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Eigenvalues of the Laplace Operator on Quantum Graphs

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Special thanks to Dr. David Borthwick and Dr. Evans Harrell for their infinite patience and guidance.

Thanks to my mom, Ying Yang, for her slightly less than infinite patience.

An abstract of
A dissertation submitted to the Faculty of the
James T. Laney School of Graduate Studies of Emory University
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy
in Mathematics
2023

Abstract

Eigenvalues of the Laplace Operator on Quantum Graphs
By Haozhe Yu

This thesis focuses on estimates of eigenvalues on quantum graphs. The thesis is separated into five chapters,

- Chapter 1: Introduction to quantum graphs, the Laplace operator, and summary of the main results.
- Chapter 2: The upper bound of eigenvalues based on the Davies inequality.
- Chapter 3: The upper bound of the gap between the first two eigenvalues on tree graphs.
- Chapter 4: Generalizes results in the previous chapter on graphs which are modifications of trees.
- Chapter 5: The lower bound of the gap between the first two eigenvalues on quantum graphs.

Contents

CHAPTER 1

Introduction

1. Motivation

Our work is inspired by 2 classical questions in spectral theory: one involves the first positive eigenvalue and geometric quantity, the other concerns the gaps between eigenvalues. Studying operators of Schrödinger type on metric graphs is a growing subfield of mathematical physics. Quantum graphs have been used to understand a plethora of physical objects and to study complex phenomena of quantum mechanics including universality of spectral statistics, nodal statistics, resonances and much more. Metric graphs can serve as a mathematical approximation for networks where there is a well defined distance function between nodes in the network. Coming back to the importance of the first positive eigenvalue, it cannot be overstated. In mathematical physics it is the energy level associated with the ground state of the system, or the first excited state, if Neumann conditions are imposed; in the latter case λ_1 is therefore often referred to as the spectral gap. Similarly, the gap between eigenvalues is associated with the difference between energy level.

2. Preliminaries

Let Γ be a connected graph with the vertex set V and the edge set E. The cardinality of the sets V and E will be denoted $|V|$ and $|E|$, respectively. For $v \in V$ and $e \in E$, e and v are said to be adjacent if v is connected to e, expressed as $e \sim v$. For all vertices $v \in V$, the number of edges connected to v is called the degree of v denoted d_v. If $d_v = 2$, then v is called an artificial vertex. Artificial vertices do not affect the underlying topology of Γ and are often added or removed for convenience or as part of a technique to better understand the graph Γ. However one must be careful when adding or deleting artificial vertices, because Γ is constructed by its set of edges and vertices; changing these sets technically changes the graph Γ to some new graph Γ'.

We turn Γ into a metric graph by identifying each edge $e \in E$ with the interval $[0, |e|]$, where $|e| > 0$ is the length of the edge e, which introduces a coordinate system along e. When using a coordinate system, a point along e will be denoted x_e or simply x if e is clear. Edges have no direction, so the orientation of any coordinate system is arbitrary.

A bond in Γ is an oriented edge, denoted by $\vec{e}$, so that each edge contains exactly two bonds. Each bond contains the initial vertex $\partial^-(\vec{e})$ and final vertex $\partial^+(\vec{e})$. Two bonds $\vec{e_1}$ and $\vec{e_2}$ are consecutive if $\partial^+(\vec{e_1}) = \partial^-(\vec{e_2})$. We now define a path γ along the graph Γ.

Definition 2.1. A path γ consists of a pair of vertices connected by an ordered sequence of consecutive bonds, i.e.,

$$\gamma = \{v_-, \vec{e_1}, \dots, \vec{e_n}, v_+\},$$

where $v_- = \partial^-(\vec{e_1})$ and $v_+ = \partial^+(\vec{e_n})$ and its length is defined as $|\gamma| = \sum_{i=1}^{n} |\vec{e_i}|$.

For two vertices v and w, the distance $d(v,w)$ is defined as the minimal length of the path connecting them. Since along each edge the distance is determined by the coordinate x, we can define the distance $d(x,y)$ between two points x and y by considering them as artificial vertices, i.e. vertices of degree 2.

DEFINITION 2.2. The total length of the Γ is defined as $L(\Gamma) = \sum_{e_i \in E} |e_i|$.

A function f on the metric graph Γ is defined as a collection of functions $f_e : [0,|e|] \to \mathbb{C}$ for each edge $e \in E$, so the Hilbert space associated to Γ is defined as

$$L^2(\Gamma) := \bigoplus_{e \in E} L^2(0,|e|),$$

where

$$L^2(0,|e|) = \{f_e | \int_0^{|e|} f_e(x)^2 dx < \infty\}.$$

Similarly, we also define the Sobolev space $H^1(\Gamma)$ as

$$H^1(\Gamma) := \{f \in \bigoplus_{e \in E} H^1(0,|e|) : f \text{ is continuous across vertices}\},$$

with inner product

$$\|f\|^2_{H^1(\Gamma)} := \sum_{e \in E} \|f_e\|^2_{H^1}.$$

The continuity condition imposed on functions from $H^1(\Gamma)$ means that any function f from this space assumes the same value at a vertex v on all edges adjacent to v, and thus $f(v)$ has the unique value. We can generalize the definition from H^1 to H^k, but we do not employ the vertex condition. We denote by $\widetilde{H}^k(\Gamma)$ the space

$$\widetilde{H}^k(\Gamma) = \bigoplus_{e \in E} H^k(0,|e|),$$

which consists of functions f on Γ that on each edge belong to $H^k(0,|e|)$ and such that

$$\|f\|^2_{\widetilde{H}^k(\Gamma)} := \sum_{e \in E} \|f_e\|^2_{H^k} < \infty.$$

We refer readers to section 1.3 in [**3**] for more details.

A quantum graph is a metric graph equipped with a quantum mechanical Hamiltonian operator. In this paper, we focus on the Laplace operator $-\Delta$. In particular, we can write $-\Delta = -\frac{d^2}{dx^2}$ since $-\Delta$ acts as the differential operator $-\frac{d^2}{dx^2}$ on each edge. Here, as before, x is the coordinate x along an edge. Notice that for the Laplacian operator the direction of the edge is irrelevant. This is not true if one wants to consider first order operators like $\frac{d}{dx}$. We refer readers to section 2.2 in [**3**] for the first order operator cases.

To have the standard self-adjoint extension of $-\Delta$, we can construct the domain as in spectral theory

$$D(-\Delta_N) := \{u \in H^1(\Gamma) : \langle u, \cdot \rangle_{H^1} \text{ extends to } L^2(\Gamma) \text{ as a bounded functional}\}. \tag{2.1}$$

$-\Delta_N$ is self-adjoint on the domain (2.1). Moreover, if a function f belongs to the domain (2.1), then f_e is in $C^1[0,|e|]$ on each edge e. Furthermore, at each vertex $v \in V$

$$\sum_{e \sim v} f'_e(v) = 0,$$

where the derivative is taken outward from v along e. The condition that the sum of derivative vanish at the vertex is called standard vertex condition. The set of vertices imposed by standard condition is denoted by V_N.

Furthermore, we want to consider $-\Delta$ with Dirichlet (zero) vertices on a subset $V_0 \subset V$. Therefore, we need to construct the following function space

$$H_0^1(\Gamma) := \{u \in H^1(\Gamma) : \ u = 0 \text{ on all vertices } v \in V_0\}. \tag{2.2}$$

For the relationship between the vertex condition and self-adjointness, we refer readers to Theorem 1.4.4 in [**3**].

In this paper, we assume Γ is compact which means it consists of finitely many edges of finite length, we have the following result from Theorem 3.1.1 in [**3**] for the spectrum of $-\Delta(\Gamma)$ which is denoted by $\sigma(-\Delta(\Gamma))$, or $\sigma(-\Delta)$ when Γ is clear.

THEOREM 2.3. *If Γ is a compact quantum graph, then $\sigma(-\Delta)$ only contains isolated eigenvalues with finite multiplicity and as $j \to \infty$, $\lambda_j \to \infty$.*

The above theorem implies that for all $\lambda_j \in \sigma(-\Delta)$, there exists a function $\phi_j \in D(-\Delta)$ such that

$$(-\Delta - \lambda_j)\phi_j = 0.$$

We will refer to λ_j and ϕ_j as eigenvalue, eigenfunction correspondingly. Because $-\Delta$ is self-adjoint on $D(-\Delta)$ all eigenvalues must be real. We order the eigenvalues such that $\lambda_j \le \lambda_{j+1}$ for all $j \in \{1, 2, ...\}$. It is well known (see [**26**]) that all eigenfunctions are contained in $C^\infty(\Gamma) := \{f \in \bigoplus_{e\in E} C^\infty(0, |e|) : f \text{ is continuous across vertices}\}$ and for eigenvalues such that $\lambda_i \neq \lambda_j$, the eigenfunctions ϕ_i and ϕ_j are orthogonal, meaning

$$\int_\Gamma \phi_j \phi_i = 0. \tag{2.3}$$

We can ensure that all eigenfunctions are orthogonal. Let n be the dimension of the eigenspace associated with the eigenvalue λ_j, call this space S_j. By the Gram-Schmidt process we can choose n functions $\{\phi_1, \phi_2, ..., \phi_n\}$ in S_j such that each function ϕ_j is orthogonal to ϕ_i, $i \neq j$. We then choose $\{\phi_1, \phi_2, ..., \phi_n\}$ to be the eigenfunctions associated with the eigenspace S_j.

The Rayleigh quotient on Γ is defined as

$$R(f) := \frac{||f'||^2_{L^2}}{||f||^2_{L^2}}. \tag{2.4}$$

Because all of the eigenfunctions can be made to be orthogonal and the set $\{\phi_j\}_{j=1}^\infty$ is a basis for $L^2(\Gamma)$, we can use the min-max formula to express all eigenvalues

$$\lambda_j = \min_{\substack{X \subset H^1 \\ dim(X)=j}} \{\max_{f \in X \setminus 0} \{R(f)\}\}. \tag{2.5}$$

The first non-trivial eigenvalue λ_1 is known as the spectral gap. If all vertices of Γ are imposed by standard condition, we will denote the eigenvalues by $0 = \lambda_1^N < \lambda_2^N \le \dots$, where the smallest non-trivial eigenvalue is given by

$$\lambda_2^N(\Gamma) = \inf \left\{ \frac{\int_\Gamma |u'|^2}{\int_\Gamma |u|^2} : u \neq 0 \in H^1(\Gamma), \int_\Gamma u = 0 \right\}.$$

If at least one of vertices of Γ is imposed by Dirichlet condition, we will denote the eigenvalues by $0 < \lambda_1 \le \lambda_2 \le \dots$, where the first non-trivial eigenvalue is given by

$$\lambda_1(\Gamma) = \inf\left\{ \frac{\int_\Gamma |u'|^2}{\int_\Gamma |u|^2} : u \in H_0^1(\Gamma) \right\}.$$

In this paper, vertices of Γ will be imposed by either standard condition or Dirichlet condition, i.e. $V = V_N \cup V_0$.

3. Main results

In this paper, we obtain following main results on quantum graphs:

- The upper bound of λ_2^N
- The upper bound of $\lambda_2 - \lambda_1$
- The lower bound of $\lambda_2 - \lambda_1$

The Davies inequality [**6**] has been used to estimate eigenvalues on compact Riemannian manifolds in [**13**]. Post listed this result on compact metric graph in [**8**], but he did not provide a proof. We prove it here.

THEOREM 3.1. *Let Γ be a compact quantum graph and $A_1, A_2, \dots, A_k$ be disjoint closed subgraphs on Γ. The total length of Γ is denoted by L. Denote*

$$d := \min_{i\neq j} d(A_i, A_j).$$

Then

$$\lambda_k^N(\Gamma) \le \frac{4}{d^2} \max_{i\neq j} \left(\log \frac{2L}{\sqrt{|A_i||A_j|}} \right)^2, \tag{3.1}$$

where $|A_i| = \sum_{e\in A_i} |e|$ is the size of A_i.

In particular, if we only have two subgraphs $A_1 = A$ and $A_2 = B$ then (3.1) becomes

$$\lambda_2^N(\Gamma) \le \frac{4}{d^2} \left(\log \frac{2L}{\sqrt{|A||B|}} \right)^2, \tag{3.2}$$

where $d = d(A, B)$.

In the next 2 chapters, we focus on the upper bound of $\lambda_2 - \lambda_1$ on the quantum tree with Dirichlet leaves. The main result is the following theorem.

THEOREM 3.2. *For $z \in (\lambda_n, \lambda_{n+1}]$, suppose that f is a positive function on the set $\{\lambda_1, \dots, \lambda_n\}$ such that $f(\lambda_j)/(z-\lambda_j)^2$ is nondecreasing with j. Then*

$$\sum_{j=1}^n f(\lambda_j) \le 4 \sum_{j=1}^n \frac{f(\lambda_j)}{z-\lambda_j} \lambda_j.$$

In the last chapter, we study the lower bound of $\lambda_2 - \lambda_1$ on quantum graphs. By adapting the idea of the weighted Cheeger constant in [**14**], we establish a lower bound of $\lambda_2 - \lambda_1$ in terms of ϕ_1. By estimating ϕ_1, we obtain the following inequality as our main result.

