paths $\mathcal{P}_{\mathcal{L}}$ such that:*
  - *For every $P \in \mathcal{P}_{\mathcal{L}}$, the path P has exactly $h+2$ vertices.*
  - *For every path $P \in \mathcal{P}_{\mathcal{L}}$, the endpoints u and v of P are such that $\{u,v\} \in \mathcal{L}$.*
  - *For every $P \in \mathcal{P}_{\mathcal{L}}$, the path P does not have an internal vertex in $V(\mathcal{L})$.*
  - *For every $\{u,v\} \in \mathcal{L}$, exactly $\log n$ paths in $\mathcal{P}_{\mathcal{L}}$ have endpoints u and v.*
  - *For every pair of paths P and $\widehat{P}$ in $\mathcal{P}_{\mathcal{L}}$, the paths P and $\widehat{P}$ are edge-disjoint.*
  - *For every $P \in \mathcal{P}_{\mathcal{L}}$, the path P has exactly one internal vertex in Z.*
- *We say that F is* (h,n)-cluster free *if F does not contain an $(\mathcal{L}, Z, h, \log n)$-cluster for every $\mathcal{L} \subset \binom{V(F)}{2}$ and $Z \subset V(F)$ with $|\mathcal{L}| \leqslant n(\log n)^{-6h}$ and $|Z| = h^2|\mathcal{L}|$.*

It follows from this definition that the graph obtained by taking the union of the paths in an $(\mathcal{L}, Z, h, \log n)$-cluster has at most $2|\mathcal{L}|+|Z|+|\mathcal{L}|(\log n)(h-1)$ vertices and exactly $|\mathcal{L}|(\log n)(h+1)$ edges, as well as a very specific structure. Also, observe that if F is (h, n)-cluster free, then any subgraph $\widehat{F}$ of F will be (h, n)-cluster free as well.

Lemma 3.12 (Existence Lemma)**.** *For all $h, \ell \in \mathbb{Z}^+$ and $\delta \in \mathbb{R}^+$, there exists $\varepsilon \in \mathbb{R}^+$ such that, for any $t \in \mathbb{Z}^+$, there exists $n_2 \in \mathbb{Z}$ for which the following holds. For any $n \geqslant n_2$,*

$$q := 4(\log n)^2 n^{-1+1/(h+1)}, \qquad N := tn, \qquad \text{and} \qquad p := 4\ell q,$$

there exists a graph G on N vertices satisfying all of the following properties:

(i) *Every vertex in G has degree at most $(\log n)^3 n^{1/(h+1)}$.*

(ii) *G is (h, n)-cluster free.*

(iii) *$G \in \mathcal{I}(N, p)$.*

(iv) *For all disjoint subsets $X_1, X_2, \ldots, X_{h+1} \subset V(G)$, every (not necessarily induced) subgraphs H on $\bigsqcup_{i=1}^{h+1} X_i$ with $H \in \mathcal{H}(h, n, \varepsilon, q)$ is $(1-\delta, \log n)$-path abundant.*

Observe that if G is any graph satisfying property (iii) in the Existence Lemma then, by the Coloring Lemma, any ℓ-coloring of G yields a monochromatic copy of some $H \in \mathcal{H}(h, n, \varepsilon, q)$. Moreover, if G also satisfies property (iv) in the Existence Lemma, then the monochromatic copy of H must be path abundant. Additionally, if G satisfies properties (i) and (ii) in the Existence Lemma, then the path abundant monochromatic H must also satisfy properties (i) and (ii) in the Existence Lemma. Such a graph H is described by the following definition. Note that this definition has no dependency on ε.

Definition 3.13. *Let $\mathcal{J}(h,n,\delta)$ be the set of all graphs H on $\bigsqcup_{i=1}^{h+1} X_i$ that are in $\mathcal{H}(h,n)$ and satisfy all the following:*

(i) Every vertex in H has degree at most $(\log n)^3 n^{1/(h+1)}$.

(ii) H is (n,h)-cluster free.

(iii) H is $(1-\delta,\log n)$-path abundant.

Our final lemma establishes that every $H \in \mathcal{J}(h,n,\delta)$ has the desired universal property to slightly smaller graphs provided δ is sufficiently small.

Lemma 3.14 (Embedding Lemma)**.** *For all $h,d \in \mathbb{Z}^+$, there exist $\delta \in \mathbb{R}^+$ and $n_3 \in \mathbb{Z}^+$ such that, for all $n \geqslant n_3$, the following holds. Every graph H on $\bigsqcup_{i=1}^{h+1} X_i$ with $H \in \mathcal{J}(h,n,\delta)$ is universal to the set of graphs*

$$\left\{S^{(h)} : |V(S)| = \frac{n}{(\log n)^{7h}} \text{ and } \Delta(S) \leqslant d\right\}.$$

Proof of Theorem 3.2

We will now prove our main result based upon the three lemmas we have stated.

Proof of Theorem 3.2. Consider any $h,d,\ell \in \mathbb{Z}^+$. Recall that Lemmas 3.14, 3.12, and 3.9 are quantified as follows.

$$\begin{aligned} L3.14 :&\quad \forall h,d \quad \exists \delta, n_3 \\ L3.12 :&\quad \forall h,\ell,\delta \;\; \exists \varepsilon \;\; \forall t \;\; \exists n_2 \\ L3.9 :&\quad \forall h,\ell,\varepsilon, \;\; \exists t, n_1 \end{aligned}$$

A sequential application of Lemmas 3.14, 3.12, 3.9, and 3.12 yields

$$\delta := \delta^{L3.14}(h,d), \quad n_3 := n_3^{L3.14}(h,d),$$

$$\varepsilon := \varepsilon^{L3.12}(h, \ell, \delta),$$

$$t := t^{L3.9}(h, \ell, \varepsilon), \quad n_1 := n_1^{L3.9}(h, \ell, \varepsilon),$$

$$n_2 := n_2^{L3.12}(h, \ell, \delta, \varepsilon, t).$$

Set $s_0 := \max\{n_1, n_2, n_3, e^t\}$ and consider any $s \geqslant s_0$. Take

$$n := (\log s)^{8h} s, \quad N := nt, \quad q := 4(\log n)^2 n^{-1+1/(h+1)}, \quad \text{and} \quad p := 4\ell q.$$

Observe that $n \geqslant s \geqslant s_0$. From the Existence Lemma (Lemma 3.12), we obtain a graph G on N vertices that satisfies the properties (i)–(iv) in the Existence Lemma. We will now show that G has the desired universal Ramsey property. That is, consider any ℓ-coloring of the edges of G. We will show that G contains a monochromatic copy of $S^{(h)}$ for every graph S with $|V(S)| = s$ and $\Delta(S) \leqslant d$.

Since $G \in \mathcal{I}(N, p)$, by the Coloring Lemma (Lemma 3.9), this coloring of G yields disjoint vertex subsets $X_1, X_2, \ldots, X_{h+1} \subset V(G)$ and a monochromatic subgraph H on $\bigsqcup_{i=1}^{h+1} X_i$ with $H \in \mathcal{H}(h, n, \varepsilon, q)$. Since G also exhibits properties (i)–(iv) in the Existence Lemma, the monochromatic subgraph H on $\bigsqcup_{i=1}^{h+1} X_i$ must be a member of the class $\mathcal{J}(h, n, \delta)$. By the Embedding Lemma (Lemma 3.14), the monochromatic subgraph H is universal to the family of graphs $\{S^{(h)} : |V(S)| = n(\log n)^{-7h} \text{ and } \Delta(S) \leqslant d\}$. Since $n = (\log s)^{8h} s$ was chosen so that $s \leqslant n(\log n)^{-7h}$, this gives that H is also universal to $\{S^{(h)} : |V(S)| = s \text{ and } \Delta(S) \leqslant d\}$, as desired.

Having established that G has the desired universal Ramsey property, we will now count the number of edges in G. Based upon the maximum degree in G being at most $(\log n)^3 n^{1/(h+1)}$ (and using $\log n \leqslant (\log s)^2$, $1 + 1/(h+1) \leqslant 3/2$, and $n \geqslant 2^t$), the number of edges in G is at most

$$\begin{aligned}(\log n)^3 n^{1/(h+1)} N &\leqslant (\log n)^4 n^{1+1/(h+1)} \\ &\leqslant ((\log s)^2)^4 ((\log s)^{8h})^{3/2} s^{1+1/(h+1)} \leqslant (\log s)^{20h} s^{1+1/(h+1)}.\end{aligned}$$

This completes the proof of Theorem 3.2. □

3.3 Proof of the Coloring Lemma

This section is devoted to proving Lemma 3.9. For the remainder of this section, fix $\varepsilon \in \mathbb{R}^+$ and $h, \ell \in \mathbb{Z}^+$ and set

$$q(n) := 4(\log n)^2 n^{-1+1/(h+1)} \qquad \text{and} \qquad p(n) := 4\ell q.$$

We must show there exists an integer t so that for sufficiently large n and $N := tn$, any ℓ-coloring of any graph $G \in \mathcal{I}(N,p)$ yields disjoint vertex subsets $X_1, X_2, \ldots, X_{h+1} \subset V(G)$ and a monochromatic subgraph H on $\bigsqcup_{i=1}^{h+1} X_i$ with $H \in \mathcal{H}(h,n,\varepsilon,q)$ (see Definitions 3.8 and 3.7).

Our approach to finding a monochromatic subgraph $H \in \mathcal{H}(h,n,\varepsilon,q)$ will be to first find several intermediate classes of graphs. The main idea will be to first find a monochromatic subgraph H_2 (in the class $\mathcal{H}_2$ defined below) in which the number of vertices and edges are controlled but not yet exactly correct. We then transition to a subgraph $H_1 \subset H_2$ (in the class $\mathcal{H}_1$ defined below) in which the number of vertices is precisely as desired and the number of edges is still controlled. Finally, we will obtain a subgraph $H \subset H_1$ with $H \in \mathcal{H}$ in which both the number of vertices and the number of edges are exactly as desired.

To define the intermediate classes of graphs, we need the following pair of definitions.

Definition 3.15 ((η)-regular). *For $\eta \in \mathbb{R}^+$, the bipartite graph $E(X_i, X_{i+1})$ is (η)-regular if, for every $\widehat{X}_i \subset X_i$ and $\widehat{X}_{i+1} \subset X_{i+1}$ with $|\widehat{X}_i| \geqslant \eta|X_i|$ and $|\widehat{X}_{i+1}| \geqslant \eta|X_{i+1}|$,*

$$(1-\eta)\frac{e(X_1,X_{i+1})}{|X_i||X_{i+1}|} \leqslant \frac{e(\widehat{X}_i,\widehat{X}_{i+1})}{|\widehat{X}_i||\widehat{X}_{i+1}|} \leqslant (1+\eta)\frac{e(X_1,X_{i+1})}{|X_i||X_{i+1}|}.$$

Definition 3.16 (Density)**.** *We say that the bipartite graph $E(X_i, X_{i+1})$ has density*

$$d_i := \frac{e(X_i, X_{i+1})}{|X_i||X_{i+1}|}.$$

Definition 3.17 (Intermediate Graph Classes)**.**

- $\mathcal{H}_2(h, n, \varepsilon_2, q)$: *A graph H_2 on $\bigsqcup_{i=1}^{h+1} W_i$ is in $\mathcal{H}_2(h, n, \varepsilon_2, q)$ if, for some integer m satisfying $4n \leqslant m \leqslant n \log n$, all the following hold:*

  (i) $|W_i| = m$ *for all* $i \in [h+1]$.

  (ii) $E(H) = \bigsqcup_{i=1}^{h+1} E_H(W_i, W_{i+1})$.

  (iii) For each $i \in [h+1]$, *the bipartite graph* $E(W_i, W_{i+1})$ *is* (ε_2)*-regular.*

  (iv) For each $i \in [h+1]$, *the bipartite graph* $E(W_i, W_{i+1})$ *has density* d_i *satisfying* $2q \leqslant d_i \leqslant 8\ell q$.

- $\mathcal{H}_1(h, n, \varepsilon_1, q)$: *A graph H_1 on $\bigsqcup_{i=1}^{h+1} X_i$ is in $\mathcal{H}_1(h, n, \varepsilon_1, q)$ if all the following hold:*

  (i) $|X_i| = n$ *for all* $i \in [h+1]$.

  (ii) $E(H) = \bigsqcup_{i=1}^{h+1} E_H(X_i, X_{i+1})$.

  (iii) For each $i \in [h+1]$, *the bipartite graph* $E(X_i, X_{i+1})$ *is* (ε_1)*-regular.*

  (iv) For each $i \in [h+1]$, *the bipartite graph* $E(X_i, X_{i+1})$ *has density* d_i *satisfying* $(3/2)q \leqslant d_i \leqslant 12\ell q$.

- $\mathcal{H}(h, n, \varepsilon, q)$: *Recall that $\mathcal{H}(h, n, \varepsilon, q)$ was introduced in Definition 3.7. It follows from this definition that a graph H on $\bigsqcup_{i=1}^{h+1} X_i$ is in $\mathcal{H}(h, n, \varepsilon, q)$ if all the following hold:*

  (i) $|X_i| = n$ *for all* $i \in [h+1]$.

  (ii) $E(H) = \bigsqcup_{i=1}^{h+1} E_H(X_i, X_{i+1})$.

  (iii) For each $i \in [h+1]$, *the bipartite graph* $E(X_i, X_{i+1})$ *is* (ε)*-regular.*

(iv) *For each $i \in [h+1]$, the bipartite graph $E(X_i, X_{i+1})$ has density d_i satisfying $d_i = q$.*

We will now state three claims. The first claim (Claim 3.18) will establish that, for appropriate parameters, any ℓ-coloring of any graph $G \in \mathcal{I}(N,p)$ contains a monochromatic subgraph $H_2 \in \mathcal{H}_2(h,n,\varepsilon_2,q)$. The next claim (Claim 3.19) will establish that, for appropriate parameters, any graph $H_2 \in \mathcal{H}_2(h,n,\varepsilon_2,q)$ contains a subgraph $H_1 \in \mathcal{H}_1(h,n,\varepsilon_1,q)$. The final claim (Claim 3.20) will establish that, for appropriate parameters, any graph $H_1 \in \mathcal{H}_1(h,n,\varepsilon_1,q)$ contains a subgraph in $H \in \mathcal{H}(h,n,\varepsilon,q)$. These claims will then be used to prove the Coloring Lemma.

Claim 3.18. *For any $\varepsilon_2 \in \mathbb{R}^+$, there exists $t \in \mathbb{Z}^+$ such that, for every sufficiently large integer n and $N := tn$, every graph $G \in \mathcal{I}(N,p)$ has the following property. Any ℓ-coloring of the edges of G yields disjoint vertex subsets $W_1, W_2, \ldots, W_{h+1} \subset V(G)$ and a monochromatic subgraph H_2 on $\bigsqcup_{i=1}^{h+1} W_i$ with $H_2 \in \mathcal{H}_2(h,n,\varepsilon_2,q)$.*

Claim 3.19. *For any $\varepsilon_1 \in \mathbb{R}^+$, there exist $\varepsilon_2 \in \mathbb{R}^+$ such that, for every sufficiently large integer n the following holds. Every graph H_2 on $\bigsqcup_{i=1}^{h+1} W_i$ with $H_2 \in \mathcal{H}_2(h,n,\varepsilon_2,q)$ contains vertex subsets $X_i \subset W_i$ and a subgraph $H_1 \subset H_2$ on $\bigsqcup_{i=1}^{h+1} X_i$ such that $H_1 \in \mathcal{H}_1(h,n,\varepsilon_1,q)$.*

Claim 3.20. *For any $\varepsilon \in \mathbb{R}^+$, there exist $\varepsilon_1 \in \mathbb{R}^+$ such that, for all sufficiently large n, the following holds. Every graph H_1 on $\bigsqcup_{i=1}^{h+1} X_i$ with $H_1 \in \mathcal{H}_1(h,n,\varepsilon_1,q)$ has a monochromatic subgraph H on $\bigsqcup_{i=1}^{h+1} X_i$ such that $H \in \mathcal{H}(h,n,\varepsilon,q)$.*

The proofs of Claims 3.18, 3.19, and 3.20 will be provided in Subsections 3.3.1, 3.3.2, and 3.3.3 respectively. We will now show how these claims establish the Coloring Lemma. Recall that we have already fixed ε, h, and ℓ and defined $q(n)$ and $p(n)$ at the beginning of this section. Fix

$$\varepsilon_1 := \varepsilon_1^{C3.20}(\varepsilon), \qquad \varepsilon_2 := \varepsilon_2^{C3.19}(\varepsilon_1), \qquad \text{and} \qquad t^{C3.18} := t(\varepsilon_2).$$

Let n be any sufficiently large integer and define $N := tn$. Consider any ℓ-coloring of any graph $G \in \mathcal{I}(N, p)$. Claim 3.18 yields disjoint vertex subsets $W_1, W_2, \ldots, W_{h+1} \subset V(G)$ and a monochromatic subgraph H_2 on $\bigsqcup_{i=1}^{h+1} W_i$ with $H_2 \in \mathcal{H}_2(h, n, \varepsilon_2, q)$. Claim 3.19 gives vertex subsets $X_i \subset W_i$ and a subgraph $H_1 \subset H_2$ on $\bigsqcup_{i=1}^{h+1} X_i$ such that $H_1 \in \mathcal{H}_1(h, n, \varepsilon_1, q)$. Claim 3.20 gives that the graph H_1 on $\bigsqcup_{i=1}^{h+1} \widehat{X}_i$ contains a subgraph H on $\bigsqcup_{i=1}^{h+1} X_i$ with $H \in \mathcal{H}(h, n, \varepsilon, q)$. This completes the proof of the Coloring Lemma.

3.3.1 Proof of Claim 3.18

Proof of Claim 3.18. Consider any $\varepsilon_2 \in \mathbb{R}^+$. We must show that there exists $t \in \mathbb{Z}^+$ such that, for every sufficiently large integer n and $N := tn$, every graph $G \in \mathcal{I}(N, p)$ has the following property. Any ℓ-coloring of the edges of G yields a monochromatic subgraph in $\mathcal{H}_2(h, n, \varepsilon_2, q)$.

Let $r_\ell(K_{h+1})$ denote the ℓ-color Ramsey number for K_{h+1}, i.e. the least integer j such that every ℓ-coloring of the edges of the complete graph K_j yields a monochromatic copy of K_{h+1}. Set

$$r := r_\ell(K_{h+1}), \qquad \varepsilon_{reg} := \min\{1/r^2, \varepsilon_2/2\ell\}, \qquad \text{and} \qquad k_{min} := r.$$

Observe that every graph on $k \geqslant k_{min}$ vertices with at least $(1 - \varepsilon_{reg})\binom{k}{2}$ edges contains a copy of K_r. Having defined ε_{reg} and k_{min} and having fixed the integer ℓ at the beginning of this section, we will procure the integers k_{max}, N_0, and D_0 from the sparse regularity lemma. Its statement requires the following definition.

Definition 3.21 ((η, ρ)-regular)**.** *We say that the bipartite graph $E(X_i, X_{i+1})$ is (η, ρ)-regular if, for every $\widehat{X}_i \subset X_i$ and $\widehat{X}_{i+1} \subset X_{i+1}$ with $|\widehat{X}_i| \geqslant \eta |X_i|$*

and $|\widehat{X}_{i+1}| \geqslant \eta |X_{i+1}|$,

$$\left| \frac{e(X_1, X_{i+1})}{|X_i||X_{i+1}|} - \frac{e(\widehat{X}_i, \widehat{X}_{i+1})}{|\widehat{X}_i||\widehat{X}_{i+1}|} \right| \leqslant \eta\rho. \tag{3.4}$$

The following is a suitable variant of Szemerédi's regularity lemma for sparse graphs [47, 48] (see also [39, 68]).

Fact 3.22 (Sparse Regularity Lemma). *For every $\varepsilon_{reg} \in \mathbb{R}^+$ and integers $k_{min}, \ell \in \mathbb{Z}^+$, there exist $k_{max}, N_0, D_0 \in \mathbb{Z}^+$ such that the following holds. Consider any integer $N \geqslant N_0$ and real number p with $pN \geqslant D_0$, and any set of graphs $G_1, G_2, \ldots, G_\ell$ on the same vertex set $[N]$ that each satisfy property (i) in the definition of $\mathcal{I}(N,p)$ (Definition 3.8). Then there exists an integer k satisfying $k_{min} \leqslant k \leqslant k_{max}$ and a vertex partition $[N] = V_1 \cup V_2 \cdots \cup V_k$ that has the following properties.*

- *For all $i \in [k]$, we have $|V_i| = N/k$.*
- *For at least $(1 - \varepsilon_{reg})\binom{k}{2}$ of the pairs $\{i, j\} \in \binom{[k]}{2}$, all the bipartite graphs $E_{G_{\ell'}}(V_i, V_j)$, where $\ell' \in [\ell]$, are (ε_{reg}, p)-regular.*

Having obtained k_{max}, N_0, and D_0 from the above lemma, set

$$t := 4k_{max}.$$

Let n be any integer large enough so that

$$N = nt \geqslant N_0 \qquad \text{and} \qquad pN = 4t(\log n)^2 n^{1/(h+1)} \geqslant D_0.$$

Consider any graph $G \in \mathcal{I}(N,p)$ and any ℓ-coloring of G. Our goal is to show that this arbitrary edge coloring of G yields a monochromatic subgraph in $\mathcal{H}_2(h, n, \varepsilon_2, q)$.

Observe that this coloring corresponds to a partition of $E(G)$ into subgraphs $G_1, G_2, \ldots, G_\ell$ which each inherit property (i) in the definition of $\mathcal{I}(N,p)$. Hence, by the Sparse Regularity Lemma, there exists an integer k satisfying $k_{min} \leqslant k \leqslant k_{max}$ and a vertex partition $V(G) = V_1 \cup V_2 \cdots \cup V_k$ into classes of size $m := N/k$ such that for at least $(1-\varepsilon_{reg})\binom{k}{2}$ of the pairs $\{i,j\} \in \binom{[k]}{2}$, the bipartite graph $E(V_i, V_j)$ is (ε_{reg}, p)-regular with respect to every color class.

Define an auxiliary *cluster graph* on $[k]$ by joining vertex i to vertex j if the bipartite graph $E(V_i, V_j)$ is (ε_{reg}, p)-regular with respect to every color class. The cluster graph has $k \geqslant k_{min}$ vertices and at least $(1-\varepsilon_{reg})\binom{k}{2}$ edges, implying that the cluster graph contains a copy of K_r.

