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On Graphs With a Given Endomorphism Monoid

By

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Mathematics

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An abstract of
A dissertation submitted to the Faculty of the Graduate School
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ABSTRACT

On Graphs With A Given Endomorphism Monoid

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Hedrlín and Pultr proved that for any monoid $\mathbf{M}$ there exists a graph G with endomorphism monoid isomorphic to $\mathbf{M}$. We will give a construction $G(\mathbf{M})$ for a graph with prescribed endomorphism monoid $\mathbf{M}$. Using this construction we derive bounds on the minimum number of vertices and edges required to produce a graph with a given endomorphism monoid for various classes of finite monoids. We state bounds for the class of all monoids as well as for certain subclasses – groups, k-cancellative monoids, commutative 3-nilpotent monoids, rectangular groups, completely simple monoids, a variety of strong semillatices and others.

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For my brother, my keeper, Yadin.

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Chapter 1

Introduction

This work is concerned with properties of finite graphs. We shall assume the reader is familiar with the basic definitions and concepts of Graph Theory [W]. The following will be of central importance.

Definition. An endomorphism of a (finite) graph G is a function $f : V(G) \to V(G)$ where $f(x)f(y) \in E(G)$ whenever $xy \in E(G)$.

We recall some algebraic concepts: given a nonempty set S and an associative binary operation $\cdot$ on S, $\mathbf{S} = (S, \cdot)$ is a semigroup. If there exists an element $e \in S$ such that $xe = ex = x$ for all $x \in S$ then we say that e is the unity of $\mathbf{S}$ and $\mathbf{S}$ is a monoid. We recall that a transformation monoid on a set X is a set of mappings $f : X \longrightarrow X$ closed under composition and containing the identity mapping. For a monoid we shall write $\mathbf{M} = (X, \cdot, e)$ where e is the unity of $\mathbf{M}$. If $\mathbf{S} = (S, \cdot)$ is a semigroup then $\mathbf{S}^1 = (S \cup \{1\}, \cdot, 1)$ is a monoid where 1 is a new element (i.e. $1 \notin S$) and the operation $\cdot$ is extended to $S \cup \{1\}$ such that $1 \cdot 1 = 1$ and $x1 = 1x = x$ for all $x \in S$. The set of all endomorphisms of any graph G, along with the identity endomorphism, form a monoid under composition (the so called endomorphism monoid of G).

In [F1] and [F2] Frucht proved that every finite group is isomorphic to the automorphism group of some finite graph. The analogous result for monoids was given by Hedrlín and Pultr [HP1,HP2] (or [PT]) who proved that for any monoid $\mathbf{M}$ there exists a graph G such that $\mathrm{End}(G)$ is isomorphic to $\mathbf{M}$ and if $\mathbf{M}$ is finite then we can assume that G is finite as well. This result inspired many analogous theorems for restricted classes of graphs. For a good survey see the monograph by Pultr and Trnková [PT].

Our main goal is to give bounds on the minimum number of edges and vertices required to realize a given monoid as the endomorphism monoid of a graph. For a monoid $\mathbf{M}$ let $\nu(\mathbf{M})$ denote the least number n such that there exists a graph G of order n with $\mathrm{End}(G) \cong \mathbf{M}$. Analogously, for $\alpha \in (0,1]$ we define $\nu(\mathbf{M}, \alpha)$ with the added restriction that G has at most $n^{1+\alpha}$ edges. Observe that $\nu(\mathbf{M}) = \nu(\mathbf{M}, 1)$. Likewise, let $\varepsilon(\mathbf{M})$ denote the least number s such that there exists a graph G with s edges and $\mathrm{End}(G) \cong \mathbf{M}$. For a class of monoids $\mathcal{M}$ we set

$$\begin{aligned}
\nu_{\mathcal{M}}(m) &= \max_{|X|=m} \{\nu(\mathbf{M}) \mid \mathbf{M} = (X, \cdot, 1) \in \mathcal{M}\},\\
\nu_{\mathcal{M}}(m, \alpha) &= \max_{|X|=m} \{\nu(\mathbf{M}, \alpha) \mid \mathbf{M} = (X, \cdot, 1) \in \mathcal{M}\},\\
\varepsilon_{\mathcal{M}}(m) &= \max_{|X|=m} \{\varepsilon(\mathbf{M}) \mid \mathbf{M} = (X, \cdot, 1) \in \mathcal{M}\}.
\end{aligned}$$

If $\mathcal{M}$ is the class of all monoids then we shall omit the index $\mathcal{M}$.

From a result of Hedrlín and Pultr [HP3], it follows that $\nu(m) \leq m$ for any infinite cardinal m and by a counting argument we immediately obtain that, under GCH, $\nu(m) = m$ for any infinite cardinal m. It is an open question whether there is a model of set theory in which $\nu(m) < m$ for some infinite cardinal m.

In what follows we will restrict ourselves to the finite case. From results of Hedrlín and Pultr [HP1] it follows that $\nu(m) \leq cm^2$ for some $c > 0$. On the other hand, Babai [B1] showed that for any finite group $\mathbf{H}$ distinct from the cyclic groups of order 4, 5 and 6, there exists a finite graph G such that the group of all automorphisms of G is isomorphic to $\mathbf{H}$ and $|V(G)| = 2|\mathbf{H}| + 1$. This motivated the question, posed by Babai and Nešetřil, of whether there exists a constant c such that $\nu(m) \leq cm$. This was answered in the negative in [KR1] by showing that $\nu(m) \geq (\sqrt{2}+o(1))(m\sqrt{\log m})$. In fact, one can show an analogous result for a subclass of monoids. A monoid $\mathbf{M} = (X, \cdot, e)$ is called <u>three-nilpotent</u> if $\mathbf{M}$ has a zero 0 and $abc = 0$ for all $a, b, c \in X \setminus \{e\}$. Let $\mathcal{CN}$ denote the class of all finite, commutative three-nilpotent monoids. We show that $\nu_{\mathcal{CN}}(m) \geq (1 + o(1))m\sqrt{\log m}$.

An upper bound on ν was obtained by Babai [B2] and, independently, by Koubek and Rödl in [KR1], where $\nu(m) \leq (\sqrt{2} + o(1))m^{\frac{3}{2}}$ was attained (the Babai bound is of the same order). In [KR1] it was also proved that $\nu_{\mathcal{N}}(m) \leq O(m \log m)$ where $\mathcal{N}$ denotes the class of all finite three-nilpotent monoids. Our aim is to give some related results.

In Chapter 2 we give a lower bound on $\varepsilon(m)$, as well as some other related lower bounds which follow from basic counting arguments. Specifically, we show that:

Theorem 1. For all m, $\frac{m^2}{2}(1 + o(1)) \leq \varepsilon(m) \leq (1 + o(1))m^2$.

Theorem 2. For all m, $(1 + o(1)) \left(\frac{1+\alpha}{2}\right)^{\frac{1}{1+\alpha}} m^{\frac{2}{1+\alpha}} \leq \nu(m, \alpha) \leq (1 + o(1))m^{\frac{2}{1+\alpha}}$
where the lower bound holds for $\alpha \in (0, 1]$ and the upper bound holds for $\alpha \in (0, \frac{1}{3})$.

Note that while the upper and lower bounds on $\nu(m)$ are still far apart, the bounds on $\varepsilon(m)$ and $\nu(m, \alpha)$ are only separated by a constant.

In Chapter 3 we introduce some *ad hoc* tools and give an explicit construction for a graph on $n \leq O(m^{\frac{3}{2}})$ vertices with prescribed endomorphism monoid. The upper bounds in Theorems 1–6 are all a consequence of said construction. We also give an example of a monoid which shows that the upper bound of $O(m^{\frac{3}{2}})$ cannot be improved using our construction.

We were unable to improve the general upper bound on $\nu(m)$. However, it is known that for 3-nilpotent monoids, which almost all monoids are (see [KR2]), the upper bound can be significantly improved (see [KR1]). This led us to consider other special classes. This discussion begins in Chapter 4 where classes of monoids based on groups are considered. Let $\mathcal{G}$ denote the class of all finite groups. We prove that

Theorem 3. $\nu_{\mathcal{G}}(m) \leq (2 + o(1))m\sqrt{\log \log m}$.

We observe that such graphs, with endomorphism monoid isomorphic to a group, are cores. A comparison of this result with Babai's result [B1] for automorphism groups leads to the following open problem.

Problem: Is there a constant c such that $\nu_{\mathcal{G}}(m) \leq cm$?

We investigated two generalizations of groups. Given nonempty sets B and C and a group $\mathbf{H}$ let $S = (B \times H \times C, \cdot)$ and define a binary operation $\cdot$ on S by $(b, h, c) \cdot (b', h', c') = (b, hh', c')$. Then $\mathbf{S} = (S, \cdot)$ is a monoid known, unfortunately, as a <u>rectangular</u> <u>group</u> (see monographs of semigroups [CP] and [H]).Let $\mathcal{RG}$ be the class of all monoids $\mathbf{M} = \mathbf{S}^1$ where $\mathbf{S}$ is a finite rectangular group. Then:

Theorem 4. For all m, $\nu_{\mathcal{RG}}(m) \leq (2 + o(1))m\sqrt{\log m}$.

The second generalization of groups are k-right cancellative monoids. For a natural number $k \geq 1$ we say that a monoid $\mathbf{M} = (X, \cdot, e)$ is k-

right cancellative if $|\{y \in X \mid xy = x'\}| \leq k$ for all $x, x' \in X$ and weakly k-right cancellative if for some generating set A, $|\{a \in A \mid xa = x'\}| \leq k$ for all $x, x' \in X$. Clearly, every k-right cancellative monoid is weakly k-right cancellative. Also, it is well-known that H is a group if and only if it is 1-cancellative. Let $\mathcal{C}_k$ be the class of all finite, weakly k-right cancellative monoids. It follows from a simple probabilistic argument that:

Theorem 5. For all m, $\nu_{\mathcal{C}_k}(m) \leq (5 + o(1))m\sqrt{k \ln m}$.

Note that this result is weaker for finite groups than Theorem 3. In the last section of Chapter 4 we investigate the class $\mathcal{CS}$ of all monoids $\mathbf{M} = \mathbf{S}^1$ where $\mathbf{S}$ is a finite, completely simple semigroup (the precise definition is given in the relevant section). Completely simple semigroups – see [CP], the third chapter, or [H] – are the basic building blocks used in a structural description of semigroups. We prove:

Theorem 6. For all m, $\nu_{\mathcal{CS}}(m) \leq (2 + o(1))m^{\frac{7}{6}}$.

In Chapter 5 we will generalize the $\mathcal{P}$-graph construction (to be defined) from Chapter 3 in order to derive bounds for new monoid classes. Among these are the so called "strong semilattices of C-semigroups" where C is one of the following: groups, abelian groups, rectangular groups or completely simple semigroups. Our general approach will be as follows. Suppose $\{M^{(i)} \mid i \in I\}$ is a collection of monoids with $\nu(M^{(i)})$ "small" for all $i \in I$ and M is a subdirect product of the $M^{(i)}$. Then we can extend the $\mathcal{P}$-graph construction to products and derive a bound on $\nu(M)$ in terms of the $\nu(M^{(i)})$ which substantially improves the bound we would have gotten from applying our earlier construction directly to M. We will apply these tools to three particular classes; finite abelian groups denoted $\mathcal{A}$, finite normal band monoids denoted

$\mathcal{NB}$, and finite semilattice monoids denoted $\mathcal{L}$. Formal definitions for each class will be given in Chapter 5. For now we point out that finite abelian groups and semilattices are subdirect products of cyclic groups and two-element semilattices, respectively. Normal bands, like semilattices, are also subdirect products with factors from a finite collection to be described later.

Theorem 7. Let $\mathcal{M}$ be any of the classes $\mathcal{L}$, $\mathcal{A}$, $\mathcal{NB}$. Then

$$\nu_{\mathcal{M}}(n) = O(n).$$

In Chapter 5 we will determine more precise constants for each class.

The result for abelian groups modifies the result

$$\nu_{\mathcal{G}}(n) \leq (2 + o(1))n\sqrt{\log\log n}$$

of Theorem 3 for the class $\mathcal{G}$ of finite groups. Note, also, that our result for abelian groups is distinct from the Babai result [B1] mentioned above as we require that our graph have no proper endomorphisms.

The last chapter is devoted to classes of monoids built out of semigroups and semilattices. This construction was intensely studied in the monograph by Petrich [P]. We will show that the resulting object, the so called strong semilattice of semigroups, can be handled by our construction.

For a collection $\mathcal{C}$ of semigroups we shall write $\mathfrak{S}(\mathcal{C})$ for the class of monoids which are either strong semilattices of $\mathcal{C}$-semigroups or are strong semilattices of $\mathcal{C}$-semigroups with added new identity. We will consider several cases for $\mathcal{C}$: groups $\mathcal{G}$, abelian groups $\mathcal{A}$, rectangular groups $\mathcal{RG}$ and, completely simple semigroups $\mathcal{CS}$. We shall define these classes precisely in Chapter 5.

In the last chapter we will prove:

Theorem 8. The following bounds hold for strong semilattices of $\mathcal{C}$-semigroups

$$\nu_{\mathfrak{S}(\mathcal{A})}(n) \le 21n, \qquad \nu_{\mathfrak{S}(\mathcal{G})}(n) \le (2+o(1))n\sqrt{\log\log n},$$
$$\nu_{\mathfrak{S}(\mathcal{RG})}(n) \le (2+o(1))n\sqrt{\log n}, \qquad \nu_{\mathfrak{S}(\mathcal{CS})}(n) = (2+o(1))n^{\frac{7}{6}}.$$

We conclude our introduction with Figure 1.1, a diagram for the classes of semigroups considered here ordered by inclusion.

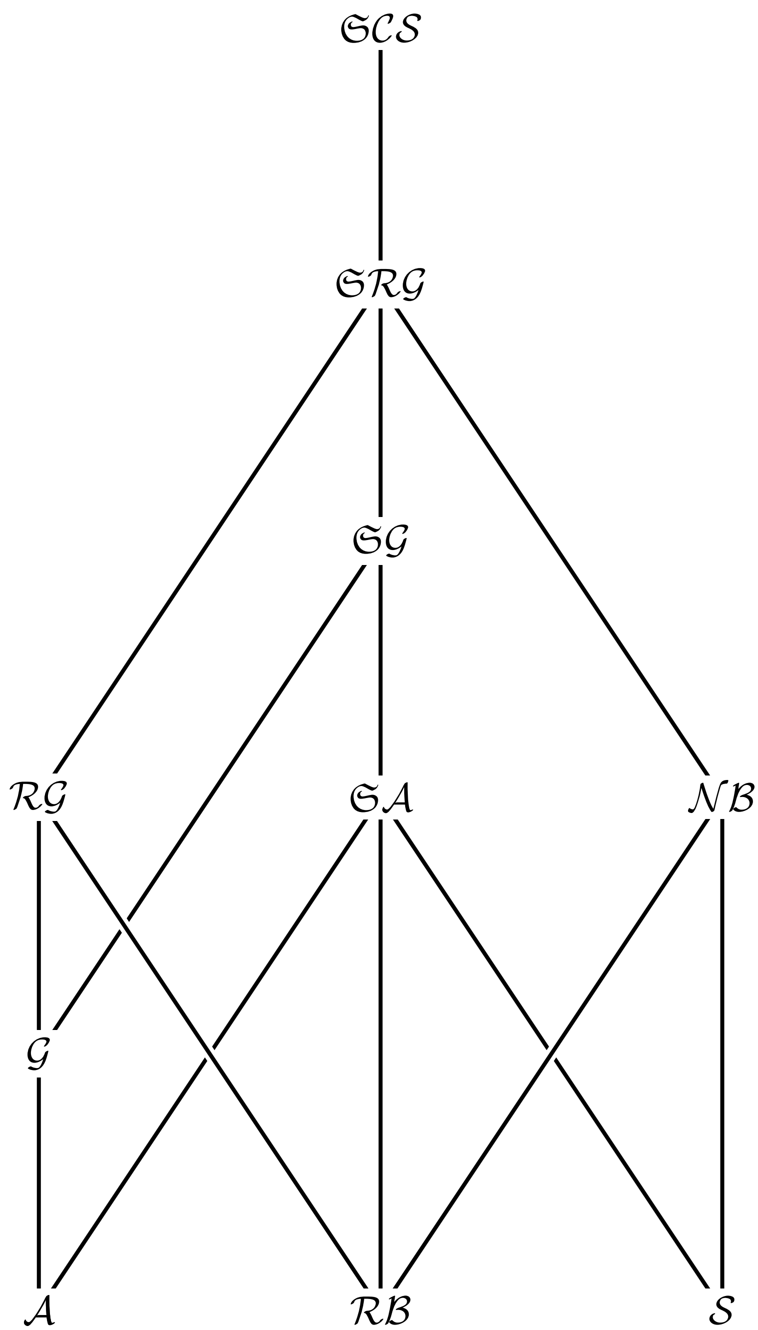

Figure 1.1: Poset of semigroup classes

Chapter 2

Lower Bounds

We begin by proving the lower bounds from Theorems 1 and 2.

Theorem 1 – Lower Bound. For all m, $\varepsilon(m) \geq \frac{m^2}{2}(1+o(1))$.

Proof: We begin by observing that a graph with m endomorphisms has at most t isolated vertices whenever $t^t > m$ (which implies $t = (1+o(1))\frac{\log m}{\log\log m}$).Let G be a graph with $\mathrm{End}(G) \cong \mathbf{M} = (X, \cdot, e)$ where $|X| = m$ and $|E(G)| = q$. By our observation $|V(G)| < 2q + t$. We will use this to bound q from below.

The number of (labeled) graphs with at most q edges and at most $2q+t$ vertices is

$$\sum_{j=0}^{2q+t} \sum_{i=0}^{q} \binom{\binom{j}{2}}{i} < (2q+t)q\binom{\binom{2q+t}{2}}{q}$$
$$< (2q+t)q\left[\frac{(2q+t)^2 e}{2q}\right]^q \qquad (*)$$

As any m-element monoid can be represented by a graph counted in $(*)$ this quantity must exceed $m^{m^2(1+o(1))}$, the number of monoids on m elements [KRS] (see also [KR2]). Taking logarithms we get,

$$\log[(2q+t)q] + q\log\left[\frac{(2q+t)^2 e}{2q}\right] > m^2 \log m(1+o(1)) \qquad (**)$$

substituting $q = \frac{m^2}{2}(1+o(1))$ and $t = (1+o(1))\frac{\log m}{\log\log m}$ the left-hand side of $(**)$ becomes

$$\log \frac{m^2}{4} + \frac{m^2}{2}\log(m^2)(1+o(1)) = m^2 \log m(1+o(1))$$

as required. Thus, $\epsilon(m)$ which is at least the minimum q satisfying the above inequality, is greater than $\frac{m^2}{2}(1+o(1))$. $\square$

Theorem 2 – Lower Bound. For all m and $\alpha \in (0,1]$, $\nu(m,\alpha) \geq (1+o(1))\left(\frac{1+\alpha}{2}\right)^{\frac{1}{1+\alpha}} m^{\frac{2}{1+\alpha}}$.