THEOREM 3.3. *Let Γ be a metric graph with mixed standard and Dirichlet vertex conditions as described above. There exists a constant $C(L, \ell_0) > 0$, depending only on the total length L and minimum edge length ℓ_0, such that*

$$\lambda_2 - \lambda_1 \ge C(L, \ell_0). \tag{3.3}$$

CHAPTER 2

The Upper Bound of The Spectral Gap

1. Introduction

Let $-\Delta := -d^2/dx^2$ be the Laplace operator which acts on the L^2 space of functions on the edges of Γ. In this chapter, all vertices of Γ are imposed by standard condition. Therefore, the domain for $-\Delta$ are functions which are $H^1(\Gamma)$ and satisfy standard vertex condition, so we can write the quadratic form of $-\Delta$ as

$$h(f,f) := \|f'\|^2_{L^2(\Gamma)}.$$

As mentioned above, the smallest non-trivial eigenvalue (also called the spectral gap) is given by

$$\lambda_2^N(\Gamma) = \inf\left\{\frac{\int_\Gamma |u'|^2}{\int_\Gamma |u|^2} : u \neq 0 \in H^1(\Gamma), \int_\Gamma u = 0\right\}.$$

Before we discuss the Davies inequality on quantum graphs, we provide some example graphs and their spectral gap. More details can be found in section 3 of [**4**].

- *Path Graph* I(L): Consists of a single edge of length L and two degree one vertices. Because both vertices are degree one the second vertex condition forces the path graph to coincide with a Neumann interval of length L. Hence,

$$\lambda_2^N(I(L)) = \frac{\pi^2}{L^2}.$$

- *Symmetric Star Graph* S(L,E): The symmetric star graph has a single central vertex and $|E|$ edges of equal length $\frac{L}{|E|}$.

$$\lambda_2^N(S(L,E)) = \frac{\pi^2|E|^2}{4L^2}.$$

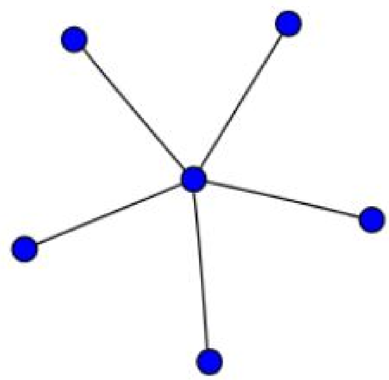

FIGURE 1. A symmetric star graph with 5 edges.

- *Symmetric Flower Graph* F(L,E): A symmetric flower graph also has a single central vertex and $|E| \geq 2$ edges of length $\frac{L}{|E|}$. However, all edges of a flower graph are loops, meaning both end points are attached to the central vertex.
$$\lambda_2^N(F(L,E)) = \frac{\pi^2|E|^2}{L^2};$$
If $|E| = 1$, then we have a loop of length L, which has $\lambda_2^N = \frac{4\pi^2}{L^2}$.

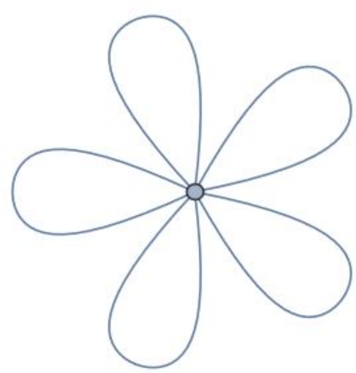

FIGURE 2. A symmetric flower graph with 5 edges.

- *Symmetric Pumpkin Graph* K(L,E): A symmetric pumpkin graph has two vertices and $|E|$ edges. The vertices are the end points for each edge in $K(L,E)$, all edges have length $\frac{L}{|E|}$.
$$\lambda_2^N(F(L,E)) = \frac{\pi^2|E|^2}{L^2}.$$

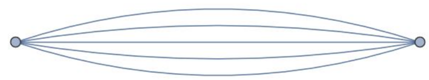

FIGURE 3. A symmetric pumpkin graph with 5 edges.

Our goal in this chapter is to find an upper bound of λ_2^N for graph Γ in terms of geometric properties of Γ. To do so, we need to fix a characteristic of the graph, otherwise the spectral gap might be arbitrarily large. For example, consider the sequence of symmetric star graphs $\{S(L,n)\}_{n=1}^{\infty}$, each graph in the sequence has the same total length, L, and the number of edges increases for each graph in the sequence. The spectral gap for this sequence

$$\lambda_2^N(S(L,n)) = \frac{\pi^2 n^2}{L^2}$$

is divergent. However, if we fix L and $|E|$, we obtain a sharp upper bound on λ_2^N in terms of the arithmetic mean value $\mathcal{A} = \frac{L}{|E|}$ of the edge length. We list Theorem 4.2 in [**4**] below and readers can find the proof in the same paper.

THEOREM 1.1. *Let Γ be a quantum graph having length $L > 0$ and $|E| \geq 2$ edges. Then*

$$\lambda_2^N(\Gamma) \leq \frac{\pi^2 |E|^2}{L^2} = \frac{\pi^2}{\mathcal{A}^2},$$

with equality if and only if Γ is an equilateral pumpkin or flower graph. If $E = 1$, then

$$\lambda_2^N(\Gamma) = \begin{cases} \frac{4\pi^2}{L^2} & \text{if } \Gamma \text{ is a loop} \\ \frac{\pi^2}{L^2} & \text{if } \Gamma \text{ is a path.} \end{cases}$$

In the papers [**4, 9, 10, 12**], we can find more estimates of λ_2^N.

2. Application of The Davies Inequality

The Davies inequality was introduced by Professor Davies in his paper [**6**] and reviewed by Grigor'yan in [**7**]. We follow the procedure in [**7**] to provide the upper bound of λ_2^N on quantum graphs.

Let Γ be a compact connected quantum graph. Consider the initial value problem on Γ for $f(x) \in H^1(\Gamma)$, where along each edge $e \in \Gamma$

$$\begin{cases} \Delta u(t,x) = \frac{\partial u(t,x)}{\partial t}, \\ u(0,x) = f(x). \end{cases} \tag{2.1}$$

And $u(t,x)$ is continuous on Γ and satisfies standard vertex condition

$$\sum_{e \sim v} \frac{\partial u(t,x)}{\partial x} = 0$$

on each vertex $v \in V$ for all $t > 0$. We can find a solution for $u(t,x)$,

$$u(t,x) = e^{t\Delta} f.$$

The eigenfunctions of $-\Delta$ form a basis for $L^2(\Gamma)$. Let $\{\phi_n\}_{n=0}^{\infty}$ be this basis of eigenfunctions where ϕ_i is the eigenfunction with the eigenvalue λ_i. In this basis, f can be expressed as

$$f = \sum_{n=0}^{\infty} A_n \phi_n,$$

where

$$A_n = \int_\Gamma f(y)\phi_n(y)dy.$$

Then $e^{t\Delta} f$ is

$$\begin{aligned} e^{t\Delta} f &= \sum_{n=0}^{\infty} A_n e^{-t\lambda_k} \phi_n(x) \\ &= \sum_{n=0}^{\infty} \{\int_\Gamma f(y)\phi_n(y)dy\} e^{-t\lambda_k}\phi_n(x) \\ &= \int_\Gamma \{\sum_{n=0}^{\infty} e^{-t\lambda_k}\phi_n(x)\phi_n(y)\} f(y)dy. \end{aligned}$$

The kernel in the curly brackets is called the heat kernel on Γ and will be denoted by $p_\Gamma(t,x,y)$ or simply $p(t,x,y)$ when Γ is clear, so

$$p(t,x,y) = \sum_{n=0}^{\infty} e^{-t\lambda_k}\phi_n(x)\phi_n(y).$$

More details about the heat kernel can be found in section 2.2 in [**7**].

To prove the Davies inequality, we need the following lemma from [**5**].

Lemma 2.1. *(*Essentially Aronson, [**5**]) *Let Γ be a metric graph with a finite number of edges. The set of edges is denoted by E. Suppose that u satisfies the heat equation on the edges of a metric graph while being continuous at the vertices and satisfying Kirchhoff conditions there, and that $\xi(t,x)$ is an absolutely continuous function on Γ, satisfying*

$$\xi_t + \frac{1}{2}\xi_x^2 \le 0 \tag{2.2}$$

a.e. on edges. (This formula does not depend on the orientation of the edge.) Then the function

$$J(t) := \sum_{e\in E} \int_e u^2(t,x) e^{\xi(t,x)} dx \tag{2.3}$$

is non-increasing in t.

Proof. Given any orientation for the edges, we calculate

$$
\begin{aligned}
J'(t) &= \sum_{e\in E}\int_e (2uu_t e^{\xi} + u^2\xi_t e^{\xi})dx\\
&\le \sum_{e\in E}\int_e (2uu_{xx}e^{\xi} - \frac{1}{2}u^2\xi_x^2 e^{\xi})dx\\
&= \sum_{e\in E}\left(2\int_e ((uu_x e^{\xi})_x - u_x^2 e^{\xi} - uu_x e^{\xi}\xi_x)dx - \frac{1}{2}\int_e u^2\xi_x^2 e^{\xi}dx\right)\\
&= 2\sum_{e\in E}\int_e ((uu_x e^{\xi})_x - \sum_{e\in E}\frac{1}{2}\left(\int_e (4u_x^2 e^{\xi} + 4uu_x e^{\xi}\xi_x + u^2\xi_x^2 e^{\xi})dx\right)\\
&=: BT - \frac{1}{2}\sum_{e\in E}\int_e e^{\xi}\left(2u_x + u\xi_x\right)^2 dx
\end{aligned}
$$

$$
\le BT. \tag{2.4}
$$

The new feature is the term BT, which although independent of edge orientations is best elucidated by specifying them. To this end, let us evaluate the sum of the contributions at a given vertex v by orienting all of its adjacent edges outward. The sum of the contributions at v then has the form

$$
-2u(t,v)e^{\xi(t,v)}\sum_e u_x(t,v^+) = 0,
$$

because of the Kirchhoff conditions at v. Now summing over all vertices, we find that $J'(t)\le 0$. □

Remark 2.2. Inviting choices for ξ are expressions that involve the distance function in the graph $d(x,y)$, which can fail to be differentiable at a finite set of points intertior to an edge, *viz.* at cut points, where there are multiple paths of the same length connecting x and y. This is why the lemma was phrased with relaxed regularity assumptions. An alternative would have been to regard cut points as degree-two "artificial" vertices.

The next result is the inequality of Davies quoted as Theorem 3.2 in [**7**], and adapted to metric graphs:

Given Lemma 2.1, the "first proof" of this result in [**7**] needs no change.

Theorem 2.3. *(*Essentially Davies, [**7**]) *Let A and B be two nonintersecting measurable sets in Γ. Then*

$$
\int_A\int_B p(t,x,y)dxdy \le \sqrt{|A||B|}\exp\left(-\frac{d(A,B)^2}{4t}\right), \tag{2.5}
$$

where $d(A,B)$ is the distance between A and B defined as minimal $d(x,y)$ with $x\in A$ and $y\in B$, $|A|,|B|$ are the sizes of A, B.

Proof. Let A and B be two disjoint compact subsets in graph Γ. Denote $d(A,B)$ as the distance between A and B. Consider the function $u(t,x) = e^{t\Delta}1_A$. We can write

$$
\begin{aligned}
\int_B\int_A p(t,x,y)dydx &= \int_B\big(\int_\Gamma p(t,x,y)1_A dy\big)dx\\
&= \int_B u(t,x)dx
\end{aligned}
$$

$$
\le |B|^{1/2}\left(\int_B u^2(t,x)dx\right)^{1/2}. \tag{2.6}
$$

Let us set, for some $\alpha > 0$,

$$\xi(t,x) := \alpha d(x,A) - \frac{\alpha^2}{2}t$$

and it is easy to check that

$$\xi_t = -\frac{1}{2}|\xi_x|^2 = -\frac{\alpha^2}{2}.$$

Now, we can consider the function

$$J(t) := \int_\Gamma u^2(t,x)e^{\xi(t,x)}dx$$

which is non-increasing by Lemma 2.1 in $t > 0$. If $x \in B$ then

$$\xi(t,x) \geq \alpha d(B,A) - \frac{\alpha^2}{2}t,$$

whence

$$\begin{aligned} J(t) &\geq \int_B u^2(t,x)e^{\xi(t,x)}dx \\ &\geq \exp\left(\alpha d(A,B) - \frac{\alpha^2}{2}t\right)\int_B u^2(t,x)dx. \end{aligned} \tag{2.7}$$

On the other hand, if $x \in A$ then $\xi(0,x) = 0$. By the continuity of $J(t)$ at $t = 0+$, we have

$$J(t) \leq J(0) = \int_\Gamma e^{\xi(0,x)}1_A dx = |A|. \tag{2.8}$$

Combining these results, we obtain

$$\int_B\int_A p(t,x,y)dydx \leq \sqrt{|A||B|}\exp\left(-\frac{\alpha}{2}d(A,B) + \frac{\alpha^2}{4}t\right).$$

Setting $\alpha = d(A,B)/t$ we finish the proof.