Define a coloring of this copy of K_r in the cluster graph with the color set $[\ell]$ as follows. Color the edge ij with color $\ell' \in [\ell]$ if the bipartite graph $E(V_i, V_j)$ has density at least $2q$ in color ℓ'. Edges may be colored with multiple colors, but every edge will receive at least one color because condition *(ii)* in the definition of $\mathcal{I}(n,p)$ guarantees that the bipartite graph $E(V_i, V_j)$ has density at least $(1/2)p = 2\ell q$. By the definition of the Ramsey number r, this ℓ-coloring of K_r contains a monochromatic copy of K_{h+1}, and hence a monochromatic copy of the cycle C_{h+1} in some color ℓ'. This corresponds to sets $W_1, W_2, \ldots, W_{h+1}$ of size $m = N/k$ so that, for each $i \in [h+1]$, the bipartite graph $E_{G_{\ell'}}(W_i, W_{i+1})$ is (ε_{reg}, p)-regular with density d_i satisfying $2q \leqslant d_i \leqslant 8\ell q$, where the upper bound on d_i follows from condition *(ii)* in the definition of $\mathcal{I}(n,p)$. Observe that $m = N/k \geqslant N/k_{max} = 4n$ and that $m \leqslant N < n \log n$. To complete the proof, we must only demonstrate that every (ε_{reg}, p)-regular graph $E(W_i, W_{i+1})$ having density d_i satisfying $2q \leqslant d_i \leqslant 8\ell q$ is also (ε_2)-regular. To this end, consider any subsets $\widehat{W}_i \subset W_i$ and $\widehat{W}_{i+1} \subset W_{i+1}$ with $|\widehat{W}_i|, |\widehat{W}_{i+1}| \geqslant \varepsilon_2 m$. Since $E(W_i, W_{i+1})$ is (ε_{reg}, p)-regular and $|\widehat{W}_i|, |\widehat{W}_{i+1}| \geqslant \varepsilon_2 m \geqslant \varepsilon_{reg} m$, it follows from Definition 3.21 that

$$\left|\frac{e(W_1, W_{i+1})}{|W_i||W_{i+1}|} - \frac{e(\widehat{W}_i, \widehat{W}_{i+1})}{|\widehat{W}_i||\widehat{W}_{i+1}|}\right| \leqslant \varepsilon_{reg} p.$$

Furthermore, since $d_i \geqslant 2q = p/2\ell$ and $\varepsilon_{reg} \leqslant \varepsilon_2/2\ell$, this gives that

$$\left|\frac{e(W_1, W_{i+1})}{|W_i||W_{i+1}|} - \frac{e(\widehat{W}_i, \widehat{W}_{i+1})}{|\widehat{W}_i||\widehat{W}_{i+1}|}\right| \leqslant \varepsilon_{reg} p \leqslant \frac{\varepsilon_2}{2\ell}(2\ell d_i) = \varepsilon_2 \frac{e(W_1, W_{i+1})}{|W_i||W_{i+1}|},$$

which implies

$$(1 - \varepsilon_2)\frac{e(W_1, W_{i+1})}{|W_i||W_{i+1}|} \leqslant \frac{e(\widehat{W}_i, \widehat{W}_{i+1})}{|\widehat{W}_i||\widehat{W}_{i+1}|} \leqslant (1 + \varepsilon_2)\frac{e(W_1, W_{i+1})}{|W_i||W_{i+1}|}.$$

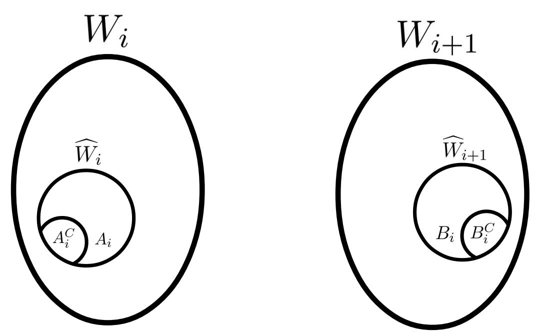
□

3.3.2 Proof of Claim 3.19

Proof of Claim 3.19. Consider any $\varepsilon_1 \in \mathbb{R}^+$. We must show that there exist $\varepsilon_2 \in \mathbb{R}^+$ such that, for every sufficiently large integer n, every graph in $\mathcal{H}_2(h, n, \varepsilon_2, q)$ contains a subgraph in $\mathcal{H}_1(h, n, \varepsilon_1, q)$.

Set $\beta := 1/2$ and $\widehat{\varepsilon}_1 := \varepsilon_1/2$. We obtain the positive real number ε_2 and the constant c from the following lemma. Roughly speaking, the lemma asserts that most induced subgraphs of a (ε_2)-regular bipartite graph can be made (ε_1)-regular by the deletion of only a few vertices provided that $\varepsilon_2 \ll \varepsilon_1$. This basic idea of the lemma is shown in Figure 3.2.

Figure 3.2: Given an (ε_2)-regular bipartite graph $E_i = E(W_i, W_{i+1})$, the induced bipartite graph $E_{E_i}(\widehat{W}_i, \widehat{W}_{i+1})$ is in $\mathcal{G}$ if there exists small subsets $A_i^C \subseteq \widehat{W}_i$ and $B_i^C \subseteq \widehat{W}_{i+1}$ such that, for $A_i := \widehat{W}_{i+1} \backslash A_i^C$ and $B_i := \widehat{W}_{i+1} \backslash B_i^C$, the induced bipartite graph $E_{E_i}(A_i, B_i)$ is $(\widehat{\varepsilon}_1)$-regular with appropriate density.



Fact 3.23 (Corollary 3.9 in [38])**.** *For all $0 < \beta < 1$ and $\widehat{\varepsilon}_1 > 0$, there exists $\varepsilon_2, c > 0$ such that the following holds for any (ε_2)-regular bipartite graph $E_i = E(W_i, W_{i+1})$ with density d_i satisfying $2n \geqslant c d_i^{-1}$.*

- *Let $\mathcal{G}$ be the set of induced subgraphs $E_{E_i}(\widehat{W}_i, \widehat{W}_{i+1}) \subset E(W_i, W_{i+1})$ which have the following property: There exist $A_i \subset \widehat{W}_i$ and $B_i \subset \widehat{W}_{i+1}$ with $|A_i| \geqslant (1 - \widehat{\varepsilon}_1)|\widehat{W}_i|$ and $|B_i| \geqslant (1 - \widehat{\varepsilon}_1)|\widehat{W}_{i+1}|$ such that the induced bipartite graph $E_{E_i}(A_i, B_i)$ is $(\widehat{\varepsilon}_1)$-regular with density $\widehat{d}_i$ satisfying $(1 - \widehat{\varepsilon}_1)d_i \leqslant \widehat{d}_i \leqslant (1 + \widehat{\varepsilon}_1)d_i$.*

Then the number of induced subgraphs $E_{E_i}(\widehat{W}_i, \widehat{W}_{i+1})$ with $\widehat{W}_i \in \binom{W_i}{2n}$ and $\widehat{W}_{i+1} \in \binom{W_{i+1}}{2n}$ that are not in $\mathcal{G}$ is at most $\beta^{2n}\binom{|W_i|}{2n}\binom{|W_{i+1}|}{2n}$.

Having obtained ε_2 and c from the above lemma, let n by any integer large enough so that $2n \geqslant cq^{-1}$. Now consider any graph H_2 on $\bigsqcup_{i=1}^{h+1} W_i$ with $H_2 \in \mathcal{H}_2(h, n, \varepsilon_2, q)$. For some fixed integer m satisfying $4n \leqslant m \leqslant n \log n$, we have that $|W_i| = m$ for all $i \in [h+1]$. Recall that our aim is to show that there exist a collection of n element subsets $\{X_i \subset W_i : i \in [h+1]\}$ so that, for each $i \in [h+1]$, the induced bipartite graph $E(X_i, X_{i+1})$ is (ε_1)-regular with density between $(3/2)q$ and $12\ell q$.

To this end, we first consider a random selection of $2n$ element subsets $\{\widehat{W}_i \subset W_i : i \in [h+1]\}$. By the union bound and Fact 3.23 (applied with $|W_i| = |W_{i+1}| = m$ and having $\beta = 1/2$), with probability at least $1-(h+1)(1/2)^{2n} > 0$, this random selection of subsets will have the property that, for each $i \in [h+1]$, the bipartite graph $E_i := E(\widehat{W}_i, \widehat{W}_{i+1})$ is in $\mathcal{G}$ (as defined in Fact 3.23). Hence, we may fix such a selection $\{\widehat{W}_i \subset W_i : i \in [h+1]\}$ of $2n$ element subsets such that each of the bipartite graphs $E_i = E(\widehat{W}_i, \widehat{W}_{i+1})$ are in $\mathcal{G}$. Now, for each $i \in [h+1]$ and associated bipartite graph $E_i = E(\widehat{W}_i, \widehat{W}_{i+1})$, we may find subsets $A_i \subset \widehat{W}_i$ and $B_i \subset \widehat{W}_{i+1}$ with $|A_i|, |B_i| \geqslant (1-\widehat{\varepsilon}_1)|2n|$ such that $E_{E_i}(A_i, B_i)$ is $(\widehat{\varepsilon}_1)$-regular with density $\widehat{d}_i$ satisfying $(1-\widehat{\varepsilon}_1)d_i \leqslant \widehat{d}_i \leqslant (1+\widehat{\varepsilon}_1)d_i$. Thus for the set $\widehat{W}_i$, we have selected subsets $A_i \subset \widehat{W}_i$ and $B_{i-1} \subset \widehat{W}_i$ with respect to the bipartite graphs $E_i = E(\widehat{W}_i, \widehat{W}_{i+1})$ and $E_{i-1} = E(\widehat{W}_{i-1}, \widehat{W}_i)$ respectively. For each $\widehat{W}_i$, let X_i be any subset of $A_i \cap B_{i-1}$ of size n.

For each $i \in [h+1]$, the bipartite graph $E(X_i, X_{i+1})$ is (ε_1)-regular as desired since:

- $E(X_i, X_{i+1})$ is a subgraph of the $(\widehat{\varepsilon}_1)$-regular bipartite graph $E(A_i, B_i)$.
- $(1-\widehat{\varepsilon}_1)2n \leqslant |A_i| \leqslant 2n$ and $(1-\widehat{\varepsilon}_1)2n \leqslant |B_i| \leqslant 2n$.
- $|X_i| = |X_{i+1}| = n$.
- $\widehat{\varepsilon}_1 = \varepsilon_1/2$.

Also, $E(X_i, X_{i+1})$ has density between $(3/2)q$ and $12\ell q$ since:

- $E(X_i, X_{i+1})$ is a subgraph of the $(\widehat{\varepsilon}_1)$-regular bipartite graph $E(A_i, B_i)$ of density $\widehat{d}_i$ satisfying $(1-\widehat{\varepsilon}_2)2q \leqslant \widehat{d}_i \leqslant (1+\widehat{\varepsilon}_2)8\ell q$.
- $|X_i| \geqslant \widehat{\varepsilon}_1|A_i|$ and $|X_{i+1}| \geqslant \widehat{\varepsilon}_1|B_i|$.

□

3.3.3 Proof of Claim 3.20

Proof of Claim 3.20. Consider any $\varepsilon \in \mathbb{R}^+$. Take $\varepsilon_1 := \varepsilon/2$ and let n be any sufficiently large integer. Consider any graph H_1 on $\bigsqcup_{i=1}^{h+1} X_i$ with $H_1 \in \mathcal{H}_1(h, n, \varepsilon_1, q)$. We must show that H_1 has a monochromatic subgraph H on $\bigsqcup_{i=1}^{h+1} X_i$ with $H \in \mathcal{H}(h, n, \varepsilon, q)$.

For each $i \in [h+1]$, consider a random selection $R_i \subset E(X_i, X_{i+1})$ of qn^2 edges. We claim that the random subgraph $R := \bigcup_{i \in [h+1]} R_i$ will have the desired property $R \in \mathcal{H}(h, n, \varepsilon, q)$ with positive probability. Indeed, this probability can be easily bounded using the hypergometric distribution (See Lemma 3.34), keeping in mind that $(3/2)qn^2 \leqslant e(X_i, X_{i+1}) \leqslant 12\ell qn^2$. This establishes the existence of the desired subgraph $H \in \mathcal{H}(h, n, \varepsilon, q)$. □

3.4 Proof of the Existence Lemma

This section of the paper proves Lemma 3.12, which asserts the existence of a sparse graph G with certain properties. It suffices to prove the following lemma.

Lemma 3.24. *For all constants $h, \ell \in \mathbb{Z}^+$ and any constant $\delta \in \mathbb{R}^+$, there exists a constant $\varepsilon \in \mathbb{R}^+$ such that, for any constant $t \in \mathbb{Z}^+$,*

$$q := 4(\log n)^2 n^{-1+1/(h+1)}, \qquad N := tn, \qquad \text{and} \qquad p := 4\ell q,$$

an instance G of the random graph $G(N, p)$ asymptotically almost surely has each of the following properties:

(i) Every vertex in G has degree at most $(\log n)^3 n^{1/(h+1)}$.

(ii) G is (h, n)-cluster free (see Definition 3.11).

(iii) $G \in \mathcal{I}(N, p)$ *(see Definition 3.8).*

(iv) For all disjoint subsets $X_1, X_2, \ldots, X_{h+1} \subset V(G)$*, every (not necessarily induced) subgraphs* H *on* $\bigsqcup_{i=1}^{h+1} X_i$ *with* $H \in \mathcal{H}(h, n, \varepsilon, q)$ *is* $(1-\delta, \log n)$*-path abundant (see Definitions 3.7 and 3.10).*

In the statement of the previous lemma and elsewhere in this section, we say that a number is a *constant* if it does not depend on n and that a statement holds *asymptotically almost surely* (*a.a.s.*) if the probability the statement is true approaches 1 as $n \to \infty$.

The first subsection contains Claims 3.25, 3.26, and 3.29, which respectively establish that properties (i), (ii), and (iii) in Lemma 3.24 each hold a.a.s. Notice that these properties do not depend upon ε. The second and most substantial subsection will establish a lemma (Lemma 3.31) derived from a result in [38]. In Subsection 3.4.3, Claim 3.42 will then use this lemma to establish the existence of an ε for which the property (iv) in Lemma 3.24 holds a.a.s. These claims together constitute a proof of Lemma 3.24.

3.4.1 Properties (i), (ii), and (iii) in Lemma 3.24

In this subsection, we prove Claims 3.25, 3.26, and 3.29, which correspond to properties (i), (ii), and (iii) in Lemma 3.24.

Claim 3.25. *For any constants* $h, t, \ell \in \mathbb{Z}^+$*, let* $N := tn$ *and let* $p := 4\ell (\log n)^2 n^{-1+1/(h+1)}$*. Then a.a.s. the random graph* $G(N, p)$ *has maximum degree less than* $(\log n)^3 n^{1/(h+1)}$*.*

Proof of Claim 3.25. It is a well-known fact that the random graph $G(N, p)$ a.a.s. has maximum degree less than $2pN$ for all $p \gg (\log n)/n$, say. Moreover,

$$2pN = 2 \cdot 4\ell(\log n)^2 n^{-1+1/(h+1)} \cdot tn < (\log n)^3 n^{1/(h+1)}.$$

□

Claim 3.26. *For any constants $h, t, \ell \in \mathbb{Z}^+$, let $N := tn$ and let $p := 4\ell(\log n)^2 n^{-1+11/(h+1)}$. Then a.a.s. the random graph $G(N,p)$ is (h,n)-cluster free.*

Proof of Claim 3.26. Recall the definition of an $(\mathcal{L}, Z, h, \log n)$-cluster given in Definition 3.11. It follows that in the complete graph on N vertices, each $(\mathcal{L}, Z, h, \log n)$-cluster is defined by:

- Specifying a size of L for $\mathcal{L}$.
- Picking a set $\mathcal{L}$ of L pairs of vertices.
- Picking a set Z of vertices.
- For each $\{u, v\} \in \mathcal{L}$, picking a set of $\log n$ paths, each of which can be specified by:
  - Picking a vertex in Z to appear in the interior of the path.
  - Picking $h - 1$ other vertices to appear in the interior of the path.
  - Ordering the h internal vertices on the path.

It follows that in $G(N,p)$, the expected number of $(\mathcal{L}, Z, h, \log n)$-clusters for $\mathcal{L} \subset \binom{[N]}{2}$ and $Z \subset [N]$ with $|\mathcal{L}| \leqslant n(\log n)^{-6h}$ and $|Z| = h^2|\mathcal{L}|$ is bounded above for sufficiently large n by

$$\begin{aligned}
&\sum_{L=1}^{n(\log n)^{-6h}} \binom{N^2}{L}\binom{N}{h^2L}\left(h^2L\cdot\binom{N}{h-1}\cdot h!\right)^{(\log n)L} p^{(\log n)(h+1)L}\\
&\quad\leqslant \sum_{L=1}^{n(\log n)^{-6h}} N^{3h^2L}\left(h^{h+2}LN^{h-1}p^{h+1}\right)^{(\log n)L}\\
&\quad\leqslant \sum_{L=1}^{n(\log n)^{-6h}} N^{3h^2L}\left(h^{h+2}n(\log n)^{-6h}(nt)^{h-1}(4\ell(\log n)^2)^{h+1}n^{-h}\right)^{(\log n)L}\\
&\quad= \sum_{L=1}^{n(\log n)^{-6h}} N^{3h^2L}\left(h^{h+2}t^{h-1}(4\ell)^{h+1}(\log n)^{2-4h}\right)^{(\log n)L}\\
&\quad\leqslant \sum_{L=1}^{n} N^{3h^2L}\left(\frac{1}{\log n}\right)^{(\log n)L} \leqslant \sum_{L=1}^{n}\left(\frac{(nt)^{3h^2}}{(\log n)^{\log n}}\right)^{L} \leqslant n\cdot\frac{(nt)^{3h^2}}{(\log n)^{\log n}},
\end{aligned}$$

which goes to 0 as $n\to\infty$. Because we have that the expected number of forbidden $(\mathcal{L}, Z, h, \log n)$-clusters that $G(N,p)$ contains goes to 0, a.a.s. $G(N,p)$ is (h,n)-cluster free. $\square$

Before we state the next claim, we introduce a definition and an external lemma that are needed in its proof.

Definition 3.27. *We say that a graph G is (p,a)-uniform if*

$$\left|e(V_1,V_2)-p|V_1||V_2|\right| \leqslant a\sqrt{p|V(G)||V_1||V_2|}$$

for all disjoint sets $V_1, V_2 \subset V(G)$ such that $1\leqslant |V_1|\leqslant |V_2|\leqslant p|V(G)||V_1|$.

Fact 3.28 (Lemma 3.8 in [44])**.** *For every $p=p(N)$, $0<p\leqslant 1$, a.a.s. the random graph $G(N,p)$ is $(p, e^2\sqrt{6})$-uniform.*

Claim 3.29. *For any constants $h,t,\ell\in\mathbb{Z}^+$ and $N := tn$, we have that for $p := 4\ell(\log n)^2 n^{-1+1/(h+1)}$, a.a.s. the random graph $G(N,p)$ is in $\mathcal{I}(N,p)$.*

Proof of Claim 3.29. By Fact 3.28 stated above, a.a.s. we have that

$$e(V_1, V_2) \leqslant p|V_1||V_2| + e^2\sqrt{6} \cdot \sqrt{pN|V_1||V_2|},$$

for all disjoint sets $V_1, V_2 \subset V(G(N,p))$ with $1 \leqslant |V_1| \leqslant |V_2| \leqslant pN|V_1|$. This is exactly the first condition given in the definition of $\mathcal{I}(N,p)$. The other condition given in the definition of $\mathcal{I}(N,p)$ states that a.a.s.

$$(1/2) \cdot p|V_1||V_2| \leqslant e(V_1, V_2) \leqslant 2 \cdot p|V_1||V_2|$$

for all disjoint sets $V_1, V_2 \subset V(G(N,p))$ with $|V_1|, |V_2| \geqslant N(\log N)^{-1}$. This can easily be established by the union bound. $\square$

3.4.2 Proof of Lemma

For the remainder of this subsection, let $X_1, X_2, \ldots, X_{h+1}$ be fixed (labeled) sets each of size n. The follow class of describes the graphs on $\bigsqcup_{i=1}^{h+1} X_i$ that do not have the desired path abundance property.

Definition 3.30. *Let $\mathcal{B}(h,n,\varepsilon,q,\delta)$ be the set of all graphs B on $\bigsqcup_{i=1}^{h+1} X_i$ such that $B \in \mathcal{H}(h,n,\varepsilon,q)$ and B is not $(1-\delta, \log n)$-path abundant.*

Lemma 3.31. *For any constant $h \in \mathbb{Z}^+$ and any constants $\delta, \beta \in \mathbb{R}^+$, there exist constants $\varepsilon, n_4 \in \mathbb{R}^+$ such that that following holds. For any $n \geqslant n_4$ and $q := 4(\log n)^2 n^{-1+1/(h+1)}$, we have that*

$$|\mathcal{B}(h,n,\varepsilon,q,\delta)| \leqslant \beta^{qn^2} \binom{n^2}{qn^2}^{h+1}.$$

In Subsection 3.4.3, Lemma 3.31 will be used to establish Claim 3.42, which states that the random graph $G(N,p)$ a.a.s. has the property that it does not contain any selection of disjoint vertex subsets $X_1, X_2, \ldots, X_{h+1}$ and subgraph B on $\bigsqcup_{i=1}^{h+1} X_i$ with $B \in \mathcal{B}(h, n, \varepsilon, q, \delta)$. In other words, Claim 3.42 implies that a.a.s. $G(N, p)$ has the property that for every section of disjoint vertex subsets $X_1, X_2, \ldots, X_{h+1}$ and subgraph H on $\bigsqcup_{i=1}^{h+1} X_i$, the graph H is $(1-\delta, \log n)$-path abundant if $H \in \mathcal{H}(h, n, \varepsilon, q)$, which is exactly property (iv) in Lemma 3.24. Keep in mind that although Claim 3.42 concerns any selection of disjoint vertex subsets $X_1, X_2, \ldots, X_{h+1}$ in $G(N, p)$, for the time being in this section we are only counting the graphs in $\mathcal{B}(h, n, \varepsilon, q, \delta)$ on already determined vertex sets $X_1, X_2, \ldots, X_{h+1}$.

Essentially, we are trying to show that all but exponentially few graphs on $\bigsqcup_{i=1}^{h+1} X_i$ in $\mathcal{H}(h, n, \varepsilon, q)$ (see Definition 3.7) have the property that almost all pairs of vertices in X_1 are joined by $\log n$ transversal paths. The key external lemma we will use establishes that all but exponentially few graphs in $\mathcal{H}(h, n, \widehat{\varepsilon}, q/4\log n)$ (again see Definition 3.7) have the property that most pairs of vertices in X_1 are connected by at least one path. This lemma will be related to the result we are trying to prove by a double counting argument in which a set $\mathcal{F}$ of 'bad families' of graphs (see Definition 3.35) is considered. We now introduce not only the key external lemma and a related definition, but also the standard Hypergeometic Bound. This will be followed by a proof of Lemma 3.31.