Proof: The lower bound follows from a counting argument analogous to the one used in proving Theorem 1. Let $n = \nu(M,\alpha)$. The number of sufficiently sparse graphs on at most n vertices is given by

$$\sum_{j=0}^{n}\sum_{i=0}^{j^{1+\alpha}}\binom{\binom{j}{2}}{i} \leq n^{2+\alpha}\left(\frac{n^2}{2n^{1+\alpha}}e\right)^{n^{1+\alpha}}$$

which, for sufficiently large n, is less than $n^{n^{1+\alpha}}$. By assumption every monoid of size at most m is represented by one of the $n^{n^{1+\alpha}}$ graphs and so

$$n^{n^{1+\alpha}} \geq m^{m^2(1+o(1))}$$

which is equivalent to

$$n(\log n)^{\frac{1}{1+\alpha}} \geq (\log m)^{\frac{1}{1+\alpha}} m^{\frac{2}{1+\alpha}}(1+o(1)). \qquad (*)$$

For each m, the minimum real number $n_0 = n_0(m)$ for which $(*)$ holds satisfies

$$\frac{\log m}{\log n_0} = \frac{1+\alpha}{2}(1+o(1)).$$

Combining this equality with $(*)$ yields that for each m

$$n \geq n_0(m) = (1+o(1))m^{\frac{2}{1+\alpha}}\left(\frac{\log m}{\log n_0}\right)^{\frac{1}{1+\alpha}} = (1+o(1))\left(\frac{1+\alpha}{2}\right)^{\frac{1}{1+\alpha}} m^{\frac{2}{1+\alpha}}. \square$$

Remark: The only other result in this work regarding edges is Proposition 3.6. There we improved the constant term from 15 to 7 in the following result of Hedrlín and Pultr. For any monoid $\mathbf{M} = (X, \cdot, 1)$ there exists a graph G with at most $(15 + o(1))m$ edges such that $\mathrm{End}(G) \cong \mathbf{M}$. It may be of interest to find the smallest constant with this property.

Finally, we modify the lower bound on $\nu_{\mathcal{N}}(m)$ from [KR1] for commutative monoids. We recall that a monoid $\mathbf{M} = (X, \cdot, 1)$ is commutative 3-nilpotent if $\mathbf{M}$ has a zero 0 and $abc = 0$ for all triples $\{a, b, c\}$ where $a, b, c \in X \setminus \{1\}$ and $ab = ba$ for all $a, b \in X$. Let $\mathcal{CN}$ be the class of all finite, commutative 3-nilpotent monoids. Then:

Theorem 2.1. For all m, $\nu_{\mathcal{CN}}(m) \geq (1 + o(1))m\sqrt{\log m}$.

Proof: The proof is a simple modification of the proof in [KR1] where a similar argument was used to bound the number of all (not necessarily commutative) semigroups. Let X be an m-element set and $e \in X$. Let $X \setminus \{e\} = A \cup B$ be a partition. Select an element $0 \in B$ and a mapping $g : A \times A \longrightarrow B$ such that $g(x, y) = g(y, x)$ for all $x, y \in A$. We define a binary operation by

$$x \cdot y = \begin{cases} x & \text{if } y = e, \\ y & \text{if } x = e, \\ 0 & \text{if } x \in B \text{ or } y \in B, \\ g(x, y) & \text{if } x, y \in A. \end{cases}$$

A routine verification yields that $(X, \cdot, e)$ is a 3-nilpotent commutative monoid and for distinct mappings g_1 and g_2 we obtain distinct monoids.

For a fixed A with $|A| = t$ there exist $(m-t-1)^{\binom{t}{2}+t}$ symmetric mappings and hence we have at least

$$\frac{1}{m!}m\binom{m-1}{t}(m-t-1)^{\binom{t}{2}+t} > \frac{1}{m!}(m-t-1)^{\frac{1}{2}t^2}$$

non-isomorphic monoids of this type.

Setting $t = m - \frac{m}{\log m}$ then yields at least

$$\frac{1}{m!}(\frac{m}{\log m}-1)^{\frac{1}{2}(m-\frac{m}{\log m})^2} = 2^{(\frac{1}{2}+o(1))m^2\log m}$$

non-isomorphic monoids. As in Theorems 1 and 2 we compare this quantity with the number of all graphs on an n-element set, obtaining

$$2^{\binom{n}{2}} \geq 2^{\frac{m^2}{2}\log m(1+o(1))}$$

which yields

$$\nu_{\mathcal{CN}}(m) = n \geq (1+o(1))m\sqrt{\log m}.\ \square$$

The remainder of this work is concerned with upper bounds on $\nu(m)$. In what follows we describe a construction for a graph $G = G(\mathbf{M})$ on less than $O(m^{\frac{3}{2}})$ vertices with a prescribed endomorphism monoid $\mathbf{M}$ and at most m^2 edges.

Chapter 3

Construction

3.1 A Reduction to $\mathcal{P}$-graphs

Before giving the details of our construction we introduce the notions of $\mathcal{P}$-graph and $\mathcal{P}$-endomorphism. We will prove that the problem of representing a monoid as the endomorphism monoid of a graph can be reduced to the problem of representation by a $\mathcal{P}$-endomorphism monoid of a $\mathcal{P}$-graph.

Definition. A partition graph (shortly a $\mathcal{P}$-graph) is a graph $F_{\mathcal{P}}$ where F is an undirected graph equipped with a partition (coloring) $\mathcal{P}$ of the vertex set $V(F) = \cup_{i=0}^{t} V_i$ where each V_i is an independent set in F.

Definition. A $\mathcal{P}$-endomorphism of $F_{\mathcal{P}}$ is a graph homomorphism of F such that $f(V_i) \subseteq V_i$ for all $i \in [0, t]$.

By $\mathrm{End}_{\mathcal{P}}(F_{\mathcal{P}})$ we denote the set of all $\mathcal{P}$-endomorphisms. Then $\mathrm{End}_{\mathcal{P}}(F_{\mathcal{P}})$ contains the identity map on $V(F)$ and is closed under composition of mappings. Thus, $\mathrm{End}_{\mathcal{P}}(F_{\mathcal{P}})$ is a transformation monoid.

Our construction has three parts: a $\mathcal{P}$-graph $F_{\mathcal{P}}$ (to be defined in Section 3.3), a rigid graph H (that is a graph with no non-trivial endomorphisms) and an amalgamation $*$ of the two.

The Rigid Graph. Similar graphs were considered in [HN]. We present a somewhat simplified version from [PT]. For every $q \geq 3$ there exists a rigid graph $H = H_q$ on $4q+1$ vertices and less than $4q^2$ edges such that every $z \in V(H)$ is contained in a clique of size q. Formally, we set

- $V(H) = \{0, 1, \dots, 4q\}$
- $E(H) = \{\{z, z'\} \mid z, z' \in Z, |z - z'| < q\} \cup \{\{0, 4q\}, \{0, 3q-1\}\}$.

From [PT] we know that H is rigid and, clearly, every vertex $z \in V(H)$ belongs to a clique of size q. Moreover, no $x \in \{3q, 3q+1, \dots, 4q-2\}$ is adjacent to 0 or $2q-1$ and there exists no joint neighbor of 0 and $2q-1$. These facts are exploited in the following construction.

The Amalgamation. Let $F_{\mathcal{P}}$ be a $\mathcal{P}$-graph with $V(F) = \cup_{i=0}^{t} V_i$ and suppose that for every $i = 1, 2, \dots, t$ each $v \in V_i$ has a neighbor from V_0 and every $v \in V_0$ has at most one neighbor from V_i for all $i = 1, 2, \dots, t$. We set $q = t + 2$ and let $H = H_q$ be the rigid graph defined above. We define the graph $G = F_{\mathcal{P}} * H$ (see Figure 3.2) as follows: set $V(G) = V(F) \cup V(H)$ and $E(G) = E(F) \cup E(H) \cup L$ where

$$L = \bigcup_{i=1}^{t} \{\{3q-1+i, v\} \mid v \in V_i\} \cup \{\{0, v\}, \{2q-1, v\} \mid v \in V_0\}.$$

The next theorem reduces a representation problem of monoids by endomorphism monoids of graphs to a representation of monoids by $\mathcal{P}$-endomorphism monoids.

Proposition 3.1. Let $G = F_{\mathcal{P}} * H$ be the graph described above. Then $\mathrm{End}_{\mathcal{P}}(F_{\mathcal{P}}) \cong \mathrm{End}\, G$.

Proof: We show that $g : V(G) \to V(G)$ is an endomorphism of G if and only if g is the identity on H and restricts to a $\mathcal{P}$-endomorphism of $F_{\mathcal{P}}$. Consider a $\mathcal{P}$-endomorphism $f : V(F) \to V(F)$ of $F_{\mathcal{P}}$. Define $g : V(G) \to V(G)$ by

$$g(w) = \begin{cases} f(w) & \text{if } w \in V \\ w & \text{if } w \notin V. \end{cases}$$

Then g preserves $E(F)$ and $E(H)$ and, as $f(V_i) \subseteq V_i$ for all $i \in [0, t]$, g preserves L as well. Hence, g is an endomorphism of G.

Conversely, let g be an endomorphism of G and pick $z \in V(H)$. Since every vertex $z \in V(H)$ belongs to a clique of G of size $q = t + 2$ and every clique of G containing $v \in V(F)$ has size at most $t + 1$ we have $g(z) \in V(H)$. Since H is rigid we infer that $g(z) = z$ for all $z \in V(H)$. Next we prove that $g(V_0) \subseteq V_0$. Let $v \in V_0$, then $\{0, v\}$ and $\{2q-1, v\}$ are edges of G. On the other hand, the neighborhoods of 0 and $2q - 1$ are disjoint in H. Hence $g(v) \notin \{0, 1, \ldots, 4q\}$. Since $\{0, v\} \in E(G)$ and $g(0) = 0$ but $\{0, y\} \in E(G)$ for no $i \in I \setminus \{0\}$ and $y \in V_i$ we infer that $g(V_0) \subseteq V_0$. Finally, we prove that $g(V_i) \subseteq V_i$ for all $i \in I$. Consider $i \in I \setminus \{0\}$ then for every $y \in V_i$ there exists $x \in V_0$ with $\{x, y\} \in E(G)$. Then $\{g(x), g(y)\} \in E(G)$ and hence $y \in V(F_{\mathcal{P}}) \cup \{0, 2q-1\}$. Further $\{3q - 1 + i, v\} \in E(G)$ which implies $\{3q - 1 + i, g(v)\} \in E(G)$. Since $\{0, 3q - 1 + i\}, \{2q - 1, 3q - 1 + i\} \notin E(G)$ and since $z \in V(F_{\mathcal{P}})$ is a neighbor of $3q-1+i$ exactly when $z \in V_i$ we conclude that $g(v) \in V_i$ (see Figure 3.1). Thus $g(V_i) \subseteq V_i$ for all $i \in I$. Since g is an endomorphism of G we infer that $\{g(x), g(y)\} \in E(F_{\mathcal{P}})$ for all $\{x, y\} \in E(F_{\mathcal{P}})$, and therefore, the restriction of g to $V(F_{\mathcal{P}})$ is a $\mathcal{P}$-endomorphism of $F_{\mathcal{P}}$, as

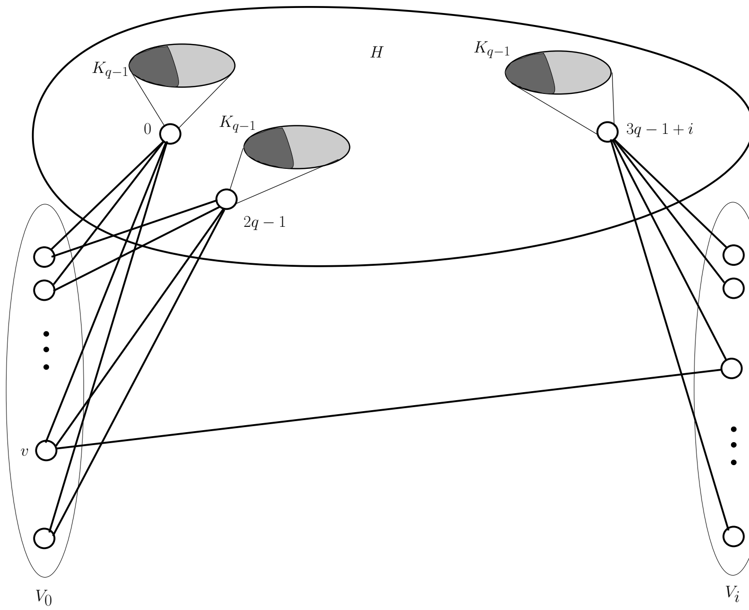

Figure 3.1: Amalgamation

required. □

3.2 Translations of Monoids

Before giving the remaining details of our construction we discuss the set of left (right) translations of a monoid, a notion upon which almost all representations of a monoid as an endomorphism monoid are based.

Let $\mathbf{M} = (X, \cdot, 1)$ be a monoid and $x \in X$. Then a mapping $l : X \to X$ is called a left translation by x if $l(y) = xy$ for all $y \in X$. By $\mathbf{M}^L$ we denote the set of all left translations of $\mathbf{M}$ under composition. Analogously, a right translation $r \in \mathbf{M}^R$ is a mapping given by $y \mapsto yx$. The following statement was originally formulated for semigroups. We adjust it to our purposes as follows.

Lemma 3.2. [CP] Let $\mathbf{M} = (X, \cdot, 1)$ and $\mathbf{M}' = (X, \odot, 1)$ be monoids where $x \odot y = y \cdot x$ for all $x, y \in X$. Then $\mathbf{M}^L \cong \mathbf{M}$ and $\mathbf{M}^R \cong \mathbf{M}'$. $\square$

The following characterization will be useful to us.

Lemma 3.3. [GH] Let $\mathbf{M} = (X, \cdot, 1)$ be a monoid. A mapping $g : X \to X$ is a left (or right) translation by some $x \in X$ if and only if $g \circ r = r \circ g$ for each right (or left) translation $r \in \mathbf{M}^R$ (or $r \in \mathbf{M}^L$, resp.). $\square$

These statements provide a basis for identifying and realizing monoids.

3.3 The graph $F_{\mathcal{P}}$

We fix the following notation for Chapters 3 and 4. Fix a natural number $t > 1$ and let $\binom{[t]}{2} = \{(i, j) \mid 1 \leq i < j \leq t\}$. For a given set A let $\mathfrak{P}(A) = \{B \mid B \subseteq A\}$. Fix a monoid $\mathbf{M} = (X, \cdot, 1)$ where $|X| = m$, a set of generators A and a map $\phi : \binom{[t]}{2} \to \mathfrak{P}(A)$ via $(i, j) \mapsto B_{ij}$. We define a $\mathcal{P}$-graph $F_{\mathcal{P}} = F_{\mathcal{P}}(\mathbf{M}, \phi)$ as follows:

- $V(F) = \bigcup_{i=0}^{t} V_i$ where $V_i = X \times \{i\}$;
- $\mathcal{P} = \{V_i \mid i \in [0, t]\}$;
- $E_0 = \{\{(x, 0), (x, i)\} \mid x \in X, i \in [1, t]\}$;
- $E_{ij} = \{\{(x, i), (xa, j)\} \mid x \in X, a \in B_{ij}\}$ for all $(i, j) \in \binom{[t]}{2}$;
- $E = E_0 \cup (\bigcup \{E_{ij} \mid (i, j) \in \binom{[t]}{2}\})$.

Furthermore, let $R(x, x') = \{a \in X \mid xa = x'\}$ and set $B(x, x') = \bigcup_{ij} \{B_{ij} \mid B_{ij} \cap R(x, x') = \emptyset\}$. Finally, suppose ϕ satisfies the following:

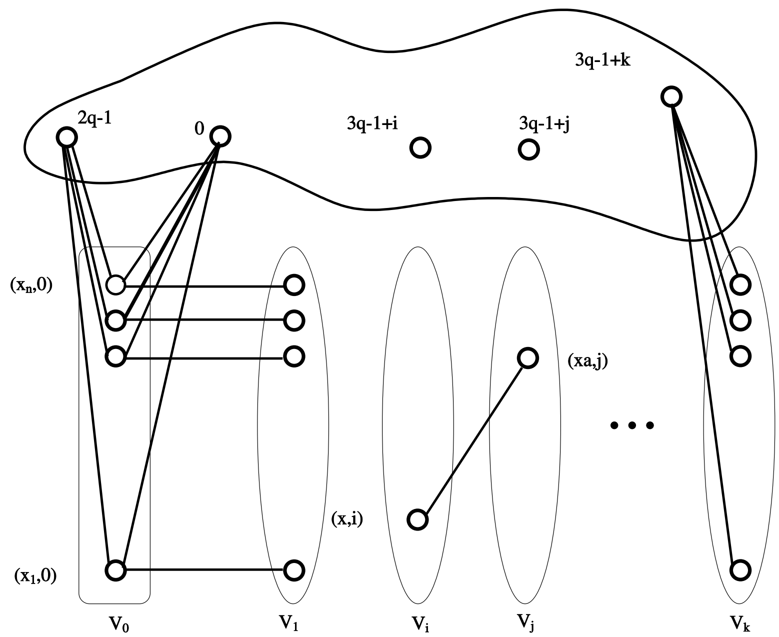

Figure 3.2: $G = F_{\mathcal{P}} * H$

Condition (P). $B(x, x') = A \setminus R(x, x')$ for every $x, x' \in X$.

Remark 1. Note that the graph $F_{\mathcal{P}} = G(\mathbf{M}, \phi)$ satisfies the two requirements made in the previous section:
(i) all sets V_i are independent;
(ii) the edges between V_0 and V_i form a perfect matching for all $i \in [1, t]$.

Remark 2. By definition $B(x, x') \subseteq A \setminus R(x, x')$. Therefore, in order to verify condition (**P**), it will be sufficient to prove the opposite inclusion.

Remark 3. Note that (**P**) implies

$$\bigcup_{ij} B_{ij} = A.$$

If $|X| = 1$ this is trivial. Otherwise, for every $a \in A$ we have $a \notin R(1, a')$ for any $a' \neq a$. Hence, by $(\mathbf{P})$ we get that $a \in B(1, a')$ and thus $a \in B_{ij}$ for some $(i,j) \in \binom{[t]}{2}$. If

$$\bigcup_{ij} B_{ij} = A \quad \text{and} \quad |B_{ij}| \leq 1 \quad \text{for all } (i,j) \in \binom{[t]}{2}$$

then $(\mathbf{P})$ is true. Hence, if $t \geq \sqrt{2|A|}$ then we can satisfy $(\mathbf{P})$ because $\binom{t}{2} \geq |A|$.

With the above construction we are able to state our central technical result.

Proposition 3.4. Let $F_{\mathcal{P}}$ be the graph defined above. Then the graph $G = G(\mathbf{M}, \phi) = F_{\mathcal{P}} * H$ introduced in Proposition 3.1 (see Figure 3.2) has the following properties:

(i) $|V(G)| = (t+1)m + 4t + 9 < (t+1)(m+4) + 5$,

(ii)

$$\begin{aligned}|E(G)| = &|E(F)| + |V(F)| + |V_0| + |E(H)| \leq \\ &\left(\sum_{(i,j)\in\binom{[t]}{2}} |B_{ij}|\right) m + 2mt + 2m + 4(t+2)^2,\end{aligned}$$

(iii) if $(\mathbf{P})$ holds then $\mathrm{End}(G) \cong \mathbf{M}$.