□

Now, we are ready to provide an upper bound for λ_2^N in Theorem 4.1 in [**7**] on quantum graphs which is equivalent to Theorem 6.4 in [**8**]. Post provided the following upper bound for λ_2^N in [**8**] without a proof, so we prove it here for the completeness.

THEOREM 2.4. *Let Γ be a compact quantum graph and $A_1, A_2, \ldots, A_k$ be disjoint closed set on Γ. The total length of Γ is denoted by L. Denote*

$$d := \min_{i\neq j} d(A_i, A_j).$$

Then

$$\lambda_k^N(\Gamma) \leq \frac{4}{d^2}\max_{i\neq j}\left(\log\frac{2L}{\sqrt{|A_i||A_j|}}\right)^2, \tag{2.9}$$

where $|A_i| = \sum_{e\in A_i}|e|$ is the size of A_i.

In particular, if we only have two sets $A_1 = A$ and $A_2 = B$ then (2.9) becomes

$$\lambda_2^N(\Gamma) \leq \frac{4}{d^2}\left(\log\frac{2L}{\sqrt{|A||B|}}\right)^2, \tag{2.10}$$

where $d = d(A,B)$.

Proof. We first prove (2.10). By the eigenfunction expansion,

$$p(t,x,y) := \sum_{i=1}^{\infty} e^{-t\lambda_i^N} \phi_i(x)\phi_i(y),$$

we can write, for any $t > 0$,

$$\int_A \int_B p(t,x,y)dydx = \sum_{i=1}^{\infty} e^{-t\lambda_i^N} \int_A \phi_i(x)dx \int_B \phi_i(y)dy.$$

Denote

$$a_i := \int_A \phi_i(x)dx = \langle 1_A, \phi_i\rangle_{L^2(\Gamma)}, \quad b_i := \langle 1_B, \phi_i\rangle_{L^2(\Gamma)}$$

and observe that

$$\sum_{i=1}^{\infty} a_i^2 = |1_A|^2_{L^2(\Gamma)} = |A| \quad \text{and} \quad \sum_{i=1}^{\infty} b_i^2 = |B|.$$

Taking $\phi_1 \equiv \frac{1}{\sqrt{L}}$, we obtain

$$a_1 = \langle 1_A, \frac{1}{\sqrt{L}}\rangle_{L^2(\Gamma)} = \frac{|A|}{\sqrt{L}} \quad \text{and} \quad b_1 = \frac{|B|}{\sqrt{L}}.$$

Thus, we have

$$\begin{aligned}\int_A \int_B p(t,x,y)dydx &= a_1 b_1 + \sum_{i=2}^{\infty} e^{-t\lambda_i^N} a_i b_i \\ &\geq a_1 b_1 - e^{-t\lambda_2^N} \left(\sum_{i=2}^{\infty} a_i^2\right)^{1/2} \left(\sum_{i=2}^{\infty} b_i^2\right)^{1/2} \\ &\geq \frac{|A||B|}{L} - e^{-t\lambda_2^N} \sqrt{|A||B|}.\end{aligned}$$

By the Davies inequality, we obtain

$$\sqrt{|A||B|} e^{-d^2/4t} \geq \frac{|A||B|}{L} - e^{-t\lambda_2^N} \sqrt{|A||B|}$$

and

$$e^{-t\lambda_2^N} \geq \frac{\sqrt{|A||B|}}{L} - e^{-d^2/4t}.$$

Choosing t so that

$$e^{-d^2/4t} = \frac{\sqrt{|A||B|}}{2L},$$

we conclude that

$$\lambda_2^N \leq \frac{1}{t} \log \frac{2L}{\sqrt{|A||B|}} = \frac{4}{d^2} \left(\log \frac{2L}{\sqrt{|A||B|}}\right)^2.$$

Now we are going to prove for the case $k > 2$. Consider

$$J_{lm} = \int_{A_l} \int_{A_m} p(t,x,y)dydx$$

and denote

$$a_i^{(l)} := \langle 1_{A_l}, \phi_i\rangle.$$

Then by the same process, we have

$$
\begin{aligned}
J_{lm} &= \sum_{i=1}^{\infty} a_i^{(l)} a_j^{(m)} \\
&= \frac{|A_l||A_m|}{L} + \sum_{i=2}^{k-1} e^{-\lambda_i^N t} a_i^{(l)} a_i^{(m)} + \sum_{i=k}^{\infty} e^{-\lambda_i^N t} a_i^{(l)} a_i^{(m)} \\
&\geq \frac{|A_l||A_m|}{L} + \sum_{i=2}^{k-1} e^{-\lambda_i^N t} a_i^{(l)} a_i^{(m)} - e^{-\lambda_k^N t} \sqrt{|A_l||A_m|}. \qquad (2.11)
\end{aligned}
$$

On the other hand, by the Davies inequality,

$$J_{lm} \leq \sqrt{|A_l||A_m|} e^{-d^2/4t}. \qquad (2.12)$$

Therefore, we can make the same argument as in the case $k = 2$ as long as

$$\sum_{i=2}^{k-1} e^{\lambda_i^N t} a_i^{(l)} a_i^{(m)} \geq 0. \qquad (2.13)$$

Let us show that the inequality above can be achieved by choosing l, m. Let us interpret the sequence $a^{(j)} := \left(a_2^{(j)}, \dots, a_{k-1}^{(j)}\right)$ as $(k-2)$-dimensional vector in $\mathbb{R}^{k-2}$. Here j ranges from 1 to k so that we have k vectors $a^{(j)}$ in $\mathbb{R}^{k-2}$. Let us introduce the inner product of vectors $u = (u_2, \dots, u_{k-1})$ and $v = (v_2, \dots, v_{k-1})$ in $\mathbb{R}^{k-2}$ by

$$(u, v)_t := \sum_{i=2}^{k-1} e^{-\lambda_i^N t} u_i v_i$$

and apply the following lemma

Lemma 2.5. *From any $n+2$ vectors in n-dimensional Euclidean space, it is possible to choose two vectors with non-negative inner product.*

Therefore, we can find l, m so that $\left(a^{(l)}, a^{(m)}\right)_t \geq 0$ and (2.13) holds. Then (2.12) and (2.11) yield

$$e^{-\lambda_k^N t} \geq \frac{\sqrt{|A_l||A_m|}}{L} - e^{-d^2/4t},$$

and we are left to choose t. However, t should not depend on l, m because we use t to define the inner product before choosing l, m. So, we first write

$$e^{-\lambda_k^N t} \geq \min_{i \neq j} \frac{\sqrt{|A_i||A_j|}}{L} - e^{-d^2/4t},$$

and then define t by

$$e^{-d^2/4t} = \frac{1}{2} \min_{i \neq j} \frac{\sqrt{|A_i||A_j|}}{L},$$

we conclude

$$\lambda_k^N \leq \frac{4}{d^2} \max_{i \neq j} \left(\log \frac{2L}{\sqrt{|A_i||A_j|}} \right)^2.$$

$\square$

In the following example, we will compare different estimates of λ_k^N.

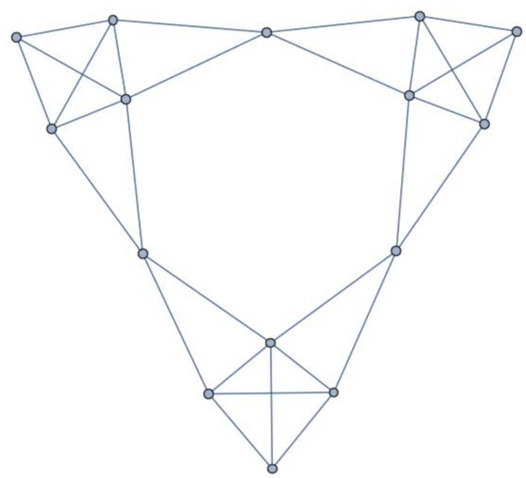

FIGURE 4. "K4 necklace" Graph

EXAMPLE 2.6. Assume we have a "K4 necklace" graph Γ which is constructed by three K4 graphs with edges of length a and paths between them with length $4a$ (so the length of each edge on the path $2a$). The total length of Γ is L. In [**9**], authors provided an estimate

$$\lambda_k(G) \leq \left(k - 2 + \beta + |V_0| + \frac{|N| + \beta}{2}\right)^2 \frac{\pi^2}{L^2}, \tag{2.14}$$

where $\beta = |E| - |V| + 1$ is the Betti number, N is the set of Neumann vertices of degree one and V_0 is set of Dirichlet vertices. Let's find the Betti number β for the graph Γ at first:

$$\beta = |E| - |V| + 1 = (6 + 4) * 3 - (4 + 1) * 3 + 1 = 16.$$

Then estimate (2.14) will be

$$\lambda_3 \leq (1 + \frac{3}{2} * 16)^2 \frac{\pi^2}{L^2} \approx \frac{6168}{L^2}.$$

Now, we are going to calculate our estimate (2.9) in terms of L. Note that the distance between three K4 subgraph is $d = 4a = 4 * \frac{L}{42}$ and the size of each K4 is $m = 6 * \frac{L}{42}$. Therefore,

$$\lambda_3 \leq \frac{21^2}{L^2} \left(\log (2 * 7)\right)^2 \approx \frac{3071}{L^2}.$$

CHAPTER 3

The Upper Bound for Trees

This chapter and the next chapter are based on our work [**24**].

1. Introduction

Let u be a solution of

$$\Delta u + \lambda u = 0,$$

in a region D in $\mathbb{R}^m$, $m > 1$, subject to the Dirichlet boundary condition. D is a region such that the spectrum is discrete, i.e. $0 < \lambda_1 \leq \lambda_2 \leq \dots$. Payne, Pólya and Weinberger [**19**] showed that for domains in $\mathbb{R}^2$, the inequality

$$\lambda_{n+1} \leq \lambda_n + \frac{4}{n} \sum_{i=1}^{n} \lambda_i,$$

where $n = 1, 2, \dots$, is satisfied as long as the spectrum of D is discrete. Furthermore, in the paper [**18**, Theorem 1], Hile and Protter established the following result.

THEOREM 1.1. *Let σ be the unique solution on (λ_n, ∞) of the equation*

$$\sum_{i=1}^{n} \frac{\lambda_i}{\sigma - \lambda_i} = \frac{mn}{4}.$$

Then

$$\lambda_{n+1} \leq \sigma. \tag{1.1}$$

If V_0 is not empty then we obtain a lower bound

$$\lambda_k \geq \frac{k^2 \pi^2}{4L^2}.$$

As for upper bounds, a simple test function argument yields

$$\lambda_1 \leq \frac{\pi^2}{\ell_{\max}^2}, \tag{1.2}$$

where $\ell_{\max}$ is the maximum edge length. Similarly, Berkolaiko-Kennedy-Kurasov-Mugnolo [**11**, Thm. 1.3] proved an upper bound

$$\lambda_1 \leq \frac{\pi^2}{girth(\Gamma)^2}, \tag{1.3}$$

where the *girth* is defined as the minimum cycle length of the graph formed from Γ by identifying all Dirichlet vertices. For higher eigenvalues, the same authors also proved upper bounds in terms of the Betti number $\beta = |E| - |V| + 1$, which counts the number of independent cycles in Γ. In our notation, this yields [**9**, Thm. 4.9]

$$\lambda_k \leq \left(k - \frac{1}{2} + \frac{3}{2}|E| - |V_N| - \frac{1}{2}|V_0|\right)^2 \frac{\pi^2}{L^2}. \tag{1.4}$$

Combining these results, we have a lower bound

$$\frac{\lambda_{n+1}}{\lambda_n} \leq (2n + 1 + 3|E| - 2|V_N| - |V_0|)^2, \tag{1.5}$$

provided Γ is not a cycle graph.

Notice that the ratio λ_2/λ_1 can be arbitrarily large in general. We can see this in the following example [**22**, Example 1.2]:

Example 1.2. Consider an equilateral "balloon" graph, consisting of a pumpkin graph with k edges of length 1, with a pendant edge attached at one vertex. Dirichlet conditions are imposed at the endpoint of the free edge. The lowest two eigenvalues are equal on all edges of the pumpkin, so we can use the linear parameter $x \in [0, 2]$, with the range $[0, 1]$ corresponding to the free edge. For an eigenfunction of the form

$$\phi(x) = \begin{cases} \sin(\sigma x), & x \in [0,1], \\ c \cos \sigma(2 - x), & x \in [1,2], \end{cases}$$

the continuity and vertex conditions give $(\tan \sigma)^2 = 1/k$. The lowest two solutions give

$$\lambda_1 = (\arctan(1/\sqrt{k}))^2, \qquad \lambda_2 = (\pi - \arctan(1/\sqrt{k}))^2,$$

and hence

$$\frac{\lambda_2}{\lambda_1} = \left(\frac{\pi}{\arctan(1/\sqrt{k})} - 1\right)^2.$$

This gives $\lambda_2/\lambda_1 = 25$ for $k = 3$, and $\lambda_2/\lambda_1 \sim \pi^2 k$ as $k \to \infty$.