Definition 3.32 (Path Dense)**.** *A graph H on on $\bigsqcup_{i=1}^{h+1} X_i$ with $H \in \mathcal{H}(h, n)$ is* $(1-\eta)$-path dense *if at least $(1-\eta)\binom{n}{2}$ pairs of vertices $\{u, v\} \in \binom{X_1}{2}$ are joined by at least one transversal path (transversal paths are defined in Definition 3.10).*

The next lemma is a corollary of Lemma 5.9 in [38]. (To obtain Fact 3.33 below, one sets the parameters in Lemma 5.9 as follows: $\ell = h+2$, $\beta = \widehat{\beta}$, $\delta = \delta/4$, $\gamma = \delta/4$, $\nu = \delta/2$, $q = 4(\log n)^2 n^{-1+1/(h+1)}$, $m = qn^2/(4\log n)$ and noticing that $n^{h+2} \ll m^{h+1}$.)

Fact 3.33. *For any $\widehat{\beta}, \delta \in \mathbb{R}^+$, there exists $\widehat{\varepsilon} \in \mathbb{R}^+$ so that the following holds. For $q = 4(\log n)^2 n^{-1+1/(h+1)}$, $m := qn^2/(4\log n)$, and sufficiently large n, the total number of graphs E on $\bigsqcup_{i=1}^{h+1} X_i$ with $E \in \mathcal{H}(h, n, \widehat{\varepsilon}, m/n^2)$ that are not $(1-\delta/2)$-path dense is at most*

$$\widehat{\beta}^m \binom{n^2}{m}^{h+1}. \tag{3.5}$$

The following is a well-known bound on the hypergeometic distribution (see, e.g., Theorem 2.10 and Equation (2.12) in [45]).

Fact 3.34 (Hypergeometic Bound)**.** *Let Y be a set and $\widehat{Y}$ be a subset of Y. Suppose that $M \subset Y$ is a subset of size m chosen at random from Y and let the random variable X denote the number of elements in $M \cap \widehat{Y}$. Then*

$$\Pr\left(\left|X - \frac{m|\widehat{Y}|}{|Y|}\right| \leqslant t\right) \geqslant 1 - 2\exp\left\{-\frac{2t^2}{|Y|}\right\}.$$

We will now prove Lemma 3.31.

Proof of Lemma 3.31. Consider any $h \in \mathbb{Z}^+$ and $\beta, \delta \in \mathbb{R}^+$ and define $q := 4(\log n)^2 n^{-1+1/(h+1)}$. We must show that there exists an $\varepsilon \in \mathbb{R}^+$ such that for sufficiently large n we have

$$|\mathcal{B}(h, n, \varepsilon, q, \delta)| \leqslant \beta^{qn^2} \binom{n^2}{qn^2}^{h+1}.$$

Making use of Fact 3.33, set

$$\widehat{\beta} := \tfrac{\beta^2}{9^{2(h+1)}}, \qquad \widehat{\varepsilon} := \varepsilon^{F3.33}(\widehat{\beta}, \delta), \qquad \varepsilon := \widehat{\varepsilon}/2, \qquad \text{and} \qquad m := \tfrac{qn^2}{4\log n}.$$

As mentioned before, the fundamental idea in our proof is to relate the bound in Fact 3.33 to $|\mathcal{B}(h, n, \varepsilon, q, \delta)|$ by counting the number of 'bad families,' which are defined as follows.

Definition 3.35 (Bad Family)**.** *A set of graphs* $F = \{E_1, E_2, \ldots, E_{4\log n}\}$ *is called a* bad family *if both the following hold:*

- *Every* $E \in F$ *is a graph on* $\bigsqcup_{i=1}^{h+1} X_i$ *with* $E \in \mathcal{H}(h, n, \widehat{\varepsilon}, m/n^2)$*.*
- *Fewer than half of the graphs* $E \in F$ *are* $(1 - \delta/2)$*-path dense.*

Let $\mathcal{F}$ *be the set of all bad families of graphs.*

Proposition 3.36.

$$|\mathcal{F}| \leqslant \left(\widehat{\beta}^m \binom{n^2}{m}^{h+1}\right)^{2\log n} \left(\binom{n^2}{m}^{h+1}\right)^{2\log n}.$$

Proof of Proposition 3.36. To verify Proposition 3.36, we use that for each $F \in \mathcal{F}$, there are $2\log n$ graphs $E \in F$ in $\mathcal{H}(h, n, \widehat{\varepsilon}, m/n^2)$ that are not $(1 - \delta/2)$-path dense. By Fact 3.33, the number of graphs of this type is at most as in (3.5). This readily yields the bound in Proposition 3.36. $\square$

The next definition refers to $\mathcal{H}(h, n, 1, m/n^2)$, which is the set of graphs in $\mathcal{H}(h, n)$ on $\bigsqcup_{i=1}^{h+1} X_i$ in which all of the bipartite graph (X_i, X_{i+1}) have m/n^2 edges (i.e., the choice of $\varepsilon = 1$ in Definition 3.7 imposes no uniformity restriction).

Definition 3.37 (Associated Family)**.** *For each graph* $B \in \mathcal{B}(h, n, \varepsilon, q, \delta)$*, we call the set of edge-disjoint graphs* $A = \{E_1, E_2, \ldots, E_{4\log n}\}$ *an* associated family *to* B *if both the following hold:*

- *Every* $E \in A$ *is a graph o on* $\bigsqcup_{i=1}^{h+1} X_i$ *with* $E \in \mathcal{H}(h, n, 1, m/n^2)$*.*
- $B = \bigcup_{i=1}^{4\log n} E_i$*.*

Since for each $B \in \mathcal{B}(h, n, \varepsilon, q, \delta)$ an associated family A is obtained by partitioning the qn^2 edges in each of the $h + 1$ bipartite graphs into $4\log n$

classes of size m, it follows that each B is associated to

$$\binom{qn^2}{m, m, \ldots, m}^{h+1} ((4 \log n)!)^{-1}$$

associated families. Moreover, no two distinct graphs $B_1, B_2 \in \mathcal{B}(h, n, \varepsilon, q, \delta)$ will yield a common associated family. The next claim gives a lower bound for the size of $\mathcal{F}$ and will be proved by establishing that, for each $B \in \mathcal{B}(h, n, \varepsilon, q, \delta)$, half of its associated families are bad families.

Proposition 3.38.

$$|\mathcal{F}| \geqslant |\mathcal{B}(h, n, \varepsilon, q, \delta)| \frac{1}{2} \binom{qn^2}{m, m, \ldots, m}^{h+1} ((4 \log n)!)^{-1}$$

Proof of Proposition 3.38. As discussed before the proposition, it suffices to show that at least half the associated families for any $B \in \mathcal{B}(h, n, \varepsilon, q, \delta)$ are bad families. Hence, to prove Proposition 3.38, it suffices to show the following two subpropositions.

Subproposition 3.39. *For every $B \in \mathcal{B}(h, n, \varepsilon, q, \delta)$ and every associated family $A = \{E_1, E_2, \ldots, E_{4 \log n}\}$, fewer than half of the graphs $E \in A$ are $(1-\delta/2)$-path dense.*

Subproposition 3.40. *For every $B \in \mathcal{B}(h, n, \varepsilon, q, \delta)$, at least half the associated families $A = \{E_1, E_2, \ldots, E_{4 \log n}\}$ have the property that all $E \in A$ are in $\mathcal{H}(h, n, \widehat{\varepsilon}, m/n^2)$.*

Proof of Subproposition 3.39. We prove the contrapositive by arguing that if at least $2 \log n$ of the graphs $E \in A$ are $(1 - \delta/2)$-path dense, then B is $(1 - \delta, \log n)$-path abundant. To this end, fix a set of $2 \log n$ graphs $E \in A$ that are $(1 - \delta/2)$-path dense. For each of theses graphs, fix one transversal path for each of the $(1 - \delta/2)\binom{n}{2}$ pairs of vertices $\{u, v\} \in \binom{X_1}{2}$ that are joined by traversal paths. Let $\mathcal{P}$ be the set of paths obtained by this process, so that

$$|\mathcal{P}| = (2\log n)(1-\delta/2)\binom{n}{2}. \tag{3.6}$$

Also, observe that each pair of vertices $\{u,v\} \in \binom{X_1}{2}$ is joined by at most $2\log n$ paths in $\mathcal{P}$. Now suppose that exactly $\alpha\binom{n}{2}$ pairs of vertices in $\binom{X_1}{2}$ are joined by at least $\log n$ transversal paths in $\mathcal{P}$. It follows that

$$|\mathcal{P}| \leqslant \alpha\binom{n}{2}2\log n + (1-\alpha)\binom{n}{2}\log n. \tag{3.7}$$

From (3.6) and (3.7),

$$(2\log n)(1-\delta/2)\binom{n}{2} \leqslant \alpha\binom{n}{2}2\log n + (1-\alpha)\binom{n}{2}\log n,$$

which implies

$$2-\delta \leqslant 2\alpha + (1-\alpha),$$

giving that $\alpha \geqslant 1-\delta$. This establishes that B is $(1-\delta, \log n)$-path abundant, completing the proof of Subproposition 3.39. □

Proof of Subproposition 3.40. Consider any $B \in \mathcal{B}(h,n,\varepsilon,q,\delta)$. For any $\widehat{X}_i \subset X_i$ and $\widehat{X}_{i+1} \subset X_{i+1}$ each of size $|\widehat{X}_i|, |\widehat{X}_{i+1}| \geqslant \widehat{\varepsilon}n \geqslant \varepsilon n$, by definition of $\mathcal{B}(h,n,\varepsilon,q,\delta)$ we have that

$$\left|e_B(\widehat{X}_i,\widehat{X}_{i+1}) - q|\widehat{X}_i||\widehat{X}_{i+1}|\right| \leqslant \varepsilon q|\widehat{X}_i||\widehat{X}_{i+1}|,$$

or equivalently

$$\left|\frac{e_B(\widehat{X}_i,\widehat{X}_{i+1})}{4\log n} - \frac{q}{4\log n}|\widehat{X}_i||\widehat{X}_{i+1}|\right| \leqslant \varepsilon\frac{q}{4\log n}|\widehat{X}_i||\widehat{X}_{i+1}|. \tag{3.8}$$

Now if M is a random subgraph on $m = qn^2/(4\log n)$ edges of the bipartite graph $E_B(X_i, X_{i+1})$ on qn^2 edges, then the hypergeometric bound stated in Lemma 3.34 (applied with $Y = E_B(X_i, X_{i+1})$ and $\widehat{Y} = E_B(\widehat{X}_i, \widehat{X}_{i+1})$) gives that

$$\left| e_M(\widehat{X}_i, \widehat{X}_{i+1}) - \frac{e_B(\widehat{X}_i, \widehat{X}_{i+1})}{4\log n} \right| \leqslant \varepsilon \frac{q}{4\log n} |\widehat{X}_i||\widehat{X}_{i+1}| \tag{3.9}$$

holds with probability at least

$$\begin{aligned} 1 - 2\exp\left\{ -\frac{2(\varepsilon q|\widehat{X}_i||\widehat{X}_{i+1}|/4\log n)^2}{qn^2} \right\} &\geqslant 1 - 2\exp\left\{ -\frac{\varepsilon^6 qn^2}{8(\log n)^2} \right\} \\ &\geqslant 1 - 2\exp\left\{ -2^{-1}\varepsilon^6 n^{1+\frac{1}{h+1}} \right\}. \end{aligned}$$

From the triangle equality applied to (3.8) and (3.9) (and fact that $\varepsilon + \varepsilon = \widehat{\varepsilon}$), this gives

$$\left| e_M(\widehat{X}_i, \widehat{X}_{i+1}) - \frac{q}{4\log n} |\widehat{X}_i||\widehat{X}_{i+1}| \right| \leqslant \widehat{\varepsilon} \frac{q}{4\log n} |\widehat{X}_i||\widehat{X}_{i+1}| \tag{3.10}$$

with probability at least

$$1 - 2\exp\left\{ -2^{-1}\varepsilon^6 n^{1+1/(h+1)} \right\}. \tag{3.11}$$

Now consider a random partition of B into an associated family

$$A = \{E_1, E_2, \ldots, E_{4\log n}\}.$$

The associated family A will have the desired property that all of the graphs $E \in A$ are in $\mathcal{H}(h, n, \widehat{\varepsilon}, m/n^2) = \mathcal{H}(h, n, \widehat{\varepsilon}, q/(4\log n))$ if inequality (3.10) is satisfied for every choice of $M = E_j$ for $j \in [4\log n]$, every choice of $i \in [h+1]$, and every choice of $\widehat{X}_i \subset X_i$ and $\widehat{X}_{i+1} \subset X_{i+1}$. It follows from (3.11) and the union bound that this will occur with probability at least

$$1-(4\log n)\cdot(h+1)\cdot 2^n\cdot 2^n\cdot 2\exp\left\{-2^{-1}\varepsilon^6 n^{1+1/(h+1)}\right\},$$

which tends to 1 as $n\to\infty$. This establishes that a random partition of B into an associated family $A=\{E_1,E_2,\dots,E_{4\log n}\}$ will have the property that all of the graphs $E\in F$ are in $\mathcal{H}(h,n,\widehat{\varepsilon},m/n^2)$ with probability at least 1/2 for sufficiently large n. It follows that at least half of the associated families $A=\{E_1,E_2,\dots,E_{4\log n}\}$ to any $B\in\mathcal{B}(h,n,\varepsilon,q,\delta)$ have the property that all of the graphs $E\in F$ are in $\mathcal{H}(h,n,\widehat{\varepsilon},m/n^2)$, which completes the proof of Subproposition 3.40. □

Hence, we have proved Proposition 3.38. □

We now return to the proof of Lemma 3.31, recalling that we would like to show

$$|\mathcal{B}(h,n,\varepsilon,q,\delta)|\leqslant\beta^{qn^2}\binom{n^2}{qn^2}^{h+1}.$$

Propositions 3.38 and 3.36, which we have already established, together give that

$$\begin{aligned}&|\mathcal{B}(h,n,\varepsilon,q,\delta)|\\&\quad\leqslant\left(\widehat{\beta}^m\binom{n^2}{m}^{h+1}\right)^{2\log n}\left(\binom{n^2}{m}^{h+1}\right)^{2\log n}\cdot 2\binom{qn^2}{m,m,\dots,m}^{-(h+1)}(4\log n)!.\end{aligned}$$

Thus to establish Lemma 3.31, it suffices to prove the following.

Proposition 3.41.

$$\widehat{\beta}^{2m\log n}\binom{n^2}{m}^{4(h+1)\log n}\cdot 2\binom{qn^2}{m,m,\dots,m}^{-(h+1)}(4\log n)!\leqslant\beta^{qn^2}\binom{n^2}{qn^2}^{h+1}.$$

Proof of Proposition 3.41. Keeping in mind that

$$qn^2 = 4(\log n)m, \qquad \widehat{\beta} = \beta^2 9^{-2(h+1)}, \qquad \binom{a}{b} \leqslant (\frac{ea}{b})^b,$$

$$\binom{a}{b, b, \ldots, b} \geqslant \left(\frac{a}{be}\right)^a, \quad \text{and} \quad \left(\frac{a}{b}\right)^b \leqslant \binom{a}{b},$$

we see that

$$\begin{aligned}
&\widehat{\beta}^{2m\log n}\binom{n^2}{m}^{4(h+1)\log n} \cdot 2\binom{qn^2}{m, m, \ldots, m}^{-(h+1)} \cdot (4\log n)! \\
&\leqslant \left(\frac{\beta^2}{9^{2(h+1)}}\right)^{2m\log n} \left(\frac{n^2 e}{m}\right)^{m4(h+1)\log n} \cdot 2\left(\frac{qn^2}{me}\right)^{-qn^2(h+1)} \cdot (4\log n)! \\
&= \beta^{qn^2}\left(\frac{n^2 e}{9m}\right)^{4(h+1)m\log n} \cdot 2\left(\frac{me}{qn^2}\right)^{qn^2(h+1)} \cdot (4\log n)! \\
&= 2\beta^{qn^2}\left(\frac{n^2 e}{9m} \cdot \frac{me}{qn^2}\right)^{qn^2(h+1)} \cdot (4\log n)! \\
&\leqslant \beta^{qn^2}\left(\frac{n^2}{qn^2}\right)^{qn^2(h+1)} \cdot \left(\frac{e^2}{9}\right)^{qn^2(h+1)} 2(4\log n)! \\
&\leqslant \beta^{qn^2}\left(\frac{n^2}{qn^2}\right)^{qn^2(h+1)} \leqslant \beta^{qn^2}\left(\frac{n^2}{qn^2}\right)^{h+1}.
\end{aligned}$$

□

This completes the proof of Lemma 3.31. □

3.4.3 Property (*iv*) in Lemma 3.24

In this subsection, we will prove Claim 3.42, which correspond to property (*iv*) in Lemma 3.24.

Claim 3.42. *For all constants $h, \ell \in \mathbb{Z}^+$ and $\delta \in \mathbb{R}^+$, there exists a constant $\varepsilon \in \mathbb{R}^+$ such that the following holds. For any constant $t \in \mathbb{Z}^+$,*

$$q := 4(\log n)^2 n^{-1+1/(h+1)}, \qquad N := tn, \qquad \textit{and} \qquad p := 4\ell q,$$

the random graph $G(N,p)$ a.a.s. has the following property. For any selection of disjoint subsets $X_1, X_2, \ldots, X_{h+1} \subset V(G)$, every (not necessarily induced) subgraphs H on $\bigsqcup_{i=1}^{h+1} X_i$ with $H \in \mathcal{H}(h,n,\varepsilon,q)$ is $(1-\delta, \log n)$-path abundant.

Proof. Consider any $h, \ell \in \mathbb{Z}^+$ and $\delta \in \mathbb{R}^+$. Let

$$\beta := (24\ell)^{-(h+1)}.$$

By Lemma 3.31, we may now fix

$$\varepsilon := \varepsilon^{L3.31}(h,\delta,\beta) \qquad \text{and} \qquad n_4 := n_4^{L3.31}(h,\delta,\beta),$$

and without loss of generality assume that $\varepsilon < 1/2$. Now consider any integer $t \in \mathbb{Z}^+$.

To show that a.a.s. every subgraph $H \in \mathcal{H}(h,n,\varepsilon,q)$ appearing in $G(N,p)$ is $(1-\delta,\log n)$-path abundant, as we previously remarked, it suffices to show that a.a.s. $G(N,p)$ does not contain disjoint subsets $X_1, X_2, \ldots, X_{h+1} \subset V(G)$ and a subgraph B on $\bigsqcup_{i=1}^{h+1} X_i$ with $B \in \mathcal{B}(h,n,\varepsilon,q,\delta)$. By Lemma 3.31, for all $n \geqslant n_4$, the expected total number of subgraphs $B \in \mathcal{B}(h,n,\varepsilon,q,\delta)$ appearing in $G(N,p)$ over all choices of subsets is bounded above by

$$\begin{aligned}
\binom{N}{(h+1)n}((h+1)n)! \cdot \beta^{qn^2} \binom{n^2}{qn^2}^{(h+1)} \cdot p^{qn^2(h+1)} \\
&\leqslant N^{(h+1)n} \cdot \beta^{qn^2} \left(\frac{en^2}{qn^2(h+1)}\right)^{qn^2(h+1)} p^{qn^2(h+1)} \\
&\leqslant 2^{(h+1)n\log N} \cdot \left(\frac{\beta^{1/(h+1)} e(4\ell q)}{q(h+1)}\right)^{qn^2(h+1)} \\
&\leqslant 2^{(h+1)n\log N} \cdot \left(\beta^{1/(h+1)} e 4\ell\right)^{qn^2(h+1)} \\
&\leqslant 2^{(h+1)n\log tn} \cdot \left(\frac{1}{2}\right)^{qn^2},
\end{aligned}$$

which tends to 0 as $n \to \infty$. Therefore the probability that $G(N,p)$ contains disjoint vertex subsets $X_1, X_2, \ldots, X_{h+1} \subset V(G)$ and a subgraph B on $\bigsqcup_{i=1}^{h+1} X_i$ with $B \in \mathcal{B}(h, n, \varepsilon, q, \delta)$ also tends to 0 as $n \to \infty$, completing the proof of Claim 3.26. □

3.5 Proof of the Embedding Lemma

In this section, we prove Lemma 3.14, which states that for certain parameters every $J \in \mathcal{J}(h, n, \delta)$ (see Definition 3.13) is universal to the set of graphs $\{S^{(h)} : |V(S)| = n(\log n)^{-7h} \text{ and } \Delta(S) \leqslant d\}$. The proof will be divided into two subsections, which are preceded by the following sketch of the proof.

Consider any graph $J \in \mathcal{J}(h, n, \delta)$ on $\bigsqcup_{i=1}^{h+1} X_i$ and any graph S with $|V(S)| = n(\log n)^{-7h}$ and $\Delta(S) \leqslant d$. Our aim will be to find a mapping $\phi : V(S) \to X_1$ such that each edge $uv \in E(S)$ can be paired with a transversal path (see Definition 3.10) between $\phi(u)$ and $\phi(v)$. Observe that if the set of transversal paths selected are internally vertex-disjoint, this will correspond to an embedding of the subdivided graph $S^{(h)}$ into J. Roughly speaking, this will be accomplished by first finding an embedding $\phi : V(S) \to X_1$ and associating each edge $uv \in E(S)$ with not one associated transversal path, but a family of many transversal paths between $\phi(u)$ and $\phi(v)$. This will be done so that all the paths in all the associated families are edge-disjoint. We then will select one path from each associated family to obtain the desired collection of internally vertex-disjoint paths.

We will now elaborate upon this sketch. For the graph J, we say that two vertices $u, v \in X_1$ are $(\log n)$-*path connected* if u and v are joined by at least $\log n$ pairwise edge-disjoint transversal paths in J. Since J is $(1-\delta, \log n)$-path abundant (see Definition 3.10), at least $(1-\delta)\binom{n}{2}$ pairs of vertices in X_1 are $(\log n)$-path connected (see Definition 3.10). Define an auxiliary graph A by

$$V(A) := X_1 \quad \text{and} \quad E(A) := \{uv : u \text{ and } v \text{ are } (\log n)\text{-path connected in } J\}.$$

For each $uv \in E(A)$, let Π_{uv} be a fixed set of $\log n$ pairwise edge-disjoint transversal paths in J with endpoints u and v. We say the distinct edges $e_1, e_2 \in E(A)$ are *incompatible* if there exist paths $\pi_{e_1} \in \Pi_{e_1}$ and $\pi_{e_2} \in \Pi_{e_2}$ such that π_{e_1} and π_{e_2} have an edge in common. Define the *incompatibility function* $f\colon E(A) \to \mathcal{P}(E(A))$ by

$$f(e_1) := \{e_2 : e_1 \text{ and } e_2 \text{ are incompatible}\}.$$

Given this set-up, the proof has two steps:

- Find a graph embedding $\phi\colon S \to A$ such that $\phi(e_1) \notin f(\phi(e_2))$ for every $e_1, e_2 \in E(S)$.
- For each edge $e \in E(S)$, select a path $\pi_{\phi(e)} \in \Pi_{\phi(e)}$ so that for all pairs of edges $e_1, e_2 \in E(S)$, the paths $\pi_{\phi(e_1)}$ and $\pi_{\phi(e_2)}$ are internally vertex-disjoint.

The key to the first of these two steps is the following lemma. Although stated in a general context, when we apply the lemma the function f will be the incompatibility function defined above.