Proof: Since properties (i) and (ii) immediately follow from the construction (because $|E_{ij}| = |B_{ij}|m$ for all $(i,j) \in \binom{[t]}{2}$, and $|E_0| = |\bigcup_{i=1}^t V_i| = tm$) we will verify (iii) only. By Proposition 3.1, $\mathrm{End}(G) \cong \mathrm{End}_{\mathcal{P}}(F_{\mathcal{P}})$ and, to establish the isomorphism, it remains only to prove that $\mathrm{End}_{\mathcal{P}}(F_{\mathcal{P}}) \cong \mathbf{M}$.

Lemma 3.5. $\mathrm{End}_{\mathcal{P}}(F_{\mathcal{P}}) \cong \mathbf{M}$.

We introduce the following notation. Let $f \in \mathrm{End}_{\mathcal{P}}(F_{\mathcal{P}})$. Then $f(V_i) \subseteq V_i$ for $i \in [0,t]$ and consequently $f(x,i) = (x',i)$ for some $x' \in X$. For each $f \in \mathrm{End}_{\mathcal{P}}(F_{\mathcal{P}})$ let $\{f^i\}_{i=0}^t$ be the class of maps where $f^i : X \to X$ via $x \mapsto x'$.

Fact. For each $f \in \mathrm{End}_{\mathcal{P}}(F_{\mathcal{P}})$ and for all $0 \leq i,j \leq t$, $f^i = f^j$ holds.

Proof of Fact: Fix $x \in X$, $f \in \mathrm{End}_{\mathcal{P}}(F_{\mathcal{P}})$ and $i \in [1,t]$. It suffices to show that $f^i = f^0$. From $f(V_0) \subseteq V_0$ it follows $f(v) \in V_0$ for all $v \in V_0$. Thus $\{f(v), f(w)\} \in E_0$ for all $\{v,w\} \in E_0$. As $\{v,w\} \in E_0$ if and only if $v = (x,0)$ and $w = (x,i)$ for some $x \in X$ and $i \in [1,t]$, we conclude that $\{(f^0(x),0),(f^i(x),i)\} \in E_0$ for all $x \in X$ and $i \in [1,t]$ and hence, $f^0(x) = f^i(x)$ for all $x \in X$ and $i \in [1,t]$. $\square$

Since f^i depends only on f (and, in particular, is independent of i) there is a well defined map on X given by $\tilde{f} = f^i$, $i \in [0,t]$. Let $\mathcal{D} = \{\tilde{f} \mid f \in \mathrm{End}_{\mathcal{P}}(F_{\mathcal{P}})\}$. Since $\mathcal{D}$ is closed under composition and contains the identity mapping it is a transformation monoid. We shall sometimes refer to elements of $\mathcal{D}$ as determining mappings.

Proof of Lemma 3.5: By Lemma 3.2 it suffices to show that $\mathrm{End}_{\mathcal{P}}(F_{\mathcal{P}}) \cong \mathbf{M}^L$. We do this in two stages – first showing $\mathrm{End}_{\mathcal{P}}(F_{\mathcal{P}}) \cong \mathcal{D}$ and then showing $\mathcal{D} \cong \mathbf{M}^L$.

Claim 3.51. The map $\psi : \mathrm{End}_{\mathcal{P}}(F_{\mathcal{P}}) \to \mathcal{D}$ sending $f \mapsto \tilde{f}$ is an isomorphism.

Proof: The map is onto by definition. We need to show it is one-to-one and respects composition. Let $f,g \in \mathrm{End}_{\mathcal{P}}(F_{\mathcal{P}})$ be distinct and pick $v \in V(F)$ such that $f(v) \neq g(v)$. There exists $x \in X$ and $i \in 0 \dots t$ such that $v = (x, i)$. Thus, $(f^i(x), i) = f(x, i) \neq g(x, i) = (g^i(x), i)$ and so $\tilde{f}(x) \neq \tilde{g}(x)$ and consequently the map is one-to-one. Secondly, we have

$$(\tilde{f}\tilde{g}(x), i) = f(\tilde{g}(x), i) = f(g(x, i)) = (f \circ g)(x, i) = (\widetilde{fg}(x), i).$$

Thus, $\widetilde{fg} = \tilde{f}\tilde{g}$ and ψ is an isomorphism. $\square$

Claim 3.52. $\mathcal{D} = \mathbf{M}^L$.

Proof: We show both inclusions. To establish $\mathbf{M}^L \subseteq \mathcal{D}$ we first need to characterize the maps in $\mathcal{D}$.

Lemma 3.53. Let $g : X \to X$. Then $g \in \mathcal{D}$ if and only if for every $(i, j) \in \binom{[t]}{2}$, every $x \in X$ and every $a \in B_{ij}$ there exists $b \in B_{ij}$ with $g(xa) = g(x)b$.

Proof: Assume $g \in \mathcal{D}$. Then there exists $f \in \mathrm{End}_{\mathcal{P}}(F_{\mathcal{P}})$ with $g = \tilde{f}$. Choose $x \in X$, $(i, j) \in \binom{[t]}{2}$ and $a \in B_{ij}$. Since $\{(x, i), (xa, j)\} \in E_{ij}$ we get that $\{f(x, i), f(xa, j)\} \in E_{ij}$ where $f(x, i) = (g(x), i)$ and $f(xa, j) = (g(xa), j)$. Hence there exists $b \in B_{ij}$ with $g(xa) = \tilde{f}(xa) = \tilde{f}(x)b = g(x)b$ and the condition is satisfied.

Conversely, let $g : X \to X$ and assume that for every $x \in X$, every $(i, j) \in \binom{[t]}{2}$ and every $a \in B_{ij}$ there exists $b \in B_{ij}$ with $g(xa) = g(x)b$. We define $f : V(F) \to V(F)$ by $f(x, i) = (g(x), i)$ for all $x \in X$ and all $i \in [0, t]$ and we show that f is a $\mathcal{P}$-endomorphism of $F_{\mathcal{P}}$. By the definition of f we have $f(V_i) \subseteq V_i$ for all $i \in [0, t]$ and consequently $\{f(v), f(w)\} \in E_0$ for all $\{v, w\} \in E_0$. It remains to prove that for every pair $(i, j), 1 \leq i < j \leq t$ and every $\{v, w\} \in E_{ij}$ we have $\{f(v), f(w)\} \in$

E_{ij}. Select $(i,j) \in \binom{[t]}{2}$ and an edge $\{v,w\} \in E_{ij}$. There exists $x \in X$ and $a \in B_{ij}$ with $\{v,w\} = \{(x,i),(xa,j)\}$. By assumption there exists $b \in B_{ij}$ with $g(xa) = g(x)b$. From the definition of f it follows that $f(xa,j) = (g(xa),j) = (g(x)b,j)$ and, as $\{(g(x),i),(g(x)b,j)\} \in E_{ij}$ we get that $\{f(x,i),f(xa,j)\} \in E_{ij}$, as required. $\square$

Proof of 3.52 (continued): We first show $\mathbf{M}^L \subseteq \mathcal{D}$. Let r_a be a right translation by some $a \in A$. If $g : X \longrightarrow X$ is a left translation of $\mathbf{M}$ then, by Lemma 3.3, for every $x \in X$ and $a \in A$ we have $g(xa) = g \circ r_a(x) = r_a \circ g(x) = g(x)a$. Thus, g satisfies the conditions of Lemma 3.53 with $b = a$ and we get that $\mathbf{M}^L \subseteq \mathcal{D}$. It remains to show the reverse inclusion $\mathcal{D} \subseteq \mathbf{M}^L$. Let $g \in \mathcal{D}$ and $f \in \mathrm{End}_{\mathcal{P}}(F_{\mathcal{P}})$ with $g = \tilde{f}$ and choose $a \in A$. We prove that g commutes with the right translation r_a of a. Indeed, if $g(x)a = r_a(g(x)) \neq g(r_a(x)) = g(xa)$, then $a \notin R(g(x),g(xa))$ which implies, by ($\mathbf{P}$), that $a \in B(g(x),g(xa))$. It follows that there exists $(i,j) \in \binom{[t]}{2}$ with $a \in B_{ij}$ and $B_{ij} \cap R(g(x),g(xa)) = \emptyset$. Thus, $\{(x,i),(xa,j)\} \in E_{ij}$ while $\{f(x,i),f(xa,j)\} = \{(g(x),i),(g(xa),j)\} \notin E_{ij}$ as $g(xa) \neq g(x)b$ for all $b \in B_{ij}$, contradicting that f is an endomorphism. Consequently g commutes with r_a and as r_a was arbitrary (among right translations by generators) we get that $g \in \mathbf{M}^L$, as desired. $\square$

This completes the proof of Lemma 3.5 and, hence the proof of Proposition 3.4 as well. $\square$

Observe that the upper bounds in Theorems 1 and 2 follow:

Theorem 1 – Upper Bound. For all m, $\varepsilon(m) \leq (1+o(1))m^2$.

Proof: In order to satisfy condition (**P**) it suffices, by Remark 3, to set each B_{ij} equal to a singleton such that

$$\bigcup_{(i,j)\in\binom{[t]}{2}} B_{ij} = A \quad \text{and} \quad \sum_{(i,j)\in\binom{[t]}{2}} |B_{ij}| \leq \binom{t}{2}.$$

By (iii) in Proposition 3.4, $G(\mathbf{M}, \phi) \cong \mathbf{M}$. Hence, by (ii), we have $|E(G)| \leq \binom{t}{2}m + 2mt + 2m + 4(t+2)^2$ which, from $t = \sqrt{2m} + 1$ (see Remark 3) yields $|E(G)| \leq m^2(1 + o(1))$. $\square$

Theorem 2 – Upper Bound. For all m and $\alpha \in (0, \frac{1}{3})$, $\nu(m, \alpha) \leq (1 + o(1))m^{\frac{2}{1+\alpha}}$

Proof: For a given monoid $\mathbf{M}$ and $\alpha \in (0, \frac{1}{3})$ set $t = (1 + o(1))m^{\frac{1-\alpha}{1+\alpha}}$. Then $\binom{t}{2} > m$ and hence, by Remark 3, condition (**P**) is satisfied. Thus, by Proposition 3.4, there exists a graph G with $|V(G)| = (t + 1)(m + 4) + 5 = (1 + o(1))m^{\frac{2}{1+\alpha}}$ while

$$\begin{aligned}|E(G)| \leq & m^2 + (2 + o(1))m^{\frac{2}{1+\alpha}} + 2m + (4 + o(1))m^{\frac{2-2\alpha}{1+\alpha}} = \\ & m^2(1 + o(1)) = |V(G)|^{1+\alpha}\end{aligned}$$

as required. $\square$

Finally, we derive a bound for graphs with bounded average degree.

Proposition 3.6. For every monoid $\mathbf{M} = (X, \cdot, 1)$ with $|X| = m$ there exists a graph G such that $\mathrm{End}(G) \cong \mathbf{M}$, $|E(G)| \leq (7+o(1))|V(G)|$ and $|V(G)| = m^2 + 5m + 9$. Thus the average degree is less than 14.

Proof: If we set $t = m$ then, by Remark 3, we can satisfy (**P**) and $|V(G)| = m^2+5m+9$. By Proposition 3.4, $|E(G)| \leq (m-1)m+2m^2 + 2m + 4(m+2)^2 = (7 + o(1))m^2 \leq (7 + o(1))|V(G)|$. $\square$

3.4 Example

In this section we will construct a monoid $\mathbf{M}(n)$ of size $m = 2n+3$ such that every graph $G(\mathbf{M}(n), \phi)$ with $\mathrm{End}(G(\mathbf{M}(n), \phi)) \cong \mathbf{M}(n)$ satisfies $\binom{t}{2} \geq n$. Consequently every such graph has $\Theta(n^{\frac{3}{2}})$ vertices.

Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be disjoint n-element sets. Let 0, c and 1 be pairwise distinct elements with $0, c, 1 \notin X \cup Y$ and $E = \{\{x_i, y_j\} \mid i \neq j\}$. We define a binary operation $\odot$ on the set $Z = X \cup Y \cup \{0, c, 1\}$ by

$$z \odot w = \begin{cases} z & \text{if } w = 1, \\ w & \text{if } z = 1, \\ c & \text{if } \{z, w\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

One can easily verify that $\mathbf{M}(n) = (Z, \odot, 1)$ is a monoid. Clearly, $X \cup Y$ is a minimal set of generators for $\mathbf{M}(n)$. We prove the following bound.

Proposition 3.7. For every $n > 1$, if $\phi : \binom{[t]}{2} \longrightarrow \mathfrak{P}(X \cup Y)$ is a mapping such that $\mathrm{End}(G(\mathbf{M}(n), \phi))$ is isomorphic to $\mathbf{M}(n)$, then $\binom{t}{2} \geq n$ and $|V(G(\mathbf{M}(n), \phi))| \geq (2n+3)\sqrt{n} = \Omega(m^{\frac{3}{2}})$.

Proof: Let $\phi : \binom{[t]}{2} \longrightarrow \mathfrak{P}(X \cup Y)$ with $\phi(j, k) = B_{jk}$ for all $(j, k) \in \binom{[t]}{2}$. Let us denote $\mathbf{M} = \mathbf{M}(n)$. The statement easily follows from the following fact.

Fact. If $\mathrm{End}(G(\mathbf{M}, \phi)) \cong \mathbf{M}$, then for all $i \in [1, n]$ there exist $(j, k), (j', k') \in \binom{[t]}{2}$ such that $B_{jk} \cap X = \{x_i\}$ and $B_{j'k'} \cap Y = \{y_i\}$.

Indeed, if $\mathrm{End}(G(\mathbf{M}, \phi)) \cong \mathbf{M}$, then for every $i \in [1, n]$ there exists $(j, k) \in \binom{[t]}{2}$ with $B_{jk} \cap X = \{x_i\}$. Consequently, $|\binom{[t]}{2}| \geq n$ and hence, the underlying set of $G(\mathbf{M}, \phi)$) has cardinality at least $(2n+3)\sqrt{2n} = \Omega(m^{\frac{3}{2}})$. $\square$

Proof of Fact: We argue by contraposition, that is, suppose there exists $i \in [1, n]$ with $\{x_i\} \neq B_{jk} \cap X$ for all $(j, k) \in \binom{[t]}{2}$. If $\{y_i\} \neq B_{j'k'} \cap Y$ for all $(j', k') \in \binom{[t]}{2}$ then the proof is analogous.

By Claim 3.54, every left translation of $\mathbf{M}$ is a determining mapping of $G(\mathbf{M}, \phi)$, i.e. $\mathbf{M}^L \subseteq \mathcal{D}$. Since, by Lemma 3.2, $\mathbf{M}^L \cong \mathbf{M}$ it suffices to show that there exists a determining mapping $g \in \mathcal{D}$ which is not a left translation of $\mathbf{M}$. Consider a mapping $g : Z \to Z$ such that

$$g(z) = \begin{cases} y_i & \text{if } z = 1, \\ c & \text{if } z \in X, \\ 0 & \text{if } z \in Y \cup \{0, c\}. \end{cases}$$

Observe that g is not a left translation of $\mathbf{M}$. Indeed, from $g(1) = y_i$ it follows that if g is a left translation then it is a left translation by y_i. But $g(x_i) = c \neq 0 = y_i \odot x_i$ and hence g is not a left translation by y_i.

We show that g is a determining mapping using Lemma 3.53. Fix $z \in Z$, $(j, k) \in \binom{[t]}{2}$ and $a \in \phi(j, k)$. First assume $z \neq 1$. Then $za \in \{c, 0\}$ and hence $g(za) = 0$. Since $g(z) \in \{c, 0\}$ and $a \neq 1$ we conclude that $g(z)a = 0$ and the condition from Lemma 3.53 is fulfilled with $b = a$. Secondly, assume that $z = 1$. If $a \in Y$ then $g(z)a = 0$ and $g(za) = g(a) = 0$ and again it suffices to set $b = a$. If $a \in X$ then $g(za) = g(a) = c$. From $\{x_i\} \neq B_{jk} \cap X$ it follows that there exists $a' \in B_{jk} \cap X$ with $a' \neq x_i$. Then $g(z)a' = y_i a' = c$ and the condition of Lemma 3.53 is fulfilled. Thus, g is a determining mapping and $\mathrm{End}(G(\mathbf{M}, \phi))$ is not isomorphic to $\mathbf{M}$. $\square$

Chapter 4

Special Classes of Monoids

4.1 Groups and Generalizations

For a given monoid $\mathbf{M}$ there may be several choices for the map ϕ, other than a singleton map, yielding $\mathrm{End}(G(\mathbf{M}, \phi)) \cong \mathbf{M}$. In some cases we can exploit the algebraic structure of a monoid to construct a more "efficient" map. We will require the following technical proposition.

Proposition 4.1. For every finite non-empty set A there exists a family $\{A_i \mid i \in I\}$ of subsets of A such that $\{a\} = \bigcap\{A_i \mid i \in I,\, a \in A_i\}$ for all $a \in A$ and $|I| = 2\lceil \log |A| \rceil$.

Proof: Let $k = \lceil \log |A| \rceil$. Then there exists an injective mapping $f : A \to \{0,1\}^k$ for $k = \lceil \log |A| \rceil$. For each projection $\pi_i : \{0,1\}^k \to \{0,1\}$ with $i \in [1,k]$ let $A_{ij} = (\pi_i \circ f)^{-1}(j),\ j = 0,1$. Then the family $\{A_{ij} \mid i \in [1,k], j = 0,1\}$ has the required properties. □

Proposition 4.2. Let $\mathbf{H}$ be a group with a set A of generators. Then there exists a family $\mathcal{F} = \{B_{ij} \mid (i,j) \in \binom{[t]}{2}\}$ of subsets of A such that $\mathrm{End}(G(\mathbf{H}, \phi))$ is isomorphic to $\mathbf{H}$ and $t \leq 2\sqrt{\lceil \log |A| \rceil} + 2$.

Proof: We show that condition (**P**), required for Proposition 3.4(iii), is satisfied by a suitable choice of B_{ij}. Choose $x, x' \in \mathbf{H}$. Since $\mathbf{H}$ is a group $R(x, x')$ is a singleton. Suppose, that $R(x, x') = \{a_0\}$. By Proposition 4.1, there exists a family $\mathcal{F} = \{B_{ij} \mid (i,j) \in \binom{[t]}{2}\}$ of subsets of A such that $\{a\} = \bigcap\{B_{ij} \mid B_{ij} \in \mathcal{F},\, a \in B_{ij}\}$ for all $a \in A$. Hence, for $a \in A \setminus \{a_0\}$ there exists $(i,j) \in \binom{[t]}{2}$ with $a \in B_{ij}$ and $a_0 \notin B_{ij}$. Then $a \in B(x, x')$ and as $a \in A \setminus \{a_0\}$ was arbitrary $B(x, x') = A \setminus \{a_0\} = A \setminus R(x, x')$, that is, condition (**P**) is satisfied. Hence, by Proposition 3.4, $\mathrm{End}(G(\mathbf{H}, \phi)) \cong \mathbf{H}$. Since in order to use the construction we require $\binom{t}{2} \geq |\mathcal{F}| = 2\lceil \log |A| \rceil$ it suffices to take $t \leq 2\sqrt{\lceil \log |A| \rceil} + 2$. □

Let $\mathcal{G}$ be the family of finite groups.

Theorem 3. $\nu_{\mathcal{G}}(m) \leq (2 + o(1))m\sqrt{\log\log m}$.