In addition to showing that λ_2/λ_1 can be arbitrarily large, Example 1.2 shows the need for the cycle restriction in (1.5), as the balloon is a cycle graph. The right side of (1.5) reduces to $(3k + 1)^2$ for this case, and so the inequality fails as the ratio approaches π^2 for large k

Throughout this chapter, we assume that Γ is a metric tree, with V_0 consisting of the external vertices of degree one and V_N consisting of the internal vertices. On Γ, Nicaise [**16**] obtained the same inequality (1.1) by using the ideas developed in [**18**]. In particular,

$$\lambda_{n+1} - \lambda_n \leq \frac{4}{n} \sum_{j=1}^{n} \lambda_j. \tag{1.6}$$

and

$$\frac{\lambda_2}{\lambda_1} \leq 2 + \sqrt{5} \tag{1.7}$$

are satisfied on Γ. Our goal in this chapter is to adapt the techniques of Harrell-Stubbe [**15**] to produce a general eigenvalue estimate on tree Γ which generalizes the Nicaise bound [**16**]. To obtain the main result, we need the following definition.

Definition 1.3. An affine function on a metric graph is interpreted as a continuous function which is linear on each edge.

2. Main Results

Now, we are ready to produce a family of affine functions whose derivatives cover Γ uniformly in an average sense. Let $\mathcal{A}(\Gamma)$ denote the space of affine functions on Γ which satisfy standard vertex conditions at inner points of Γ. No vertex condition are imposed at the points of V_0. If we interpret Γ as an electric circuit, with each edge assigned a resistance equal to its length, then functions $\mathcal{A}(\Gamma)$ corresponds precisely to a voltage function satisfying the Ohm's and Kirchhoff's circuit laws. As Kirchhoff [**25**] demonstrated in 1847, there exists a voltage function for any combination of external voltages applied at the points in V_0. Hence, $\mathcal{A}(\Gamma)$ contains non-constant functions provided V_0 contains at least two points.

From Nicaise [**16**, Lemma 4.2] we quote the following result. The original result did not include a proof, so we will give one here. A similar result was derived independently in Demirel-Harrell [**22**, Thm. 2.9], but without the restriction to three functions.

Lemma 2.1. *For a metric tree Γ, there exist functions $g_\alpha \in \mathcal{A}(\Gamma)$ for $\alpha \in \{1,2,3\}$ such that on all edges, $|g'_\alpha|$ equals 0 or 1 and*

$$\sum_{\alpha=1}^{3} |g'_\alpha| = 2. \tag{2.1}$$

Proof. Let us refer to a subgraph of Γ consisting of a vertex with two adjoining external edges (leaves) as a *leaf-pair*. Trimming a leaf pair from a vertex of degree ≥ 3 reduces its degree by 2. If all possible leaf-pairs are trimmed from a given vertex, the result either an artificial (degree 2) or an external vertex (degree 1). By carrying out this trimming process as far as possible at each vertex, we eventually reduce Γ to a single segment. Hence Γ can be constructed by starting from a single segment and attaching leaf-pairs successively. Each leaf-pair is added by gluing its vertex to any point on the graph, which could be an existing vertex or an edge point.

The family $\{g_\alpha\}$ is constructed by induction, using this decomposition of Γ. For the initial segment, we may choose an arbitrary parametrization x and set $g_1(x) = x$, $g_2(x) = -x$, and $g_3(x) = 0$.

Now suppose the family $\{g_\alpha\}$ has been defined with the desired properties for a tree Γ. Let $\widetilde{\Gamma}$ be a graph obtained by adding a single leaf-pair to Γ. The extensions $\widetilde{g}_\alpha$ may be defined as follows:

(1) Suppose the leaf-pair is attached at an internal vertex of Γ (possibly artificial), so that each g_α already satisfies standard vertex conditions at this point. We can extend the family so that both $\widetilde{g}'_1$ and $\widetilde{g}'_2$ alternate ± 1 on edges of the leaf-pair, while $\widetilde{g}_3$ is constant on these edges.
(2) If the leaf-pair is attached an an external vertex of Γ, then by construction two of the g_α, say $\alpha = 1,2$ will have derivatives ± 1 at this vertex and g_3 will be constant. We extend the family so that $\widetilde{g}'_1$ is zero on one leaf of the pair, $\widetilde{g}'_2$ is zero on the other, and the derivatives on the other leaves are chosen to satisfy the vertex condition. The third function, whose derivative vanishes into the vertex, is extended so that $\widetilde{g}'_3$ alternates ± 1 on the new leaves.

At every stage of the induction, each function satisfies the vertex conditions and $|g'_\alpha| = 1$ for two values of α and $|g'_\alpha| = 0$ for the third. $\square$

Using the collection $\{g_\alpha\}$, we can prove the main result of this section, a version of the general eigenvalue inequality from Theorem 5 in [**15**].

THEOREM 2.2. *For $z \in (\lambda_n, \lambda_{n+1}]$, suppose that f is a positive function on the set $\{\lambda_1, \dots, \lambda_n\}$ such that $f(\lambda_j)/(z-\lambda_j)^2$ is nondecreasing with j. Then*

$$\sum_{j=1}^{n} f(\lambda_j) \le 4 \sum_{j=1}^{n} \frac{f(\lambda_j)}{z-\lambda_j} \lambda_j.$$

PROOF. Let G_α be the multiplication operator on $L^2(\Gamma)$ associated to g_α. We define a corresponding set of first-order differential operators

$$D_\alpha := \frac{1}{2}[\Delta, G_\alpha].$$

On each edge, we can write $D_\alpha = g'_\alpha \frac{d}{dx}$ by direct computing. Similarly, on each edge $[D_\alpha, G_\alpha] = 2(g'_\alpha)^2$ by direct computing, so $[D_\alpha, G_\alpha]$ is the projection onto the support of g'_α. Since g'_α takes values in $\{0, \pm 1\}$, by the the construction in Lemma 2.1,

$$\sum_{\alpha=1}^{3} [D_\alpha, G_\alpha] = 2.$$

Thus we can write

$$\begin{aligned}\sum_{j=1}^{n} f(\lambda_j) &= \operatorname{tr}(P_n f(-\Delta)) \\ &= \frac{1}{2} \sum_{\alpha=1}^{3} \operatorname{tr}(P_n f(-\Delta)[D_\alpha, G_\alpha]),\end{aligned}$$

where P_n denotes the spectral projection onto the eigenspace for $\{\lambda_1, \dots, \lambda_n\}$. As in the proof of [**15**, Thm. 1], we expand $\operatorname{tr}(P_n f(-\Delta)[D_\alpha, G_\alpha])$ to obtain the following result

$$\begin{aligned}\operatorname{tr}(P_n f(-\Delta)[D_\alpha, G_\alpha]) &= \operatorname{tr}(P_n f(-\Delta)[D_\alpha(P_n + P_{n^c})G_\alpha - G_\alpha(P_n + P_{n^c})D_\alpha]) \\ &= \operatorname{tr}(P_n f(-\Delta)[D_\alpha P_n G_\alpha - G_\alpha P_n D_\alpha]) + \operatorname{tr}(P_n f(-\Delta)[D_\alpha P_{n^c} G_\alpha - G_\alpha P_{n^c} D_\alpha]),\end{aligned}$$

where P_{n^c} is the spectral projection onto the eigenspace for $\{\lambda_{n+1}, \cdots\}$. Using the gap formula

$$\begin{aligned}\langle [-\Delta, G_\alpha]\phi_j, \phi_m\rangle &= \langle -\Delta G_\alpha \phi_j, \phi_m\rangle - \langle G_\alpha(-\Delta\phi_j), \phi_m\rangle \\ &= \lambda_m \langle G_\alpha \phi_j, \phi_m\rangle - \lambda_j \langle G_\alpha \phi_j, \phi_m\rangle \\ &= (\lambda_m - \lambda_j)\langle G_\alpha \phi_j, \phi_m\rangle.\end{aligned}$$

Therefore,

$$
\begin{aligned}
\operatorname{tr}(P_n f(-\Delta)[D_\alpha P_n G_\alpha - G_\alpha P_n D_\alpha]) &= \sum_{j=1}^n \langle f(\lambda_j)(D_\alpha P_n G_\alpha - G_\alpha P_n D_\alpha)\phi_j, \phi_j\rangle \\
&= \sum_{j=1}^n f(\lambda_j)[\langle D_\alpha P_n G_\alpha \phi_j, \phi_j\rangle - \langle G_\alpha P_n D_\alpha \phi_j, \phi_j\rangle] \\
&= \sum_{j=1}^n f(\lambda_j)[-\langle P_n G_\alpha \phi_j, D_\alpha \phi_j\rangle - \langle G_\alpha P_n D_\alpha \phi_j, \phi_j\rangle] \\
&= \sum_{j=1}^n f(\lambda_j)[-\langle \sum_{m=1}^n \langle G_\alpha \phi_j, \phi_m\rangle \phi_m, D_\alpha \phi_j\rangle - \langle \sum_{m=1}^n \langle D_\alpha \phi_j, \phi_m\rangle \phi_m, G_\alpha \phi_j\rangle] \\
&= -2 \sum_{j=1}^n \sum_{m=1, \lambda_j \neq \lambda_m}^n f(\lambda_j) \langle G_\alpha \phi_j, \phi_m\rangle \langle D_\alpha \phi_j, \phi_m\rangle \\
&= 2 \sum_{j=1}^n \sum_{m=1, \lambda_j \neq \lambda_m}^n f(\lambda_j) \frac{\langle [-\Delta, G_\alpha]\phi_j, \phi_m\rangle \langle D_\alpha \phi_j, \phi_m\rangle}{\lambda_j - \lambda_m} \\
&= \sum_{j=1}^n \sum_{m=1, \lambda_j \neq \lambda_m}^n \frac{f(\lambda_j) - f(\lambda_m)}{\lambda_j - \lambda_m} \langle [-\Delta, G_\alpha]\phi_j, \phi_m\rangle \langle D_\alpha \phi_j, \phi_m\rangle.
\end{aligned}
$$

We obtain the similar result for $\operatorname{tr}(P_n f(-\Delta)[D_\alpha P_{n^c} G_\alpha - G_\alpha P_{n^c} D_\alpha])$, except that symmetrization does not apply. Therefore,

$$
\begin{aligned}
\sum_{j=1}^n f(\lambda_j) = &- \sum_{\alpha=1}^3 \sum_{j=1}^n \sum_{\substack{m=1 \\ \lambda_m \neq \lambda_j}}^n \frac{f(\lambda_j) - f(\lambda_m)}{\lambda_j - \lambda_m} |\langle D_\alpha \phi_j, \phi_m\rangle|^2 \\
&+ 2 \sum_{\alpha=1}^3 \sum_{j=1}^n \sum_{q=n+1}^\infty \frac{f(\lambda_j)}{\lambda_q - \lambda_j} |\langle D_\alpha \phi_j, \phi_q\rangle|^2.
\end{aligned}
$$

By the hypotheses on f, the argument from proof of [**15**, Thm. 5] then applies directly to give

$$
\sum_{j=1}^n f(\lambda_j) \leq 2 \sum_{\alpha=1}^3 \sum_{j=1}^n \frac{f(\lambda_j)}{z - \lambda_j} \|D_\alpha \phi_j\|^2.
$$

By the construction of g_α,

$$
\sum_{\alpha=1}^3 \|D_\alpha \phi_j\|_{L^2}^2 = 2\|\phi_j'\|_{L^2}^2 = 2\lambda_j,
$$

which completes the proof.

□

Taking $f = 1$ in Theorem 2.2 yields the Hile-Protter bound (1.1) obtained by Nicaise. Using $f(\lambda) = (z - \lambda)^2$ gives an inequality

$$
\sum_{j=1}^n (z - \lambda_j)(z - 5\lambda_j) \leq 0, \tag{2.2}
$$

for $z \in [\lambda_n, \lambda_{n+1}]$, which was obtained previously by Demirel-Harrell [**22**, Eq. (3.15)]. The discriminant of the quadratic polynomial on the left side of (2.2) is positive and the roots must lie outside the interval $(\lambda_n, \lambda_{n+1})$. This yields the following result, analogous to [**15**, Prop. 6]:

THEOREM 2.3. *For Γ a metric tree with Dirichlet vertices on the external vertices, the quantity*

$$D_n := \left(\frac{3}{n}\sum_{j=1}^{n}\lambda_j\right)^2 - \frac{5}{n}\sum_{j=1}^{n}\lambda_j^2$$

satisfies $D_n \geq 0$ for all $n \geq 2$. Furthermore, the eigenvalues satisfy the inequalities

$$\lambda_n \geq \frac{3}{n}\sum_{j=1}^{n}\lambda_j - \sqrt{D_n}, \qquad \lambda_{n+1} \leq \frac{3}{n}\sum_{j=1}^{n}\lambda_j + \sqrt{D_n},$$

and hence

$$\lambda_{n+1} - \lambda_n \leq 2\sqrt{D_n}.$$

For $n = 1$ we have $D_1 = 4\lambda_1^2$, so the estimate reduces to $\lambda_2/\lambda_1 \leq 5$, equivalent to (1.6) and (1.1) but weaker than (1.7) . To compare the estimates for λ_3, let us define the moments for the first pair of eigenvalues,

$$a_1 := \frac{1}{2}(\lambda_1 + \lambda_2), \qquad a_2 := \frac{1}{2}(\lambda_1^2 + \lambda_2^2).$$

Then the estimate from (2.3) reads

$$\lambda_3 \leq 3a_1 + \sqrt{9a_1^2 - 5a_2}. \tag{2.3}$$

On the other hand, the Hile-Protter type bound (1.1) reduces to

$$\lambda_3 \leq 3a_1 + \sqrt{9a_1^2 - 5\lambda_1\lambda_2}.$$

Since $\lambda_1\lambda_2 \leq a_2$, by the geometric mean inequality, the bound (2.3) is stronger.