Lemma 3.43. *Let d and n be positive integers. Let A be a graph such that:*

(i) $|V(A)| = n$.

(ii) Every vertex in A has degree at least $(1 - 1/6d)n$.

Let S be a graph such that:

(iii) $|V(S)| \leqslant n/6$.

(iv) Every vertex in S has degree at most d.

Let $f : E(A) \to \mathcal{P}(E(A))$ be a function that maps each edge $e \in E(A)$ to a set of edges $f(e) \subset E(A)$ such that:

(v) $|f(e)| \leqslant n/6^3 d^4$ for all $e \in E(A)$.

(vi) $e_1 \in f(e_2)$ if and only if $e_2 \in f(e_1)$.

(vii) $e \notin f(e)$ for all $e \in E(A)$.

Then there is an embedding $\phi\colon S \to A$ such that

$$\phi(E(S)) \cap f(\phi(E(S))) = \varnothing, \tag{3.12}$$

where $f(\phi(E(S))) := \bigcup_{e \in \phi(E(S))} f(e)$.

To select a system of internally vertex-disjoint paths $\pi_{\phi(e)} \in \Pi_{\phi(e)}$ for the edges $e \in S$, we will make use of J being (h, n)-cluster free, that for distinct edges $e_1, e_2 \in S$ the families $\pi_{\phi(e_1)}$ and $\pi_{\phi(e_2)}$ consist of pairwise edge-disjoint paths, and the following result of Aharoni and Haxell.

Fact 3.44 ([2]). *Let X be a finite set and let $\widehat{\Pi}_1, \dots, \widehat{\Pi}_m \subset \binom{X}{h}$ be families of h-subsets of X. Suppose that, for every non-empty $L \subset [m]$, there are more than $h(|L| - 1)$ pairwise disjoint h-sets in $\bigcup_{l \in L} \widehat{\Pi}_l$. Then there exist $\widehat{\pi}_1, \dots, \widehat{\pi}_m$ with $\widehat{\pi}_i \in \widehat{\Pi}_i$ for every $i \in [m]$ such that $\widehat{\pi}_i \cap \widehat{\pi}_j = \varnothing$ for every distinct $i, j \in [m]$. We call $\{\widehat{\pi}_i : i \in [m]\}$ a* system of disjoint representatives *for $\{\widehat{\Pi}_i : i \in [m]\}$.*

The remaining part of this section is divided into two subsections. The first subsection contains a proof of Lemma 3.43 and the second subsection contains a proof of Lemma 3.14 based upon Lemma 3.43 and Fact 3.44.

3.5.1 Proof of Lemma 3.43

Proof. Let n, d, A, S, and f be as in the statement of Lemma 3.43. To prove the lemma, we introduce some terminology and then present an embedding algorithm.

Definition 3.45 (Dangerous Vertex)**.**

- *We call edges e_1 and e_2 in $E(A)$* incompatible *if $e_1 \in f(e_2)$.*
- *We call a pair of incident edges $xy, yz \in E(A)$ that are incompatible a* useless P_3*. We call y the* center vertex *of the useless P_3 and the pair x, z the* end vertices *of the useless P_3.*
- *We call a pair of vertices $\{u, v\} \in \binom{V(A)}{2}$ a* dangerous pair *if u, v are the end vertices of at least $n/6\binom{d}{2}$ useless P_3.*
- *We call a vertex $v \in V(A)$ a* dangerous vertex *if it is in at least $n/6d^2$ dangerous pairs.*

We now work to obtain an upper bound for the number of dangerous vertices in A. Recalling that each edge is incompatible with at most $n/6^3d^4$ other edges, the number of useless P_3 is at most

$$\frac{n}{6^3d^4} \cdot \binom{n}{2} \leqslant \frac{n^3}{2^1 6^3 d^4}.$$

It follows that the number of dangerous pairs of vertices is at most

$$\frac{n^3}{2^1 6^3 d^4} \cdot \frac{6\binom{d}{2}}{n} \leqslant \frac{n^2}{2^2 6^2 d^2}.$$

Finally, the number of dangerous vertices is at most

$$2 \cdot \frac{n^2}{2^2 6^2 d^2} \cdot \frac{6d^2}{n} \leqslant \frac{n}{12}. \tag{3.13}$$

Set J_0 to be the set of dangerous vertices in A.

Definition 3.46 (Guilty Vertex)**.** *Suppose S' is an induced subgraph of S, A' is an induced subgraph of A, and ϕ' is an embedding of the graph S' into A'.*

- *We call $e \in E(A')$ a* forbidden edge *if $e \in f(\phi'(E(S')))$.*
- *We will call a vertex $v \in \phi'(V(S'))$* guilty by association*, or simply* guilty*, if v is incident to at least $n/6d$ forbidden edges.*

That is, a forbidden edge in A is incompatible with an edge that has already been used in the embedding, and a vertex is guilty by association if it is incident to too many forbidden edges.

Definition 3.47 (Safe and Legal Embeddings)**.** *Suppose S' is an induced subgraph of S, the graph A' is an induced subgraph of the graph A, and ϕ' is an embedding of the graph S' into the graph A'.*

- *We say that the embedding ϕ' is* legal *if $\phi'(E(S')) \cap f(\phi'(E(S')) = \varnothing$.*
- *We say vertices s_1, s_2 in S are* P_3-connected *if $s_1v, s_2v \in E(S)$ for some $v \in V(S)$.*
- *We say that the embedding ϕ' is* safe *if none of the pairs $\{\phi'(s_1), \phi'(s_2)\}$ of vertices in A is dangerous for vertices $s_1, s_2 \in V(S')$ that are P_3-connected in S.*

That is, an embedding is legal if it has not used any pair of incompatible edges, and an embedding is safe if for each $s \in S$ and any pair of vertices $s_1, s_2 \in N(s)$, the embedding ϕ' has not mapped s_1 and s_2 onto a dangerous pair of vertices.

Before formally stating our embedding algorithm, we present the main idea, which is as follows. We keep a set J of 'jailed' vertices. We initially send all the dangerous vertices to jail. We then construct a legal and safe partial embedding ϕ' of an induced subgraph $S' \subset S$ into $A \backslash J$ by sequentially embedding vertices. As edges are added to the embedding, however, the number of forbidden edges may increase and already embedded vertices may become guilty by association. This is problematic because guilty vertices may prevent the embedding from being extended in a legal manner later. To resolve this, whenever guilty vertices appear in A', we send them to jail and remove them from the embedding. (Therefore, the domain S' of the partial embedding ϕ' may decrease in size as the algorithm progresses.) We will show that not too many vertices end up in jail and that when no guilty vertices are present, a legal and safe embedding can always be augmented to form a larger legal and safe embedding.

Algorithm: Initially take

$$S' := \varnothing, \qquad\qquad J := J_0, \qquad\qquad A' := A \backslash J,$$

and set $\phi' \colon S' \to A'$ to be the empty function. As we proceed through the algorithm, we will update these sets and this function.

STEP 1: If there exists a vertex $v \in \phi'(V(S'))$ that is guilty in the current embedding, replace J by $J \cup \{v\}$, replace S' by $S' \backslash \{\phi'^{-1}(v)\}$, update the function ϕ' by removing the pair $(\phi'^{-1}(v), v)$, update A' to $A \backslash J$, and repeat STEP 1. Otherwise, go to STEP 2.

STEP 2: Arbitrarily pick a vertex $s \in V(S) \backslash V(S')$ and extend ϕ' to s by mapping s to some vertex $v \in V(A') \backslash \phi(V(S'))$ so that the new embedding is both legal and safe. Also, replace S' by $S' \cup \{s\}$ and add (s, v) to ϕ'. If $S' = S$, terminate the algorithm; otherwise, go to STEP 1.

We make the following observations about this algorithm:

- Once a vertex is placed into J, it will always remain in J.

- The set of dangerous pairs and the set of dangerous vertices are both fixed from the beginning and do not change.
- Extending an embedding by adding a new vertex (and up to d edges) may make a vertex $v \in \phi'(V(S'))$ guilty.
- At the start of STEP 2, there are no guilty vertices and the current embedding is both legal and safe.

It remains to show that STEP 2 is always possible and that the algorithm will successfully terminate. This will be accomplished by the following two facts.

Proposition 3.48. *The size of the set J will never reach $n/6$.*

Proof of Proposition 3.48. Towards contradiction, consider the first moment in the execution of the algorithm at which $|J| = n/6$. Let B be the set of edges that were forbidden at any point in time up to this stopping point. That is, B is the set of edges that appeared in $f(\phi'(E(S'))$ for any partial embedding ϕ' the algorithm considered over its run time. We will reach a contradiction by considering the size of B.

To obtain an upper bound for the size of B, notice that whenever a vertex was added to the embedding, up to d edges were added to the embedding as well, and thus at most $d \cdot n/(6^3 d^4)$ forbidden edges were added to B for each vertex embedded. Since the number of vertices added to the embedding is at most

$$|J| - |J_0| + |S| \leqslant \frac{n}{6} + \frac{n}{6} \leqslant \frac{n}{3},$$

it follows that

$$|B| \leqslant \frac{n}{3} \cdot d \cdot \frac{n}{6^3 d^4} \leqslant \frac{n^2}{6^3 d^3}. \tag{3.14}$$

We now obtain a lower bound for the size of B. Notice that each guilty vertex that was added to J was incident to at least $n/6d$ forbidden edges in A'. Moreover, since vertices in J remain in J, this set of $n/6d$ forbidden edges will never again appear in A'. This gives

$$|B| \geqslant (|J| - |J_0|) \cdot \frac{n}{6d} \geqslant \left(\frac{n}{6} - \frac{n}{12}\right) \cdot \frac{n}{6d} = \frac{n^2}{72d}. \tag{3.15}$$

Equalities (3.14) and (3.15) yield the contradiction

$$\frac{n^2}{72d} \leqslant |B| \leqslant \frac{n^2}{6^3 d^3},$$

completing the proof of Proposition 3.48. □

Proposition 3.49. *STEP 2 is always possible.*

Proof of Proposition 3.49. Arbitrarily pick a vertex $s \in V(S) \backslash V(S')$ to extend the embedding to. We must find a vertex $v \in A'$ so that extending ϕ' to include the pair (s, v) will produce an embedding that is both legal and safe. We will now list six cases in which such a vertex $v \in A$ will not produce an embedding that is both legal and safe. Cases 1, 2, and 3 correspond to the map not being an embedding into A'; Case 4 corresponds to the embedding using an edge incompatible with an edge already used (and thus not being legal); Case 5 corresponds to the embedding using two new edges that are incompatible with each other (and thus not being legal); and Case 6 corresponds to the embedding not being safe.

1. The vertex v belongs to $\phi'(S')$.

2. The vertex v belongs to J.

3. For some $s' \in S'$ with $ss' \in E(S)$, the edge $\phi(s')v$ is not in $E(A)$.

4. For some $s' \in S'$ with $ss' \in E(S)$ and $e' \in E(S')$, the edge $\phi(s')v$ is in $f(\phi(e'))$.

5. For some $s_1, s_2 \in S'$ with $ss_1, ss_2 \in E(S)$, the edges $\phi(s_1)v$ and $\phi(s_2)v$ are incompatible.

6. For some $s' \in S'$ that is P_3-connected in S to s, the pair $\{\phi'(s'), v\}$ is dangerous.

Observe that if none of (1)–(6) holds, then extending ϕ to include (s, v) will produce an embedding that is both legal and safe.

The number of vertices in A in Cases 1 and 2 is at most

$$|S| + |J| \leqslant \left(\frac{n}{6} - 1\right) + \frac{n}{6} \leqslant \frac{2n}{6} - 1.$$

To count the number of vertices in A in Case 3, observe that s has at most d neighbors in S'. Hence, there are at most d choices for s'. Also, from hypothesis each s' is not adjacent to at most $n/6d$ vertices. Hence, the number of vertices in Case 3 at most

$$d \cdot \frac{n}{6d} \leqslant \frac{n}{6}.$$

Similarly, to count the number of vertices in A in Case 4, again recall that s has at most d neighbors in S'. Also for each such neighbor s', it follows from the fact that $\phi'(s')$ is not guilty by association that $\phi(s')$ is incident to at most $n/6d$ forbidden edges. Hence, the total number of vertices in Case 4 is at most

$$d \cdot \frac{n}{6d} = \frac{n}{6}.$$

To count the number of vertices in A in Case 5, observe that there are are at most $\binom{d}{2}$ choices for s_1 and s_2, and for any choice of s_1, s_2, since the embedding is safe, there are at most $n/6\binom{d}{2}$ vertices v that are part of a useless P_3 with $\phi'(s_1)$ and $\phi'(s_2)$. Hence the total number of vertices in

Case 5 is at most

$$\binom{d}{2} \cdot \frac{n}{6\binom{d}{2}} \leqslant \frac{n}{6}.$$

Finally, to count the number of vertices that are in Case 6, observe that in the graph S, the vertex s is distance two away from at most d^2 other vertices. Since each of the images of these vertices is not dangerous, the images are each in at most $n/6d^2$ dangerous pairs. Hence, the total number of vertices $v \in A$ that are in Case 6 is at most

$$d^2 \cdot \frac{n}{6d^2} = \frac{n}{6}.$$

In conclusion, there must be at least

$$n - \left(\frac{2n}{6} - 1\right) - 4 \cdot \frac{n}{6} \geqslant 1$$

vertices $v \in A$ such that the map obtained by extending ϕ' to include (s, v) will produce both a legal and safe embedding. This completes the proof of Proposition 3.49. □

This concludes the proof of Lemma 3.43. □

3.5.2 Proof of Lemma 3.14

Consider any pair of positive integers h and d. We will make use of the following simple fact.

Fact 3.50. *For every $\nu > 0$ there exist $\delta > 0$ and n_6 such that for every integer $n \geqslant n_6$ the following holds. If A is a graph on n vertices with at least $(1-\delta)\binom{n}{2}$ edges, then there exists a subgraph $\widehat{A}$ with $|V(\widehat{A})| \geqslant (1-\nu)n$ and with minimum degree at most $(1-\nu)|V(\widehat{A})|$.*

With $\nu := 1/6d$, choose δ and n_6 in accordance with the previous fact.

Choose $n_3 \geqslant n_6$ so that the second inequality in (3.17) below is satisfied for all $n \geqslant n_3$. Now consider any $n \geqslant n_3$, any $J \in \mathcal{J}(h, n, \delta)$, and any graph S with $|V(S)| = n(\log n)^{-7h}$ and $\Delta(S) \leqslant d$. We must show that $S^{(h)} \subseteq J$.

As at the beginning of Section 3.5, define the auxiliary graph A by

$$V(A) := X_1 \quad \text{and} \quad E(A) := \{uv : u \text{ and } v \text{ are } (\log n)\text{-path connected in } J\}.$$

Let $\widehat{A}$ be a subgraph of A on $\widehat{n}$ vertices such that $\widehat{n} \geqslant n/2$ and every vertex in $\widehat{A}$ has degree at least $(1 - 1/6d)\widehat{n}$, guaranteed by Fact 3.50. Also, for each $uv \in E(\widehat{A})$, let Π_{uv} be a fixed set of $\log n$ transversal paths between u and v in J that are pairwise edge-disjoint. As before, we say that a pair of distinct edges $e_1, e_2 \in E(A)$ are *incompatible* if there exist paths $\pi_{e_1} \in \Pi_{e_1}$ and $\pi_{e_2} \in \Pi_{e_2}$ such that π_{e_1} and π_{e_2} have an edge in common and define

$$f(e_1) := \{e_2 : e_1 \text{ and } e_2 \text{ are incompatable}\}.$$

We will use Lemma 3.43 to embed S into $\widehat{A}$. With the set-up above, all the hypotheses other than (v) in Lemma 3.43 are clearly satisfied. To verify (v), observe that, since J has maximum degree $(\log n)^3 n^{1/(h+1)}$, the number of transversal paths any edge $e \in E(J)$ can be in is at most

$$\left((\log n)^3 n^{1/(h+1)}\right)^h \leqslant (\log n)^{3h} n^{h/(h+1)}. \tag{3.16}$$

Moreover, since for every $e \in E(A)$ the family Π_e has exactly $\log n$ edge-disjoint paths,

$$f(e) \leqslant (\log n) \cdot (h+1) \cdot (\log n)^{3h} n^{h/(h+1)} < \frac{n/2}{6^3 d^4}, \tag{3.17}$$

where the second inequality follows from $n \geqslant n_3$. Thus, by Lemma 3.43, there exists an embedding ϕ of S into $\widehat{A}$ such that the image of $E(S)$ under ϕ

contains no pair of incompatible edges.

Finally, to select a system of internally pairwise vertex-disjoint paths from the families $\{\Pi_{\phi(e)} : e \in E(S)\}$, the result of Aharoni and Haxell (Fact 3.44) will be used. Take $X := \bigcup_{i=2}^{h+1} X_i$, and set

$$\widehat{\Pi}_e := \{V(\pi) \cap X : \pi \in \Pi_e\},$$

so that each element in $\widehat{\Pi}_e$ is a set of vertices in X that corresponds to the interior of a path in Π_e. Thus a system of disjoint representatives for the set of families $\{\widehat{\Pi}_{\phi(e)} : e \in E(S)\}$ corresponds to an embedding of $S^{(h)}$ into J. Clearly,

$$|\{\widehat{\Pi}_{\phi(e)} : e \in E(S)\}| = |E(S)| \leqslant dn(\log n)^{-7} \leqslant n(\log n)^{-6}. \tag{3.18}$$

We claim that the hypothesis of Fact 3.44 holds. Towards contradiction, assume that there exists a set $\mathcal{L}$ of $L \leqslant n(\log n)^{-6}$ edges in $\phi(E(S)) \subseteq A$ such that there are at most $h(L-1)$ pairwise disjoint h-sets in $\bigcup_{l\in\mathcal{L}} \widehat{\Pi}_l$. Let Γ be a maximum set of pairwise disjoint h-sets in $\bigcup_{l\in\mathcal{L}} \widehat{\Pi}_l$. Let Z be the vertices in Γ. Observe

$$|Z| \leqslant h(L-1) \cdot h \leqslant h^2 L.$$

However, one may check that $\bigcup_{l\in\mathcal{L}} \Pi_l$ is an $(\mathcal{L}, Z, h, \log n)$-cluster of paths in the graph J. This contradicts the fact that J is (h,n)-cluster free (property (iv) in Definition 3.13). This contradiction establishes that the hypothesis of the Aharoni–Haxell theorem holds, and therefore the set of families $\{\widehat{\Pi}_{\phi(e)} : e \in E(S)\}$ has a set of disjoint representatives, yielding an embedding of $S^{(h)}$ into J. This completes the proof of Lemma 3.14.

3.6 Proof of Theorem 3.3

For brevity, we shall refer to graphs on n vertices that have maximum degree at most d as *(n,d)-graphs*. In this section, we show that if H is a graph that contains a copy of $S^{(h)}$ for every (n,d)-graph S, then H has at least $n^{1+1/(h+1)-2/d(h+1)+o(1)}$ edges. Hence, for fixed integers $h \geqslant 1$ and $d \geqslant 2$,

$$\mathrm{USR}(h,d,1,n) \geqslant n^{1+1/(h+1)-2/d(h+1)+o(1)},$$

which is the statement in Theorem 3.3.

The proof is based upon the following external lemma.

Fact 3.51 ([10], Corollary II.4.17, p. 52)**.** *Let $d \geqslant 2$ be a fixed integer and suppose that dn is even. The number $L_d(n)$ of d-regular graphs on n labeled vertices satisfies*

$$L_d(n) = (1+o(1))\sqrt{2}e^{-(d^2-1)/4}\left(\frac{d^{d/2}}{e^{d/2}d!}\right)^n n^{dn/2}.$$

Proof of Theorem 3.3. Let $L_{\leqslant d}(n)$ be the number of labeled (n,d)-graphs (recall that (n,d)-graphs have maximum degree at most d). Fact 3.51 gives that, for any fixed $d \geqslant 2$,

$$L_{\leqslant d}(n) \geqslant 2^{(d/2+o(1))n\log n}. \tag{3.19}$$

We now let $U_{\leqslant d}(n)$ be the number of *unlabeled* (n,d)-graphs, and let $U^{(h)}_{\leqslant d}(n)$ be the number of unlabeled h-subdivisions of such graphs.

We claim that

$$U^{(h)}_{\leqslant d}(n) \geqslant 2^{(d/2-1+o(1))n\log n}. \tag{3.20}$$

Indeed, first observe that, from (3.19), we have

$$U_{\leqslant d}(n) \geqslant \frac{1}{n!} \cdot 2^{(d/2+o(1))n\log n} \geqslant \frac{1}{n^n} \cdot 2^{(d/2+o(1))n\log n} \geqslant 2^{(d/2-1+o(1))n\log n}.$$

Second, observe that if two distinct unlabeled (n, d)-graphs S_1 and S_2 both have each edge subdivided h times, then the resulting graphs $S_1^{(h)}$ and $S_2^{(h)}$ are distinct unlabeled graphs. Together, these observations establish (3.20).

To complete the proof of Theorem 3.3, we use the fact that if H is a graph on m edges that contains a copy of every unlabeled h-subdivision of (n, d)-graphs, then it must be the case that

$$\sum_{i=0}^{nd(h+1)/2} \binom{m}{i} \geqslant U_{\leqslant d}^{(h)}(n) \geqslant 2^{(d/2-1+o(1))n\log n}. \tag{3.21}$$

If $m \leqslant nd(h+1)$, then the left hand side of (3.21) is at most $2^{nd(h+1)}$, which yields a contradiction to the inequality in (3.21). We therefore suppose that $m \geqslant nd(h+1)$. Then, using that every binomial coefficient in (3.21) is at most $\binom{m}{nd(h+1)/2}$ and that $\binom{n}{a} \leqslant (en/a)^n$, we have

$$\sum_{i=0}^{nd(h+1)/2} \binom{m}{i} \leqslant \frac{1}{2}nd(h+1) \cdot \left(\frac{em}{nd(h+1)/2}\right)^{nd(h+1)/2}. \tag{3.22}$$

From equations (3.21) and (3.22), we have

$$\frac{1}{2}nd(h+1) \cdot \left(\frac{em}{nd(h+1)/2}\right)^{nd(h+1)/2} \geqslant 2^{(d/2-1+o(1))n\log n},$$

or, equivalently,

$$\left(\frac{m}{n}\right)^{nd(h+1)/2} \geqslant 2^{(d/2-1+o(1))n\log n}.$$

This implies that

$$\frac{m}{n} \geqslant 2^{(1/(h+1)-2/((h+1)d)+o(1))\log n},$$

giving the desired bound of

$$m \geqslant n^{1+1/(h+1)-2/((h+1)d)+o(1)}.$$

$\square$

3.7 Proof Sketch of Theorem 3.4

To prove Theorem 3.4, we must show that for any integers h and d, there exists a constant $c_{h,d}$ such that the following holds. If Q is a graph of maximum degree at most d on q vertices with the property that every pair of vertices of degree greater than 2 are distance at least $h+1$ apart, then $\widehat{r}(G) \leqslant c_{h,d}(\log q)^{20h} q^{1+1/(h+1)}$.