Proof: We show that $\nu_{\mathcal{G}}(m) \leq (2 + o(1))m\sqrt{\log\log m}$. If $\mathbf{H}$ is an m-element group then there exists a set A of generators with $|A| \leq \lceil \log m \rceil$ and, by a combination of Propositions 3.4 and 4.2, there exists a graph G on $(m+4)(2\sqrt{\lceil \log\log m \rceil} + 3) + 5 = (2 + o(1))m\sqrt{\log\log m}$ vertices such that $\mathrm{End}(G) \cong \mathbf{H}$, as required. □

Next we generalize Theorem 3 to a class of monoids generalizing groups. Let B and C be non-empty sets and $\mathbf{H}$ be a group. Then a semigroup $\mathbf{S} = \mathcal{R}(B, \mathbf{H}, C) = (B \times H \times C, \cdot)$ where $(b, h, c)(b', h', c') = (b, hh', c')$ is called a <u>rectangular</u> <u>group</u> (see monographs of semigroups [CP] and [H]). Clearly, any rectangular group is isomorphic to a product of a non-empty left-zero semigroup (that is a semigroup satisfying the identity $xy = x$), a group and a right-zero semigroup (that is a semigroup satisfying the identity $xy = y$). We shall consider monoids of the form $\mathbf{M} = \mathbf{S}^1$ where $\mathbf{S}$ is a rectangular group. First we recall a well-known folklore statement describing generating sets of rectangular groups (the proof is an easy exercise).

Lemma 4.3. Let $\mathbf{M} = \mathbf{S}^1$ be a monoid where $\mathbf{S} = \mathcal{R}(B, \mathbf{H}, C)$ is a rectangular group. Then $U \subseteq B \times H \times C$ is a set of generators if and only if there exists a set A of generators of $\mathbf{H}$ satisfying:

(a) for every $b \in B$ there exist $h \in H$ and $c \in C$ with $(b, h, c) \in U$;

(b) for every $c \in C$ there exist $h \in H$ and $b \in B$ with $(b, h, c) \in U$;

(c) for every $a \in A$ there exist $b \in B$ and $c \in C$ with $(b, a, c) \in U$. $\square$

Using this lemma we obtain the following result.

Proposition 4.4. For every finite rectangular group $\mathbf{S} = \mathcal{R}(B, \mathbf{H}, C)$ and for every set A of generators of $\mathbf{H}$, there exist a set U of generators of $\mathbf{M} = \mathbf{S}^1$ and a family $\{B_{ij} \mid (i, j) \in \binom{[t]}{2}\}$ of subsets of U such that $\mathrm{End}(G(\mathbf{M}, \phi))$ is isomorphic to $\mathbf{M}$ and $t \leq 2\sqrt{q} + 2$ where $q = \lceil \log |B| \rceil + \lceil \log |A| \rceil + \lceil \log |C| \rceil$.

Proof: Choose $b_0 \in B$, $c_0 \in C$. Let e be the unity of $\mathbf{H}$ and let $U = U_1 \cup U_2 \cup U_3$ where

$$U_1 = \{(b, e, c_0) \mid b \in B\},$$
$$U_2 = \{(b_0, a, c_0) \mid a \in A\},$$
$$U_3 = \{(b_0, e, c) \mid c \in C\}.$$

By Lemma 4.3, U is a set of generators of $\mathbf{M}$. By Proposition 4.1, there exist families $\{B_i \mid i \in I_1\}$ of subsets of $\{(b, e, c_0) \mid b \in B\}$, $\{C_i \mid i \in I_2\}$ of subsets of $\{(b_0, e, c) \mid c \in C\}$ and $\{A_i \mid i \in I_3\}$ of subsets of $\{(b_0, a, c_0) \mid a \in A\}$ such that

- $\{(b,e,c_0)\} = \bigcap\{B_i \mid i \in I_1,\ (b,e,c_0) \in B_i\}$ for all $b \in B$;
- $\{(b_0,e,c)\} = \bigcap\{C_i \mid i \in I_2,\ (b_0,e,c) \in C_i\}$ for all $c \in C$;
- $\{(b_0,a,c_0)\} = \bigcap\{A_i \mid i \in I_3,\ (b_0,a,c_0) \in B_i\}$ for all $a \in A$,

where $|I_1| = 2\lceil \log |B| \rceil$, $|I_2| = 2\lceil \log |C| \rceil$, and $|I_3| = 2\lceil \log |A| \rceil$. Next we prove that the family

$$\{B_{ij} \mid (i,j) \in \binom{[t]}{2}\} = \{B_i \mid i \in I_1\} \cup \{C_i \mid i \in I_2\} \cup \{A_i \mid i \in I_3\}$$

with $t = \lceil 2\sqrt{q} + 1 \rceil$ where

$$q = \lceil \log |B| \rceil + \lceil \log |A| \rceil + \lceil \log |C| \rceil$$

yields $\mathrm{End}(G(\mathbf{M},\phi)) \cong \mathbf{M}$. Clearly, $|\binom{t}{2}| \geq |I_1| + |I_2| + |I_3|$. Observe that for all $u \in U$

$$\{u\} = \bigcap\{B_{ij} \mid (i,j) \in \binom{[t]}{2},\ u \in B_{ij}\}. \tag{4.1}$$

To complete the proof, by Lemma 3.5, we will verify condition (**P**). We show that for all $x, x' \in (B \times H \times C) \cup \{1\}$

$$B(x,x') = U \setminus R(x,x'). \tag{4.2}$$

By Remark 2, it is sufficient to show that $U \setminus R(x,x') \subseteq B(x,x')$.

First observe that if $|R(x,x') \cap U| \leq 1$ then, by (1), for every $u \in U \setminus R(x,x')$ there exists $(i,j) \in \binom{[t]}{2}$ with $u \in B_{ij}$ and $B_{ij} \cap R(x,x') = \emptyset$. Thus equation (2) holds. To complete the argument we must consider the case $|R(x,x') \cap U| > 1$. Therefore we describe $R(x,x')$ for $x, x' \in X$. Clearly,

$$R(x,x') = \begin{cases} \{x'\} & \text{if } x = 1, \\ \emptyset & \text{if } x \neq 1 \text{ and } x' = 1 \text{ or} \\ & x = (b,h,c), x' = (b',h',c'),\ b \neq b', \\ \{(\bar{b}, h^{-1}h', c') \mid \bar{b} \in B\} & \text{if } x = (b,h,c) \text{ and } x' = (b,h',c'). \end{cases}$$

Observe that $|R(x,x')| > 1$ only if $x = (b,h,c)$ and $x' = (b,h',c')$. Next observe that if $c' \neq c_0$, then, by the definition of U, $R(x,x') \subseteq U_3$ and hence $|R(x,x') \cap U| \leq 1$ (since c' in $R(x,x')$ is fixed). Similarly, if $c' = c_0$ and $h^{-1}h' \neq e$ we obtain $R(x,x') \subseteq U_2$ and hence $|R(x,x') \cap U| \leq 1$ holds (since $h^{-1}h'$ in $R(x,x')$ is fixed) as well. Thus $|R(x,x') \cap U| > 1$ if and only if $c' = c_0$ and $h^{-1}h' = e$ (this is equivalent to $h = h'$) in which case $R(x,x') = U_1$. Consequently (since $\{(b_0,e,c_0)\} = U_1 \cap U_2 \cap U_3$)

$$U \setminus R(x,x') = U_2 \cup U_3 \setminus \{(b_0,e,c_0)\}.$$

If $u = (b_0,e,c) \in U_3 \setminus \{(b_0,e,c_0)\}$ then, by the property of $\{C_i \mid i \in I_2\}$, there exists $(i,j) \in \binom{[t]}{2}$ with $u \in B_{ij} \subseteq \{(b_0,e,c) \mid c \in C\}$ and $(b_0,e,c_0) \notin B_{ij}$. Hence $B_{ij} \cap R(x,x') = \emptyset$ and thus $u \in B(x,x')$. If $u = (b_0,a,c_0) \in U_2 \setminus \{(b_0,e,c_0)\}$ then, by the property of $\{A_i \mid i \in I_3\}$, there exists $(i,j) \in \binom{[t]}{2}$ with $u \in B_{ij} \subseteq \{(b_0,a,c_0) \mid a \in A\}$ and $(b_0,e,c_0) \notin B_{ij}$. Hence $B_{ij} \cap R(x,x') = \emptyset$ and thus $u \in B(x,x')$ and (2) holds. $\square$

Let $\mathcal{RG}$ denote the class of monoids $\mathbf{M} = \mathbf{S}^1$ where $\mathbf{S}$ is a finite rectangular group.

Theorem 4. For all m, $\nu_{\mathcal{RG}}(m) \leq (2+o(1))m\sqrt{\log m}$.

Proof: We need to show that $\nu_{\mathcal{RG}}(m) \leq (2+o(1))m\sqrt{\log m}$. If $\mathbf{M} = \mathbf{S}^1$ is a monoid on an m-element set where $\mathbf{S} = \mathcal{R}(B,\mathbf{H},C)$ is a rectangular group then, by Propositions 4.4 and 3.4, there exists a graph G on a set of size $(m+4)(2\sqrt{q}+3)+5$ with $q = \lceil \log |B| \rceil + \lceil \log |A| \rceil + \lceil \log |C| \rceil$ where A is a set of generators of $\mathbf{H}$ such that $\mathrm{End}(G)$ is isomorphic to $\mathbf{M}$. Clearly, $q \leq 3 + \log m$ and, hence, $(m+4)(2\sqrt{\log m + 3} + 3) + 5 \leq (2+o(1))m\sqrt{\log m}$, from which the statement follows. $\square$

We recall that for the class $\mathcal{RZ}$ of all monoids $\mathbf{M} = \mathbf{R}^1$ where $\mathbf{R}$ is a finite right-zero semigroup (i.e. $xy = y$ for all elements x and y) one can show $\nu_{\mathcal{RZ}}(m) \leq 6m + 9$. This follows from the construction in the proof of Theorem 3 in [KR3].

For a natural number $k \geq 1$, we say that a monoid $\mathbf{M} = (X, \cdot, e)$ with a set of generators A is <u>weakly</u> k-<u>right</u> <u>cancellative</u> if $|\{a \in A \mid xa = x'\}| \leq k$ for all $x, x' \in X$. Recall that a finite monoid is a group if and only if it is 1-right cancellative. In this sense it is a generalization of a group.

Proposition 4.5. Let $\mathbf{M} = (X, \cdot, e)$ with $|X| = m$ be a finite weakly k-right cancellative monoid with respect to a set of generators A. Then there exists a family $\{B_{ij} \mid (i,j) \in \binom{[t]}{2}\}$ of subsets of A such that $\mathrm{End}(G(\mathbf{M}, \phi))$ is isomorphic to $\mathbf{M}$ with $t = \lceil \sqrt{2|A|} + 1 \rceil$.

Moreover, for $s = \min\{m^2|A|, \binom{|A|}{k}|A|\}$ if $|A| \geq (k+1)e \ln s$ then $t \leq \sqrt{2(k+1)e \ln s} + 1$.

Proof: We show that there exists a family of subsets of A satisfying condition $(\mathbf{P})$. Then we can apply Proposition 3.4. Specifically, we show that, with t as above, there exists a family $\{B_{ij} \mid (i,j) \in \binom{[t]}{2}\}$ such that for every pair $x, x' \in X$ and every $a \in A$ with $a \notin R(x, x')$ there exists $(i,j) \in \binom{[t]}{2}$ with

(i) $a \in B_{ij}$;

(ii) $B_{ij} \cap R(x, x') = \emptyset$.

First, observe that if $t = \lceil \sqrt{2|A|} + 1 \rceil$ then $\binom{t}{2} \geq |A|$ and we can satisfy the above conditions with each B_{ij} a singleton, that is, by taking $B_{ij} = \{a\}$ where a runs through A and (i,j) runs through $\binom{[t]}{2}$.

If, moreover $|A| \geq (k+1)e \ln s$ we can improve the bound on t using the following probabilistic argument: for each $(i,j) \in \binom{[t]}{2}$ a set B_{ij} is a random subset of A obtained by placing each $a \in A$, randomly, in B_{ij} with probability $\frac{1}{k+1}$. The choices are made independently for $(i,j) \in \binom{[t]}{2}$. Fix a set $R \subseteq A$ with $|R| \leq k$ and $a \in A \setminus R$. For each $(i,j) \in \binom{[t]}{2}$ let E_{ij} denote the event that $a \in B_{ij}$ and $R \cap B_{ij} = \emptyset$. Since

$$\mathrm{Prob}(R \cap B_{ij} = \emptyset) = \left(1 - \frac{1}{k+1}\right)^{|R|} \geq \left(1 - \frac{1}{k+1}\right)^{k} \geq \frac{1}{e}$$

we conclude that $\mathrm{Prob}(E_{ij}) \geq \frac{1}{(k+1)e}$ which, in turn, yields

$$\mathrm{Prob}\Big(\bigwedge_{1 \leq i < j \leq t} \neg E_{ij}\Big) \leq \left(1 - \frac{1}{(k+1)e}\right)^{\binom{t}{2}} \qquad (*)$$

In order to ensure condition (**P**) we need to satisfy $\bigwedge_{1 \leq i < j \leq t} \neg E_{ij}$ for all $R = R(x,x')$ and $a \in A \setminus R(x,x')$. We consider two cases:

Case 1. If $m^2 < \binom{|A|}{k}$ then, in view of $(*)$, we need

$$\left(1 - \frac{1}{(k+1)e}\right)^{\binom{t}{2}} m^2|A| < e^{-\frac{\binom{t}{2}}{(k+1)e}} m^2|A| < 1$$

which is satisfied if $t \geq \sqrt{2(k+1)e \ln(m^2|A|)} + 1$.

Case 2. If $m^2 \geq \binom{|A|}{k}$ we use the fact that $\mathbf{M}$ is weakly k-right concellative with respect to the set of generators A. In order to verify condition (**P**), it suffices to show conditions (i) and (ii) hold with $\{R(x,x') \mid x,x' \in X\}$ replaced by the broader family $\binom{[A]}{k}$, the set of all k-tuples of A. This is because for each x,x' and $a \notin R(x,x')$ we may verify (i) and (ii) for $R \in \binom{A}{k}$ with $a \notin R$ and $R(x,x') \subseteq R$. Thus, we need

$$\left(1 - \frac{1}{(k+1)e}\right)^{\binom{t}{2}} \binom{|A|}{k} |A| < 1$$

which is satisfied if $t \geq \sqrt{2(k+1)e\ln(\binom{|A|}{k}|A|)} + 1$. $\square$

For a natural number $k \geq 1$, let $\mathcal{C}_k$ consist of all finite, weakly k-right cancellative monoids.

Theorem 5. For all m, $\nu_{\mathcal{C}_k}(m) \leq (5 + o(1))m\sqrt{k \ln m}$.

Proof: We will show that $\nu_{\mathcal{C}_k}(m) \leq 5m\sqrt{k \ln m}$ for $m \geq m_0$. By Proposition 3.4, there exists a graph G with endomorphism monoid $\mathbf{M}$ on at most $(t+1)(m+4)$ vertices. Since $\ln s \leq 3 \ln m$, Proposition 4.5 gives $t \leq \sqrt{6(k+1)e \ln m + 1}$. Consequently, since $k \geq 2$ implies $\sqrt{6e(k+1)} < 5\sqrt{k}$ we get, for sufficiently large m, that $\nu_{\mathcal{C}_k}(m) \leq (m+4)(t+1) < 5m\sqrt{k \ln m}$. $\square$

4.2 Completely Simple Monoids

The aim of this section is to prove Theorem 6. One of the most important classes of semigroups generalizing groups is the class of completely simple semigroups, see [CP].

Definition 4.6. [CP] A semigroup $\mathbf{S}$ is <u>completely</u> <u>simple</u> if and only if there exists no proper two-sided ideal of $\mathbf{S}$ and there exist no distinct idempotents e and f of $\mathbf{S}$ with $ef = fe = e$.

While this is the standard definition we use an equivalent (see Rees Theorem – Theorem 1.3.2 in [H] or Theorem 3.5 in [CP]), more combinatorial description which will allow us to parallel the techniques in Section 3. Let B and C be sets, $\mathbf{H} = (H, \cdot, e)$ be a group with identity e and let $P = \{p_{c,b}\}_{c \in C, b \in B}$ be a $|C| \times |B|$ matrix with entries from H.

Then $\mathcal{R}(B, \mathbf{H}, C; P)$ is a semigroup $(B \times H \times C, \odot)$ where

$$(b, h, c) \odot (b', h', c') = (b, hp_{c,b'}h', c')$$

(here the product $hp_{c,b'}h'$ is considered in the group $\mathbf{H}$). It is well-known that $\mathcal{R}(B, \mathbf{H}, C; P)$ is a semigroup. Rees' Theorem says that a semigroup $\mathbf{S}$ is completely simple if and only if it is isomorphic to $\mathcal{R}(B, \mathbf{H}, C; P)$ for some sets B and C, a group $\mathbf{H}$ and a $|C| \times |B|$ matrix P with entries from $\mathbf{H}$.

Note that a semigroup $\mathbf{S}$ is a rectangular group, discussed in the previous section, if and only if $\mathbf{S}$ is isomorphic to $\mathcal{R}(B, \mathbf{H}, C; P)$ where $p_{b,c} = e$ for all $b \in B$ and $c \in C$. One can easily verify that if B or C is a singleton set then $\mathcal{R}(B, \mathbf{H}, C; P)$ is a rectangular group. This follows from Proposition 4.7. Consequently in view of Theorem 4 we can restrict ourselves to the case when both B and C have cardinality at least 2. We recall the important properties of matrix representations.

Proposition 4.7. [CP] Let sets B, C and $b_0 \in B$, $c_0 \in C$ be given. Let $\mathbf{H}$ be a group and P be a $|C| \times |B|$-matrix over $\mathbf{H}$. Then there exists a $|C| \times |B|$-matrix P' over $\mathbf{H}$ such that

(i) $p'_{c,b_0} = p'_{c_0,b} = e$ for all $b \in B$ and $c \in C$;

(ii) the semigroups $\mathcal{R}(B, \mathbf{H}, C; P)$, and $\mathcal{R}(B, \mathbf{H}, C; P')$ are isomorphic. $\square$

We say that a $|C| \times |B|$ matrix P over a group $\mathbf{H}$ is (c_0, b_0)-<u>standardized</u> for $b_0 \in B$ and $c_0 \in C$ if $p_{c,b_0} = p_{c_0,b} = e$ for all $b \in B$ and $c \in C$.

We say that a monoid $\mathbf{M}$ is <u>completely</u> <u>simple</u> if $\mathbf{M}$ is isomorphic to $\mathbf{S}^1$ for a completely simple semigroup $\mathbf{S}$. Let $\mathcal{CS}$ denote the class of all finite, completely simple monoids.

In what follows we assume that finite, non-empty sets B and C, elements $b_0 \in B$, $c_0 \in C$, a finite group $\mathbf{H}$ and a (c_0, b_0)-standardized $|C| \times$

$|B|$ matrix P over $\mathbf{H}$ are given. Let $\mathbf{M} = (X, \cdot, 1) = \mathcal{R}(B, \mathbf{H}, C; P)^1$ be a completely simple monoid such that $X = \{B \times H \times C\} \cup \{1\}$ and $|X| = m = |B||H||C| + 1$. First we give a folklore statement regarding generators of $\mathbf{M}$ (generalizing Lemma 4.3).