CHAPTER 4

Extensions of The Upper Bound

1. Main Results

In this chapter we investigate the possibility for extending the upper bounds discussed in the previous chapter to graphs which are modifications trees. We will consider two possibilities: adding edges between existing vertices of the tree and attaching pendant graphs.

A set of affine functions satisfying (2.1) is not necessarily available on a general graph. However, it is worth noting that the existence of a single function $h \in \mathcal{A}(\Gamma)$ gives a bound in terms of ϕ_1. Returning to the electric circuit analogy from Chapter 3, let us define a (scalar) *current* as a function $\eta : \Gamma \to [0, \infty)$ which is constant on each edge and which, under some choice of edge orientations, satisfies Kirchhoff's current law. This is equivalent to the condition

$$\eta = |h'| \text{ for some } h \in \mathcal{A}(\Gamma). \tag{1.1}$$

LEMMA 1.1. *If Γ admits a non-zero current function η, then*

$$\lambda_2 - \lambda_1 \leq 4 \frac{\|\eta \phi_1'\|_{L^2}^2}{\|\eta \phi_1\|_{L^2}^2}. \tag{1.2}$$

PROOF. Given η, choose h according to (1.1). By shifting h by a constant if necessary, we can assume that $u := h\phi_1$ is orthogonal to ϕ_1. Then min-max gives the estimate

$$\lambda_2 \leq \frac{\|u'\|_{L^2}^2}{\|u\|_{L^2}^2}. \tag{1.3}$$

Because u satisfies the vertex conditions, by the assumptions on h, we can integrate by parts to compute

$$\begin{aligned} \|u'\|_{L^2}^2 &= \langle u, -\Delta u \rangle \\ &= \langle u, \lambda_1 u - 2h'\phi_1' \rangle \\ &= \lambda_1 \|u\|_{L^2}^2 - 2\langle u, h'\phi_1' \rangle. \end{aligned}$$

By (1.3) this gives

$$\lambda_2 - \lambda_1 \leq \frac{A}{\|u\|_{L^2}^2}, \tag{1.4}$$

where

$$A := -2\langle u, h'\phi_1' \rangle.$$

The Cauchy-Schwarz estimate gives

$$A^2 \leq 4 \|u\|_{L^2}^2 \|\eta \phi_1'\|_{L^2}^2. \tag{1.5}$$

On the other hand, we can compute A using integration by parts,

$$\begin{aligned}
A &= -\frac{1}{2}\int_\Gamma (h^2)'(\phi_1^2)' \\
&= -\frac{1}{2}\left(\sum_{v\in V}\sum_{e\sim v}(h_e^2(v))'\phi_1^2(v) - \int_\Gamma \phi_1^2\cdot(h^2)''\right) \\
&= -\frac{1}{2}\left(\sum_{v\in V}\sum_{e\sim v}2h_e(v)\phi_1^2(v)\cdot h_e'(v) - \int_\Gamma \phi_1^2\cdot(2(h')^2+2h\cdot h'')\right) \\
&= \int_\Gamma (h')^2\cdot\phi_1^2 \\
&= \int_\Gamma \eta^2\cdot\phi_1^2.
\end{aligned}$$

Rewriting (1.5) in form of

$$\frac{A}{\|u\|_{L^2}^2} \le 4\frac{\|\eta\phi_1'\|_{L^2}^2}{A},$$

so we have the following result

$$\frac{A}{\|u\|_{L^2}^2} \le 4\frac{\|\eta\phi_1'\|_{L^2}^2}{\|\eta\phi_1\|_{L^2}^2},$$

and the result follows from (1.4). □

As noted in Chapter 3, any graph Γ for which V_0 has at least two vertices will admit non-zero current functions. However, to obtain universal bounds from (1.2) we need either some uniformity in the choice of η or some control over the behavior of ϕ_1.

2. Saguaro Graphs

The first observation is that the conclusion of Lemma 2.1 continues to hold for graphs constructed from trees by adding edges uniformly. That is, suppose Γ is constructed from a tree by replacing each internal edge by a pumpkin with $k > 1$ edges of the same length, and each external edge by a star with k edges of the same length. We will call the result a *saguaro* graph, after the tree-like cactus.

Given a saguaro graph Γ based on the tree Γ_{T}, we can apply Lemma 2.1 to produce a trio of affine functions $\{g_\alpha\}$ on Γ_{T}. These functions can be extended to elements of $\mathcal{A}(\Gamma)$ satisfying (2.1) by simply replicating the values on parallel edges. The existence of this tree implies that all of the universal tree bounds carry over to Γ, including

$$\frac{\lambda_2}{\lambda_1} \le 2+\sqrt{5}, \qquad \lambda_{n+1}-\lambda_n \le \frac{4}{n}\sum_{j=1}^{n}\lambda_j,$$

from Nicaise [**16**], along with the general bounds from Theorem 2.2.

We can also consider the case of an irregular saguaro graph, where the number of edges of each pumpkin or star varies between values $k_{\min}$ and $k_{\max}$. Suppose $\{g_\alpha\}$ denotes the trio of affine functions produced by Lemma 2.1 for the underlying tree Γ_{T}. On Γ we can produce a corresponding set of currents $\{\eta_\alpha\}$ by subdividing the current g_α' among the parallel edges in each segment. That is, on a segment (pumpkin or star) with k edges, we set $\eta_\alpha := |g_\alpha'|/k$.

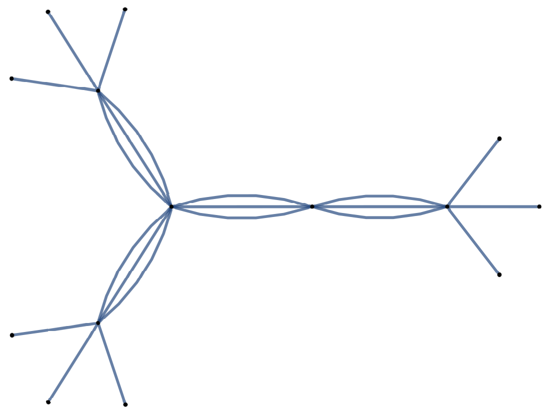

FIGURE 1. A saguaro graph.

Because $\sum_\alpha |g'_\alpha|^2 = 1$ on Γ_T, this construction gives

$$\frac{1}{k_{\max}^2} \le \sum_{\alpha=1}^{3} \eta_\alpha^2 \le \frac{1}{k_{\min}^2}$$

By summing the inequality

$$(\lambda_2 - \lambda_1)\|\eta_\alpha \phi_1\|_{L^2}^2 \le 4\|\eta_\alpha \phi_1'\|_{L^2}^2$$

over α, we obtain

$$\frac{\lambda_2}{\lambda_1} \le 1 + 4\frac{k_{\max}^2}{k_{\min}^2}.$$

3. Ornamented Trees

Suppose Γ is constructed from a tree graph Γ_T by attaching pendant graphs P_j, $j = 1, \ldots m$, to internal (and possibly artificial) vertices of Γ_T. Let $q_j \in V_N$ denote the attachment vertex for P_j. We also introduce the following estimates for eigenvalues.

In terms of the total length $L := |\Gamma|$, the standard eigenvalues satisfy

$$\lambda_k^N \ge \frac{k^2\pi^2}{4L^2} \tag{3.1}$$

for all $k \ge 2$. This lower bound was proven for $k = 2$ by Nicaise [**16**, Thm. 3.1], for $k \ge 2$ by Friedlander [**20**, Thm. 1], and independently by Kurasov-Naboko [**21**] for k even. The bound is sharp, and equality for some k implies that Γ is a segment if $k = 2$ and an equilateral k-star if $k > 2$. Returning to the general case, if V_0 is not empty then the arguments leading to (3.1) imply a lower bound

$$\lambda_k \ge \frac{k^2\pi^2}{4L^2}. \tag{3.2}$$

This was proven for $k = 1$ in [**16**, Thm. 3.1] and is implicit in the proof of [**20**, Thm. 1] for $k \ge 1$. As for upper bounds, a simple test function argument yields

$$\lambda_1 \le \frac{\pi^2}{\ell_{\max}^2}, \tag{3.3}$$

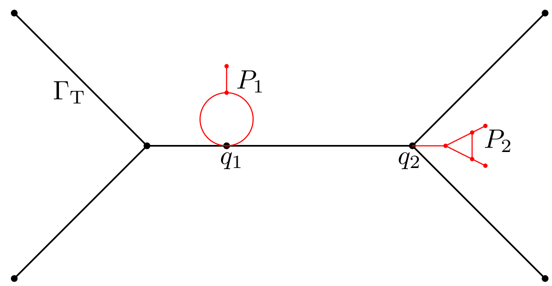

FIGURE 2. An ornamented tree with two pendants.

where ℓ_{max} is the maximum edge length.

PROPOSITION 3.1. *Let Γ be an ornamented tree consisting of metric tree Γ_{T}, with Dirichlet conditions at exterior vertices, and pendants $P_1, \dots, P_m$ each containing at least one Dirichlet vertex. Suppose that*

$$\lambda_1 \le \lambda_1(P_j)$$

for each j, where $\{\lambda_n\}$ denotes the spectrum of Γ and $\lambda_1(P_j)$ is the first eigenvalue of P_j defined by assigning standard boundary conditions at the attachment point q_j. Then

$$\frac{\lambda_2}{\lambda_1} \le 5.$$

In particular, this inequality holds if

$$|P_j| \le \frac{1}{2}\ell_{\max}(\Gamma_T) \tag{3.4}$$

for each j.

PROOF. Let $\{g_\alpha\} \in \mathcal{A}(\Gamma_{\mathrm{T}})$ be the trio of functions obtained by Lemma 2.1. We define corresponding current functions on Γ by setting $\eta_\alpha = |g'_\alpha|$ on edges of Γ_T and $\eta_\alpha = 0$ on each pendant P_j. This gives

$$\frac{1}{2}\sum_{\alpha=1}^{3} \|\eta_\alpha \phi_1\|^2_{L^2(\Gamma_T)} = \int_{\Gamma_{\mathrm{T}}} \phi_1^2,$$

and

$$\frac{1}{2}\sum_{\alpha=1}^{3} \|\eta_\alpha \phi'_1\|^2_{L^2(\Gamma_T)} = \int_{\Gamma_{\mathrm{T}}} (\phi'_1)^2.$$

Thus, by Lemma 1.1,

$$\lambda_2 - \lambda_1 \le 4\frac{\int_{\Gamma_{\mathrm{T}}} (\phi'_1)^2}{\int_{\Gamma_{\mathrm{T}}} \phi_1^2}. \tag{3.5}$$

An integration by parts gives

$$\int_{\Gamma_{\mathrm{T}}} (\phi'_1)^2 = \lambda_1 \int_{\Gamma_{\mathrm{T}}} \phi_1^2 + \sum_{j=1}^{m} \phi_1(q_j)\partial_{P_j}\phi_1(q_j), \tag{3.6}$$

where $\partial_{P_j}\phi_1(q_j)$ denotes the sum of derivatives of ϕ_1 at q_j into the incident edges of P_j. Note that sum of derivatives of ϕ_1 at other interior vertices of Γ_{T} is 0 by the standard vertex condition.