To accomplish this, we first define the 'super subdivision' of a graph. We then show that for any graph Q as in Theorem 3.4, there exists a graph S such that the super subdivision of S contains Q as a subgraph. It will then suffice to demonstrate how our main Theorem 3.2 concerning subdivisions can be extended to super subdivisions.

Definition 3.52 (Super Subdivision $S^{(*)}$). *Give a graph S and integers h and d, we define the* super subdivision *$S^{(*)}$ of S to be the graph obtained by replacing each edge uv in S by a system of $d(h+1)$ paths from u to v, of which exactly d paths have length k for each $k \in \{h+1, h+2, \ldots, 2h+1\}$.*

Proposition 3.53. *Let Q be any graph with $|V(Q)| = q$ and $\Delta(Q) \leqslant d$ with the property that every two vertices of degree greater than 2 are distance at least $h+1$ apart. Then there exists a graph S with $|V(S)| \leqslant q$ and $\Delta(S) \leqslant d$ such that $Q \subset S^{(*)}$.*

Proof of Proposition 3.53. For vertices $x_1, x_2 \in Q$, let $dist_Q(x_1, x_2)$ be the minimum number of edges in a path with endpoints x_1 and x_2. Let X be a maximal subset of vertices in Q that satisfies both of the following properties:

- All vertices of degree greater than 2 are contained in X.
- All pairs of vertices $x_1, x_2 \in X$ satisfy $dist_Q(x_1, x_2) > h$.

Now construct a graph S by taking $V(S) = X$ and joining vertices $x_1, x_2 \in S$ if $dist_Q(x_1, x_2) < 2h + 2$. It follows that $\Delta(S) \leqslant \Delta(Q)$ and that $Q \subseteq S^{(h^*)}$. □

In view of Proposition 3.53, to establish Theorem 3.4 it suffices to establish the following lemma.

Lemma 3.54. *For any $h, d \in \mathbb{Z}^+$, there exists a constant $c_{h,d}$ such that for every graph S with $|V(S)| = s$ and $\Delta(S) \leqslant d$,*

$$\hat{r}(S^{(*)}) \leqslant c_{h,d} (\log s)^{20h} s^{1+1/(h+1)}.$$

To prove Lemma 3.54, we consider another way of obtaining the super subdivision $S^{(*)}$ from the graph S. Begin by fixing a proper edge coloring $\chi : E(S) \to [d+1]$ which exists since $\delta(S) \leqslant d$. For integers $i \in [d+1]$, $j \in [d]$, and $k \in \{h+1, h+2, \ldots, 2h+1\}$, let $M_{i,j,k} := \chi^{-1}(i)$; it follows that $M_{i,j,k} = M_{i,j',k'}$ for all $j, j' \in [d]$ and $k, k' \in \{h+1, h+2, \ldots, 2h+1\}$. Define the multiset of matchings

$$\mathcal{M} := \big\{M_{i,j,k} : i \in [d+1],\ j \in [d],\ k \in \{h+1, h+2, \ldots, 2h+1\}\big\}.$$

We construct $S^{(*)}$ on $V(S)$ by the following procedure. For every $M_{i,j,k} \in \mathcal{M}$ and every $xy \in M_{i,j,k}$, add a path of length k between x and y. Consequently, for any $xy \in E(S)$, there are d paths of length k between x and y for each $k \in \{h+1, h+2, \ldots, 2h+1\}$. It follows that the resulting graph is the super subdivision $S^{(*)}$ of S.

Since the full proof is notationally cumbersome, we first demonstrate the main ideas in the context of two propositions that allows for simpler notation. These propositions consider the simpler case where the set $\mathcal{M}$ of multiple matchings is replaced by a pair of matchings.

Definition 3.55 ($S^{(M_1,M_2,k_1,k_2)}$). *Let S be a graph and $M_1, M_2 \subset E(S)$ be not necessarily disjoint matchings with $M_1 \cup M_2 = E(S)$. Let k_1 and k_2 be integers. Define $S^{(M_1,M_2,k_1,k_2)}$ to be the graph on $V(S)$ obtained by adding a path of length k_1 between x and y for every edge $xy \in M_1$ and a path of length k_2 between x and y for every edge $xy \in M_2$. (Since M_1 and M_2 need not be disjoint, some edges in $E(S)$ may be replaced by two paths.)*

Proposition 3.56. *For any $h \in \mathbb{Z}^+$, there exists a constant c_h such that if S is a graph with $|V(S)| = s$ and M_1 and M_2 are matchings such that $M_1 \cup M_2 = E(S)$, then*

$$\hat{r}(S^{(M_1,M_2,h+1,h+2)}) \leqslant c_h(\log s)^{20h} s^{1+1/(h+1)}.$$

Proof of Proposition 3.56. We will make three claims that are similar to the Coloring Lemma, Existence Lemma, and Embedding Lemma used in the proof of Theorem 3.2. Before stating the first of these claims, we introduce a couple definitions. The second of which is demonstrated in Figure 3.3.

Definition 3.57 ($C_{h+1,h+2}$). *Let $C_{h+1,h+2}$ be the graph on $2h+2$ vertices obtained from the cycle C_{h+1} with cyclically ordered vertices $x_1^1, x_2^1, \ldots, x_{h+1}^1$ and a copy of the cycle C_{h+2} with cyclically ordered vertices $x_1^2, x_2^2, \ldots, x_{h+2}^2$ and association $x_1 := x_1^1 = x_1^2$.*

Definition 3.58 (Incomplete Blowup of $C_{h+1,h+2}$). *A incomplete blowup H of $C_{h+1,h+2}$ is obtained by replacing each vertex x_j^i with a independent set X_j^i of n vertices and each edge by a (not necessarily complete) bipartite graph. Also, define $H^1 := H[\cup_{\alpha \in [h+1]} X_\alpha^1]$ and $H^2 := H[\cup_{\alpha \in [h+2]} X_\alpha^1]$.*

Recall that in the proof of Theorem 3.2 the class $\mathcal{H}(h, n, \varepsilon, q)$ was the set of incomplete blowups of C_{h+1} in which the bipartite graph had exactly qn^2

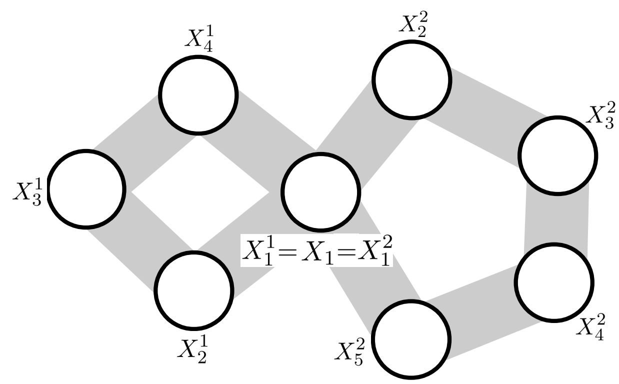


Figure 3.3: An incomplete blowup of $C_{h+1,h+2}$ for $h = 3$.

edges and were (ε, q)-regular (as in Definition 3.21). We now define an analogous concept.

Definition 3.59. *Let $\mathcal{H}^*(h, n, \varepsilon, q)$ be the set of all graphs that are incomplete blowup of $C_{h+1,h+2}$ where every edge in $C_{h+1,h+1}$ corresponds to an (ε, q)-regular bipartite graph with exactly qn^2 edges.*

The next claim is analogous to the Coloring Lemma.

Claim 3.60. *For any $\varepsilon \in \mathbb{R}^+$ and $h, \ell \in \mathbb{Z}^+$, there exist $t, n_1 \in \mathbb{Z}^+$ such that for all $n \geqslant n_1$,*

$$q := 4(\log n)^2 n^{-1+1/(h+1)}, \qquad N := tn, \qquad \text{and} \qquad p := 4\ell q,$$

every graph $G \in \mathcal{I}(N, p)$ has the following property. Any ℓ-coloring of the edges of G yields a monochromatic subgraph $H \in \mathcal{H}^*(h, n, \varepsilon, q)$.

Proof of Claim 3.60. In the proof of the Coloring Lemma (Lemma 3.9), we defined a cluster graph that had vertices corresponding to the vertex classes obtained from an application of the Regularity Lemma and edges corresponding to pairs that exhibited regularity. The edges of the cluster graph were ℓ-colored by the majority color in the corresponding partition. We previously argued that the cluster graph contained a monochromatic clique of size $h+1$, and hence a copy of C_{h+1}). By taking t sufficiently larger and an appropriate modification of the parameters in the proof, we can instead find a monochromatic clique of size $2h+2$, and hence a copy of $C_{h+1,h+2}$. This will yield a monochromatic $H \in \mathcal{H}^*(h, n, \varepsilon, q)$. □

Our next claim will be analogous to the Existence Lemma. To state it, we first need a modified notion of path abundance.

Definition 3.61 (Transversal Paths for $\mathcal{H}^*$)**.** *Let H be a partial blowup of $C_{h+1,h+2}$.*

- *For a pair of vertices $u, v \in X_1^1$, a* transversal path *between u and v in H^1 is the same as described in Definition 3.10.*
- *For a pair of vertices $u \in X_1^2$ and $v \in X_{h+2}^2$, a* transversal path between u and v *in H^2 is a path P of length $h+1$ with exactly one vertex in X_i^2 for each $i \in [h+2]$.*

Definition 3.62 (Path Abundance for $\mathcal{H}^*$)**.** *Let H be a partial blowup of $C_{h+1,h+2}$. We say that the graph H is* $(1-\delta, \log n)$-path abundant *if both of the following hold:*

- *The graph H^1 is path abundant (as defined in Definition 3.10).*
- *The graph H^2 has the property that for at least $(1-\delta)n^2$ pairs of vertices $u \in X_1^2$ and $v \in X_{h+2}^2$, there are at least $\log n$ transversal paths between u and v that are pairwise edge-disjoint (as defined in Definition 3.61).*

We now state the next claim that is analogous to the Existence Lemma.

Claim 3.63. *For all $h, \ell \in \mathbb{Z}^+$ and $\delta \in \mathbb{R}^+$, there exists $\varepsilon \in \mathbb{R}^+$ such that for any $t \in \mathbb{Z}^+$ there exists $n_2 \in \mathbb{Z}$ such that the following holds. For any $n \geqslant n_2$ and*

$$q := 4(\log n)^2 n^{-1+1/(h+1)}, \qquad N := tn, \qquad \textit{and} \qquad p := 4\ell q,$$

there exists a graph G on N vertices satisfying all of the following properties:

(i) Every vertex in G has degree at most $(\log n)^3 n^{1/(h+1)}$.

(ii) G is (h,n)-cluster free.

(iii) $G \in \mathcal{I}(N,p)$.

(iv) Every (not necessarily induced) subgraph $H \in \mathcal{H}^*(h,n,\varepsilon,q)$ of G is $(1-\delta, \log n)$-path abundant.

Proof of Claim 3.63. Properties (i)-(iii) are the same as in the Existence Lemma and the modified notion of path abundance in Property (iv) is proved analogously. □

After stating one more definition, we state a claim analogous to the Embedding Lemma.

Definition 3.64. *Let $\mathcal{J}^*(h,n,\delta)$ be the set of all graphs J that are partial blowups of $C_{h+1,h+2}$ such that*

(i) Every vertex in J has degree at most $(\log n)^3 n^{1/(h+1)}$.

(ii) J is (n,h)-cluster free (as defined in Definition 3.11).

(iii) J is $(1-\delta, \log n)$-path abundant (as defined in Definition 3.62).

(iv) There is a matching of size $(1-\delta)n$ between X^2_{h+2} and X^2_1.

As in the proof of Theorem 3.2, the Coloring Lemma and Existence Lemma together yield a monochromatic $H \in \mathcal{J}^*(h,n,\delta)$. Note that the additional Property (iv) follows from the fact that $H \in \mathcal{H}^*(h,n,\varepsilon,q)$ and hence the bipartite graph of H induced between X^2_{h+2} and X^2_1 is (ε,p)-regular. The next claim in analogous to the Embedding Lemma.

Claim 3.65. *For all $h \in \mathbb{Z}^+$, there exist $\delta \in \mathbb{R}^+$ and $n_3 \in \mathbb{Z}^+$ such that for all $n \geqslant n_3$ the following holds. Every graph $H \in \mathcal{J}^*(h, n, \delta)$ is universal to the set of graphs*

$$\left\{ S^{(M_1, M_2, h+1, h+2)} : |V(S)| = \frac{n}{(\log n)^{7h}} \right\}.$$

Proof of Claim 3.65. The proof of this claim follows the lines of the argument used to establish the Embedding Lemma where $S^{(h)}$ was embedded into $J \in \mathcal{J}(h, n, \delta)$. Recall that the main steps in this argument were:

- Considering an auxiliary graph A with vertex set X_1 where vertices $x, y \in X_1$ were joined if x and y were path connected (i.e. if there was a set Π_{xy} of $\log n$ edge-disjoint transversal paths between x and y).

- Defining an incompatibility function $f : E(A) \to \mathcal{P}(E(A))$ where each edge was incomparable with certain other edges.

- Finding an embedding ϕ of S into A such that $f(\phi(S)) \cap \phi(S) = \varnothing$.

- Showing that for every edge $xy \in \phi(S)$, a path $\pi_{xy} \in \Pi_{xy}$ could be selected so that the set of paths selected $\{\pi_{xy} : xy \in \phi(S)\}$ were pairwise internally disjoint. This corresponded to embedding $S^{(h)}$ into J.

The proof of Claim 3.65 is similar, so we only mention where it differs. We begin by fixing a matching Γ between X^2_{h+2} and X_1 of size at least $(1 - \delta)n$. For a vertex $v \in X_1$, denote the vertex it is matched to in X^2_{h+2} under Γ by $\widehat{v}$. Now fix an ordering $v_1, v_2, \ldots, v_n$ of the vertices in X_1. Given this setup, we introduce the following definition.

Definition 3.66. *(Path Linked) For $i < j$, the vertices $v_i, v_j \in X_1$ are* path linked *in H^2 (see Definition 3.58) if v_i and $\widehat{v}_j$ are path connected (i.e. if there exists a set Π_{ij} of $\log n$ edge-disjoint transversal paths between v_i and $\widehat{v}_j$).*

If v_j is not incident to an edge in Γ, then $\widehat{v}_j$ is not defined and v_i and v_j are not path linked. The concept is illustrated in Figure 3.4.

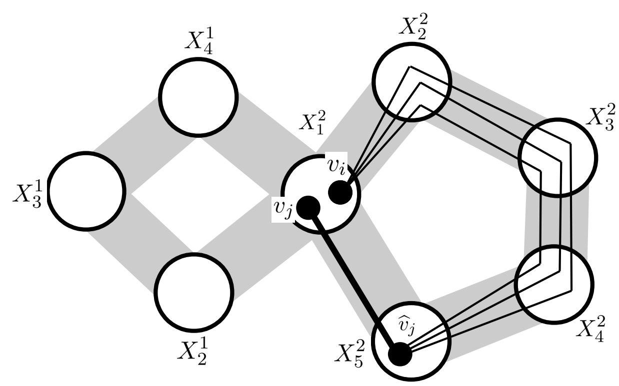


Figure 3.4: Vertices $v_i, v_j \in X_1^2$ are path linked in H^2 if there are many edge-disjoint paths between v_i and $\widehat{v}_j$.

Observe that since most pairs of vertices $v_i \in X_1$ and $\widehat{v}_j \in X_{h+2}^2$ are path connected, most pairs of vertices $v_i, v_j \in X_1$ are path linked. Now, for all path linked pairs $v_i \in X_1$ and $v_j \in X_1$, fix a set Π_{ij}^2 of $\log n$ edge-disjoint transversal paths between v_i and $\widehat{v}_j$ in H^2. Also, as in the original proof, fix a set Π_{ij}^1 of edge-disjoint transversal paths in H^1 for all path linked pairs $v_i \in X_1$ and $v_j \in X_1$. The proof now continues to follow the lines of the argument used to establish the Embedding Lemma with the following modifications:

- Define A by joining two vertices if and only if they are path connected in H^1 and path linked in H^2. Observe that, as before, AUX will be an 'almost complete' graph.

- Define the edges v_iv_j and v_kv_l in A to be incompatible if either of the following to conditions are met:

- There exists paths $\pi_{ij} \in \Pi^1_{ij}$ and $\pi_{kl} \in \Pi^1_{kl}$ such that π_{ij} and π_{kl} have an edge in common. (This is the same notion of incompatibility as used in the proof of the Embedding Lemma.)
- There exists paths $\pi_{ij} \in \Pi^2_{ij}$ and $\pi_{kl} \in \Pi^2_{kl}$ such that π_{ij} and π_{kl} have an edge in common.

- As before, we find an embedding ϕ of S into A such that $f(\phi(S)) \cap \phi(S) = \varnothing$. This is possible since S has bounded degree, the graph A is almost complete, and each edge is still incompatible with at most $o(n)$ other edges.

- Finally, for each edge $xy \in \phi(M)$, we select a path $\pi_{xy} \in \Pi^2_{xy}$ of length $h+1$ so that the sets of paths chosen $\{\pi_{xy} : xy \in \phi(M)\}$ are pairwise vertex-disjoint. Appending the appropriate matching edge in Γ to each path gives a desired set of paths of length $h+2$ in H^2. The paths of length $h+1$ are found in H^1 in the same manner as in our previous proof.

This completes the proof of Claim 3.65. □

We have now proved three claims analogous to the Coloring Lemma, Existence Lemma, and Embedding Lemma. The proof of Proposition 3.56 now follows the lines of the proof of Theorem 3.2.

□

Our second proposition describes how the situation changes if the edges in the matching M are divided one additional time.

Proposition 3.67. *For any $h \in \mathbb{Z}^+$, there exists a constant c_h such that for every graph S with $|V(S)| = s$ satisfies*

$$\hat{r}(S^{(M_1,M_2,h+1,h+3)}) \leqslant c_h(\log s)^{20h} s^{1+1/(h+1)}.$$

Proof of Proposition 3.67. The proof of this proposition differs from the previous proof as follows. In place of $C_{h+1,h+2}$, we take $C_{h+1,h+3}$ where the vertices are labeled $x_1^1, x_2^1, \ldots, x_{h+1}^1$ and $x_1^2, x_2^2, \ldots, x_{h+3}^2$ with $x_1 := x_1^1 = x_1^2$. We also require that 'almost perfect matchings' exist in both of the bipartite graphs (X_{h+1}^2, X_{h+2}^2) and (X_{h+2}^2, X_1^2).

We now begin the embedding process by fixing two such perfect matchings. These matchings together yield a collection of disjoint paths on three vertices that cover almost all vertices in $X_{h+1}^2 \cup X_{h+2}^2 \cup X_1^2$. For a vertex $v \in X_1$ which is covered by one of these paths of length two, define the vertex $\hat{v} \in X_{h+1}^2$ to be the corresponding vertex it is joined to in X_1^2 under the fixed collection of P_3. The remaining part of the proof is analogous to the proof of Claim 3.65. □

Having demonstrated the main idea of Lemma 3.54 in Propositions 3.56 and 3.67, we now briefly remark on how the proof of Lemma 3.54 differs.

Proof of Lemma 3.54. In Propositions 3.56 and 3.67, the two matchings were accommodated by replacing C_{h+1} by $C_{h+1,h+2}$ and $C_{h+1,h+3}$ respectively. Here, we will 'append' a cycle of length k for each of the matchings $M_{i,j,k} \in \mathcal{M}$. More formally, let C_* be the graph obtained by by following process. Take $d(d+1)$ disjoint cycles of each of the lengths $k \in \{h+1, h+2, \ldots, 2h+1\}$, for a total of $d(d+1)(h+1)$ cycles. From these cycles, C_* results by identifying one common vertex from all the cycles.

Propositions 3.56 and 3.67 has already demonstrated the main ideas involved embedding matchings in two cycles simultaneously. These ideas easily generalize to $d(d+1)(h+1)$ matchings associated to finite lengths of at least $h+1$. □

Chapter 4

Ramsey numbers involving large girth graphs and hypergraphs

4.1 Introduction

Recall that for a positive integer ℓ and graphs H and G, we write $H \to (G)_\ell$ if every ℓ-coloring of the edges of H yields a monochromatic copy of G. If $H \to (G)_\ell$, we say that H is *Ramsey for G* for ℓ colors. Ramsey's theorem establishes that, for every graph G and every positive integer ℓ, there exists a graph H such that $H \to (G)_\ell$. In this chapter, we consider three Ramsey-type problems that pertain to cycles in graphs and hypergraphs.

4.1.1 Cycles in Graphs

The first of these results has its roots in a problem suggested by Paul Erdős, which asks if for every pair of positive integers ℓ and k, there exists a graph H having girth$(H) = k$ and the Ramsey property $H \to (C_k)_\ell$. (Erdős explicitly stated a weaker form of this problem in [27].) The existence of such graphs was first established in [62], and the following theorem addresses the associated numerical problem.

Theorem 4.1. *Let $r = r_\ell(C_k)$ denote the least integer m such that $K_m \to (C_k)_\ell$. Then for all integers $k \geqslant 4$ and $\ell \geqslant 2$, there exists a graph H satisfying*

$$\mathrm{girth}(H) = k, \quad H \to (C_k)_\ell, \quad \textit{and} \quad |V(H)| \leqslant r^{40k^2} k^{40k^3}.$$

In Theorem 4.1, the exponential dependency of $|V(H)|$ on k is unavoidable. This follows from the observation that a minimal graph H with the desired properties must have minimum degree greater than ℓ and girth at least k. Also, note that $r_\ell(C_k)$ is known to be polynomial in ℓ for fixed even k, and for fixed odd k, satisfies the exponential relation $c_1^\ell \leqslant r_\ell(C_k) \leqslant c_2^{\ell \log \ell}$ for some positive constants c_1 and c_2 (see, e.g., [31]). This leads to the following corollary.

Corollary 4.2. *For every integer $k \geqslant 3$, there exist constants c_1 and c_2 such that for every integer $\ell \geqslant 2$, there exists a graph H such that* $\mathrm{girth}(H) = k$ *and $H \to (C_k)_\ell$, which satisfies $|V(H)| \leqslant \ell^{c_1}$ if k is even and $|V(H)| \leqslant c_2^{\ell \log \ell}$ if k is odd.*

In Section 4.6, we will further expand upon Theorem 4.1. In particular, we prove a lower bound and give a simpler proof for the cases $k = 4$ and $k = 6$.