Lemma 4.8. A set $U \subseteq B \times H \times C$ is a set of generators of $\mathbf{M}$ whenever there exists a set A of generators of $\mathbf{H}$ such that

1. for every $b \in B$ there exist $h \in H$ and $c \in C$ with $(b, h, c) \in U$;
2. for every $c \in C$ there exist $h \in H$ and $b \in B$ with $(b, h, c) \in U$;
3. for every $a \in A$ there exist $b \in B$ and $c \in C$ with $(b, a, c) \in U$. $\square$

Let $U \subseteq A \times H \times B$ be a set of generators and let $\phi : \binom{[t]}{2} \to \mathfrak{P}(U)$ be a mapping with $(i, j) \mapsto B_{ij}$. Instead of using condition (**P**), we will use Lemma 4.9 which gives three conditions on ϕ that imply (**P**). Then, for a suitable t, we construct ϕ satisfying these conditions. The construction then yields an upper estimate on t.

Lemma 4.9. Suppose the family $\{B_{ij} \mid (i, j) \in \binom{[t]}{2}\}$ satisfies:

(h1) For every $c, c' \in C$ with $c \neq c'$ there exists $(i, j) \in \binom{[t]}{2}$ such that $U \cap (B \times H \times \{c\}) \subseteq B_{ij}$ and $B_{ij} \cap (B \times H \times \{c'\}) = \emptyset$.

(h2) For every $(b, h, c) \in U$ and every $(b', h') \in B \times H$ with $(b, h) \neq (b', h')$ there exists $(i, j) \in \binom{[t]}{2}$ such that $(b, h, c) \in B_{ij}$ and $(b', h', c) \notin B_{ij}$.

(h3) For every $(b, h, c) \in U$, every $c' \in C$ and every $h' \in H$ with $p_{c',b}h \neq h'$ there exists $(i, j) \in \binom{[t]}{2}$ such that $(b, h, c) \in B_{ij}$ and if $(\bar{b}, \bar{h}, c) \in B_{ij}$ then $p_{c',\bar{b}}\bar{h} \neq h'$.

Then ϕ satisfies condition (**P**) and, thus, $\mathrm{End}(F_{\mathcal{P}} * H) \cong \mathbf{M}$.

Proof: Recall that we need to show for every pair $x_1, x_2 \in X$

$$B(x_1, x_2) = \bigcup_{ij} \{B_{ij} \mid B_{ij} \cap R(x_1, x_2) = \emptyset\} = U \setminus R(x_1, x_2).$$

As $B(x_1, x_2) \subseteq U \setminus R(x_1, x_2)$ is clear, we show only the reverse inclusion. We consider several cases:

(a) If $x_2 = 1$ then $R(x_1, x_2) \cap U = \emptyset$. Consequently, condition (**P**) becomes $B(x_1, x_2) = U$. To show this equality fix $(b, h, c) \in U$. By (h1) (applied with c' arbitrary), there exists $(i, j) \in \binom{[t]}{2}$ such that $U \cap (B \times H \times \{c\}) \subseteq B_{ij}$ and thus $(b, h, c) \in B_{ij}$. Since $B_{ij} \cap R(x_1, x_2) = \emptyset$, by assumption, $(b, h, c) \in B(x_1, x_2)$. As (b, h, c) was an arbitrary element from U we get $B(x_1, x_2) = U$.

(b) If $x_1 = 1$ then $R(x_1, x_2) = \{x_2\}$. Consequently, condition (**P**) becomes $B(x_1, x_2) = U \setminus \{x_2\}$. As above, one containment is clear and we need to show only $B(x_1, x_2) \supset U \setminus \{x_2\}$. To this end set $x_2 = (b_2, h_2, c_2)$, we are going to show that an arbitrary $x = (b, h, c) \in U \setminus \{x_2\}$ belongs to $B(x_1, x_2)$. We distinguish two subcases: namely, $c \neq c_2$ and $c = c_2$.

If $c \neq c_2$ then, by (h1), there exists $(i, j) \in \binom{[t]}{2}$ such that $U \cap (B \times H \times \{c\}) \subseteq B_{ij}$ and $B_{ij} \cap B \times H \times \{c_2\} = \emptyset$. Thus $(b, h, c) \in B_{ij}$ and $B_{ij} \cap R(x_1, x_2) = \emptyset$. Hence, $(b, h, c) \in B(x_1, x_2)$.

If $c = c_2$ then $(b, h) \neq (b_2, h_2)$ and so, by (h2), there exists $(i, j) \in \binom{[t]}{2}$ such that $x_2 \notin B_{i,j}$ and $(b, h, c) = (b, h, c_2) \in B_{ij}$. Consequently, $R(x_1, x_2) \cap B_{ij} = \emptyset$ and $(b, h, c) \in B(x_1, x_2)$, as required.

(c) The remaining case is $x_1 = (b_1, h_1, c_1),\ x_2 = (b_2, h_2, c_2)$. Then

$$R(x_1, x_2) = \{(\bar{b}, \bar{h}, c_2) \mid \bar{b} \in B, \bar{h} \in H, h_1 p_{c_1, \bar{b}} \bar{h} = h_2\}. \tag{4.3}$$

Again, we need to show

$$U \setminus \{(\bar{b}, \bar{h}, c_2) \mid \bar{b} \in B, \bar{h} \in H, h_2 = h_1 p_{c_1, \bar{b}} \bar{h}\} \subset B(x_1, x_2).$$

Assume, therefore, that $(b,h,c) \in U \setminus R(x_1,x_2)$. If $c \neq c_2$ then, by (h1) (applied with $c' = c_2$), $(b,h,c) \in B(x_1,x_2)$. Thus we can assume that $c = c_2$ which implies that $h_2 \neq h_1 p_{c_1,b} h$. Consequently $p_{c_1,b} h \neq h_1^{-1} h_2$ and, by (h3) (applied with $h' = h_1^{-1} h_2$ and $c' = c_1$), there exists $(i,j) \in \binom{[t]}{2}$ with $(b,h,c) \in B_{ij}$ and if $(\bar{b},\bar{h},c) \in B_{ij}$ then $p_{c_1,\bar{b}} \bar{h} \neq h_1^{-1} h_2$ (i.e. $h_1 p_{c_1,\bar{b}} \bar{h} \neq h_2$). Since we have just established that $(b,h,c) \in B_{ij}$ it remains to show that $B_{ij} \cap R(x_1,x_2) = \emptyset$. To this end let $(\bar{b},\bar{h},\bar{c}) \in B_{ij}$. If $\bar{c} \neq c_2$ then, by (3), $(\bar{b},\bar{h},\bar{c}) \notin R(x_1,x_2)$. If $\bar{c} = c_2 (= c)$ then $h_1 p_{c_1,\bar{b}} \bar{h} \neq h_2$ and, by (3), $(\bar{b},\bar{h},\bar{c}) \notin R(x_1,x_2)$ as well. In either case $B_{ij} \cap R(x_1,x_2) = \emptyset$ follows. Consequently, $B(x_1,x_2) = U \setminus R(x_1,x_2)$, completing the proof. $\square$

Observe that for $i = 1,2,3$ if a family $\mathcal{F}_i$ satisfies condition (hi) then the family $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ satisfies (h1), (h2) and (h3). In Lemma 4.10 we will construct families satisfying (h1) and (h2).

Lemma 4.10. For every set $U \subseteq B \times H \times C$ of generators of $\mathbf{M}$ there exist a family $\mathcal{F}_1 = \{Y_i \mid i \in I_1\}$ of subsets of U satisfying (h1) with $|I_1| = 2\lceil \log |C| \rceil$ and a family $\mathcal{F}_2 = \{X_i \mid i \in I_2\}$ of subsets of U satisfying (h2) with $|I_2| = 2\lceil \log |B \times H| \rceil$.

Proof: Let $U \subseteq B \times H \times C$ be a set of generators. By Proposition 4.1, there exists

- a family $\{T_i \mid i \in I_1\}$ of subsets of C such that $|I_1| = 2\lceil \log |C| \rceil$ and $\{c\} = \bigcap \{T_i \mid i \in I_1,\, c \in T_i\}$ for all $c \in C$.
- a family $\{W_i \mid i \in I_2\}$ of subsets of $B \times H$ such that $|I_2| = \lceil \log |B \times H| \rceil$ and $\{y\} = \bigcap \{W_i \mid i \in I_2,\, y \in W_i\}$ for all $y \in B \times H$.

Define $Y_i = \{(b,h,c) \in U \mid c \in T_i\}$ for all $i \in I_1$, $X_i = \{(b,h,c) \in U \mid (b,h) \in W_i\}$ for all $i \in I_2$ and consider the families $\mathcal{F}_1 = \{Y_i \mid i \in I_1\}$

and $\mathcal{F}_2 = \{X_i \mid i \in I_2\}$. If c_1 and c_2 are distinct elements of C then there exists $i \in I_1$ with $c_1 \in T_i$ and $c_2 \notin T_i$. Hence $U \cap (B \times H \times \{c_1\}) \subseteq Y_i$ and $(B \times H \times \{c_2\}) \cap Y_i = \emptyset$. Thus $\mathcal{F}_1$ satisfies (h1). If $(b, h, c) \in U$ and $(b', h') \in B \times H$ with $(b, h) \neq (b', h')$ then there exists $i \in I_2$ with $(b, h) \in W_i$ and $(b', h') \notin W_i$. Then $(b, h, c) \in \{(b, h)\} \times C \subseteq X_i$ and $(\{(b', h')\} \times C) \cap X_i = \emptyset$, thus $(b', h', c) \notin X_i$. Whence $\mathcal{F}_2$ satisfies (h2). $\square$

Next we give two constructions for a family satisfying condition (h3). Then, depending on the size of C, we choose the smaller family $\mathcal{F}_3$ and show that it has size $O(m^{\frac{1}{3}})$ yielding $|G(\mathbf{M}, \phi)| = O(m^{\frac{7}{6}})$.

Lemma 4.11. For every set $U \subseteq B \times H \times C$ of generators of $\mathbf{M}$ there exists a family $\mathcal{F}_3$ of subsets of U satisfying (h3) of size $2|C|\lceil \log |H| \rceil$.

Proof: By Lemma 4.1, there exists a family $\{H_i \mid i \in I_3\}$ of subsets of H such that $\{h\} = \bigcap\{H_i \mid i \in I_3,\ h \in H_i\}$ for all $h \in H$ and $|I_3| = 2\lceil \log(|H|) \rceil$. For every $c \in C$ and $i \in I_3$ define a set $D_{i,c} = \{(\bar{b}, \bar{h}, \bar{c}) \in U \mid p_{c,\bar{b}}\bar{h} \in H_i\}$. Let $(b, h, c) \in U$, $c' \in C$ and $h' \in H$ with $p_{c',b}h \neq h'$ be given. Then there exists $i \in I_3$ with $p_{c',b}h \in H_i$ and $h' \notin H_i$. We prove that $(b, h, c) \in D_{i,c'}$ and if $(\bar{b}, \bar{h}, \bar{c}) \in D_{i,c'}$ then $p_{c',\bar{b}}\bar{h} \neq h'$. Thus $D_{i,c'}$ satisfies the statement stronger than condition (h3) for given (b, h, c), c' and h'. Since $p_{c',b}h \in H_i$ we conclude that $(b, h, c) \in D_{i,c'}$. If $(\bar{b}, \bar{h}, \bar{c}) \in D_{i,c'}$ then $p_{c',\bar{b}}\bar{h} \in H_i$. On the other hand, $h' \notin H_i$ and we infer that $p_{c',\bar{b}}\bar{h} \neq h'$. Whence the family $\{D_{i,c} \mid i \in I_3,\ c \in C\}$ satisfies condition (h3). $\square$

Lemma 4.12. There exists a set $U \subseteq B \times H \times C$ of generators of $\mathbf{M}$ and a family $\mathcal{F}_3$ of subsets of U satisfying (h3) of size $\left\lceil \frac{|B|}{|C|} \right\rceil + 2\lceil \log \log |H| \rceil$.

Proof: First we construct a set $U \subseteq B \times H \times C$ of generators. Clearly, there exists a family $\{B_i \mid i \in I_3\}$ of subsets of B such that $B = \bigcup_{i \in I_3} B_i$, $I_3 = \left\lceil \frac{|B|}{|C|} \right\rceil$ and $|B_i| \leq |C|$ for all $i \in I_3$. For every $i \in I_3$ choose a surjective mapping $\psi_i : C \to B_i$. Let A be a set of generators of $\mathbf{H}$ with $|A| \leq \log |H|$ and $e \notin A$ (e is the unity of $\mathbf{H}$). Then $U = \{(\psi_i(c), e, c) \mid c \in C,\, i \in I_3\} \cup \{(b_0, a, c_0) \mid a \in A\}$. By Lemma 4.8, since $B = \bigcup \{\mathrm{Im}(\psi_i) \mid i \in I_3\}$, U is a set of generators. Next we shall construct a family of subsets of U. By Proposition 4.1, there exists a family $\{A_i \mid i \in I_4\}$ of subsets of A such that $\{a\} = \bigcap \{A_i \mid i \in I_4,\, a \in A_i\}$ for all $a \in A$ and $|I_4| = 2\lceil \log |A| \rceil \leq 2\lceil \log \log |H| \rceil$. Without loss of generality we can assume that $I_3 \cap I_4 = \emptyset$. Now define

- $D_i = \{(\psi_i(c), e, c) \mid c \in C\}$ for $i \in I_3$,
- $D_i = \{(b_0, a, c_0) \mid a \in A_i\}$ for $i \in I_4$,
- $\mathcal{F}_3 = \{D_i \mid i \in I_3 \cup I_4\}$.

Since $|I_3 \cup I_4| \leq \left\lceil \frac{|B|}{|C|} \right\rceil + 2\lceil \log \log |H| \rceil$ it suffices to prove that condition (h3) is satisfied. For this purpose assume that $(b, h, c) \in U$, $c' \in C$ and $h' \in H$ with $p_{c',b} h \neq h'$ are given. First assume that $h = e$. Then, $h \notin A$, there exists $i \in I_3$ with $b = \psi_i(c)$. Clearly, then $(b, h, c) \in D_i$ and if $(\bar{b}, \bar{h}, c) \in D_i$ then $(\bar{b}, \bar{h}, c) = (b, h, c)$. Thus $(\bar{b}, \bar{h}, c) \in D_i$ implies $p_{c',\bar{b}} \bar{h} \neq h'$ and thus in this case condition (h3) is satisfied. Secondly, if $h \neq e$ then $h \in A$, $b = b_0$ and $c = c_0$ and there exists $i \in I_4$ with $h \in A_i$ and $h' \notin A_i$. Hence $(b, h, c) \in D_i$ and if $(\bar{b}, \bar{h}, \bar{c}) \in D_i$ then $\bar{b} = b_0$, $\bar{c} = c_0$ and $\bar{h} \in A_i$. Since P is (c_0, b_0)-standardized $p_{c',\bar{b}} = e$ and consequently $p_{c',\bar{b}} \bar{h} = \bar{h}$. On the other hand, as $\bar{h} \in A_i$ and $h' \notin A_i$, we have $p_{c',\bar{b}} \bar{h} = \bar{h} \neq h'$. Thus (h3) is satisfied. $\square$

We recall that $\mathcal{CS}$ is the class of all monoids $\mathbf{M} = \mathbf{S}^1$ where $\mathbf{S}$ is a finite, completely simple semigroup.

Theorem 6. For all m, $\nu_{\mathcal{CS}}(m) \leq (2+o(1))m^{\frac{7}{6}}$.

Proof: We need to show that $\nu_{\mathcal{CS}}(m) \leq (2+o(1))m^{\frac{7}{6}}$. Let $\mathbf{M} = (X, \cdot, 1)$ be a completely simple monoid isomorphic to $\mathcal{R}(B, \mathbf{H}, C; P)^1$ for some finite non-empty sets B and C, a finite group $\mathbf{H}$ and a $|C| \times |B|$ matrix P over $\mathbf{H}$. Let $|X| = m$. If $|C| \leq \frac{m^{\frac{1}{3}}}{\lceil \log |H| \rceil}$ then we can apply Lemmas 4.10 and 4.11 to a set $U \subseteq B \times H \times C$ of generators of $\mathbf{M}$ yielding a family $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ of subsets of U satisfying conditions (h1), (h2) and (h3) with

$$\begin{aligned}
|\mathcal{F}| &\leq 2\lceil \log |C| \rceil + 2\lceil \log |B \times H| \rceil + \left\lceil \frac{m^{\frac{1}{3}}}{\lceil \log |H| \rceil} 2\lceil \log |H| \rceil \right\rceil \\
&\leq 2\left(\lceil m^{\frac{1}{3}} \rceil + \lceil \log |B \times H \times C| \rceil + 1 \right) \\
&= (2+o(1))m^{\frac{1}{3}}.
\end{aligned}$$

If $|C| \geq \frac{m^{\frac{1}{3}}}{\lceil \log |H| \rceil}$ then we can apply Lemmas 4.10 and 4.12 to obtain a set $U \subseteq B \times H \times C$ of generators of $\mathbf{M}$ and a family $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ of subsets of U satisfying conditions (h1), (h2) and (h3). First we bound the size of $\mathcal{F}_3$. Indeed, by Lemma 4.12

$$\begin{aligned}
|\mathcal{F}_3| &= \left\lceil \frac{|B|}{|C|} \right\rceil + 2\lceil \log \log |H| \rceil = \left\lceil \frac{|B||C||H|}{|C|^2|H|} \right\rceil + 2\lceil \log \log |H| \rceil \\
&\leq \frac{m \log^2 |H|}{m^{\frac{2}{3}}|H|}(1+o(1)) = \frac{m^{\frac{1}{3}} \log^2 |H|}{|H|}(1+o(1)) \\
&\leq 1.2m^{\frac{1}{3}}(1+o(1))
\end{aligned}$$

where the last inequality holds since $\frac{\log^2 x}{x} \leq 1.2$ for all $x \in \mathbb{N}$.

By Lemma 4.10 we have

$$|\mathcal{F}_1 \cup \mathcal{F}_2| \leq 2\lceil \log |B \times H| \rceil + 2\lceil \log |C| \rceil \leq o(m^{\frac{1}{3}})$$

and consequently $|\mathcal{F}| \leq (2 + o(1))m^{\frac{1}{3}}$.

Finally, since t is the smallest natural number such that $\binom{t}{2} \geq |\mathcal{F}|$ we get, in either of the above cases, that $t = (2 + o(1))m^{\frac{1}{6}}$ and, using Lemma 4.9, we conclude that $\nu_{\mathcal{CS}}(m) = (2 + o(1))m^{\frac{7}{6}}$. $\square$

Chapter 5

Generalizing the $\mathcal{P}$-graph

5.1 A general construction

The aim of this section is to extend the (generalized) $\mathcal{P}$-graph construction, described below, to subdirect products of monoids. We reformalize our definition of $\mathcal{P}$-graph as follows.

Definition. A pair $\mathbf{G} = (G, \mathcal{P})$ is a partition-graph (or a $\mathcal{P}$-graph for short) if $G = (V, E)$ is a graph and $\mathcal{P} = \{V_i \mid i \in I\}$ is a partition of V into independent sets and there exists $0 \in I$ (we shall distinguish the set V_0 by the term "axis") such that for any $i \in I \setminus \{0\}$ the graph induced on $V_0 \cup V_i$ is a star forest saturating V_i. More explicitly the following holds:

(d1) for every $i \in I \setminus \{0\}$ and for every $x \in V_i$ there exists $y \in V_0$ with $\{x, y\} \in E$;

(d2) for every $i \in I$ and every $y \in V_0$ there exists at most one $x \in V_i$ with $\{x, y\} \in E$.