Now for each j let u_j denote the first eigenfunction of P_j, with standard vertex conditions imposed at q_j, so that $u_j \geq 0$ on P_j and

$$-\Delta u_j = \lambda_1(P_j)u_j.$$

By Green's identity, and the fact that $\partial_{P_j} u_j(q_j) = 0$,

$$\begin{aligned}\int_{P_j} (-\phi_1 \Delta u_j + u_j \Delta \phi_1) &= \phi_1(q_j)\partial_{P_j} u_j(q_j) - u_j(q_j)\partial_{P_j}\phi_1(q_j) \\ &= -u_j(q_j)\partial_{P_j}\phi_1(q_j).\end{aligned}$$

Note that the direction of $\partial_{P_j}\phi_1(q_j)$ at q_j into the incident edges of P_j. On the other hand,

$$\int_{P_j} (-\phi_1 \Delta u_j + u_j \Delta \phi_1) = (\lambda_1(P_j) - \lambda_1) \int_{P_j} \phi_1 u_j.$$

Since the eigenfunctions are positive, we conclude that $\lambda_1(P_j) - \lambda_1 \geq 0$ implies that

$$\partial_{P_j}\phi_1(q_j) \leq 0.$$

Under this assumption, (3.6) gives

$$\int_{\Gamma_{\mathrm{T}}} (\phi_1')^2 \leq \lambda_1 \int_{\Gamma_{\mathrm{T}}} \phi_1^2,$$

and it follows from (3.5) that $\lambda_2/\lambda_1 \leq 5$. By the general bounds (3.1) and (3.3), the eigenvalue condition will hold provided

$$|P_j| \leq \frac{1}{2}\ell_{\max}(\Gamma).$$

Clearly it suffices to compute $\ell_{\max}$ over Γ_{T} rather than Γ.

$\square$

CHAPTER 5

The Lower Bound

This chapter is based on our work [**24**].

1. The Weighted Cheeger Constant

As in the introduction, Γ denotes a compact, connected metric graph, and the Laplacian $-\Delta$ is defined with vertex conditions according to the decomposition $V = V_N \cup V_0$. The eigenvalues $\{\lambda_j\}$ are written in increasing order, starting from $\lambda_1 > 0$ by assuming $V_0 \neq \varnothing$. We may assume that the corresponding eigenfunctions ϕ_j are real and that $\phi_1 > 0$ away from V_0.

In section 3 in [**16**], Nicaise introduced a Cheeger-type constant for quantum graphs by

$$h(\Gamma) := \inf \frac{\# S}{\min(|Y_1|, |Y_2|)}, \tag{1.1}$$

with the infimum taken over finite sets S such that $\Gamma \smallsetminus S$ is a disjoint union of non-empty open sets Y_1 and Y_2. This decomposition is called a Cheeger cut. With this Cheeger constant, Nicaise established a lower bound,

$$\lambda_2^N \geq \frac{1}{4} h(\Gamma)^2. \tag{1.2}$$

Note that if $V_0 = \varnothing$ then the inequality (1.2) is what we want since $\lambda_1^N = 0$. In the same paper, Nicaise also established

$$\lambda_2^N \geq \frac{\pi^2}{4L^2}.$$

The general formula

$$\lambda_k^N \geq \frac{k^2 \pi^2}{4L^2}, \tag{1.3}$$

for all $k \geq 2$, was proven by Friedlander in Theorem 1 in [**20**] and independently by Kurasov-Naboko [**21**] for k even. The bound is sharp, and equality for some k implies that Γ is a segment if $k = 2$ and an equilateral k-star if $k > 2$.

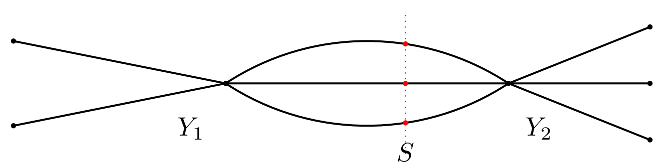

FIGURE 1. A Cheeger cut dividing the graph into two components.

In the case $V_0 \neq \varnothing$, a lower bound

$$\lambda_k \geq \frac{k^2\pi^2}{4L^2}. \tag{1.4}$$

was proven for $k = 1$ by Nicaise in [**16**, Thm. 3.1] and is implicit in the proof of [**20**, Thm. 1] for $k \geq 1$. Berkolaiko-Kennedy-Kurasov-Mugnolo [**9**, Thm. 4.7] showed that, if Γ is not a cycle the bound can be improved for $k \geq |E| - |V_0| + 1$ to

$$\lambda_k \geq \left(k - \frac{1}{2}(|E| - |V_0| + 1)\right)^2 \frac{\pi^2}{L^2}.$$

The argument for the Cheeger estimate (1.2) from [**16**, Thm. 3.2] also yields

$$\lambda_1 \geq \frac{1}{4} h(\Gamma)^2.$$

In this section we will establish a lower bound for $\lambda_2 - \lambda_1$ by adapting the weighted Cheeger constant from Cheng and Oden [**14**] to the metric graph setting. Given a continuous function $\phi : \Gamma \to \mathbb{R}$, we define

$$h_\phi(\Gamma) := \inf \frac{\sum_S \phi^2}{\min(\int_{Y_1} \phi^2, \int_{Y_2} \phi^2)}, \tag{1.5}$$

where $S \subset \Gamma$ is a finite subset such that $\Gamma \backslash S$ is a disjoint union of non-empty open sets Y_1 and Y_2.

We need the co-area formula [**17**, section 3] in our proof. For convenience, we list the formula here.

Theorem 1.1. *Suppose Ω is is an open set in $\mathbb{R}^n$ and g is a real-valued Lipschitz function on Ω. Then, for an L^1 function ϕ,*

$$\int_\Omega \phi(x)|\nabla g| d^n x = \int_{\mathbb{R}} \int_{g^{-1}(t)} \phi(x) dH_{n-1}(x) dt,$$

where H_{n-1} is the $n-1$-dimensional Hausdorff measure.

Now, we are ready for the main result.

Theorem 1.2. *In the setting described above,*

$$\lambda_2 - \lambda_1 \geq \frac{1}{4} h_{\phi_1}(\Gamma)^2.$$

The key step in the proof is the following estimate.

Lemma 1.3. *Let f be a piecewise C^1 function on Γ, and for $\phi : \Gamma \to \mathbb{R}$ continuous, suppose that*

$$\int_\Gamma f\phi^2 = 0. \tag{1.6}$$

Then

$$\|f'\phi\|_{L^2(\Gamma)} \geq \frac{1}{2} h_\phi(\Gamma)\, \|f\phi\|_{L^2(\Gamma)}.$$

Proof. Set

$$k := \sup\left\{t : \int_{\{f \leq t\}} \phi^2 \leq \tfrac{1}{2} \int_\Gamma \phi^2\right\},$$

and define

$$g_+ = (f - k)^2 \chi_{\{f > k\}}.$$

The level set $\{g_+ = t\}$ is finite for almost every $t > 0$, so we can apply co-area formula on each edge $e \in \Gamma$ which gives

$$\int_e \phi^2 |g_+'| dx = \int_0^\infty \left(\sum_{\{g_+ = t\} \cap e} \phi^2 \right) dt.$$

Therefore,

$$\begin{aligned} \int_\Gamma \phi^2 |g_+'| dx &= \sum_{e \in \Gamma} \int_e \phi^2 |g_+'| dx \\ &= \sum_{e \in \Gamma} \int_0^\infty \left(\sum_{\{g_+ = t\} \cap e} \phi^2 \right) dt \\ &= \int_0^\infty \left(\sum_{\{g_+ = t\}} \phi^2 \right) dt. \end{aligned} \tag{1.7}$$

Note that the integral on the left side is one-dimensional so we have discrete sum in the bracket on the right side. Note also that, by the choice of k,

$$\int_{\{g_+ > t\}} \phi^2 \leq \int_{\{g_+ < t\}} \phi^2$$

for all $t > 0$. Thus, for t such that $\{g_+ = t\}$ is finite,

$$\sum_{\{g_+ = t\}} \phi^2 = \frac{\sum_{\{g_+ = t\}} \phi^2}{\int_{\{g_+ > t\}} \phi^2} \int_{\{g_+ > t\}} \phi^2 \geq h_\phi(\Gamma) \int_{\{g_+ > t\}} \phi^2.$$

Plugging this back into (1.7) gives

$$\begin{aligned} \int_\Gamma |g_+'| \phi^2 &\geq h_\phi(\Gamma) \int_0^\infty \left(\int_{\{g_+ > t\}} \phi^2 \right) dt \\ &= h_\phi(\Gamma) \int_\Gamma g_+ \phi^2. \end{aligned} \tag{1.8}$$

Similarly, for $g_- := (f - k)^2 \chi_{\{f < k\}}$ we obtain

$$\int_\Gamma |g_-'| \phi^2 \geq h_\phi(\Gamma) \int_\Gamma g_- \phi^2. \tag{1.9}$$

Since $(f - k)^2 = g_+ + g_-$, adding (1.8) and (1.9) gives

$$\int_\Gamma (|g_+'| + |g_-'|) \phi^2 \geq h_\phi(\Gamma) \int_\Gamma (f - k)^2 \phi^2. \tag{1.10}$$

After computing

$$|g_+'| + |g_-'| = 2|f - k| \cdot |f'|,$$

we can apply Cauchy-Schwarz to (1.10) to obtain

$$\begin{aligned} h_\phi(\Gamma) \|(f - k)\phi\|_{L^2(\Gamma)}^2 &\leq 2 \int_\Gamma |f - k| \cdot |f'| \phi^2 \\ &\leq 2 \|(f - k)\phi\|_{L^2(\Gamma)} \cdot \|f'\phi\|_{L^2(\Gamma)}. \end{aligned}$$

Hence

$$\|f'\phi\|_{L^2(\Gamma)} \geq \frac{1}{2} h_\phi(\Gamma) \|(f - k)\phi\|_{L^2(\Gamma)}.$$

The final step is to note that the hypothesis (1.6) implies that

$$\|(f - k)\phi\|_{L^2(\Gamma)}^2 = \|f\phi\|_{L^2(\Gamma)}^2 + k^2 \|\phi\|_{L^2(\Gamma)}^2 \geq \|f\phi\|_{L^2(\Gamma)}^2.$$

□

Using Lemma 1.3, we use a straightforward spectral gap estimate to establish the weighted Cheeger bound.

PROOF OF THEOREM 1.2. Let $f = \phi_2/\phi_1$, which is smooth away from the vertices and satisfies

$$\int_\Gamma f\phi_1^2 = \langle \phi_2, \phi_1\rangle = 0.$$

Assuming the eigenfunctions are normalized,

$$\begin{aligned}\lambda_2 &= \|\phi_2'\|_{L^2(\Gamma)}^2 \\ &= \|(f\cdot\phi_1)'\|_{L^2(\Gamma)}^2 \\ &= \|f'\phi_1 + f\phi_1'\|_{L^2(\Gamma)}^2 \\ &= \|f'\phi_1\|_{L^2(\Gamma)}^2 + \|f\phi_1'\|_{L^2(\Gamma)}^2 + \frac{1}{2}\int_\Gamma (f^2)'(\phi_1^2)'.\end{aligned} \tag{1.11}$$

Since both f^2 and ϕ_1^2 satisfy the vertex conditions, we can integrate by parts to obtain

$$\begin{aligned}\frac{1}{2}\int_\Gamma (f^2)'(\phi_1^2)' &= -\frac{1}{2}\int_\Gamma f^2\Delta(\phi_1^2) \\ &= -\frac{1}{2}\int_\Gamma f^2(2\phi_1\phi_1')' \\ &= -\int_\Gamma f^2\left((\phi_1')^2 + \phi_1\cdot\phi_1''\right) \\ &= \int_\Gamma \lambda_1\phi_1^2\cdot f^2 - f^2(\phi_1')^2 \\ &= \int_\Gamma \lambda_1\phi_2^2 - f^2(\phi_1')^2 \\ &= \lambda_1 - \|f\phi_1'\|_{L^2(\Gamma)}^2.\end{aligned}$$

This simplifies (1.11) to

$$\lambda_2 - \lambda_1 = \|f'\phi_1\|_{L^2(\Gamma)}^2. \tag{1.12}$$

The result then follows from Lemma 1.3, since $f\phi_1 = \phi_2$. □

2. Estimates of the first eigenfunction

To make use of Theorem 1.2 in the case $V_0 \neq \varnothing$, we need some control over the range of the first eigenfunction ϕ_1. In particular, we will establish a lower bound on a subset that excludes the Dirichlet vertices.

For each vertex $v_j \in V_0$, parametrize the edge incident to v_j by $x_j \in [0, \ell_j]$, with $x_j = 0$ at v_j. Within this edge define the interval,

$$I_j := \sup\{x_j \in [0,\ell_j) : \phi_1'(x_j) > 0\}, \tag{2.1}$$

which includes the full interior of the edge unless ϕ_1 has a local maximum. Then let

$$\Gamma_1 := \Gamma\backslash\left(\cup_{v_j\in V_0} I_j\right). \tag{2.2}$$

If ϕ_1 has no local maxima in external edges, then Γ_1 is the subgraph obtained by trimming from Γ all edges incident on V_0. If artificial vertices are added to Γ at points where if ϕ_1

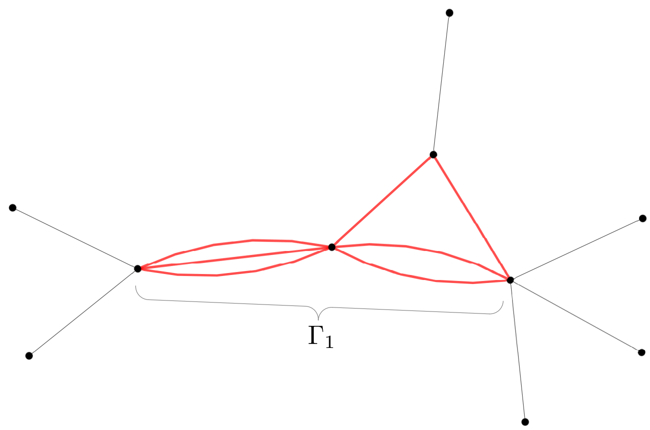

FIGURE 2. The subgraph Γ_1 created by trimming external edges.

does have local maxima within an outer edge, then we would need to first add artificial vertices at these maxima before trimming.