4.1.2 Arithmetic Progressions

For a subset $S \subset \mathbb{N}$ and integers $k \geqslant 3$ and $\ell \geqslant 2$, we write $S \to (AP_k)_\ell$ if every ℓ-coloring of the integers in S yields a monochromatic arithmetic progression of length k. Van der Waerden's Theorem establishes that for all $k \geqslant 3$ and $\ell \geqslant 2$, there exists some integer N such that $[N] \to (AP_k)_\ell$, where $[N] = \{1, 2, \ldots, N\}$. Many generalizations of this well-known theorem have been considered. One generalization suggested by Erdős [28], asks if for all $k \geqslant 3$ and $\ell \geqslant 2$, there exists an AP_{k+1}-free set $S \subset \mathbb{N}$ that has the Ramsey property $S \to (AP_k)_\ell$, where a set is AP_{k+1}-free if it does not contain an arithmetic progression of length $k+1$. This was answered independently by Spencer [72] and by Nešetřil and Rödl [54]. Moreover, Graham and Nešetřil [41] showed that there exist arbitrarily large AP_{k+1}-free sets S that have the property $S \to (AP_k)_\ell$ and are minimal in the sense that, for every $s \in S$, the subset $S' = S \backslash \{s\}$ does not have the property $S' \to (AP_k)_\ell$.

Furthermore, one may want to restrict the structure of the arithmetic progressions of length k in a set $S \subset \mathbb{N}$, but keep the Ramsey property. That is, consider the *system of copies* of arithmetic progressions of length k in S, which is the k-uniform hypergraph $(S, \binom{S}{AP_k})$ on the vertex set S with edge set $\binom{S}{AP_k}$ consisting of the k element subsets of S that form arithmetic progressions of length k. For a simpler notation, it will be convenient to identify this hypergraph just by its edge set. Moreover, we denote its chromatic number simply by $\chi\binom{S}{AP_k}$ instead of $\chi(\binom{S}{AP_k})$ and suppress the outer pair of parenthesis for other numerical hypergraph parameters as well.

Observe that $S \to (AP_k)_\ell$ if and only if the chromatic number satisfies $\chi\binom{S}{AP_k} > \ell$. Hence, van der Waerden's Theorem establishes that for fixed k, the $\chi\binom{[N]}{AP_k} \to \infty$ as N tends to infinity. In view of the result of Erdős and Hajnal [32], which establishes the existence of hypergraphs having both large chromatic number and large girth, it is naturally to ask the following. Does for all k, $g \geqslant 3$ and $\ell \geqslant 2$ exist a set $S \subset \mathbb{N}$ so that the hypergraph $\binom{S}{AP_k}$ satisfies both properties

(P1) $\chi\binom{S}{AP_k} > \ell$,

(P2) $\mathrm{girth}\binom{S}{AP_k} \geqslant g$?

As usual we say a k-uniform hypergraph has girth at least g if, for any integer h with $2 \leqslant h < g$, any subset of h edges span at least $(k-1)h+1$ vertices. In particular, $\mathrm{girth}\binom{S}{AP_k} \geqslant 3$ implies that no two arithmetic progressions can intersect in more than one point, which implies that S is AP_{k+1}-free. The existence of sets $S \subset \mathbb{N}$ satisfying properties (P1) and (P2) was established in [61] (see also [62]) and our next result gives an upper bounds for the size of the smallest such set S.

Theorem 4.3. *Let $w = w_\ell(k)$ denote the least integer N such that $[N] \to (AP_k)_\ell$. Then for all integers k, $g \geqslant 3$, and $\ell \geqslant 2$, there exists a set $S \subset \mathbb{N}$ such that*

$$\chi\binom{S}{AP_k} > \ell, \quad \operatorname{girth}\binom{S}{AP_k} \geqslant g, \quad \text{and} \quad |S| \leqslant k^{400k^2(k+g)} w^{400k(k+g)} g^{8kg}.$$

To illustrate the result, consider the special case $k = 3$ for fixed $g \geqslant 3$. A result of Sanders [65] (see also [9]) implies that $w_\ell(3) \leqslant \exp\left(\ell^{1+o(1)}\right)$, where the error term $o(1) \to 0$ as $\ell \to \infty$. Hence, our result yields the existence of a set S of size at most $\exp\left(\ell^{1+o(1)}\right)$ such that the properties $S \to (AP_3)_\ell$ and $\binom{S}{AP_3} \geqslant g$ both hold. It follows that the added girth condition does not essentially increase the best known upper bound in this case.

4.1.3 Cliques in Graphs

Another well-known problem of Erdős and Hajnal [33] asked if, for every pair of positive integers k and ℓ, there exists a K_{k+1}-free graph H such that $H \to (K_k)_\ell$. The case $\ell = 2$ was confirmed by Folkman [36] and Nešetřil and Rödl [53] resolved the general case $\ell > 2$. Subsequently, Erdős [27] asked about a strengthened form of this result, namely the existence of a graph H with $H \to (K_k)_\ell$ in which no two copies of K_k share more than one edge, which was established in [55] (see also [56] for a generalization from cliques K_k to arbitrary graphs). .

As in the context of van der Waerden's theorem in Section 4.1.2, we may consider the structure of the cliques in H in more detail, that is, we consider the *system of copies* of K_k in H, which is the $\binom{k}{2}$-uniform hypergraph $(E(H), \binom{H}{K_k})$ having vertex set $E(H)$ and hyperedges corresponding to the edge sets of copies of K_k in H. As above we identify this hypergraph by its edge set $\binom{H}{K_k}$ and denote by $\chi\binom{H}{K_k}$ and $\mathrm{girth}\binom{H}{K_k}$ its chromatic number and its girth. Again the statement $H \to (K_k)_\ell$ is equivalent to $\chi\binom{H}{K_k} > \ell$ and that any two copies of K_k in H share at most one edge is equivalent to $\mathrm{girth}\binom{H}{K_k} \geqslant 3$. We give a new proof of the result from [55] that leads to a new upper bound on the size of the smallest such H.

Theorem 4.4. *Let $r := r_\ell(k)$ denote the least integer m such that $K_m \to (K_k)_\ell$. Then for all integers k, $g \geqslant 3$, and $\ell \geqslant 2$, there exists a graph H such that*

$$\chi\binom{E(H)}{K_k} > \ell, \quad \mathrm{girth}\binom{E(H)}{K_k} \geqslant g, \quad \textit{and} \quad |V(H)| \leqslant k^{2^{10}k^4g^2} r^{8k^2g}.$$

By reversing the dependency between g and $|V(H)|$, we obtain the following corollary.

Corollary 4.5. *For all integers $k \geqslant 3$ and $\ell \geqslant 2$, there exist $c > 0$ and n_0 such that, for every integer $n \geqslant n_0$, there exists a graph H on n vertices satisfying both $H \to (K_k)_\ell$ and $\mathrm{girth}\binom{H}{K_k} \geqslant c\sqrt{\log n}$.*

It can be shown that any graph H on n vertices satisfying $H \to (K_k)_\ell$ must also satisfy $\mathrm{girth}\binom{H}{K_k} = O(\log n)$, due to the minimum degree condition required by $\chi\binom{H}{K_k} > \ell$.

4.1.4 Organization

The proofs of Theorems 4.1, 4.3, and 4.4 rely on random constructions and the *container method* obtained independently by Balogh, Morris, and Samotij [5] and Saxton and Thomason [66]. Also, we incorporate some ideas from [52] [63]. For the numerical aspects the container result from [66] seemed to be better suited and we state it in Section 4.2. The details of the proofs of Theorems 4.1, 4.3, and 4.4 are given in Sections 4.3, 4.4, and 4.5, respectively. Section 4.6 contains some concluding remarks related to mostly to Theorem 4.1.

4.2 Hypergraph Containers

The proofs of Theorems 4.1, 4.3, and 4.4 are based upon a lemma of Saxton and Thomason. Roughly speaking, this lemma states that, if some numerical conditions are satisfied for a hypergraph $\mathcal{H}$, then there there exists a relatively 'small' set $\mathfrak{C}$ whose elements (called 'containers') are 'almost' independent sets of vertices, which together have the property that each independent set of vertices in the hypergraph $\mathcal{H}$ is a subset of one of the containers in $\mathfrak{C}$.

We now introduce some definitions and notation necessary for the precise formulation of this lemma. For a hypergraph $\mathcal{H}$, let $e(\mathcal{H})$ denote the number of edges in $\mathcal{H}$ and d denote the average degree of a vertex in $\mathcal{H}$ i.e. $d = k \cdot e(\mathcal{H})/|V(\mathcal{H})|$. For a set $S \subseteq V(\mathcal{H})$, define the degree of S by $d(S) := |\{E : E \in E(\mathcal{H}) \text{ and } E \supseteq S\}|$. Also, for each $v \in V(\mathcal{H}$, define

$$d_j(v) := \max\left\{d(S) : S \in \tbinom{V(H)}{j} \text{ and } v \in S\right\}, \qquad \text{and}$$

$$d_j := \tfrac{1}{|V(\mathcal{H})|} \textstyle\sum_{v \in V} d_j(v).$$

Container Lemma (follows from [66])**.** *Let $\mathcal{H}$ be a h-uniform hypergraph on the vertex set $[N]$ and $\tau, \varepsilon \in \mathbb{R}^+$ with $\tau < 1$. If*

$$\frac{6 \cdot h! \cdot 2^{\binom{h}{2}}}{d} \sum_{j=2}^{h} \frac{d_j}{2^{\binom{j-1}{2}} \tau^{j-1}} \leqslant \varepsilon < 1/2, \tag{4.1}$$

then there exists a collection $\mathfrak{C} \subset \mathcal{P}([N])$ of 'containers' such that all of the following hold.

i) For every independent set $I \subset V(\mathcal{H})$, there exists some $\mathcal{C} \in \mathfrak{C}$ with $\mathcal{C} \supseteq I$.

ii) For every $\mathcal{C} \in \mathfrak{C}$, the number of edges in the container satisfies $e(\mathcal{C}) \leqslant \varepsilon \cdot e(\mathcal{H})$.

iii) The number of containers satisfies $|\mathfrak{C}| \leqslant \exp\Big(1000h(h!)^3 \cdot \log{(1/\varepsilon)} \cdot N \cdot \tau \cdot \log{(1/\tau)}\Big)$.

Proof. This lemma can be deduced from Corollary 2.7 in [66]. The explicit choice of the constant c appearing in Corollary 2.7, which the statement of Corollary 2.7 guarantees only to exist, is taken to be $c := 1000h(h!)^3$; it was observed in Section 2.1 of [63] that this choice of c follows from the proof of Corollary 2.7 by noting that

$$c \leqslant \frac{288(h!)^2 h}{\ln(1/\varepsilon)} \left(1 + \frac{\ln \varepsilon}{\ln{(1 - 1/2h!)}}\right) \leqslant 1000h(h!)^3.$$

Additionally, our hypothesis on ε is equivalent to the hypothesis on ε as stated in Corollary 2.7, which can easily been seen by writing out the definition of the 'co-degree' function in Corollary 2.7 and our definition of d_j. We have also omitted the hypothesis that $\tau \leqslant 1/(144h!^2 h)$ since, for $1 > \tau > 1/(144h!^2 h)$, the conclusion follows vacuously from taking $\mathfrak{C}$ to be the set of all independent sets. □

4.3 Proof of Theorem 4.1

The objective of this section is to prove Theorem 4.1. That is, we will show that there exists a graph H with $\mathrm{girth}(H) = k$ and the Ramsey property $H \rightarrow (C_k)_\ell$ that has at most $r^{40k^2} k^{40k^3}$ vertices, where $r = r_\ell(C_k)$ is the ℓ-color Ramsey number for C_k.

Proof. Consider any integers $k \geqslant 4$ and $\ell \geqslant 2$ and set

$$n := 2^{40k^2} r^{40k^2} k^{10k^3}, \qquad c_p := 2^{12} r^{10} k^{3k+2} \log n,$$

$$p := c_p n^{-1+1/(k-1)} = c_p n^{-(k-2)/(k-1)}.$$

Let $G \sim G(n,p)$ be a random instance of the graph obtained from K_n (the complete graph on vertex set $[n]$) by selecting each edge independently with probability p. Theorem 4.1 will be an immediate consequence of the following three claims.

Claim 4.6. $\mathbb{P}\Big(G \to (C_k)_\ell\Big) \geqslant 1 - \exp\left(-2^{-1}r^{-2}p\binom{n}{2}\right)$.

Claim 4.7. $\mathbb{P}\Big(\mathrm{girth}(G) \geqslant k\Big) \geqslant \exp\left(-c_p^{k-1}n\right)$.

Claim 4.8. $\exp\left(-c_p^{k-1}n\right) - \exp\left(-2^{-1}r^{-2}p\binom{n}{2}\right) > 0$.

Indeed, in view of Claim 4.8, the probability that G has girth at least k is greater than the probability G does not have the Ramsey property $G \to (C_k)_\ell$. Hence, verifying these three claims will establish that there exists a graph having girth k that is Ramsey for C_k for ℓ-colors that has $n = 2^{40k^2}r^{40k^2}k^{10k^3} \leqslant r^{40k^2}k^{40k^3}$ vertices as desired. The first claim, which we now address, is the crux of our proof.

Proof of Claim 4.6. Consider $\mathcal{H} = \binom{E(K_n)}{C_k}$, which is the the system of all copies of C_k in K_n. It follows that $\mathcal{H}$ is a k-uniform hypergraph on $\binom{n}{2}$ vertices having $\binom{n}{k}(k-1)!/2$ edges. Set

$$\tau := r^6 n^{-1+1/(k-1)} = r^6 n^{-(k-2)/(k-1)}, \qquad \varepsilon := \frac{1}{2\ell\binom{r}{k}(k-1)!}.$$

In order to apply the Container Lemma to the deterministic hypergraph $\mathcal{H}$, we note that $0 \leqslant \varepsilon \leqslant 1/2$ and $\tau \leqslant 1$, which follows directly from the definitions of ε and τ. We will now work to verify the remaining hypothesis (4.1) of the Container Lemma. Observe that, for all $j < k$, we have $d_j = \binom{n-(j+1)}{k-(j+1)}(k-(j+1))! \leqslant n^{k-j-1}$; this is because a set S of j edges in K_n is contained in the most cycles of length k when S forms a path in K_n. Also, observe that $d = \binom{n-2}{k-2}(k-2)! \geqslant (n/2)^{k-2}$. It follows that,

$$\begin{aligned}
\frac{1}{d}\sum_{j=2}^{k-1}\frac{d_j}{2^{\binom{j-1}{2}}\tau^{j-1}} &\leqslant \frac{1}{(n/2)^{k-2}}\sum_{j=2}^{k-1}\frac{n^{k-j-1}}{\tau^{j-1}} \\
&= 2^{k-2}\sum_{j=2}^{k-1}\frac{1}{(n\tau)^{j-1}} \\
&\leqslant 2^{k-2}\cdot(k-2)\cdot\frac{1}{n\tau} \\
&= \frac{2^{k-2}(k-2)}{r^6 n^{1/(k-1)}} \leqslant \frac{2^k}{n^{1/k}}.
\end{aligned} \tag{4.2}$$

Now observe that $d_k \leqslant 1$; this is because any k edges in K_n are contained in at most one cycle C_k. Hence,

$$\begin{aligned}
\frac{2^{\binom{k}{2}}\cdot d_k}{d\cdot 2^{\binom{k-1}{2}}\cdot\tau^{k-1}} &\leqslant \frac{2^{\binom{k}{2}}\cdot 1}{(n/2)^{k-2}\cdot 2^{\binom{k-1}{2}}\cdot\tau^{k-1}} \\
&= \frac{2^{2k-3}}{n^{k-2}\tau^{k-1}} = \frac{2^{2k-3}}{r^{6k-6}}.
\end{aligned} \tag{4.3}$$

Also, making use of the fact that the Ramsey number $r = r_\ell(C_k)$ satisfies both $r > \ell$ and $r > k$ (a fact we will utilize through this section), we have that

$$\frac{6\cdot k!}{\varepsilon} = 6\cdot k!\cdot 2\ell\binom{r}{k}(k-1)! \leqslant r^{3k}. \tag{4.4}$$

Using equations (4.2), (4.3), and (4.4), we now verify equation (4.1) in the Container Lemma:

$$
\begin{aligned}
\frac{6\cdot k!\cdot 2^{\binom{k}{2}}}{d\cdot\varepsilon}\sum_{j=2}^{k}\frac{d_j}{2^{\binom{j-1}{2}}\tau^{j-1}} &\leqslant \frac{r^{3k}2^{\binom{k}{2}}}{d}\sum_{j=2}^{k}\frac{d_j}{2^{\binom{j-1}{2}}\tau^{j-1}}\\
&\leqslant r^{3k}\cdot\frac{2^{\binom{k}{2}}d_k}{d2^{\binom{k-1}{2}}\tau^{k-1}}+r^{3k}2^{\binom{k}{2}}\cdot\frac{1}{d}\sum_{j=2}^{k-1}\frac{d_j}{2^{\binom{j-1}{2}}\tau^{j-1}}\\
&\leqslant r^{3k}\cdot\frac{2^{2k-3}}{r^{6k-6}}+r^{3k}2^{\binom{k}{2}}\cdot\frac{2^k}{n^{1/k}}\\
&\leqslant \frac{2^{2k}}{r^{3k-6}}+\frac{r^{3k}2^{k^2}}{n^{1/k}}\leqslant\frac{4^k}{4^{3k-6}}+\frac{r^{3k}2^{k^2}}{n^{1/k}}<1.
\end{aligned}
$$

Having verified the hypotheses of the Container Lemma, we obtain a set $\mathfrak{C}\subseteq\mathcal{P}(E(K_n))$ of containers such that every independent set in $\mathcal{H}$ is contained in some container, each container has at most $\varepsilon|E(\mathcal{H})|=\varepsilon\binom{n}{k}(k-1)!/2$ hyperedges, and

$$
\begin{aligned}
|\mathfrak{C}| &\leqslant \exp\left(1000\cdot k\cdot(k!)^3\cdot\log(1/\varepsilon)\cdot\binom{n}{2}\cdot\tau\cdot\log(1/\tau)\right)\\
&\leqslant \exp\left(2^{10}\cdot k\cdot k^{3k}\cdot 2kr\cdot\binom{n}{2}\cdot r^6 n^{-(k-2)/(k-1)}\cdot\log n\right)\\
&= \exp\left(2^{11}k^{3k+2}r^7\binom{n}{2}(p/c_p)\log n\right)\\
&= \exp\left(\frac{1}{2r^3}p\binom{n}{2}\right)\leqslant\exp\left(\frac{1}{2\ell r^2}p\binom{n}{2}\right). \qquad (4.5)
\end{aligned}
$$

Now let $\mathcal{B}$ be the set of all ('bad') subgraphs of K_n that are not Ramsey for C_k for ℓ colors. Our goal is to show $\mathbb{P}(G\in\mathcal{B})\leqslant\exp\left(-2^{-1}r^{-2}p\binom{n}{2}\right)$.

For each $B\in\mathcal{B}$, fix an edge coloring $\chi_B: E(B)\to[\ell]$ such that each color class does not induce a monochromatic C_k. For each $i\in[\ell]$, the set $\chi^{-1}(i)$ is not only a set of edges in K_n that does not induce a cycle C_k, but also corresponds to an independent set of vertices in $\mathcal{H}$, and thus $\chi^{-1}(i)$ is contained in some container $\mathcal{C}_i \subseteq V(\mathcal{H})$. With this in mind, associate each $B \in \mathcal{B}$ to some $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\ell) \in \mathfrak{C}^\ell$ so that, for all $i \in [\ell]$, we have $\chi_B^{-1}(i) \subseteq \mathcal{C}_i$. Also, for each each $B \in \mathcal{B}$, define the associated set $D := V(\mathcal{H}) \setminus \bigcup_{i=1}^{\ell} \mathcal{C}_i$ of edges in K_n. Observe that $E(B) \cap D = \varnothing$ as $E(B) \subseteq \bigcup_{i=1}^{\ell} \mathcal{C}_i$.

Let $\mathcal{D}$ be the set of all sets D arising this way by considering each $B \in \mathcal{B}$. Thus, it is the case that, for every graph $B \in \mathcal{B}$, there exists some associated $D \in \mathcal{D}$ such that $E(B) \cap D = \varnothing$.

Consequently, by the union bound and fact that $|\mathcal{D}| \leqslant |\mathfrak{C}|^\ell$, we have

$$\begin{aligned}\mathbb{P}(G \in \mathcal{B}) &\leqslant \mathbb{P}\Big(\exists\ D \in \mathcal{D} : E(G) \cap D = \varnothing\Big) \\ &\leqslant |\mathfrak{C}|^\ell \max\Big\{\mathbb{P}(E(G) \cap D = \varnothing) : D \in \mathcal{D}\Big\}. \end{aligned} \tag{4.6}$$

We will work to deduce that $\mathbb{P}(E(G) \cap D = \varnothing)$ is small for all $D \in \mathcal{D}$ as a consequence of the fact that the number of elements in D that $E(G)$ must avoid is large.

Subclaim 4.9. *For every $D \in \mathcal{D}$, at least half of the r-elements sets $R \in \binom{[n]}{r}$ have the property that $E(K_n[R]) \cap D \neq \varnothing$, where $K_n[R]$ is the complete graph on R.*

Proof of Subclaim 4.9. Fix some $B \in \mathcal{B}$ and associated $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\ell) \in \mathfrak{C}^\ell$ that gave rise to D. To prove the claim, assume to the contrary that at least half of the r-element sets $R \in \binom{[n]}{r}$ have the property that $K_n[R] \cap D = \varnothing$. For each of these sets R, it follows that $E(K_n[R]) \subset \bigcup_{i \in [\ell]} \mathcal{C}_i$. This corresponds to an ℓ-coloring of the edges of $K_n[R]$ and, by the definition of the Ramsey number $r = r_\ell(C_k)$, this coloring yields a monochromatic C_k. That is, we have argued that half of the r-element sets $R \in \binom{[n]}{r}$ contain a cycle $C_k \in K_n[R]$ such that $E(C_k) \subset \mathcal{C}_i$ for some i, i.e. the set of edges $E(C_k)$ is a edge in the hypergraph $\mathcal{H}[\mathcal{C}_i]$. On the other hand, by the Container Lemma, we know that each container $\mathcal{C}_i$ contains at most $\varepsilon|E(\mathcal{H})| = \varepsilon\binom{n}{k}(k-1)!/2$ edges.

We thus have:

$$\frac{(1/2)\binom{n}{r}}{\binom{n-k}{r-k}} \leqslant \left|\left\{C_k : E(C_k) \subseteq \mathcal{C}_i \text{ for some } i \in [\ell]\right\}\right| \leqslant \ell\varepsilon\binom{n}{k}\frac{(k-1)!}{2} = \frac{(1/4)\binom{n}{k}}{\binom{r}{k}},$$

where the lower bound was computed using the fact that for each cycle C_k, we have that $E(C_k) \subset E(K_n[R])$ for at most at most $\binom{n-k}{r-k}$ subsets $R \in \binom{[n]}{r}$. A contradiction to the above inequality follows from the combinatorial identity that $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$. $\square$

By Subclaim 4.9 and the fact that an edge is contained in at most $\binom{n-2}{r-2}$ sets in $\binom{[n]}{r}$, for all $D \in \mathcal{D}$, we have

$$|D| \geqslant \frac{(1/2)\binom{n}{r}}{\binom{n-2}{r-2}} = \frac{\binom{n}{2}}{2\binom{r}{2}} \geqslant r^{-2}\binom{n}{2}.$$

Hence,

$$\mathbb{P}\Big(E(G) \cap D = \varnothing\Big) \leqslant (1-p)^{r^{-2}\binom{n}{2}} \leqslant e^{-r^{-2}p\binom{n}{2}}. \tag{4.7}$$

From substituting (4.7) and (4.5) into (4.6), it follows that

$$\mathbb{P}\Big(G \in \mathcal{B}\Big) \leqslant \exp\left(\frac{1}{2r^2}p\binom{n}{2} - \frac{1}{r^2}p\binom{n}{2}\right) \leqslant \exp\left(-\frac{1}{2r^2}p\binom{n}{2}\right).$$

This completes the proof of Claim 4.6.