We call $|I|$ the width of $\mathbf{G}$, and denote it by $w(\mathbf{G})$.

Definition. We say that a $\mathcal{P}$-graph $\mathbf{G}$ represents a monoid $M = (M, \cdot, e)$ if the axis V_0 can be identified with M such that:

(r1) the restriction of every $f \in \mathrm{End}_{\mathcal{P}}(\mathbf{G})$ to M is a left translation of M;

(r2) for every left translation λ of M there exists $f \in \mathrm{End}_{\mathcal{P}}(\mathbf{G})$ with $f(x) = \lambda(x)$ for all $x \in M$;

(r3) for distinct $f, f' \in \mathrm{End}_{\mathcal{P}}(\mathbf{G})$ there exists $x \in M$ with $f(x) \neq f'(x)$.

It is easy to see that $\mathcal{P}$-graphs constructed in Proposition 3.4 represent monoids in the sense of our definition.

Observe that if $\mathbf{G}$ represents a monoid M (we assume that $M = V_0$) then the restriction of a given $f \in \mathrm{End}_{\mathcal{P}}(\mathbf{G})$ to M is a left translation by $f(e)$ and hence, if f and f' are distinct $\mathcal{P}$-endomorphisms, then $f(e) \neq f'(e)$.

For a monoid M let $\Pi(M)$ denote the least natural number n such that there exists a $\mathcal{P}$-graph G representing M with $|V(G)| \leq n$ and $\mathrm{End}(G) \cong M$. For a class $\mathcal{M}$ of monoids set

$$\Pi_{\mathcal{M}}(m) = \max\{\Pi(M) \mid M \in \mathcal{M}, |M| = m\}.$$

As before, we can restrict ourselves to an investigation of $\mathcal{P}$-endomorphism monoids. Using our enhanced notation, we can restate Proposition 3.1 as follows.

Theorem 5.1. Let $\mathbf{G}$ be a $\mathcal{P}$-graph. Then there exists a graph H with $\mathrm{End}_{\mathcal{P}}(\mathbf{G}) \cong \mathrm{End}(H)$ and $|H| = |G| + 4w(\mathbf{G}) + 5$.

The above construction for the graph H yields the best known upper bound for the general case. For products, however, instead of applying the construction directly we will get an improved bound by "piecing together" the $\mathcal{P}$-graphs corresponding to the factors in an efficient way. Let $M^{(1)}, M^{(2)}, \dots, M^{(r)}$ be monoids. The elements of a product monoid $\prod_{i=1}^{r} M^{(i)}$ are r-tuples $(m^{(1)}, m^{(2)}, \dots, m^{(r)})$ with multiplication defined coordinatewise. A submonoid $M \subset \prod_{i=1}^{r} M^{(i)}$ is called a subdirect product if the projection $\pi^{(i)} : M \to M^{(i)}$ is surjective for all i.

Recall that a family $\{f^{(i)} : X \to X^{(i)} \mid i \in I\}$ of mappings is separating if for every pair $\{x, y\}$ of distinct elements of X there exists $i \in I$ with $f^{(i)}(x) \neq f^{(i)}(y)$. Observe that the set of projections $\{\pi^{(i)} \mid 1 \leq i \leq r\}$ is separating.

We can now give our construction of a (generalized) $\mathcal{P}$-graph with given endomorphism monoid.

Construction 5.2. Let us assume that

(a) $\mathfrak{G} = \{\mathbf{G}^{(i)} = (G^{(i)}, \mathcal{P}^{(i)}) \mid i \in I\}$ is a family of disjoint $\mathcal{P}$-graphs where for each $i \in I$

- $G^{(i)} = (V^{(i)}, E^{(i)})$,
- $\mathcal{P}^{(i)} = \{V_j^{(i)} \mid j \in J^{(i)}\}$,
- $0^{(i)} \in J^{(i)}$ is the axis of $\mathbf{G}^{(i)}$;

(b) V_0 is a set such that for every $i \in I$

- $0 \notin J^{(i)}$,
- $|V_0| \geq |V_{0^{(i)}}^{(i)}|$,
- $V_0 \cap V^{(i)} = \emptyset$;

(c) $\mathfrak{F} = \{f^{(i)} : V_0 \to V_{0^{(i)}}^{(i)} \mid i \in I\}$ is a family of surjective mappings.

Define a $\mathcal{P}$-graph $\mathbf{G}(\mathfrak{G}, \mathfrak{F}, V_0) = \mathbf{G} = ((V, E), \mathcal{P})$ where

- $V = V_0 \cup (\bigcup_{i \in I} V^{(i)})$,

- $\mathcal{P} = \{V_0\} \cup \{V_j^{(i)} \mid i \in I,\, j \in J^{(i)}\}$,
- V_0 is the axis of $\mathbf{G}$,
- $E = (\bigcup_{i\in I} E^{(i)}) \cup \{\{v, f^{(i)}(v)\} \mid v \in V_0, i \in I\} \cup (\bigcup_{i\in I} F^{(i)})$

where $F^{(i)} \subseteq V_0 \times V^{(i)}$ consists of all pairs $\{u, v\}$ where $u \in V_0$ such that

$$\{u, v\} \in F^{(i)} \quad \text{just when} \quad \{f^{(i)}(u), v\} \in E^{(i)}.$$

We observe that

$$|\mathbf{G}| = |V_0| + \sum_{i\in I} |\mathbf{G}^{(i)}| \quad \text{and} \quad w(\mathbf{G}) = 1 + \sum_{i\in I} w(\mathbf{G}^{(i)}). \tag{5.1}$$

Theorem 5.3 describes the $\mathcal{P}$-endomorphism monoid of $\mathbf{G}(\mathfrak{G}, \mathfrak{F}, V_0)$.

Theorem 5.3. Let $\mathfrak{F} = \{f^{(i)} : M \to M^{(i)} \mid i \in I\}$ be a separating family of surjective monoid homomorphisms and $\mathfrak{G} = \{\mathbf{G}^{(i)} \mid i \in I\}$ be a family of $\mathcal{P}$-graphs such that $\mathbf{G}^{(i)}$ represents $M^{(i)}$ for all $i \in I$. Then $\mathbf{G}(\mathfrak{G}, \mathfrak{F}, M)$ represents M, consequently $\Pi(M) \leq |M| + \sum_{i\in I} \Pi(M^{(i)})$.

Proof: Given family $\mathfrak{G}$ and $\mathfrak{F}$ set $V_0 = M$ and let $\mathbf{G} = \mathbf{G}(\mathfrak{G}, \mathfrak{F}, V_0)$ be as in Construction 5.2. Since $\mathbf{G}^{(i)}$ represents $M^{(i)}$ we conclude that $V^{(i)}_{0^{(i)}} = M^{(i)}$.

We prove that $\mathbf{G}$ represents M by verifying the three conditions (r1),(r2) and (r3). To prove (r1) consider a $\mathcal{P}$-endomorphism f of $\mathbf{G}$. Then necessarily $f(M) \subseteq M$. First we prove that

$$f^{(i)}(f(m)) = f(f^{(i)}(m)) \text{ for all } m \in M \text{ and } i \in I. \tag{5.2}$$

By definition, $\{m, f^{(i)}(m)\} \in E$ for all $m \in M$ and $i \in I$. Since f is a $\mathcal{P}$-endomorphism we obtain $\{f(m), f(f^{(i)}(m))\} \in E$ and $f(f^{(i)}(m)) \in$

$M^{(i)} = V^{(i)}_{0^{(i)}}$ because $f^{(i)}(m) \in M^{(i)}$. Since, by (d2), there exists at most one $v \in M^{(i)}$ with $\{f(m), v\} \in E$ and, by construction, $v = f^{(i)}(f(m))$ we conclude that $f(f^{(i)}(m)) = f^{(i)}(f(m))$ and (2) is proved.

Then for every $i \in I$ the restriction of f to $V^{(i)}$ is a $\mathcal{P}$-endomorphism of $\mathbf{G}^{(i)}$ and since $\mathbf{G}^{(i)}$ represents $M^{(i)}$ we obtain, by (r1) applied to $\mathbf{G}^{(i)}$, that the restriction of f to $M^{(i)}$ is $\lambda^{(i)}_{f(e^{(i)})}$. By (2), $f(e^{(i)}) = f(f^{(i)}(e)) = f^{(i)}(f(e))$. Hence, for every $v \in M$ and every $i \in I$ we have

$$f^{(i)}(f(v)) = f(f^{(i)}(v)) = \lambda^{(i)}_{f^{(i)}(f(e))}(f^{(i)}(v)) = f^{(i)}(f(e))f^{(i)}(v) = f^{(i)}(f(e)v).$$

Since $\{f^{(i)} \mid i \in I\}$ is a separating family of mappings we conclude that $f(v) = f(e)v = \lambda_{f(e)}(v)$ for all $v \in M$ and (r1) is proved.

To prove (r2) consider a left translation λ_x of M. Since $\mathbf{G}^{(i)}$ represents $M^{(i)}$ we may apply (r2) to $\lambda^{(i)}_{f^{(i)}(x)}$ to obtain a $\mathcal{P}$-endomorphism $g^{(i)}$ of $\mathbf{G}^{(i)}$ such that the restriction of $g^{(i)}$ to $M^{(i)}$ is $\lambda^{(i)}_{f^{(i)}(x)}$. Define $g : V \to V$ setting for $v \in V$

$$g(v) = \begin{cases} \lambda_x(v) & \text{if } v \in M \\ g^{(i)}(v) & \text{if } v \in V^{(i)} \text{ for some } i \in I. \end{cases}$$

We are going to verify that g is a $\mathcal{P}$-endomorphism of $\mathbf{G}$. Since $g^{(i)}$ is a $\mathcal{P}$-endomorphism of $\mathbf{G}^{(i)}$ for all $i \in I$ and since M is an independent set in G it suffices to show that $\{g(u), g(v)\} \in E(G)$ whenever $\{u, v\} \in E(G)$ for $u \in M$ and $v \in V^{(i)}$ for some $i \in I$. If $u \in M$ and $v \in V^{(i)}$ for some $i \in I$ then $\{u, v\} \in E(G)$ if and only if either $v = f^{(i)}(u)$ or $\{f^{(i)}(u), v\} \in E(G^{(i)})$. Since

$$g(f^{(i)}(u)) = g^{(i)}(f^{(i)}(u)) = \lambda^{(i)}_{f^{(i)}(x)}(f^{(i)}(u)) = f^{(i)}(x)f^{(i)}(u) = f^{(i)}(xu)$$

and $xu = \lambda_x(u) = g(u)$ we infer that $g(f^{(i)}(u)) = f^{(i)}(g(u))$. Thus if $v = f^{(i)}(u)$ then $\{g(u), g(v)\} = \{g(u), f^{(i)}(g(u))\} \in E(G)$. Assume now that $\{f^{(i)}(u), v\} \in E(G^{(i)})$. Combining $f^{(i)}(g(u)) = g(f^{(i)}(u)) =$

$g^{(i)}(f^{(i)}(u))$ and $g^{(i)}$ is a $\mathcal{P}$-endomorphism of $\mathbf{G}^{(i)}$ we infer that $\{f^{(i)}(g(u)), g(v)\} = \{f^{(i)}(g(u)), g^{(i)}(v)\} \in E(G^{(i)})$. Consequently, $\{g(u), g(v)\} \in E(G)$ and (r2) is proved.

To prove (r3) consider two $\mathcal{P}$-endomorphisms f and g of $\mathbf{G}$. We prove that if $f(m) = g(m)$ for all $m \in M$ then $f = g$. Choose $i \in I$. By (2), $f(e^{(i)}) = f(f^{(i)}(e)) = f^{(i)}(f(e))$ and $g(e^{(i)}) = g(f^{(i)}(e)) = f^{(i)}(g(e))$ for all $i \in I$. Hence if $f(e) = g(e)$ then $f(e^{(i)}) = g(e^{(i)})$ and applying (r1) to $\mathbf{G}^{(i)}$ we infer that $f(x) = g(x)$ for all $x \in M^{(i)}$. By (r3) applied to $\mathbf{G}^{(i)}$, we infer that $f(v) = g(v)$ for all $v \in V_i$. Since $i \in I$ was arbitrary we conclude that $f = g$ and (r3) is proved. Whence $\mathbf{G}(\mathfrak{G}, \mathfrak{F}, M)$ represents M as desired.□

To apply Theorem 5.3 we require a technical folklore lemma about separating families.

Lemma 5.4. Let X be a set of size $n > 1$. If $\{f^{(i)} : X \to X^{(i)} \mid i \in I\}$ is a separating family of mappings then there exists $I' \subseteq I$ such that $|I'| \leq n - 1$ and $\{f^{(i)} : X \to X^{(i)} \mid i \in I'\}$ is a separating family.

Proof: We prove the statement by induction over n. If $n = 2$ then the statement is trivial, it suffices to set $I' = \{i\}$ for $i \in I$ such that f_i is not constant. Assume that $|X| = n$ and that the statement is true for all sets Y with $|Y| < n$. Since $\{f^{(i)} : X \to X^{(i)} \mid i \in I\}$ is separating there exists $i_0 \in I$ such that $f^{(i_0)}$ is non-constant. Consequently, there exists a partition $\{Y_1, Y_2\}$ of X such that $f^{(i_0)}(y_1) \neq f^{(i_0)}(y_2)$ for all $y_1 \in Y_1$ and $y_2 \in Y_2$. Let $g^{(i)}$ be the restriction of $f^{(i)}$ to Y_1 and $h^{(i)}$ be the restriction of $f^{(i)}$ to Y_2. Then $\{g^{(i)} : Y_1 \to X^{(i)} \mid i \in I\}$ and $\{h^{(i)} : Y_2 \to X^{(i)} \mid i \in I\}$ are separating families because $\{f^{(i)} : X \to X^{(i)} \mid i \in I\}$ is a separating family. By the induction hypothesis, there exist $I_1, I_2 \subseteq I$ such that $|I_1| \leq |Y_1| - 1$, $|I_2| \leq |Y_2| - 1$, $\{g^{(i)} : Y_1 \to X^{(i)} \mid i \in I_1\}$ and

$\{h^{(i)} : Y_2 \to X^{(i)} \mid i \in I_2\}$ are separating families because $|Y_1|, |Y_2| < n$. Then $\{f^{(i)} : X \to X^{(i)} \mid i \in \{i_0\} \cup I_1 \cup I_2\}$ is a separating family and $|\{i_0\} \cup I_1 \cup I_2| \leq 1 + |I_1| + |I_2| \leq 1 + |Y_1| - 1 + |Y_2| - 1 = |X| - 1$. $\square$

Next we give a consequence for classes of monoids which are subdirect products of finitely many monoids. Let $\mathcal{M}$ be a class of monoids for which there exist natural numbers a and b such that $\Pi(M) \leq a$ for all $M \in \mathcal{M}$ and

$$b = \max_{M \in \mathcal{M}} \min\{w(\mathbf{G}) \mid \mathbf{G} \text{ represents } M \text{ and } |\mathbf{G}| \leq a\}.$$

Corollary 5.5. Let $M = (M, \cdot, e)$ be a monoid for which there exists a separating family $\{f^{(j)} : M \to M^{(j)} \mid j \in J\}$ of surjective monoid homomorphisms such that $M^{(j)} \in \mathcal{M}$ for all $j \in J$. Then there exists a $\mathcal{P}$-graph $\mathbf{G}$ representing M such that $|\mathbf{G}| \leq (a+1)|M|$ and $w(\mathbf{G}) = b|M|$.

Proof: By Lemma 5.4, we can assume that $|J| \leq |M| - 1$. We apply Construction 5.2 and Theorem 5.3 which ensures the existence of a $\mathcal{P}$-graph $\mathbf{G}$ representing M such that

$$|\mathbf{G}| \leq |M| + (|M|-1)a \leq (a+1)|M|, \quad w(\mathbf{G}) \leq 1 + (|M|-1)b \leq |M|b.$$

$\square$

5.2 Application

The aim of this section is to prove Theorem 7.

Definition. A semilattice monoid is a commutative monoid M satisfying $x^2 = x$ for all $x \in M$.

Proposition 5.6. Let $S = (S, \cdot, e)$ be a semilattice monoid. Then there exists a $\mathcal{P}$-graph $\mathbf{G}$ representing S such that $|\mathbf{G}| \leq 4|S|$ and $w(\mathbf{G}) \leq 2|S|$. Thus, $\Pi_{\mathcal{L}}(n) \leq 4n$.

Proof: Let $\mathcal{M}$ be the single element class consisting of a two-element semilattice. It is well-known that every semilattice monoid is a subdirect product of two-element semilattices [P] and thus Corollary 5.5 applies. Clearly, there exists a $\mathcal{P}$-graph $\mathbf{G}$ (an edge and an isolated vertex) representing a two-element semilattice with $|\mathbf{G}| = 3$ and $w(\mathbf{G}) = 2$. The statement then follows from Corollary 5.5. $\square$

Below we will use the following notation. For a semigroup $S = (S, \cdot)$ we write $S^1 = (S \cup \{1\}, \cdot)$ or $S^0 = (S \cup \{0\}, \cdot)$ where 1 or 0 is a new element which is not in S and $1s = s1 = s$ for all $s \in S \cup \{1\}$ and $0s = s0 = 0$ for all $s \in S \cup \{0\}$. Then S^1 is a monoid with an outer identity and S^0 is a semigroup with an outer zero.

Next we prove an auxiliary technical lemma that plays a key role in the next section. By the above definition, if $M = (M, \cdot, e)$ is a monoid then M^0 is a monoid M with adjoined outer zero.

Lemma 5.7. Let $\mathbf{G}$ be a $\mathcal{P}$-graph representing a monoid M, then there exists a $\mathcal{P}$-graph $\mathbf{H}$ representing M^0 such that

$$|\mathbf{H}| = |\mathbf{G}| + w(\mathbf{G}) + 3 \quad \text{and} \quad w(\mathbf{H}) = w(\mathbf{G}) + 2.$$

Proof: Let us assume that $\mathbf{G} = (G = (U, F), \mathcal{P})$ where $\mathcal{P} = \{U_i \mid 0 \leq i \leq l-1\}$ and U_0 is the axis of $\mathbf{G}$. Let $M = (M, \cdot, e)$ be a monoid. Assume that $0 \notin M$ and $M^0 = (M \cup \{0\}, \cdot, e)$. Below we will relabel 0 by v_0 and suppose that v_0, v_1,…,v_{l+1}, and w are pairwise distinct and $U \cap \{v_0, v_1, \ldots, v_{l+1}, w\} = \emptyset$. For i with $0 \leq i \leq l-1$ set $V_i = U_i \cup \{v_i\}$, $V_l = \{w, v_l\}$, $V_{l+1} = \{v_{l+1}\}$, and $V = \bigcup_{i=0}^{l+1} V_i$. Finally set $E = F \cup E_1 \cup E_2$ where $E_1 = \{\{v_i, v_j\} \mid 0 \leq i < j \leq l+1\}$ and $E_2 = \{\{u, w\} \mid u \in U\}$. Let $\mathbf{H} = (H = (V, E), \mathcal{R})$ where $\mathcal{R} = \{V_i \mid 0 \leq i \leq l+1\}$ and the axis is V_0. Clearly, $\mathbf{H}$ is a $\mathcal{P}$-graph with $|\mathbf{H}| = |\mathbf{G}| + w(\mathbf{G}) + 3$ and $w(\mathbf{H}) = w(\mathbf{G}) + 2$. It remains to prove that $\mathbf{H}$ represents M^0 - the details of (r1),(r2) and (r3) are below.