By construction, the maximum value of the first eigenfunction,

$$M_1 := \max_{\Gamma} \phi_1, \tag{2.3}$$

occurs at a point in Γ_1, and the minimum over Γ_1,

$$m_1 := \min_{\Gamma_1} \phi_1, \tag{2.4}$$

is strictly positive. Note that it is possible for Γ_1 to consist of only one single point v. In this case, Γ has $k = |V_0|$ edges, each connecting v to a Dirichlet vertex. The continuity condition at v implies that all of these edges have equal length. Hence Γ is a half-Dirichlet interval if $k = 1$, a full Dirichlet interval for $k = 2$, and an equilateral star graph for $k \geq 3$.

If Γ_1 contains more than one point, then m_1 occurs at a vertex $v \in V_N$, by the concavity of ϕ_1 on edges. Note that we can locally express ϕ_1 in terms of sine function and $\phi_1 > 0$ on $\Gamma \setminus V_0$. The standard vertex condition implies that the outgoing derivative of ϕ_1 at v is ≤ 0 on at least one incident edge e in Γ, otherwise the sum of derivatives at v is non-negative. By concavity, ϕ_1 is strictly decreasing in the interior of e, and so e must lie outside Γ_1. Therefore, the minimum m_1 is achieved at a vertex in V_N which is adjacent to a vertex in V_0.

The fact that ϕ_1 is strictly decreasing on an edge incident on a Dirchlet vertex implies that $\sigma\ell \leq \pi/2$, where ℓ is the length of this edge and $\sigma := \sqrt{\lambda_1}$. This gives an upper bound

$$\lambda_1 \leq \frac{\pi^2}{4\ell_0^2}, \tag{2.5}$$

provided Γ_1 contains more than one point. For the exceptional cases where Γ_1 contains a single point, we see explicitly that $\lambda_1 = \pi^2/\ell_0^2$ for the full Dirichlet interval and $\lambda_1 = \pi^2/4\ell_0^2$ for all other cases. Hence the Dirichlet interval is the only exception to the bound (2.5).

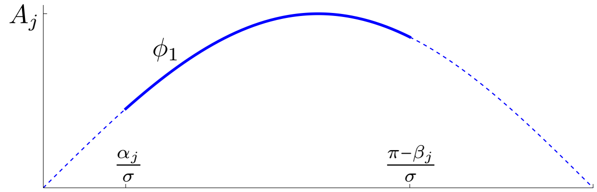

FIGURE 3. The parametrization (2.7) with phases α_j, β_j.

PROPOSITION 2.1. *Assuming that Γ_1 contains more than one point, there exists a constant $c(L, \ell_0) > 0$, depending only on $L = L(\Gamma)$ and ℓ_0 the minimum edge length, such that*

$$\frac{m_1}{M_1} \geq c(L, \ell_0). \tag{2.6}$$

PROOF. Let $\lambda_1 = \sigma^2$ for $\sigma > 0$. On each edge e_j of Γ_1, ϕ_1 is given by a positive arc of the sine function. Thus, we can choose phases $\alpha_j, \beta_j \in (0, \pi)$ such that

$$\phi_1|_{e_j}(x) = A_j \sin(\sigma x), \tag{2.7}$$

for a parametrization of e_j by

$$x \in [\frac{\alpha_j}{\sigma}, \frac{(\pi - \beta_j)}{\sigma}].$$

These phases are illustrated in Figure 3. By switching the orientation if necessary, we can assume that $\alpha_j \leq \beta_j$. These phases are

The change in $\log \phi_1$ across e_j is given by

$$\begin{aligned} \log \phi_1 \Big|_{\alpha_j/\sigma}^{(\pi-\beta_j)/\sigma} &= \log(A_j \sin(\frac{\pi - \beta_j}{\sigma})) - \log(A_j \sin(\alpha_j/\sigma)) \\ &= \log \frac{A_j \sin(\pi - \beta_j)}{A_j \sin(\alpha_j)} \\ &= \log \frac{\sin \beta_j}{\sin \alpha_j}. \end{aligned}$$

If we define

$$\delta_0 \coloneqq \min_{e_j \subset \Gamma_1} \alpha_j,$$

then for each edge e_j in Γ_1,

$$\left| \log \phi_1 \Big|_{\alpha_j/\sigma}^{(\pi-\beta_j)/\sigma} \right| \leq -\log \sin \delta_0, \tag{2.8}$$

since $\sin(\beta_j) \leq 1$. Note that $\delta_0 < \pi/2$, since $\alpha_j + \beta_j < \pi$ on each edge.

Suppose that the vertices of Γ_1 where the minimum and maximum of ϕ_1 occur can be joined by a path with at most q edges. Adding the estimate (2.8) along the path then gives

$$\log \frac{M_1}{m_1} \leq -q \log \sin \delta_0, \tag{2.9}$$

and the problem is now reduced to finding a lower bound for δ_0.

Suppose that the minimal phase δ_0 occurs at a vertex v_0 of Γ_1 which is the $x = 0$ endpoint of an edge e_j parametrized by $x \in [0, \ell_j]$. In this parametrization,

$$\phi_1(x) = A \sin(\sigma x + \delta_0).$$

Because $\delta_0 < \pi/2$, ϕ_1 does not have a local maximum at v_0, which implies that $v_0 \in V_N$.

The outward derivative into e_j from v_0 is given by

$$\phi_1'(0^+) = A\cos\delta_0.$$

Let e_k be the edge of Γ incident to v_0 for which the inward-pointing derivative at v_0 is maximal. We can parametrize e_k by $x \in [-\ell_k, 0]$, and then continue the eigenfunction as

$$\phi_1(x) = B\sin(\alpha_k - \sigma x), \quad \text{for } x \le 0, \tag{2.10}$$

for some phase $\alpha_k \in (0, \pi)$. By continuity, the amplitudes satisfy

$$A\sin\delta_0 = B\sin\alpha_k. \tag{2.11}$$

Because the vertex condition at v_0 is Neumann (standard), and by the choice of e_k, we can estimate

$$\phi_1'(0^+) \le (d_0 - 1)\phi_1'(0^-),$$

where d_0 is the degree of v_0. This gives

$$A\cos\delta_0 \le (d_0 - 1)B\cos\alpha_k.$$

Combining this with (2.11) yields a lower bound

$$\tan\delta_0 \ge \frac{1}{d_0 - 1}\tan\alpha_k. \tag{2.12}$$

To obtain a lower bound on the phase α_k from (2.10), note that since ϕ_1 cannot vanish in the interior of e_k which implies $B\sin(\alpha_k - \sigma\ell_k) > 0$, we obtain $\alpha_k \ge \sigma\ell_k$. From (2.12), we thus obtain

$$\tan\delta_0 \ge \frac{1}{d_0 - 1}\tan(\sigma\ell_k). \tag{2.13}$$

If we denote the right side of (2.13) by b, then this gives

$$\sin\delta_0 \ge \frac{b}{\sqrt{1 + b^2}},$$

and (2.9) implies that

$$m_1 \ge \left(\frac{b}{\sqrt{1 + b^2}}\right)^q M_1. \tag{2.14}$$

To complete the proof, note that $\ell_k \ge \ell_0$, both q and d_0 are bounded by L/ℓ_0 and σ was bounded below by $\pi/2L$.

□

The Harnack inequality (Proposition 2.1) implies an envelope estimate for ϕ_1, which is perhaps of independent interest. Define the function

$$\Upsilon(q) := \begin{cases} 1, & d(q, V_0) \ge \ell_0/2, \\ \sin(\frac{\pi}{\ell_0} d(q, V_0)), & d(q, V_0) < \ell_0/2. \end{cases} \tag{2.15}$$

where $d(q, V_0)$ is the distance function.

We can extend the bound (2.6) beyond Γ_1. On each interval $I_j = [0, \ell_j)$ from (2.1), the eigenfunction takes the form $\phi_1(x_j) = A_j\sin(\sigma x_j)$. The points $x_j = \ell_j$ lie in Γ_1, so

$$A_j\sin(\sigma\ell_j) \ge m_1.$$

Therefore,

$$\phi_1(x_j) \geq \frac{m_1}{\sin(\sigma\ell_j)}\sin(\sigma x_j) \tag{2.16}$$

on each interval I_j.

Now, we can prove the following theorem.

THEOREM 2.2. *Let Γ be a compact metric graph with total length L and minimum edge length ℓ_0. Suppose $V_0 \neq \varnothing$ and the first eigenfunction is normalized so that $\phi_1 \geq 0$ and $\|\phi_1\| = 1$. There exists a constant $c_1(L,\ell_0) > 0$ such that*

$$c_1(L,\ell_0)\Upsilon \leq \phi_1 \leq \sqrt{\frac{2}{\ell_0}}\Upsilon.$$

PROOF. We can assume that Γ_1 consists of more than one point, since ϕ_1 is easily computed explicitly in the exceptional cases. Next, we note that in the notation used above, the variable x_j used to parametrize I_j is equal to $d(\cdot, V_0)$ for $x_j \leq \ell_0/2$. The envelope function Υ defined in (2.15) is equal to 1 on Γ_1 and on the outer edges satisfies

$$\Upsilon|_{I_j}(x_j) = \begin{cases} 1, & x_j \geq \ell_0/2, \\ \sin(\frac{\pi}{\ell_0}x_j), & x_j < \ell_0/2. \end{cases} \tag{2.17}$$

From (2.16), (2.17), and the fact that $\sigma > \pi/2L$, we can immediately deduce the inequality,

$$\phi_1 \geq m_1 \sin(\pi\ell_0/4L)\Upsilon, \tag{2.18}$$

on all of Γ. Note that both $|\sin(\sigma\ell_j)|$ and $|\Upsilon|$ are less than or equal to 1. On the other hand, since ϕ_1 has the form $A_j\sin(\sigma x_j)$ on the outer edges and $\sigma \leq \pi/\ell_0$, it follows that

$$\phi_1 \leq M_1\Upsilon. \tag{2.19}$$

The constant M_1 satisfies the trivial inequality

$$M_1 \geq \sqrt{\frac{2}{L}}. \tag{2.20}$$

To complete the argument, we need to estimate M_1 from above. Suppose that the maximum value of ϕ_1 is achieved at a point $q_1 \in \Gamma_1$. By the concavity of ϕ_1 and the vertex conditions, there exists a segment of Γ, parametrized by $y \in [0, \pi/2\sigma]$ with $y = 0$ at q_1, on which

$$\phi_1(y) \geq M_1\cos(\sigma y).$$

Integrating ϕ_1^2 over this segment gives

$$1 \geq \int_0^{\pi/2\sigma} \phi_1^2(y)dy \geq M_1^2 \int_0^{\pi/2\sigma} \cos^2(\sigma y)dy = \pi\frac{M_1^2}{4\sigma}.$$

Since $\sigma \leq \pi/2\ell_0$ by (2.5), this proves

$$M_1 \leq \sqrt{\frac{4\sigma}{\pi}} \leq \sqrt{\frac{2}{\ell_0}}. \tag{2.21}$$

Using the inequalities (2.18), (2.19), (2.20), and (2.21), Theorem 2.2 now follows from Proposition 2.1.

$\square$

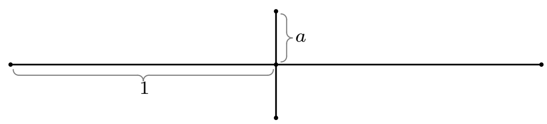

FIGURE 4. A 4-star graph as in Example 3.2.

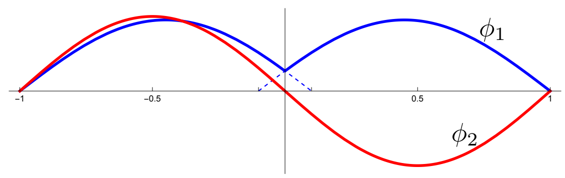

FIGURE 5. A comparison of the first two eigenfunctions for the 4-star graph of Example 3.2.

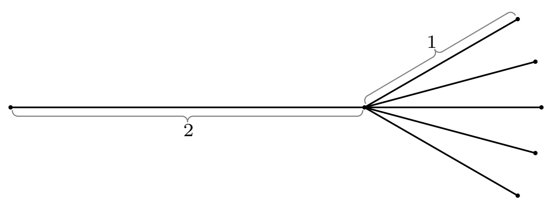

FIGURE 6. A star graph as in Example 3.3.