$\square$

Proof of Claim 4.7. To find a lower bound for the probability that $G(n,p)$ has girth at least k, we will use the standard FKG inequality as appearing in Corollary 2.13 of [45]. For this purpose, define $\mathcal{S}$ to be the set of all cycles of length less than k in K_n. For each $S \in \mathcal{S}$ and graph $\tilde{G} \subset K_n$, let $X_S(\tilde{G})$ be the indicator function with $X_S(\tilde{G}) = 1$ if S appears in $\tilde{G}$

and $X_S(\tilde{G}) = 0$ otherwise. Thus, $X(\tilde{G}) := \sum_{S \in \mathcal{S}} X_S(\tilde{G})$ counts the number of cycles of length less than k appearing in $\tilde{G}$. Moreover, since each function X_S is an increasing function of the space of all graphs on n vertices (i.e. if $\tilde{G}' \subset \tilde{G}$ then $X_S(\tilde{G}') \leqslant X_S(\tilde{G})$), by Corollary 2.13 of [45] (which follows directly from the FKG inequality), we obtain that for $G \sim G(n,p)$ we have $\mathbb{P}(X(G) = 0) \geqslant \exp\left(-\mathbb{E}(X(G))/(1-p)\right)$. We now compute

$$\begin{aligned}
\mathbb{E}\Big(X(G)\Big) = \sum_{j=3}^{k-1} \binom{n}{j} \frac{(j-1)!}{2} p^j &\leqslant \sum_{j=3}^{k-1} \frac{(pn)^j}{2j} \\
&\leqslant (k-3)\frac{(pn)^{k-1}}{2(k-1)} \leqslant (pn)^{k-1} = c_p^{k-1} n,
\end{aligned}$$

which gives

$$\mathbb{P}\Big(\mathrm{girth}(G) \geqslant k\Big) = \mathbb{P}\Big(X(G) = 0\Big) \geqslant \exp\left(-\frac{\mathbb{E}(X(G))}{1-p}\right) \geqslant \exp\left(-c_p^{k-1} n\right),$$

completing the proof of Claim 4.7.

$\square$

Proof of Claim 4.8. To show that $\exp\left(-2^{-1}r^{-2}p\binom{n}{2}\right) < \exp\left(-c_p^{k-1}n\right)$, it suffices to show that $nc_p^{k-1}/(2^{-1}r^{-2}p\binom{n}{2}) < 1$, which we now verify.

$$\begin{aligned}
\frac{nc_p^{k-1}}{2^{-1}r^{-2}p\binom{n}{2}} &= \frac{2r^2c_p^{k-2}n^2}{n^{1/(k-1)}\binom{n}{2}} \leqslant \frac{2^3r^2c_p^{k-2}}{n^{1/(k-1)}} \\
&= \frac{2^3r^2 2^{12k-24} r^{10k-20} k^{3k^2-4k-4} (\log n)^k}{2^{40k} r^{40k} k^{10k^2}} < 1.
\end{aligned}$$

$\square$

Having proved Claims 4.6,4.7, and 4.8, this completes the proof of Theorem 4.1.

$\square$

4.4 Proof of Theorem 4.3

In this section, we prove Theorem 4.3 by establishing that, for all integers $k \geqslant 3$, $\ell \geqslant 2$, and $g \geqslant 2$, there exists a set $S \subset \mathbb{N}$ of size at most $k^{400k^2(k+g)} w^{400k(k+g)} g^{8kg}$ such that the hypergraph $\binom{S}{AP_k}$ has chromatic number greater than ℓ and girth at least g, where the van der Waerden number $w = w_\ell(k)$ is the least integer N such that $[N] \to (AP_k)_\ell$.

Proof. Consider any three integers $k \geqslant 3$, $\ell \geqslant 2$, and $g \geqslant 2$. Let $w := w_\ell(k)$ be the least integer N such that $[N] \to (AP_k)_\ell$. Set

$$n := k^{400k^2(k+g)} w^{400k(k+g)} g^{8kg}, \qquad c_p := w^{25} k^{25k} \log n,$$

$$p := c_p n^{-1/(k-1)} = c_p n^{-1+(k-2)/(k-1)}, \qquad t = \frac{pn}{24w}. \tag{4.8}$$

Let $[n]_p$ denote the random set obtained by choosing each element of $[n] = \{1, 2, \ldots, n\}$ independently with probability p. (Since $c_p^{k-1} < n$, we have that $p = c_p n^{-1/(k-1)} < 1$ and is a valid choice for a probability.) We make three claims about the random hypergraph $\binom{[n]_p}{AP_k}$. The first claim asserts that $\binom{[n]_p}{AP_k}$ will have chromatic number greater than ℓ even after the deletion of any set of t vertices.

Claim 4.10. *With probability at least* $1 - 2\exp(-pn/288w)$*, the hypergraph* $\binom{[n]_p}{AP_k}$ *has the following strong Ramsey property: if* $T \subset [n]_p$ *is any subset of* t *elements, then* $\chi\binom{[n]_p \setminus T}{AP_k} > \ell$.

The next claim asserts that it is likely that $\binom{[n]_p}{AP_k}$ can, by deleting some set of t vertices, be made to have girth at least g.

Claim 4.11. *The probability that* $\binom{[n]_p}{AP_k}$ *has fewer than* t *cycles of length less that* g *is greater than* $1/2$.

The following claim compares the probabilities in the previous two claims.

Claim 4.12. $1 - 2\exp(-pn/288w) - 1/2 > 0$.

Together, these claims establish that, with positive probability, the random hypergraph $\binom{[n]_p}{AP_k}$ will have the property that there exists a set T of t vertices so that the hypergraph $\binom{[n]_p \setminus T}{AP_k}$ has girth at least g and $\chi\binom{[n]_p \setminus T}{AP_k} > \ell$. Thus, these claims together establish the existence of a graph as in Theorem 4.3. We remark that, although such a object will likely have around $pn-t$ vertices (not n vertices), this improvement is negligible. We begin by proving Claim 4.10 which is the heart of our proof. The proofs of Claims 4.11 and 4.12 will be given subsequently.

Proof of Claim 4.10. We first consider the hypergraph $\mathcal{H} := \binom{[n]}{AP_k}$ and apply the Container Lemma to this deterministic hypergraph with

$$\varepsilon = w^{-5}, \qquad c_\tau = (2^{k^2+4}k^{k+3}w^5)^{1/(k-1)}, \qquad \text{and} \qquad \tau = c_\tau n^{-1/(k-1)}.$$

Note that $\tau < 1$ and $\varepsilon < 1/2$. To verify the remaining assumption (4.1) of the Container Lemma for $\mathcal{H}$, we make a couple of observations about the average degree d and the parameter d_j, which is defined in the statement of the Container Lemma. First,

$$e(\mathcal{H}) = \sum_{i=1}^{n-k+1} \left\lfloor \frac{n-i}{k-1} \right\rfloor \geqslant \frac{n^2}{3k}; \tag{4.9}$$

here we count the number of arithmetic progressions of length k in $[n]$ by summing over the position of the first element in the progression and the above inequality holds for all $k \leqslant n/6$. Second, $d \geqslant n/3$, which follows from (4.9) and $\mathcal{H}$ being a k-uniform hypergraph. Also, in $\mathcal{H}$ the parameter d_j

(as defined in the statement of the Container Lemma) satisfies

$$d_j \leqslant d_2 \leqslant \binom{k}{2} \leqslant \frac{k^2}{2}; \tag{4.10}$$

the second inequality above follows from the fact that an arithmetic progression containing the elements $u, v \in [n]$ is determined by specifying the positions of u and v in the progression.

We now verify the remaining hypothesis (4.1) of the Container Lemma. Recalling that $d \geqslant n/3$ and $d_j \leqslant k^2/2$, we have

$$\begin{aligned}
\frac{6 \cdot k! \cdot 2^{\binom{k}{2}}}{\varepsilon \cdot d} \sum_{j=2}^{k} \frac{d_j}{2^{\binom{j-1}{2}} \tau^{j-1}} &\leqslant \frac{6 \cdot k^k \cdot 2^{k^2}}{w^{-5} \cdot n/3} \sum_{j=2}^{k} \frac{k^2/2}{\tau^{j-1}} \\
&= \frac{9k^{k+2} 2^{k^2} w^5}{n} \sum_{j=2}^{k} \frac{1}{\tau^{j-1}} \\
&\leqslant \frac{2^4 k^{k+2} 2^{k^2} w^5}{n} \cdot k \cdot \frac{1}{\tau^{k-1}} \\
&= \frac{2^{k^2+4} k^{k+3} w^5}{c_\tau^{k-1}} = 1.
\end{aligned}$$

Having verified the assumptions of the Container Lemma, we obtain a set $\mathfrak{C} \subseteq \mathcal{P}([n])$ of containers such that every independent set (i.e. AP_k-free set) is contained in some container, each container has at most $\varepsilon|E(\mathcal{H})| \leqslant$

$\varepsilon\binom{n}{2}$ hyperedges, and

$$\begin{aligned}
|\mathfrak{C}| &\leqslant \exp\big(1000 \cdot k(k!)^3 \cdot \log(1/\varepsilon) \cdot n \cdot \tau \cdot \log(1/\tau)\big) \\
&\leqslant \exp\big(2^{10} \cdot k^{3k+1} \cdot 2^3 w \cdot n \cdot \tau \cdot \log n\big) \\
&= \exp\left(\frac{pn}{2^9 w\ell} \cdot 2^{22} w^2 \ell k^{3k+1} \cdot \frac{\tau \log n}{p}\right) \\
&= \exp\left(\frac{pn}{2^9 w\ell} \cdot 2^{22} w^2 \ell k^{3k+1} \cdot \frac{(2^{k^2+4} k^{k+3} w^5)^{1/(k-1)}}{w^{25} k^{25k}}\right) \\
&\leqslant \exp\left(\frac{pn}{288 w\ell}\right). \qquad (4.11)
\end{aligned}$$

Let $\mathcal{B}$ denote the family of all sets $B \subset [n]$ with the property that there exists a subset $T \subset B$ of size t such that $(B \backslash T) \nrightarrow (AP_k)_\ell$ i.e. there exists an ℓ-coloring of $B \backslash T$ that does not yield a monochromatic AP_k. Hence $\mathcal{B}$ is the set of all ('bad') subset of $[n]$ that do not have the desired strong Ramsey property. We will work to bound $\mathbb{P}([n]_p \in \mathcal{B})$ from above.

To this end, consider any $B \in \mathcal{B}$. By the definition of $\mathcal{B}$, there exists a set T with $|T| = t$ and a partition $B \backslash T = I_1 \cup \cdots \cup I_\ell$ with the property that each I_i contains no AP_k. Moreover, each I_i is an independent set in $\mathcal{H}$, so there exists an ℓ-tuple of containers $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\ell) \in \mathfrak{C}^\ell$ such that $I_i \subset \mathcal{C}_i$ for all $i \in [\ell]$. Letting $D := [n] \backslash \bigcup_{i \in [\ell]} \mathcal{C}_i$, observe that $|B \cap D| \leqslant |T|$ since $B \backslash T = I_1 \cup \cdots \cup I_\ell \subset D^C$. Hence, we can define a function from $\mathcal{B}$ to the set

$$\mathcal{D} = \left\{D : D = [n] \backslash \bigcup_{i \in [\ell]} \mathcal{C}_i \text{ for some } (\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\ell) \in \mathfrak{C}^\ell\right\},$$

with the property that, for any associated pair $(B, D) \in (\mathcal{B}, \mathcal{D})$, we have $|B \cap D| \leqslant t$.

From this we infer that if $[n]_p \in \mathcal{B}$, then there exists some $D \in \mathcal{D}$ such that $|[n]_p \cap D| \leqslant t$. Using the union bound this yields:

$$\begin{aligned}\mathbb{P}\Big([n]_p \in \mathcal{B}\Big) &\leqslant \mathbb{P}\Big(\exists\ D \in \mathcal{D} : |[n]_p \cap D| \leqslant t\Big) \\ &\leqslant |\mathfrak{C}|^{\ell} \cdot \max\Big\{\mathbb{P}\Big(|[n]_p \cap D| \leqslant t\Big) : D \in \mathcal{D}\Big\}. \qquad (4.12)\end{aligned}$$

To show the above probability is small, we will work to show that $|D|$ is large for every $D \in \mathcal{D}$. Let $\mathcal{P}_w$ be the set of all arithmetic progressions in $[n]$ of length w.

Subclaim 4.13. *For every $D \in \mathcal{D}$, half of the elements $P \in \mathcal{P}_w$ contain an element from D.*

Proof of Subclaim 4.13. Consider any fixed $D \in \mathcal{D}$. Towards contradiction, suppose it is not the case that half of the elements $P \in \mathcal{P}_w$ contain an element from D, i.e. there are $|\mathcal{P}_w|/2$ progressions $P \in \mathcal{P}_w$ such that $P \cap D = \varnothing$.

By the definition of $\mathcal{D}$, there must exist $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\ell) \in \mathfrak{C}^\ell$ so that $D = [n]\backslash \bigcup_{i\in[\ell]} \mathcal{C}_i$. For any $P \in \mathcal{P}_w$ with $P \cap D = \varnothing$, we have $P \subset \bigcup_{i\in[\ell]} \mathcal{C}_i$. By the definition of the van der Waerden number $w = w_\ell(k) = |P|$, this implies there exists $i \in [\ell]$ and $AP_k \subset P \cap \mathcal{C}_i$. Hence, $|\mathcal{P}_w|/2$ progressions $P \in \mathcal{P}_w$ have the property that for some $i \in [\ell]$, the set $P \cap \mathcal{C}_i$ contains an AP_k. Furthermore, any AP_k is contained in at most $\binom{w}{2}$ progressions of length w (corollary of (4.10)). It follows that

$$\sum_{i\in[\ell]} e(\mathcal{C}_i) \geqslant \frac{|\mathcal{P}_w|/2}{\binom{w}{2}}. \qquad (4.13)$$

Now, observe that $|\mathcal{P}_w| \geqslant n^2/3w$ (having $w \leqslant n/6$, this is analogous to (4.9)). Also, observe that $\binom{n}{2} \geqslant e(\mathcal{H})$ because an arithmetic progression is defined by choosing its first two elements. From these two observations, inequality (4.13) gives

$$\sum_{i\in[\ell]} e(\mathcal{C}_i) \geqslant \frac{|\mathcal{P}_w|/2}{\binom{w}{2}} > \frac{(n^2/3w)/2}{w^2/2} \geqslant \frac{n^2}{3w^2} \geqslant \frac{e(\mathcal{H})}{3w^3} \geqslant \ell\varepsilon \cdot e(\mathcal{H}),$$

which contradicts property (ii) of the Container Lemma that states $e(\mathcal{C}_i) \leqslant \varepsilon \cdot e(\mathcal{H})$ for every $\mathcal{C}_i \in \mathfrak{C}$. □

Resuming the proof of Claim 4.10, we will use Subclaim 4.13, which we have just proved, to establish that $|D|$ is large for every $D \in \mathcal{D}$. To this end, note that each element in $v \in [n]$ is contained in at most $2n$ arithmetic progression of length w (there are at most w choices for which element of the AP_w intersects v and at most $n/(w-1)$ choices for the distance between consecutive elements in the AP_w). Thus, Subclaim 4.13 and (4.9) give that, for every $D \in \mathcal{D}$,

$$|D| \geqslant \frac{|\mathcal{P}_w|/2}{2n} \geqslant \frac{(n^2/3w)/2}{2n} = \frac{n}{12w}. \tag{4.14}$$

Having achieved our goal of establishing that $|D|$ is large for every $D \in \mathcal{D}$, we are now ready to prove $\mathbb{P}(|[n]_p \cap D| \leqslant t)$ is small for every $D \in \mathcal{D}$. To do this, we use a form of Chernoff's Inequality (see e.g. Corollary 2.3 in [45]) which states that for a binomially distributed random variable $X \sim Bi(n,p)$ we have that

$$\mathbb{P}(X \leqslant \mathbb{E}(X)/2) \leqslant 2\exp\left(-\mathbb{E}(X)/12\right).$$

In our application, the random variable we consider is $[n]_p \cap D$. From the inequality (4.14), we have that $\mathbb{E}(|[n]_p \cap D|) \geqslant np/12w = 2t$. Thus, Chernoff's Inequality gives that

$$\mathbb{P}\Big(|[n]_p \cap D| \leqslant t\Big) \leqslant 2 \cdot \exp\Big(-\frac{2t}{12}\Big) = 2 \cdot \exp\Big(-\frac{pn}{144w}\Big). \tag{4.15}$$

By the inequalities (4.11) and (4.15), inequality (4.12) becomes:

$$\mathbb{P}\Big([n]_p \in \mathcal{B}\Big) \leqslant \exp\left(\frac{pn}{288w}\right) \cdot 2\exp\left(-\frac{pn}{144w}\right) \leqslant 2\exp\left(-\frac{pn}{288w}\right),$$

completing the proof of Claim 4.10. □

Proof of Claim 4.11. Define a 2-cycle to be a two edges e_1, e_2 such that $|e_1 \cap e_2| > 1$. For $j > 2$, define a j-cycle to be a cyclical sequence of j edges $e_1, e_2, \ldots, e_j$ where the intersection of two consecutive edges is exactly 1, the intersection of any two nonconsecutive edges is empty, and the intersection points for each pair of consecutive edges is unique. It follows that, for $j \geqslant 2$, a j-cycle has precisely j edges, j vertices of degree 2, and $j(k-2)$ vertices of degree 1. Recalling that a k-uniform hypergraph has girth at least g if any subset of h edges $(2 \leqslant h < g)$ span at least $(k-1)h+1$ vertices, we see that a hypergraph has girth at least g if and only if it does not contain a cycle of length less than g.

Let the random variable X_j denote the number of j-cycles appearing in the random hypergraph $\binom{[n]_p}{AP_k}$. Because a 2-cycle consists of two edges intersecting in at least two points and and two points are contained in at most $\binom{k}{2}$ edges (by (4.10)), the number of 2-cycles in $\mathcal{H}$ is at most $\binom{n}{2}\binom{k}{2}^2$. Moreover, each two cycle contains at least $k+1$ points. This gives

$$\mathbb{E}(X_2) \leqslant \binom{n}{2}\binom{k}{2}^2 p^{k+1} \leqslant n^2k^4p^{k+1} = k^4c_p^{k+1}n^{(k-3)/(k-1)}.$$

Using this, we now bound $\mathbb{E}(X_2) \leqslant t/4$. In the following calculation, we will make use of the fact that $(\log n)^k \leqslant n^{1/2k}$ since $n > 2^{10k^3}$.

$$\begin{aligned}\frac{\mathbb{E}(X_2)}{t/4} &\leqslant \frac{k^4 c_p^{k+1} n^{(k-3)/(k-1)}}{c_p n^{(k-2)/(k-1)}/96w} = \frac{96wk^4 c_p^k}{n^{1/(k-1)}} \\ &\leqslant \frac{96wk^4(w^{25}k^{25k}\log n)^k}{n^{1/k}} \\ &\leqslant \frac{96wk^4(w^{25}k^{25k})^k}{n^{1/2k}} \\ &\leqslant \frac{96w^{25k+1}k^{25k^2+44}}{k^{200k(k+g)}w^{200(k+g)}g^{4g}} \leqslant 1. \end{aligned} \tag{4.16}$$

Before we count the number of j-cycles for $j > 2$, we first demonstrate that by our choice of parameters $(\log n)^{kg} \leqslant n^{1/4}$ i.e.

$$n^{1/4kg} \geqslant k^{100k}w^{100}g^2 \geqslant 4000k^4wg^2 \geqslant \log n. \tag{4.17}$$

Now to count the number of j-cycles for $j > 2$, observe that each j-cycle can be defined by choosing the j elements of degree 2, cyclically ordering the these elements, and choosing an edge containing each pair of consecutive elements. This gives that the number of j-cycles formed by AP_k in $\mathcal{H}$ is at most $\binom{n}{j}(j-1)!\binom{k}{2}^j \leqslant n^j k^{2j} p^{(k-1)j}$. Hence, we can compute (using (4.17)) that

$$
\begin{aligned}
\frac{4}{t}\cdot\sum_{j=2}^{g-1}\mathbb{E}(X_j) &\leqslant \frac{96w}{np}\sum_{j=2}^{g-1} n^j k^{2j} p^{(k-1)j} \\
&= \frac{96w}{np}\sum_{j=2}^{g-1} k^{2j} c_p^{(k-1)j} \\
&\leqslant \frac{96w}{n^{(k-2)/(k-1)}}\cdot g\cdot k^{2g}c_p^{kg} \\
&\leqslant \frac{96wgk^{2g}c_p^{kg}}{n^{1/2}} \\
&\leqslant \frac{96wgk^{2g}\cdot w^{25kg}k^{25k^2g}(\log n)^{kg}}{n^{1/2}} \\
&\leqslant \frac{96wgk^{2g}\cdot w^{25kg}k^{25k^2g}}{n^{1/4}} \\
&= \frac{96wgk^{2g}\cdot w^{25kg}k^{25k^2g}}{k^{100k^2(k+g)}w^{100k(k+g)}g^{2kg}} \leqslant 1.
\end{aligned} \tag{4.18}
$$

Thus, by (4.16) and (4.18) the expected number of cycles in $\binom{[n]_p}{AP_k}$ of length less than g is less than $t/2$. By Markov's inequality this implies that $\binom{[n]_p}{AP_k}$ will have more than t cycles of length length g with probability at most $1/2$. This proves Claim 4.11.

□

Proof of Claim 4.12. We must prove that $\exp\{-pn/288w\} < 1/2$. This readily follows from the fact that

$$pn \geqslant n^{(k-2)/(k-1)} \geqslant n^{1/2} > 288w.$$

□

This completes the proof of Theorem 4.3 □

4.5 Proof of Theorem 4.4

In this section, we establish that for all integers $\ell \geqslant 2$, $g \geqslant 2$, and $k \geqslant 3$, there exists a graph H on at most $k^{2^{10}k^4g^2}r^{2^8k^2g}$ vertices such that $\binom{H}{K_k}$ has chromatic number greater than ℓ and girth at least g, where $r := r_\ell(K_k)$.