Details of 5.7 Since $\mathbf{G}$ represents M we have $U_0 = M$ and thus, (recalling that 0 and v_0 are identified) $V_0 = M \cup \{0\} = M^0$. To prove (r1) for $\mathbf{H}$ and M^0 consider a $\mathcal{P}$-endomorphism f of $\mathbf{H}$. First observe that $\{v_i \mid 0 \leq i \leq l+1\}$ induces a unique clique of size $l+2$ in $\mathbf{H}$ and hence $f(\{v_i \mid 0 \leq i \leq l+1\}) = \{v_i \mid 0 \leq i \leq l+1\}$. For $j = 0, 1, \ldots, l+1$ we have $V_j \cap \{v_i \mid 0 \leq i \leq l+1\} = \{v_j\}$ and consequently $f(v_j) = v_j$ for all $j = 0, 1, \ldots, l+1$, in particular, $f(0) = 0$. First assume that $f(e) \in M \subseteq V_0$. Observe that for $v \in V$ we have $\{v, w\} \in E$ if and only if $v \in U$. Since $V_l = \{w, v_l\}$ we conclude that $f(w) = w$ and $f(U) \subseteq U$. Thus the restriction g of f to U is a $\mathcal{P}$-endomorphism of $\mathbf{G}$. Applying (r1) to $\mathbf{G}$ and M we conclude that $g(x) = \lambda^M_{f(e)}(x)$ for all $x \in M$ and hence $f(x) = \lambda^{M^0}_{f(e)}(x)$ for all $x \in M^0$. On the other hand, if $f(e) = 0$ then $f(w) = v_l$ because $\{e, w\}, \{0, v_l\} \in E$ and $\{0, w\} \notin E$. Since $\{u, w\} \in E$ for all $u \in U$ and since $\{x, v_l\} \in E$ exactly when $x \in \{v_i \mid 0 \leq i \leq l+1,\, i \neq l\}$ we infer that $f(V_i) = \{v_i\}$ for all $i \in \{0, 1, \ldots, l+1\}$. Thus $f(x) = \lambda^{M^0}_0(x)$ for all $x \in M^0$ and (r1) is proved for $\mathbf{H}$ and M^0.

To prove (r2) consider a left translation $\lambda^{M^0}_a : M^0 \to M^0$. First assume that $a \in M$. Applying (r2) to $\mathbf{G}$ and M there exists a $\mathcal{P}$-endomorphism g of $\mathbf{G}$ such that $g(x) = \lambda^M_a(x)$ for all $x \in M$. Define a mapping $f : V \to V$ by

$$f(v) = \begin{cases} g(v) & \text{if } v \in U \\ v & \text{if } v \in \{v_i \mid 0 \leq i \leq l+1\} \cup \{w\}. \end{cases}$$

Since g is a $\mathcal{P}$-endomorphism of $\mathbf{G}$ we obtain that f is a $\mathcal{P}$-endomorphism of $\mathbf{H}$. Clearly, $f(x) = \lambda_a^{M^0}(x)$ for all $x \in M^0$. If $a = 0$ then we define a mapping $f : V \to V$ by $f(v) = v_i$ for every $i = 0, 1, \ldots, l+1$ and every $v \in V_i$. It is easy to verify that f is a $\mathcal{P}$-endomorphism of $\mathbf{H}$ and $f(x) = \lambda_0^{M^0}(x)$ for all $x \in M^0$. Thus (r2) for $\mathbf{H}$ and M^0 is proved.

To prove (r3) consider two distinct $\mathcal{P}$-endomorphisms f and g of $\mathbf{H}$. We have already proved above that if h is a $\mathcal{P}$-endomorphism of $\mathbf{H}$ satisfying $h(e) = 0$ then $h(V_i) = \{v_i\}$ for all $i = 0, 1, \ldots, l+1$. Thus we may assume $f(e), g(e) \in M$ for otherwise $f(e) \neq g(e)$ and would be done. In the proof of (r1) we established that $f(e), g(e) \in M$ implies $f(U), g(U) \subseteq U$ and, moreover, from $f \neq g$ it follows that the restrictions f and g to U are distinct. An application of (r3) to $\mathbf{G}$ and the restrictions of f and g to U completes the proof of (r3) for $\mathbf{H}$. $\square$

Definition. A <u>normal</u> <u>band</u> <u>monoid</u> is a monoid of the form $M = S^1$ where S is an arbitrary semigroup satisfying $x^2 = x$ and $xuvx = xvux$ for all $x, u, v \in S$.

Every normal band monoid is a subdirect product of monoids from the five element class $\mathcal{M}^*$ consisting of the monoids M where

- M is a two-element semilattice monoid.
- $M = S^1$ where S is a two-element left (or right)-zero semigroup – that is a semigroup $S = S_L$ satisfying $xy = x$ for all $x, y \in S$ or $S = S_R$ satisfying $xy = y$ for all $x, y \in S$.
- $M = (S^0)^1$ where S is a two-element left (or right)-zero semigroup.

The above facts were established in [G]. Lemma 5.7 describes the $\mathcal{P}$-graphs representing each of the five aforementioned factors which may

*This is the class of subdirectly irreducible normal band monoids, that is, monoids which can not be realized as a subdirect product.

appear in a product containing a normal band monoid.

Lemma 5.8. For every subdirectly irreducible normal band monoid M there exists a $\mathcal{P}$-graph $\mathbf{G}$ representing M with $|\mathbf{G}| \le 17$ and $w(\mathbf{G}) \le 6$.

Proof: Let $M = (M, \cdot, e)$ be a subdirectly irreducible normal band monoid. If M is a semilattice monoid then there exists a $\mathcal{P}$-graph $\mathbf{G}$ representing M with $|\mathbf{G}| = 3 \le 17$ and $w(\mathbf{G}) = 2 \le 6$.

Next assume that e is an outer identity of M and $M \setminus \{e\} = \{x, y\}$ is a two-element left(or right)-zero semigroup S_L (or S_R, respectively). Now consider the $\mathcal{P}$-graph $\mathbf{G} = (G = (V, E), \mathcal{P} = \{U_i \mid i \in I\})$ where $I = \{0, 1, x, y\}$, $V = (M \times \{0, 1\}) \cup ((M \setminus \{e\}) \times \{x, y\})$, $U_i = V \cap (M \times \{i\})$ for all $i \in I$ and

$$\begin{aligned} E =& \{\{(z,0),(z,1)\} \mid z \in M\} \cup \{\{(z,0),(z,i)\} \mid z \in M \setminus \{e\},\, i \in \{x,y\}\} \cup \\ & \{\{(z,1),(zi,i)\} \mid z \in M,\, i \in \{x,y\}\}. \end{aligned}$$

Depending on whether $M = (S_L)^1$ or $M = (S_R)^1$ we obtain two $\mathcal{P}$-graphs $\mathbf{G} = \mathbf{G}_L$ and $\mathbf{G} = \mathbf{G}_R$ (see Figure 5.1 and 5.2 for $(S_L)^1$ and $\mathbf{G}_L$). Clearly, $\mathcal{P}$ and E are correctly defined and $\mathbf{G}$ is a $\mathcal{P}$-graph with $|\mathbf{G}| = 10 \le 17$ and $w(\mathbf{G}) = 4 \le 6$.

To prove that $\mathbf{G}_L$ represents $(S_L)^1$ we observe that all $\mathcal{P}$-endomorphisms of $\mathbf{G}_L$ are of the form $(u, v) \mapsto (g(u), v)$ where g is either the identity or $g(e) = g(x) = x$, $g(y) = y$ or $g(e) = g(y) = y$, $g(x) = x$. Similarly the only $\mathcal{P}$-endomorphisms of $\mathbf{G}_R$ are of the form $(u, v) \mapsto (g(u), v)$ where g is either the identity or the constant mapping with value x or y.

Finally, consider $M = \{x, y, e, 0\}$ where 0 is an outer zero of M and e is the outer identity. By Lemma 5.7 there exists $\mathcal{P}$-graph representing M with $|\mathbf{G}| = 17$ and $w(\mathbf{G}) = 6$. $\Box$

The following theorem summarizes facts about normal band monoids.

	e	x	y
e	e	x	y
x	x	x	x
y	y	y	y

Figure 5.1: multiplication table for $(S_L)^1$

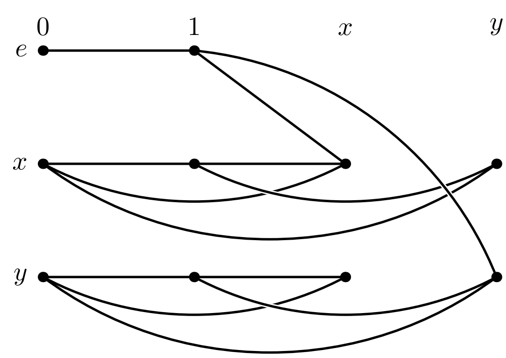

Figure 5.2: $\mathcal{P}$-graph $\mathbf{G}_L$ for $(S_L)^1$

Theorem 5.9. Let $M = (M, \cdot, e)$ be a normal band monoid. Then there exists a $\mathcal{P}$-graph $\mathbf{G}$ representing M such that $|\mathbf{G}| \leq 18|M|$ and $w(\mathbf{G}) \leq 6|M|$. Thus, $\Pi_{\mathcal{NB}}(n) \leq 18n$.

Proof: Let $\mathcal{M}$ be the class consisting of subdirectly irreducible normal band monoids. As every normal band monoid is a subdirect product of monoids from $\mathcal{M}$ we can apply Corollary 5.5. The statements follows from a combination of Lemma 5.8 and Corollary 5.5. $\Box$

Next we give an easy technical lemma about finite cyclic groups.

Lemma 5.10. For every finite cyclic group C there exists a $\mathcal{P}$-graph $\mathbf{G}$ representing C with $|\mathbf{G}| \leq 3|C|$ and $w(\mathbf{G}) = 3$.

Proof: Let C be a finite cyclic group and let $a \in C$ be a generator of C. Consider a $\mathcal{P}$-graph $\mathbf{G} = (G = (V, E), \mathcal{P} = \{U_i \mid i \in I\})$ where $I = \{0, 1, 2\}$, $V = C \times I$, $U_i = V \cap (C \times \{i\})$ for all $i \in I$ and

$$\begin{aligned} E &= E_1 \cup E_2, \quad \text{where} \\ E_1 &= \{\{(z,0),(z,i)\} \mid z \in C,\, i \in \{1,2\}\} \quad \text{and} \\ E_2 &= \{\{(z,1),(za,2)\} \mid z \in C\}. \end{aligned}$$

Clearly, $\mathcal{P}$ and E are defined so that $\mathbf{G}$ is a $\mathcal{P}$-graph with $|\mathbf{G}| = 3|C|$, $w(\mathbf{G}) = 3$. Due to the edges of E_1 for any $\mathcal{P}$-endomorphism f of $\mathbf{G}$ there exists a mapping $g : C \to C$ such that $f(z, i) = (g(z), i)$ for all $z \in C$ and $i = 0, 1, 2$. Due to the edges of E_2 we infer that $ag(x) = g(x)a$ holds for all $x \in C$. Since a is a generator of C and the only mappings that commute with the translations $\lambda_a = \rho_a$ are translations $\lambda_{a^i} = (\lambda_a)^i$ for all natural numbers $i = 0, 1, \ldots$ we deduce that $\mathrm{End}_{\mathcal{P}}(\mathbf{G}) \cong C$. $\Box$

Theorem 5.11. Let $M = (M, \cdot, e)$ be an Abelian group. Then there exists a $\mathcal{P}$-graph $\mathbf{G}$ representing M such that $|\mathbf{G}| \leq 4|M|$ and $w(\mathbf{G}) \leq 1 + 3\log|M|$. Thus, $\Pi_{\mathcal{A}}(n) \leq 4n$.

Proof: Consider a finite Abelian group $M = (M, \cdot, e)$. It is well-known that M is a direct product of finite cyclic groups, say,

$$\{M^{(i)} = (M^{(i)}, \cdot, e^{(i)}) \mid i \in I\}.$$

Clearly, we can assume that $|M^{(i)}| > 1$ for all $i \in I$ and hence $|M| = \prod_{i \in I} |M^{(i)}| \geq \sum_{i \in I} |M^{(i)}|$ and $|I| \leq \log|M|$. For each $i \in I$ let $\pi^{(i)} : M \to M^{(i)}$ be the canonical projection and set $\mathfrak{F} = \{\pi^{(i)} \mid i \in I\}$, then $\mathfrak{F}$ is a separating family of surjective monoid homomorphisms. By Lemma 5.10, there exists a family $\mathfrak{G} = \{\mathbf{G}^{(i)} \mid i \in I\}$ of $\mathcal{P}$-graphs such that $\mathbf{G}^{(i)}$ represents $M^{(i)}$, $|\mathbf{G}^{(i)}| = 3|M^{(i)}|$, $w(\mathbf{G}^{(i)}) = 3$ for all $i \in I$. We apply Construction 5.2 to obtain $\mathbf{G} = \Pi(\mathfrak{G}, \mathfrak{F}, M)$. By (1), $|\mathbf{G}| = |M| + \sum_{i \in I} 3|M^{(i)}| \leq 4|M|$ and $w(\mathbf{G}) = 1 + \sum_{i \in I} 3 = 1 + 3I \leq 1 + 3\log|M|$. By Theorem 5.3, $\mathbf{G}$ represents M and the proof is complete. $\square$

Combining Theorems 5.1, 5.9 and 5.11 with Proposition 5.4 we derive the following bounds.

Theorem 5.12. Let n be an integer. For semilattice monoids, normal band monoids, and Abelian groups the following bounds hold respectively:

$$\begin{aligned} \nu_{\mathcal{L}}(n) &\leq 12n + 5 \\ \nu_{\mathcal{NB}}(n) &\leq 42n + 5 \\ \nu_{\mathcal{A}}(n) &\leq 4n + 12\log n + 9. \end{aligned}$$

Theorem 7 follows immediately from Theorem 5.12.

Chapter 6

Semilattice extension

This section is devoted to the proof of Theorem 8. We supply the necessary concepts, omitted in Section 5.1, below. The main concept of this section is that of a strong semilattice of semigroups.

Definition. Let $(Y, \cdot)$ be a semilattice partially ordered by $y \leq z$ if and only if $yz = z$. Let $\{S_y \mid y \in Y\}$ be a family of pairwise disjoint semigroups and let $\{\phi_{y,z} : S_y \to S_z \mid y, z \in Y, y \leq z\}$ be a family of semigroup homomorphisms such that

- $\phi_{y,y}$ is the identity mapping for every $y \in Y$;
- if $x, y, z \in Y$ with $x \leq y \leq z$ then $\phi_{y,z} \circ \phi_{x,y} = \phi_{x,z}$.

Let $S = \bigcup_{y \in Y} S_y$ and for $s \in S_y \subseteq S$ and $t \in S_z \subseteq S$ where $y, z \in Y$ define

$$s \cdot t = \phi_{y,yz}(s)\phi_{z,yz}(t)$$

where the right multiplication is in S_{yz}. Then S is a semigroup which is the *strong semilattice of semigroups* $\{S_y \mid y \in Y\}$ *determined by* $\{\phi_{y,z} \mid y, z \in Y, y \leq z\}$. If $\mathcal{C}$ is a class of semigroups such that $S_y \in \mathcal{C}$ for every $y \in Y$ then we say that S is a *strong semilattice of* $\mathcal{C}$*-semigroups.*

Definition. For a class $\mathcal{C}$ of semigroups let $\mathfrak{S}(\mathcal{C})$ be the class of all finite monoids M of the following form

- M is a monoid which is a strong semilattice of $\mathcal{C}$ semigroups;
- $M = S^1$ where S is a strong semilattice of $\mathcal{C}$-semigroups (here S is not a monoid).

Recall that $\mathcal{A}$, $\mathcal{G}$, $\mathcal{RG}$, and $\mathcal{CS}$ are the classes of all abelian groups, groups, rectangular groups, and completely simple semigroups respectively.

Let $\mathcal{RG}^1$ be the class of all monoids M of the form $M = S$ or $M = S^1$ where $S \in \mathcal{RG}$. Similarly, let $\mathcal{CS}^1$ be the class of all monoids M of the form $M = S$ or $M = S^1$ where $S \in \mathcal{CS}$.

Theorem 8 gives bounds on $\nu_{\mathfrak{S}(\mathcal{C})}$ where $\mathcal{C}$ is one of the following four classes $\mathcal{A}, \mathcal{G}, \mathcal{RG}$, or $\mathcal{CS}$.

In this section we are going to prove the following more detailed form of Theorem 8.

Theorem 6.1 Let M be a monoid. Then

1. if $M \in \mathfrak{S}(\mathcal{A})$ then there exists a $\mathcal{P}$-graph $\mathbf{G}$ representing M such that $|\mathbf{G}| \leq 9|M|$ and $w(\mathbf{G}) < 3|M|$, thus $\nu_{\mathfrak{S}(\mathcal{A})}(n) \leq 21n$;
2. if $M \in \mathfrak{S}(\mathcal{G})$ then there exists a $\mathcal{P}$-graph $\mathbf{G}$ representing M such that $|\mathbf{G}| \leq 2|M|\sqrt{\log\log|M|} + 12|M|$ and $w(\mathbf{G}) < 5|M|$, thus $\nu_{\mathfrak{S}(\mathcal{G})}(n) \leq 2n\sqrt{\log\log n} + 32n$;
3. if $M \in \mathfrak{S}(\mathcal{RG})$ then there exists a $\mathcal{P}$-graph $\mathbf{G}$ representing M such that $|\mathbf{G}| \leq 12|M| + 2|M|\sqrt{\log|M|}$ and $w(\mathbf{G}) < 3|M|$, thus $\nu_{\mathfrak{S}(\mathcal{RG})}(n) \leq 24n + 2n\sqrt{\log n}$;

4. if $M \in \mathfrak{S}(\mathcal{CS})$ then there exists a $\mathcal{P}$-graph $\mathbf{G}$ representing M such that $|\mathbf{G}| \leq (2+o(1))|M|^{\frac{7}{6}}$ and $w(\mathbf{G}) \leq (4+o(1))|M|$, thus $\nu_{\mathfrak{S}(\mathcal{A})}(n) \leq (2+o(1))n^{\frac{7}{6}}$.

Proof: The proof of Theorem 6.1 is based on the special properties of strong semilattices of semigroups given in Proposition III.7.2 in [P]. From these properties follows a general representation theorem for strong semilattices of semigroups. Theorem 6.1 is obtained by a substitution of known bounds for the factors from each of the classes in the representation theorem.

Theorem 6.2 [P] Let $(Y,\cdot)$ be a semilattice and $S = (S,\cdot)$ be a strong semilattice of $\{S_y \mid y \in Y\}$ determined by $\{\phi_{y,z} : S_y \to S_z \mid y, z \in Y,\ y \leq z\}$. Assume that $0 \notin S_y$ for every $y \in Y$ and let $T_y = S_y$ if y is the greatest element of Y under $\leq$ and $T_y = (S_y)^0$ otherwise. For $y \in Y$ let us define a mapping $f_y : S \to T_y$ such that

$$f_y(s) = \begin{cases} \phi_{z,y}(s) & \text{if } s \in S_z \text{ for } z \in Y \text{ with } z \leq y \\ 0_y & \text{if } s \in S_z \text{ for } z \in Y \text{ with } z \not\leq y. \end{cases}$$

Then $\{f_y : S \to T_y \mid y \in Y\}$ is a separating family of surjective homomorphisms.