Proof. Set

$$n := k^{2^{10}k^4g^2}r^{2^8k^2g}, \quad t := \frac{p\binom{n}{2}}{2r^2}, \quad c_p := 2^{5\sqrt{\log n \log k}}r^{16}, \quad p := c_p n^{-2/(k+1)}.$$

Let $G \sim G(n,p)$ be a random instance of the graph on $[n]$ obtained by taking each edge of the complete graph on $[n]$ independently with probability p.

Claim 4.14. *With probability at least* $1/2$*, we have that* $\binom{E(G)}{K_k}$ *can be made to have girth* g *by deleting at most* t *edges.*

Claim 4.15. *With probability at least* $1 - 2\exp\left(-p\binom{n}{2}/(24r^2)\right)$*, the hypergraph* $\binom{E(G)}{K_k}$ *has the following strong Ramsey property: If* $T \subset E(G)$ *is any subset of size* t*, then* $\chi\binom{E(G)\setminus T}{K_k} > \ell$*.*

Claim 4.16. $1 - 2\exp\left(-p\binom{n}{2}/(24r^2)\right) - 1/2 > 0$.

Together, the above claims establish that, with positive probability, the random graph G will be such that $\binom{G}{K_k}$ will simultaneously have the properties in both claims. Hence, such a graph exists. From this graph, t edges can be removed to obtain a graph that has the desired properties of Theorem 4.4. It thus remains only to verify the three claims.

Proof of Claim 4.14. Recall that a 2-cycle is a pair of edges e_1, e_2 such that $|e_1 \cap e_2| > 1$ and for $j > 2$ a j-cycle is a cyclical sequence of j edges $e_1, e_2, \ldots, e_j$ where the intersection of two consecutive edges is exactly one i.e. $|e_i \cap e_{i+1}| =$

1 (addition mod j), the intersection of any two nonconsecutive edges is empty, and the intersection points for each pair of consecutive edges is unique.

Define X_j to be the number of j-cycles in the system of copies of K_k in $G(n,p)$. We first work to bound X_2. If $k=3$, we trivially have $\mathbb{E}(X_2)=0$. Otherwise for $k \geqslant 4$, a 2-cycle corresponds to two copies of K_k that intersect in more than two edges, and thus in more than two vertices. Furthermore, we see that two copies of K_k that intersect in i vertices together span exactly $2k-i$ vertices and $2\binom{k}{2}-\binom{i}{2}$ edges. With this in mind, the following bounds $\mathbb{E}(X_2) \leqslant t/4$ in $\binom{G(n,p)}{K_k}$:

$$\begin{aligned}
\frac{\mathbb{E}(X_2)}{t/4} &= \frac{8r^2}{p\binom{n}{2}} \cdot \mathbb{E}(X_2) \leqslant \frac{32r^2}{pn^2} \cdot \sum_{i=3}^{k-1} n^{2k-i} p^{2\binom{k}{2}-\binom{i}{2}} \\
&= 32r^2 n^{2k-2} p^{2\binom{k}{2}-1} \sum_{i=3}^{k-1} n^{(i^2-2i-ki)/(k+1)} c_p^{-\binom{i}{2}} \\
&\leqslant 32r^2 n^{2k-2} p^{2\binom{k}{2}-1} \cdot k \cdot \max_{3 \leqslant i \leqslant k-1} \left\{ n^{(i^2-2i-ki)/(k+1)} \right\} \\
&\leqslant 32r^2 n^{2k-2} p^{2\binom{k}{2}-1} \cdot k \cdot n^{(3-3k)/(k+1)} \\
&= \frac{32kr^2 c_p^{k^2-k-1}}{n^{(k-3)/(k+1)}} \leqslant \frac{c_p^{k^2}}{n^{1/5}} \leqslant 1.
\end{aligned}$$

We now will bound $\sum_{j=3}^{g-1} X_j$. For $j>2$, a j-cycle in $\binom{[n]}{K_k}$ consists of a cyclically ordered set of j copies of K_k such that each two consecutive copies intersect in exactly one edge of K_n. Thus, a j-cycle corresponds to a set of K_k's in K_n that span exactly $kj-2j$ vertices in K_n and $\binom{k}{2}j-j$ edges in K_n. From this, we see that, for $2<j<g$, we have

$$\mathbb{E}(X_j) \leqslant n^{kj-2j} p^{\binom{k}{2}j-j} = \left(n^{(k-2)} p^{\binom{k}{2}-1}\right)^j = c_p^{(\binom{k}{2}-1)j}.$$

Using this, we establish $\sum_{j=3}^{g-1}\mathbb{E}(X_j)\leqslant t/4$:

$$\begin{aligned}\frac{\sum_{j=3}^{g-1}\mathbb{E}(X_j)}{t/4} &\leqslant \frac{8r^2}{p\binom{n}{2}}\cdot g\cdot c_p^{(\binom{k}{2}-1)g}\\ &\leqslant \frac{32r^2g}{pn^2}c_p^{(\binom{k}{2}-1)g}\leqslant \frac{c_p^{k^2g}}{n}\leqslant 1.\end{aligned}$$

Thus, we have shown $\sum_{j=2}^{g-1}\mathbb{E}(X_j)\leqslant t/4+t/4=t/2$. By Markov's Inequality, this gives that, with probability at least 1/2, the hypergraph $\binom{G(n,p)}{K_k}$ contains fewer than t cycles of length less than g. This concludes the proof of Claim 4.14.

$\square$

Proof of Claim 4.15. We will apply the Container Lemma to the hypergraph $\mathcal{H}=\binom{K_n}{K_k}$, as was done in [63].

Let

$$n\geqslant k^{400k^4}r^{40k^2},\qquad \varepsilon=\frac{1}{2\ell\binom{r}{k}}\quad\text{and}\quad \tau=2^{4\sqrt{\log n}}r^{10/k}n^{-2/(k+1)}.$$

With the choice of n, ε and τ one can show that the condition (4.1) of the Container Lemma is satisfied; we will not explicitly show this here, as this exact statement is verified in the recent paper [63] of Rödl, Ruciński, and Schacht. Specifically, our parameters are the same as in equations (11) and (12) in [63] and $n\geqslant k^{400k^4}r^{40k^2}$, to which Claim 10 in [63] establishes that the equation (4.1) holds. Hence, we obtain a set of containers $\mathfrak{C}\subset\mathcal{P}(E(K_n))$ such that all of the following hold:

i) for every independent set $I\subset V(\mathcal{H})$, there exists some $\mathcal{C}\in\mathfrak{C}$ with $\mathcal{C}\supseteq I$,

ii) for every $\mathcal{C}\in\mathfrak{C}$, $e(\mathcal{C})\leqslant\varepsilon\cdot e(\mathcal{H})$, and

iii) $|\mathfrak{C}| \leqslant \exp\{\ell \cdot 1000 \cdot \binom{k}{2} \cdot (\binom{k}{2}!)^3 \cdot \log(1/\varepsilon) \cdot \binom{n}{2} \cdot \tau \cdot \log(1/\tau)\}$.

To further bound $|\mathfrak{C}|$, observe that

$$\begin{aligned}(12000)\tbinom{k}{2}(\tbinom{k}{2}!)^3 &\leqslant (12000) \cdot k^2 \cdot \big((k^2/2)!\big)^2 \cdot (k^2/2)! \\ &\leqslant 3^9 \cdot k^2 \cdot k^2! \cdot \frac{k^{k^2}}{2^{k^2/2}} \\ &\leqslant 3^7 \cdot k^2 \cdot k^2! \cdot k^{k^2} \\ &\leqslant k^{k^2-2} \cdot k^2 \cdot k^2! \cdot k^{k^2} \leqslant k^{2k^2} \cdot k^2! \leqslant (2k^2)!\end{aligned}$$

and thus

$$\begin{aligned}|\mathfrak{C}|^\ell &\leqslant \exp\{\ell \cdot 1000 \cdot \tbinom{k}{2} \cdot (\tbinom{k}{2}!)^3 \cdot \log(1/\varepsilon) \cdot \tbinom{n}{2} \cdot \tau \cdot \log(1/\tau)\} \\ &\leqslant \exp\left\{\ell \cdot \frac{(2k^2)!}{12} \cdot \log(1/\varepsilon) \cdot \tbinom{n}{2} \cdot \tau \cdot \log(1/\tau)\right\}.\end{aligned}$$

Additionally,

$$\ell \cdot \frac{(2k^2)!}{12} \cdot \log(1/\varepsilon) \cdot \tbinom{n}{2} \cdot \tau \cdot \log(1/\tau) \leqslant \frac{p\binom{n}{2}}{24r^2},$$

which has been verified in [63] (see the line after equation (18) on page 12). Thus, we obtain

$$|\mathfrak{C}|^\ell \leqslant \exp\left(\frac{p\binom{n}{2}}{24r^2}\right). \tag{4.19}$$

Having applied the Container Lemma, now consider the family $\mathcal{B}$ of all graphs $B \subset K_n$ with the property that there exists a subgraph $T \subset B$ of size $p\binom{n}{2}/2r^2 = t$ such that $(B\backslash T) \nrightarrow (K_k)_\ell$ i.e. there exists an ℓ-coloring of the edges of the graph $B\backslash T$ that does not contain a monochromatic K_k. Hence $\mathcal{B}$ is the set of all ('bad') graphs on n vertices that do not have the desired strong Ramsey property. We will work to bound $P(G(n,p) \in \mathcal{B})$ from above.

To this end, consider any $B \in \mathcal{B}$. By the definition of $\mathcal{B}$, there exists a set T with $|T| = t$ and a partition $B \backslash T = I_1 \cup \cdots \cup I_\ell$ with the property that each I_i contains no K_k. Moreover, each I_i is an independent set in $\mathcal{H}$ so there exists an ℓ-tuple of containers $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\ell) \in \mathfrak{C}^\ell$ such that $I_i \subset \mathcal{C}_i$ for all $i \in [\ell]$. Letting $D := [n] \backslash \bigcup_{i \in [\ell]} \mathcal{C}_i$, observe that $|B \cap D| \leqslant |T|$ since $B \backslash T = I_1 \cup \cdots \cup I_\ell \subset D^C$. Hence we can define a function from $\mathcal{B}$ to the set

$$\mathcal{D} = \Big\{ D : D = [n] \backslash \bigcup_{i \in [\ell]} \mathcal{C}_i \text{ for some } (\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\ell) \in \mathfrak{C}^\ell \Big\},$$

with the property that for any associated pair $(B, D) \in (\mathcal{B}, \mathcal{D})$ we have $|B \cap D| \leqslant t$.

Since we know that for every $B \in \mathcal{B}$ there exists some $D \in \mathcal{D}$ such that $|B \cap D| \leqslant t$, it follows that if $G(n,p) \in \mathcal{B}$, it must be the case that for some $D \in \mathcal{D}$, we have $|G(n,p) \cap D| \leqslant t$. By the union bound we obtain:

$$\begin{aligned} \mathbb{P}\Big(G(n,p) \in \mathcal{B} \Big) &\leqslant \mathbb{P}\Big(\exists\ D \in \mathcal{D} : |G(n,p) \cap D| \leqslant t \Big) \\ &\leqslant |\mathfrak{C}|^\ell \cdot \max \Big\{ \mathbb{P}(|G(n,p) \cap D| \leqslant t) : D \in \mathcal{D} \Big\}. \end{aligned} \tag{4.20}$$

To show the above probability is small, we show that $|D|$ is large for every $D \in \mathcal{D}$.

Subclaim 4.17. *For every graph $D \in \mathcal{D}$ there are at least $\frac{1}{2}\binom{n}{r}$ sets $R \subset [n]$ with $|R| = r$ such that the induced graph $D[R]$ contains an edge.*

Proof of Subclaim 4.17. Consider any fixed $D \in \mathcal{D}$ and $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\ell) \in \mathfrak{C}^\ell$ with $D = [n] \backslash \bigcup_{i \in [\ell]} \mathcal{C}_i$. Towards contradiction, suppose there are more than $\binom{n}{r}/2$ sets $R \subset [n]$ with $|R| = r$ and $D \cap K_n[R] = \varnothing$.

For any $R \subset [n]$ with $|R| = r$ and $D \cap K_n[R] = \varnothing$, we have $R \subset$

$\bigcup_{i\in[\ell]} \mathcal{C}_i$. By the definition of the Ramsey number $r = r_\ell(K_k)$, this implies there exists $i \in [\ell]$ so that $K_k \subset R \cap \mathcal{C}_i$. Further, as each K_k is contained in at most $\binom{n-k}{r-k}$ sets $R \subset [n]$ with $|R| = r$, it follows that

$$\sum_{i\in[\ell]} e(\mathcal{C}_i) > \frac{(1/2)\binom{n}{r}}{\binom{n-k}{r-k}} = \frac{(1/2)\binom{n}{k}}{\binom{r}{k}} \geqslant \ell\varepsilon\binom{n}{k}$$

which is contradiction to the second property of the Container Lemma. $\square$

In order to use Subclaim 4.17 to establish that $|D|$ is large for every $D \in \mathcal{D}$, note that each edge $e \in D$ is contained in at most $\binom{n-2}{r-2}$ sets of size r. Hence, for every $D \in \mathcal{D}$,

$$|D| \geqslant \frac{(1/2)\binom{n}{r}}{\binom{r}{2}} > \frac{1}{r^2}\binom{n}{2}. \tag{4.21}$$

Having achieved our goal of establishing that $|D|$ is large for every $D \in \mathcal{D}$, we are now ready to prove $\mathbb{P}(|[n]_p \cap D| \leqslant t)$ is small for every $D \in \mathcal{D}$. To do this, we use a form of Chernoff's inequality (see e.g. Corollary 2.3 in [45]) which states that for a binomially distributed random variable $X \sim Bi(n,p)$, we have that $\mathbb{P}\Big(X \leqslant \mathbb{E}(X)/2\Big) \leqslant 2\exp\left(-\frac{\mathbb{E}(X)}{12}\right)$. Having established $\mathbb{E}(|[n]_p \cap D|) \geqslant \frac{p}{r^2}\binom{n}{2} = 2t$ in (4.21), this yields

$$\mathbb{P}\Big(|G(n,p) \cap D| \leqslant t\Big) \leqslant 2 \cdot \exp\Big(-\frac{2t}{12}\Big) = 2 \cdot \exp\Big(-\frac{p\binom{n}{2}}{12r^2}\Big). \tag{4.22}$$

Hence, by (4.19) and (4.22), equation (4.20) becomes:

$$\mathbb{P}\Big(G(n,p) \in \mathcal{B}\Big) \leqslant \exp\Big(\frac{pn}{24r^2}\Big) \cdot 2\exp\left(-\frac{p\binom{n}{2}}{12r^2}\right) \leqslant 2\exp\left(-\frac{p\binom{n}{2}}{24r^2}\right),$$

completing the proof of Claim 4.15. $\square$

Proof of Claim 4.16. To establish that $1 - 2\exp\left(-p\binom{n}{2}/(24r^2)\right) - 1/2 > 0$, observe that

$$p\binom{n}{2} > c_p > r^{16} > 2 \cdot 24r^2.$$

$\square$

This completes the proof of Theorem 4.4 $\square$

4.6 Concluding Remarks

In view of Theorem 4.1 we consider the following function for given integers $\ell \geqslant 2$ and $k \geqslant 4$ let

$$f_\ell(k) := min\Big\{|V(H)|:\ \text{girth}(H) = k \text{ and } H \to (C_k)_\ell\Big\}.$$

Theorem 4.1 established that $f_\ell(k) \leqslant r^{40k^2}k^{40k^3}$, where $r := r_\ell(C_k)$. In view of the known upper bounds on $r_\ell(C_k)$ for even and odd k, this establishes the upper bounds in Theorems 4.18 and 4.19 stated below. These two theorems also provide complementary lower bounds.

Theorem 4.18. *There exist positive constants c_1 and c_2 such that for all $k \geqslant 2$ and $\ell \geqslant 2$,*

$$\exp\Big(c_1 k \log \ell\Big) \leqslant f_\ell(2k) \leqslant \exp\Big(c_2(k^2 \log \ell + k^3 \log k)\Big).$$

For fixed $k \geqslant 2$ Theorem 4.18 shows that $f_\ell(2k)$ is polynomial in ℓ.

Proof. We will first show that $f_\ell(2k) \leqslant \exp\Big(c_2(k^2 \log \ell + k^3 \log k)\Big)$. In [26] it was announced and in [13] it was proved that, for every integer $k \geqslant 2$, there exists a constant c such that every graph on n vertices with at least $cn^{1+1/k}$ edges contains a copy of the cycle C_{2k}. This implies that, if n is such that $\binom{n}{2}/\ell \geqslant cn^{1+1/k}$, that is, $n \geqslant c\ell^{k/(k-1)}$, then every edge coloring of K_n with ℓ colors will have a monochromatic cycle C_{2k}. Hence

$$r_\ell(C_{2k}) \leqslant c\ell^{k/(k-1)}. \tag{4.23}$$

The upper bound $f_\ell(2k) \leqslant \exp\Big(c_2(k^2\log\ell + k^3\log k)\Big)$ now follows from substituting (4.23) into Theorem 4.1.

We now turn our attention towards the lower bound in Theorem 4.18. For any $k \geqslant 2$ and $\ell \geqslant 2$ consider any graph H with $\mathrm{girth}(H) = 2k$ and the property $H \to (C_{2k})_\ell$. Let $\tilde{H} \subset H$ be edge minimal subgraph such that $\tilde{H} \to (C_{2k})_\ell$. Clearly the minimum degree of $\tilde{H}$ must be at least ℓ and $\tilde{H}$ must have girth at least $2k$. Since any graph with girth $2k$ and minimum degree ℓ must have at least $2\sum_{i=0}^{k-1}(\ell-1)^i \geqslant c\ell^{k-1}$ vertices the lower bound for $f_\ell(2k)$ follows. $\square$

The following theorem establishes similar results for the odd case.

Theorem 4.19. *There exist positive constants c_1 and c_2 such that, for all $k \geqslant 1$ and $\ell \geqslant 2$,*

$$\exp\Big(c_1k\ell\Big) \leqslant f_\ell(2k+1) \leqslant \exp\Big(c_2k^2\Big(\ell\log\ell + k\log k\Big)\Big).$$

For fixed $k \geqslant 2$ it follows that $e^{\Omega(\ell)} \leqslant f_\ell(2k+1) \leqslant e^{O(\ell\log\ell)}$.

Proof. We first show that $f_\ell(2k+1) \leqslant \exp\Big(c_2k^2\Big(\ell\log\ell + k\log k\Big)\Big)$. As established in [12],

$$2^\ell k \leqslant r_\ell(C_{2k+1}) \leqslant (\ell+2)! \cdot (2k+1). \tag{4.24}$$

The upper bound for $f_\ell(2k+1)$ follows from substituting the upper bound in (4.24) into Theorem 4.1.

To establish that $f_\ell(2k+1) \geqslant \exp\Big(c_1 k\ell\Big)$ for any $k \geqslant 1$ and $\ell \geqslant 2$, as before we begin by considering any graph H with $\text{girth}(H) = 2k+1$ and the property $H \to (C_{2k+1})_\ell$. Note that $\chi(H) > 2^\ell$, since otherwise the edges of H could be decomposed into ℓ bipartite graphs, resulting in an ℓ-coloring of $E(H)$ with no monochromatic odd cycle. Moreover, since $\chi(H) > 2^\ell$, there must be a subgraph $\tilde{H} \subset H$ with minimum degree at least 2^ℓ. Since $\tilde{H}$ has at least girth $2k+1$ and minimum degree 2^ℓ, the number of vertices in $\tilde{H}$ must be at least $1 + 2^\ell \sum_{i=1}^{k-1}(2^\ell - 1)^i \geqslant 2^{c\ell k}$ vertices for some $c > 0$. $\square$

For three special cases of k, we are able to deduce better bounds for $f_\ell(2k)$ using well known extremal constructions of graphs with girth 6, 8, and 12, respectively.

Theorem 4.20. *We have $f_\ell(6) = O(\ell^6)$, $f_\ell(8) = O(\ell^{12})$, and $f_\ell(12) = O(\ell^{30})$.*

Before proving Theorem 4.20, we first introduce some notation and state an observation upon which the proof is based. Let $\text{ex}(n; C_k)$ denote the maximum number of edges in an n vertex graph that does not contain a cycle of length k. Similarly, let $\text{ex}(n; C_3, C_4, \ldots, C_{k-1})$ denote the maximum number of edges in a graph with girth k.

Fact 4.21. *If $ex(n; C_3, C_4, \ldots, C_{2k-1}) > \ell \cdot ex(n; C_3, C_4, \ldots, C_{2k})$, then $f_\ell(2k) \leqslant n$.*

Indeed, by definition of the extremal function there exists a graph G on n vertices with girth $2k$ that has $\text{ex}(n; C_3, C_4, \ldots, C_{2k-1})$ edges. Clearly, every ℓ-coloring of G yields a monochromatic subgraph with at least

$$\text{ex}(n; C_3, C_4, \ldots, C_{2k-1})/\ell > \text{ex}(n; C_3, C_4, \ldots, C_{2k})$$

edges, which must contain a monochromatic C_{2k} since the monochromatic subgraph still has girth at least $2k$.

Proof of Theorem 4.20. To make use of this fact to prove Theorem 4.20, we use the result of Erdős and Simonovits from [34] that for every positive integer k, we have

$$\mathrm{ex}(n; C_3, C_4, \ldots, C_{2k+1}) = O(n^{1+1/k})\,.$$

Since any graph contains a bipartite subgraph with half of its edges we have

$$\begin{aligned}
\mathrm{ex}(n; C_3, C_4, C_5, C_6, \ldots, C_{2k}) &\leqslant \mathrm{ex}(n; C_4, C_6, \ldots, C_{2k}) \\
&\leqslant 2 \cdot \mathrm{ex}(n; C_3, C_4, C_5, C_6, \ldots, C_{2k+1}) \\
&= O(n^{1+1/k})\,. \qquad (4.25)
\end{aligned}$$

Erdős and Simonovits conjectured in [34] that for every positive integer $k \geqslant 2$,

$$\mathrm{ex}(n; C_3, C_4, \ldots, C_{2k-1}) = \Omega(n^{1+1/(k-1)}). \qquad (4.26)$$

This has been observed for $k = 3$ by Klein (see [25]) and follows for $k = 4$ by the work of Singleton [70], and for $k = 6$ by work of Benson [8]. For $k \in \{3, 4, 6\}$, inequalities (4.25) and (4.26) give that

$$\begin{aligned}
\mathrm{ex}(n; C_3, C_4, \ldots, C_{2k-1}) &= \Omega(n^{1+1/(k-1)})) \\
&> \ell \cdot O(n^{1+1/k}) = \ell \cdot \mathrm{ex}(n; C_3, C_4, \ldots, C_{2k}),
\end{aligned}$$

holds provided that

$$n \geqslant c' \ell^{k(k-1)}\,,$$

for some sufficiently large constant c'. Consequently, Fact 4.21 yields $f_\ell(2k) \leqslant n = \Omega(\ell^{k(k-1)})$ for $k \in \{3, 4, 6\}$ and the theorem follows. □

We remark that establishing (4.26) for all k, implies $f_\ell(2k) = O(\ell^{k(k-1)})$ for all k by the same argument.

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