Combining Construction 5.2 with Theorem 6.2 yields the following theorem.

Theorem 6.3 Let $(Y,\cdot)$ be a semilattice and $S = (S,\cdot)$ be a strong semilattice of $\{S_y \mid y \in Y\}$. Then

1. if $S = M$ is a monoid and if there exists a family $\{\mathbf{G}_y \mid y \in Y\}$ of $\mathcal{P}$-graphs such that $\mathbf{G}_y$ represents S_y for every $y \in Y$ then there exists

a $\mathcal{P}$-graph $\mathbf{G}$ representing S such that

$$|\mathbf{G}| \leq |M| + 3|Y| + \sum_{y\in Y}(|\mathbf{G}_y| + w(\mathbf{G}_y)) \leq 4|S| + 2\sum_{y\in Y}|\mathbf{G}_y|$$

and $w(\mathbf{G}) = 1 + 2|Y| + \sum_{y\in Y} w(\mathbf{G}_y)$;

2. if there exists a family $\{\mathbf{G}_y \mid y \in Y\}$ of $\mathcal{P}$-graphs such that $\mathbf{G}_y$ represents $(S_y)^1$ for every $y \in Y$ then there exists a $\mathcal{P}$-graph $\mathbf{G}$ representing $M = S^1$ such that

$$|\mathbf{G}| \leq |M| + 3|Y| + \sum_{y\in Y}(|\mathbf{G}_y| + w(\mathbf{G}_y)) \leq 4|M| + 2\sum_{y\in Y}|\mathbf{G}_y|$$

and $w(\mathbf{G}) = 1 + 2|Y| + \sum_{y\in Y} w(\mathbf{G}_y)$.

Proof: To prove the theorem we will apply Construction 5.2 and Theorem 5.3. Assume that $\{f_y : M \to S_y^0 \mid y \in Y\}$ in Case 1) and $\{f_y : M \to (S_y^1)^0 \mid y \in Y\}$ in Case 2) is a separating family of monoid homomorphisms. By assumption, there exists a family $\{\mathbf{G}_y \mid y \in Y\}$ of $\mathcal{P}$-graphs such that $\mathbf{G}_y$ represents S_y for every $y \in Y$ in Case 1) and $\mathbf{G}_y$ represents S_y^1 for every $y \in Y$ in Case 2). By Lemma 5.7, there exists a family $\{\mathbf{H}_y \mid y \in Y\}$ of $\mathcal{P}$-graphs such that $\mathbf{H}_y$ represents S_y^0 for every $y \in Y$ in Case 1) and $\mathbf{H}_y$ represents $(S_y^1)^0$ for every $y \in Y$ in Case 2). We apply Construction 5.2 on the families $\{\mathbf{H}_y \mid y \in Y\}$ and $\{f_y \mid y \in Y\}$ to obtain a $\mathcal{P}$-graph $\mathbf{G}$ such that

$$|\mathbf{G}| = |M| + \sum_{y\in Y}|\mathbf{H}_y| = |M| + 3|Y| + \sum_{y\in Y}(|\mathbf{G}_y| + w(\mathbf{G}_y))$$

and $w(\mathbf{G}) = 1 + 2|Y| + \sum_{y\in Y} w(\mathbf{G}_y)$. By Theorem 5.3, $\mathbf{G}$ represents M. Thus it remains only to find a separating family of monoid homomorphisms $\{f_y : M \to S_y^0 \mid y \in Y\}$ in Case 1) and $\{f_y : M \to (S_y^1)^0 \mid y \in Y\}$ in Case 2).

Consider Case 1). By Theorem 6.2, there exists a separating family $\{f'_y : M \to T_y \mid y \in Y\}$ of surjective semigroup homomorphisms where either $T_y = S_y$ or $T_y = (S_y)^0$. Since f'_y is a surjective semigroup homomorphism and M is a monoid we infer that T_y is a monoid and f'_y is a monoid homomorphism and thus there exists a separating family of monoid homomorphisms $\{f_y : M \to S_y^0 \mid y \in Y\}$ and the proof of Case 1) is complete.

Consider Case 2). By Theorem 6.2, there exists a separating family $\{f'_y : M \to T_y \mid y \in Y\}$ of surjective semigroup homomorphisms where either $T_y = S_y$ or $T_y = (S_y)^0$. Let us define $f_y : S^1 \to (S_y^1)^0$ for every $y \in Y$ such that $f_y(s) = f'_y(s)$ for all $s \in S$ and $f_y(1) = 1$. Then $\{f_y \mid y \in Y\}$ is a separating family because $f_y^{-1}(1) = \{1\}$ and f_y is a monoid homomorphism because f'_y is a semigroup homomorphism and 1 is an outer identity of S^1 and of $(S_y^1)^0$. Hence the proof of Case 2) follows. $\square$

Next we prove an auxiliary lemma.

Lemma 6.4 If S is a strong semilattice of groups (or abelian groups) then S^1 is also a strong semilattice of groups (or abelian groups, respectively).

Proof: Assume that S is a strong semilattice of $\{S_y \mid y \in Y\}$ determined by $\{\phi_{y,z} : S_y, \to S_z \mid y, z \in Y,\, y \leq z\}$ where $(Y, \cdot)$ is a semilattice, S_y is a group for all $y \in Y$, and e_y is the identity in S_y. Let a be an element with $a \notin Y$, then $(Y \cup \{a\}, \cdot)$ where $ay = ay = y$ for all $y \in Y \cup \{a\}$ is a semilattice. Let $S_a = e_a$ be the singleton group. For every $y \in Y \cup \{a\}$ let $\phi_{a,y}$ be the mapping such that $\phi(e_a) = e_y$. Then $\phi_{a,y} : S_a \to S_y$ is a group homomorphism for all $y \in Y \cup \{a\}$. Clearly, the family $\{\phi_{y,z} \mid y, z \in Y \cup \{a\},\, y \leq z\}$ satisfies the conditions on a family homomorphisms from the definition of strong semilattice of semigroups. Let T be a strong semilattice of $\{S_y \mid y \in Y \cup \{a\}\}$ determined by $\{\phi_{y,z} \mid y, z \in Y \cup \{a\},\ y \leq z\}$. Thus T is a strong semilattice of groups and if S_y is an abelian group for all $y \in Y$ then T is a strong semilattice of abelian groups.

It remains to prove that T is isomorphic to S^1. Clearly, $S \subseteq T$ and let $f : S^1 \to T$ be a mapping such that $f(s) = s$ for all $s \in S$ and $f(1) = e_a$. Then f is a bijection and by the definition of a strong semilattice of semigroups $f(st) = st = f(s)f(t)$ for all $s, t \in S$. For all $y \in Y \cup \{a\}$ we have $a \leq y$, $\phi_{a,y}(e_a) = e_y$ and $\phi_{y,y}$ is the identity of S_y. Thus for every $s \in S_y$ we infer that $e_a s = s e_a = s$. Since 1 is the identity of S^1 we conclude that f is a semigroup isomorphism and the proof follows. $\square$

Finally, to complete the proof of Theorem 6.1 we apply Theorem 6.3 to the four classes of monoids. Let $\mathcal{C}$ be one of the following classes of semigroups – abelian groups $\mathcal{A}$, groups $\mathcal{G}$, rectangular groups $\mathcal{RG}$, or completely simple semigroups $\mathcal{CS}$. By the definition of $\mathfrak{S}(\mathcal{C})$, if $M \in \mathfrak{S}(\mathcal{C})$ then M is a strong semilattice of semigroups $\{S_y \mid y \in Y\}$ where $S_y \in \mathcal{C}$ for all $y \in Y$ or $M = S^1$ where S is a strong semilattice of semigroups $\{S_y \mid y \in Y\}$ where $S_y \in \mathcal{C}$ for all $y \in Y$. In the first case, if there exists a family $\{\mathbf{G}_y \mid y \in Y\}$ of $\mathcal{P}$-graphs such that $\mathbf{G}_y$ represents $\mathcal{S}_y$ for all $y \in Y$ then, by Theorem 6.3(1), there exists a $\mathcal{P}$-graph $\mathbf{G}$ such that $\mathbf{G}$ represents M and

$$\begin{aligned}
|\mathbf{G}| &\leq |M| + 3|Y| + \sum_{y \in Y} (|\mathbf{G}_y| + w(\mathbf{G}_y)) \\
w(\mathbf{G}) &\leq 1 + 2|Y| + \sum_{y \in Y} w(\mathbf{G}_y).
\end{aligned}$$

In the second case, if there exists a family $\{\mathbf{G}_y \mid y \in Y\}$ of $\mathcal{P}$-graphs such that $\mathbf{G}_y$ represents $\mathcal{S}_y^1$ for all $y \in Y$ then, by Theorem 6.3(2), there exists a $\mathcal{P}$-graph $\mathbf{G}$ such that $\mathbf{G}$ represents M and

$$\begin{aligned} |\mathbf{G}| &\leq |M| + 3|Y| + \sum_{y\in Y}(|\mathbf{G}_y| + w(\mathbf{G}_y)) \\ w(\mathbf{G}) &\leq 1 + 2|Y| + \sum_{y\in Y} w(\mathbf{G}_y). \end{aligned}$$

If $\mathcal{C} = \mathcal{A}$ or $\mathcal{C} = \mathcal{G}$ then, by Lemma 6.4, we can consider only the first case and used $\mathcal{P}$-representations of groups or abelian groups. If $\mathcal{C} = \mathcal{RG}$ (or $\mathcal{C} = \mathcal{CS}$) then in the first case $S_y \in (\mathcal{RG})^1$ (or $S_y \in (\mathcal{CS})^1$) for all $y \in Y$ because S_y is a monoid for all $y \in Y$ and in the second case $S_y^1 \in (\mathcal{RG})^1$ (or $S_y^1 \in (\mathcal{CS})^1$) for all $y \in Y$. Thus it suffices to use $\mathcal{P}$-representations of monoids from $(\mathcal{RG})^1$ (or $(\mathcal{CS})^1$, respectively).

The proof is divided into four parts depending which class of semigroup is being considered. If $M = Y$ then M is a semilattice monoid and Proposition 5.6 gives the required $\mathcal{P}$-graph. Thus we can assume that $M \neq Y$ and thus $|Y| < |M|$.

- Suppose $\mathcal{C} = \mathcal{A}$. By Theorem 5.11, we can assume that $|\mathbf{G}_y| \leq 4|S_y|$ and $w(\mathbf{G}_y) \leq 1 + \log(|S_y|) \leq |S_y|$ for all $y \in Y$. Thus $|\mathbf{G}_y| + w(\mathbf{G}_y) \leq 5|S_y|$ and hence

  $$|\mathbf{G}| \leq |M| + 3|Y| + \sum_{y\in Y}(\mathbf{G}_y + w(\mathbf{G}_y) \leq 4|M| + \sum_{y\in Y} 5|S_y| \leq 9|M|.$$

  Moreover, $\sum_{y\in Y} w(\mathbf{G}_y) \leq |M|$ and hence $w(\mathbf{G}) \leq 3|M|$. From Theorem 5.1 it follows that $\nu_{\mathfrak{S}(\mathcal{A})}(n) \leq 21n$.

- Suppose $\mathcal{C} = \mathcal{G}$. By Theorem 3, we can assume that

  $$\begin{aligned} |\mathbf{G}_y| &\leq 2|S_y|(\sqrt{\log\log(|S_y|+1)} + 2) \quad \text{and} \\ w(\mathbf{G}_y) &\leq 2 + 2\sqrt{\log\log(|S_y|+1)} \end{aligned}$$

  for all $y \in Y$. Thus

  $$|\mathbf{G}_y| + w(\mathbf{G}_y) \leq (2|S_y| + 2)\sqrt{\log\log(|S_y|+1)} + 4|S_y| + 2.$$

From $x \geq \log(x+1)$ for all $x \geq 1$ we infer

$$\sum_{y \in Y} \sqrt{\log\log(|S_y|+1)} \leq \sum_{y \in Y} \sqrt{|S_y|} \leq \sum_{y \in Y} |S_y| \leq M.$$

Hence $\sum_{y \in Y}(|\mathbf{G}_y| + w(\mathbf{G}_y)) \leq 8|M| + 2|M|\sqrt{\log\log(|M|+1)}$ and consequently, $|\mathbf{G}| \leq 2|M|\sqrt{\log\log(|M|+1)} + 12|M|$. Moreover, we conclude that $\sum_{y \in Y} w(\mathbf{G}_y) \leq 2|Y| + 2|M|$ and hence $w(\mathbf{G}) \leq 6|M|$. From Theorem 5.1 it follows that $\nu_{\mathfrak{S}(\mathcal{G})}(n) \leq 2n\sqrt{\log\log n} + 36n$.

- Suppose $\mathcal{C} = \mathcal{RG}$. Since $S_y \in (\mathcal{RG})^1$ for all $y \in Y$, by theorem 4, we can assume that

$$|\mathbf{G}_y| \leq 2|S_y|(\sqrt{3 + \log|S_y|}) \quad \text{and} \quad w(\mathbf{G}_y) \leq 2\sqrt{3 + \log|S_y|}$$

for all $y \in Y$. Thus $|\mathbf{G}_y| + w(\mathbf{G}_y) \leq (2|S_y| + 2)(\sqrt{3 + \log|S_y|})$. Since

$$\sum_{y \in Y} \sqrt{3 + \log|S_y|} \leq \sum_{y \in Y} \sqrt{|S_y| + 3} \leq \sum_{y \in Y} (|S_y| + 3) \leq 4|M|$$

we infer that $|\mathbf{G}| \leq 2|M|\sqrt{3 + \log|M|} + 12|M|$. Moreover, $w(\mathbf{G}) \leq 3|M|$. From Theorem 5.1 it follows that $\nu_{\mathfrak{S}(\mathcal{RG})}(n) \leq 2n\sqrt{\log n} + 24n$.

- Suppose $\mathcal{C} = \mathcal{CS}$. Since $S_y \in (\mathcal{CS})^1$ for all $y \in Y$, by Theorem 6, we can assume that $|\mathbf{G}_y| \leq (2 + o(1))|S_y|^{\frac{7}{6}}$ and $w(\mathbf{G}_y) \leq (2 + o(1))|S_y|^{\frac{1}{6}}$ for all $y \in Y$. Thus $|\mathbf{G}_y| + w(\mathbf{G}_y) \leq (2 + o(1))|S_y|^{\frac{7}{6}}$ and hence $|\mathbf{G}| \leq (2 + o(1))|M|^{\frac{7}{6}}$. From $|S_y|^{\frac{1}{6}} < |S_y|$ for all $y \in Y$ we infer that $\sum_{y \in Y} w(\mathbf{G}_y) \leq (2 + o(1))|M|$ and hence $w(\mathbf{G}) \leq (4 + o(1))|M|$. From Theorem 7 it follows that $\nu_{\mathfrak{S}(\mathcal{CS})}(n) \leq (2 + o(1))n^{\frac{7}{6}}$.

$\square$

Summarizing we have just proved the following bounds:

Corollary 6.5

$$\Pi_{\mathfrak{S}(\mathcal{A})}(n) \leq 9n, \qquad \Pi_{\mathfrak{S}(\mathcal{G})}(n) \leq 2n\sqrt{\log\log n} + 12n,$$

$$\Pi_{\mathfrak{S}(\mathcal{RG})}(n) \leq 2n\sqrt{\log n} + 12n, \qquad \Pi_{\mathfrak{S}(\mathcal{CS})}(n) = (2 + o(1))n^{\frac{7}{6}}.$$

Remark: One of the important generalizations of groups are regular semigroups. By [P], every commutative regular semigroup is a strong semilattice of abelian groups and, thus, Theorem 6.1(1) gives an estimate on the size of a finite graph with a given commutative regular monoid as its endomorphism monoid.

Bibliography

[AS] M. Armbrust and J. Schmidt, *Zum Cayleyschen Darstellungssatz*, Math. Ann. **154** (1964), 70–72.

[B1] L. Babai, *On the minimum order of graphs with a given group*, Canad. Math. Bull. **17** (1974), 467–470.

[B2] L. Babai, *On the abstract group of automorphisms*, Combinatorics, Proc. 8th British Comb. Conf. Swansea. London Math. Soc. Lect. Notes, Cambridge Univ. Press, London, **52** (1981) 1–40.

[CP] A. H. Clifford, G. B. Preston, “Algebraic Theory of Semigroups” Math. Surveys of the Amer. Math. Soc. 7, Providence R.I. vol. I (1964), vol. II (1967).

[F1] R. Frucht, *Herstellung von Graphen mit vorgegebener abstrakter Gruppe*, Compos. Math. **6** (1938), 239–250.

[F2] R. Frucht, *Graphs of degree three with a given abstract group*, Canadian J. Math. **1** (1949), 265–278.

[F3] L. Fuchs ”Abelian Groups” Pergamon Press, New York (1960).

[G] J. A. Gerhardt *Subdirectly irreducible idempotent semigroups*, Pacif. J. Math **39** (1971), 669–676.

[GH] P. Goralčík, Z. Hedrlín, *On reconstruction of monoids from their table fragments*, Math. Z. **122** (1971), 82–92.

[H] P. M. Higgins, "Techniques of semigroup theory" Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, (1992) Section 1.3.

[HN] P. Hell and J. Nešetřil, *Graphs and k-societies*, Canad. Math. Bull. **13** (1970), 375–381.

[HP1] Z. Hedrlín and A. Pultr, *Remark on topological spaces with given semigroups*, Comment. Math. Univ. Carolinae **4** (1963), 161–163.

[HP2] Z. Hedrlín and A. Pultr, *Relations (graphs) with given finitely generated semigroups*, Monatsh. Math **68** (1964), 213–217.

[HP3] Z. Hedrlín, A. Pultr, *Relations (graphs) with given infinite semigroups*, Monatsh. Math. **68** (1964), 421–425.

[HP4] Z. Hedrlín and A. Pultr, *Symmetric relations (undirected graphs) with given semigroups*, Monatsh. Math **69** (1965), 318–322.

[KR1] V. Koubek, V. Rödl, *On the minimum order of graphs with given semigroup*, J. of Comb. Theory, Series B **36** (1984), 135–155.

[KR2] V. Koubek, V. Rödl, *Note on the number of monoids of order n*, Comment. Math. Univ. Carolinae, (1985) vol. 26, **2** (1985), 309–314.

[KR3] V. Koubek, V. Rödl, *On the number of graphs with a given endomorphism monoid*, to appear (2007).

[KRS] D.Y. Kleitman, B.P. Rothschild, J. H. Spencer, *The number of semigroups of order n*, Proc. Amer. Math. Soc. **55** (1976), 227–232.

[P] M. Petrich, ”Introduction to semigroups” Merrill, Columbus, Ohio (1973).

[PT] A. Pultr, V. Trnková, ”Combinatorial, algebraic and topological representations of groups, semigroups and categories” North-Holland Mathematical Library, 22. North-Holland Publishing Co., Amsterdam-New York, (1980).

[W] D. West, “Introduction to Graph Theory” Prentice Hall, Inc., Upper Saddle River, NJ, (1996).