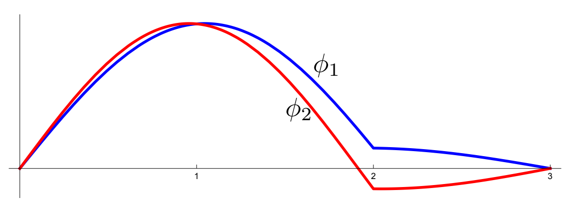

FIGURE 7. Comparison of the first two eigenfunctions from Example 3.3.

3. Estimation of the weighted Cheeger constant

With the help of Proposition 2.1, we are now ready to prove the following theorem. Again, we assume $V_0 \neq \varnothing$.

THEOREM 3.1. *Let Γ be a metric graph with mixed standard and Dirichlet vertex conditions as described above. There exists a constant $C(L, \ell_0) > 0$, depending only on the total length L and minimum edge length ℓ_0, such that*

$$\lambda_2 - \lambda_1 \geq C(L, \ell_0). \tag{3.1}$$

As we noted at the beginning of Section 2, if Γ_1 consists of a single point v, then Γ has k equilateral edges connecting v to Dirichlet vertices. In this case, we can compute eigenvalues of Γ explicitly and Theorem 3.1 holds on Γ trivially. The following cases show that dependence on both L and ℓ_0 is required for graphs with at least one Dirichlet vertex.

Example 3.2. Let Γ be a star graph with four edges, two of length 1 and two of length $a < 1$. Dirichlet conditions are imposed at the four external vertices. Note that the total length $L = 2(1+a) < 4$, $\ell_0 = a$, and the diameter is a constant $D = 2$.

Define a coordinate x so that the longer edges are parametrized by $x \in [0,1]$ and the shorter edges by $x \in [1, 1+a]$. In terms of this coordinate, the first eigenfunction can be written as

$$\phi_1(x) = \sin\left(\frac{\pi x}{1+a}\right), \tag{3.2}$$

yielding

$$\lambda_1 = \frac{\pi^2}{(1+a)^2}.$$

The second eigenfunction vanishes on the short edges and is proportional to $\sin(\pi x)$ on the long edges, so that $\lambda_2 = \pi^2$. The spectral gap is thus

$$\lambda_2 - \lambda_1 = \pi^2 \left(1 - \frac{1}{(1+a)^2}\right),$$

which is $\sim 2\pi^2 a$ as $a \to 0$.

Example 3.3. Suppose Γ is a star graph with one edge of length 2 and k edges of length 1, with Dirichlet vertex conditions on all external vertices. Here $L = 2 + k$, $\ell_0 = 1$, and the diameter is $D = 3$.

For the two lowest eigenvalues, it suffices to consider eigenfunctions which do not vanish at the central vertex, and thus must take the same values on each smaller edge. We can thus use the linear coordinate $x \in [0,3]$, with the interior vertex located at $x = 2$. Suppose an eigenfunction is given by

$$\phi(x) = \begin{cases} \sin(\sigma x), & x \in [0,2], \\ c\sin(\sigma(3-x)), & x \in [2,3]. \end{cases} \tag{3.3}$$

The continuity and vertex conditions give $\sin(2\sigma) = c \sin\sigma$ and $\cos(2\sigma) = kc\cos\sigma$. With double-angle formulas, this reduces to $(\tan\sigma)^2 = 2k+1$, yielding

$$\sigma \in \arctan\sqrt{2k+1} + \pi\mathbb{Z}.$$

On the other hand, eigenfunctions which vanish at the central vertex have eigenvalues in $(\pi\mathbb{N})^2$. The first two eigenvalues are thus given by

$$\lambda_1 = \left(\arctan\sqrt{2k+1}\right)^2, \quad \lambda_2 = \left(\pi - \arctan\sqrt{2k+1}\right)^2,$$

with values just below and above $\pi^2/4$. with values just below and above $\pi^2/4$. For large k, we have $\lambda_2 - \lambda_1 \sim 2\pi/\sqrt{k}$. In this example, the quantities $\ell_0 = 1$ and $D = 3$ are fixed, while $L = k + 2$.

In the general case, our goal is to estimate $h_{\phi_1}(\Gamma)$ in terms of the unweighted Cheeger constant $h(\Gamma)$. Since the latter satisfies the trivial bound $h(\Gamma) \geq 2/L$, this will complete the proof.

If Γ is not an interval (and $V_0 \neq \varnothing$), then its Cheeger constant also satisfies a trivial upper bound,

$$h(\Gamma) \leq \frac{1}{\ell_0}. \tag{3.4}$$

To see this, take a cut S given by a single point on an outer edge of Γ, such that Y_1 is a segment of length ℓ_0.

We can make a similar estimate for $h_{\phi_1}(\Gamma)$, in terms of the function

$$f(x) := \frac{\sin^2(\sigma x)}{\int_0^x \sin^2 \sigma t\, dt} = \frac{2\sin^2(\sigma x)}{x - \frac{1}{2\sigma}\sin(2\sigma x)}, \tag{3.5}$$

defined for $x \in (0, \pi/\sigma)$. It is easy to check that f is decreasing on this interval.

LEMMA 3.4. *If Γ_1 contains more than one point, then*

$$h_{\phi_1}(\Gamma) \le f(\ell_0). \tag{3.6}$$

PROOF. Taking a Cheeger cut S consisting of a single point $x_j = \ell_0$ in one of the intervals I_j from (2.1) gives the ratio

$$\frac{\sum_S \phi_1^2}{\int_{Y_1} \phi_1^2} = f(\ell_0), \tag{3.7}$$

where $Y_1 = \{0 \le x_j < \ell_0\}$ and Y_2 is the other component of $\Gamma \setminus S$. If V_0 contains more than one point, then we can choose j so that minimizes $\int_{Y_1} \phi_1^2$ among all I_j, guaranteeing that

$$\int_{Y_1} \phi_1^2 \le \int_{Y_2} \phi_1^2. \tag{3.8}$$

On the other hand, if $|V_0| = 1$, then, assuming that Γ_1 is not a single point, Γ contains at least one interior edge on which $\phi_1 \ge m_1$, implying that (3.8) holds also in this case. From (3.7) we thus obtain (3.6).

□

PROPOSITION 3.5. *If Γ_1 contains more than one point, then*

$$h_{\phi_1}(\Gamma) \ge \left(\frac{m_1 \sin(\sigma\ell_0/2)}{M_1}\right)^2 h(\Gamma),$$

where $\sigma = \sqrt{\lambda_1}$ and M_1, m_1 are the upper and lower bounds on ϕ_1 from Section 2.

PROOF. By Lemma 3.4, to estimate $h_{\phi_1}(\Gamma)$ we may limit our attention to Cheeger cuts S for which

$$\frac{\sum_S \phi_1^2}{\min(\int_{Y_1} \phi_1^2, \int_{Y_2} \phi_1^2)} \le f(\ell_0). \tag{3.9}$$

By relabeling if necessary, we can also assume that the minimum in the denominator is the Y_1 integral, i.e.,

$$\int_{Y_1} \phi_1^2 \le \int_{Y_2} \phi_1^2. \tag{3.10}$$

Let

$$W := \{q \in \Gamma : \mathrm{dist}(q, V_0) \ge \frac{\ell_0}{2}\}.$$

Case 1: Suppose that $S \cap W = \varnothing$. In the notation of (2.1),

$$\Gamma \setminus W = \cup_j \{0 \le x_j \le \ell_0/2\}.$$

Because ϕ_1 is increasing as a function of x_j, the convention (3.10) implies that $Y_1 \subset \Gamma \setminus W$ and $Y_2 \supset W$. Let $\mathcal{J}$ be the set of indices j for which S intersects I_j, and set

$$s_j = \max S \cap I_j.$$

If the restriction of ϕ_1 to I_j is written as $A_j \sin(\sigma x_j)$, then

$$\begin{aligned}\sum_S \phi_1^2 &= \sum_{j\in\mathcal{J}} A_j^2 \sin^2(\sigma s_j)\\ &= \sum_{j\in\mathcal{J}} f(s_j) \int_0^{s_j} A_j^2 \sin^2(\sigma x)\,dx.\end{aligned}$$

Since f is decreasing and $s_j \le \ell_0/2$, this implies

$$\sum_S \phi_1^2 \ge f(\ell_0/2) \int_{Y_1} \phi_1^2, \tag{3.11}$$

which contradicts the assumption (3.9).

Case 2: Suppose that $S \cap W \neq \varnothing$ and Y_1 contains an interval $J \subset \Gamma \setminus W$. If S' denotes the corresponding cut with the endpoints of J deleted, then Y_1 is reduced to a component $Y_1' = Y_1 \setminus J$, while $Y_2' = Y_2 \cup J$. The inequality (3.10) is still satisfied after the replacement, and

$$\sum_{\partial J} \phi_1^2 \ge f(\ell_0/2) \int_J \phi_1^2, \tag{3.12}$$

by the same argument used for (3.11).

By the assumption (3.9), and the fact that f is strictly decreasing, we obtain

$$\frac{\sum_S \phi_1^2}{\int_{Y_1}\phi_1^2} \le f(\ell_0) < f(\ell_0/2) \le \frac{\sum_{\partial J}\phi_1^2}{\int_J \phi_1^2}.$$

We write

$$\begin{aligned}\frac{\sum_{S'}\phi_1^2}{\int_{Y_1'}\phi_1^2} &= \frac{\sum_S \phi_1^2 - \sum_{\partial J}\phi_1^2}{\int_{Y_1}\phi_1^2 - \int_J \phi_1^2}\\ &= \frac{\frac{\sum_S \phi_1^2}{\int_{Y_1}\phi_1^2}\int_{Y_1}\phi_1^2 - \frac{\sum_{\partial J}\phi_1^2}{\int_J\phi_1^2}\int_J\phi_1^2}{\int_{Y_1}\phi_1^2 - \int_J\phi_1^2}\\ &= \frac{\sum_S \phi_1^2}{\int_{Y_1}\phi_1^2} - \frac{\left(\frac{\sum_{\partial J}\phi_1^2}{\int_J\phi_1^2} - \frac{\sum_S\phi_1^2}{\int_{Y_1}\phi_1^2}\right)\int_J\phi_1^2}{\int_{Y_1}\phi_1^2 - \int_J\phi_1^2}\\ &< \frac{\sum_S \phi_1^2}{\int_{Y_1}\phi_1^2}.\end{aligned}$$

Therefore,

$$\frac{\sum_S \phi_1^2}{\int_{Y_1}\phi_1^2} > \frac{\sum_{S'}\phi_1^2}{\int_{Y_1'}\phi_1^2}.$$

That is, cutting the interval J from Y_1 will reduce the Cheeger ratio.

Case 3: Suppose that $S \cap W \neq \varnothing$ and Y_1 contains no interval in $\Gamma \setminus W$. We continue to assume that the components Y_j satisfy (3.9) and (3.10), which together imply that

$$\frac{\sum_S \phi_1^2}{\int_{Y_1}\phi_1^2} \le f(\ell_0). \tag{3.13}$$

For a cut S with these properties, the points of $S \smallsetminus W$ (if any) correspond, in the parametrization (2.1), to $x_j = b_j < \ell_0/2$, such that $[0, b_j) \subset Y_2$. If we set

$$Z := \bigcap_{b_j \in S \smallsetminus W} [0, b_j],$$

then the set $S'' := S \cap W$ separates Γ into components $Y_1'' := Y_1 \cup Z$ and $Y_2'' = Y_2 \smallsetminus Z$.

By (2.16), ϕ_1 satisfies a lower bound

$$\min_W \phi_1 \geq m_1 \sin(\sigma \ell_0/2).$$

By the definition of Y_1'' and S'', we thus have

$$\begin{aligned} \frac{\sum_S \phi_1^2}{\int_{Y_1} \phi_1^2} &\geq \frac{\sum_{S''} \phi_1^2}{\int_{Y_1''} \phi_1^2} \\ &\geq \left(\frac{m_1 \sin(\sigma \ell_0/2)}{M_1} \right)^2 \frac{|S''|}{|Y_1''|}. \end{aligned} \tag{3.14}$$

For Y_2 we can write the ratio as

$$\frac{\sum_S \phi_1^2}{\int_{Y_2} \phi_1^2} = \frac{\sum_{S''} \phi_1^2 + \sum_{S \smallsetminus W} \phi_1^2}{\int_{Y_2''} \phi_1^2 + \int_Z \phi_1^2}. \tag{3.15}$$

By (3.13), and the fact

$$\sum_{S \smallsetminus W} \phi_1^2 \geq f(\ell_0/2) \int_Z \phi_1^2,$$

the decomposition (3.15) shows that

$$\frac{\sum_{S''} \phi_1^2}{\int_{Y_2''} \phi_1^2} \leq \frac{\sum_S \phi_1^2}{\int_{Y_1} \phi_1^2}.$$

The estimates of ϕ_1 then yield

$$\frac{\sum_S \phi_1^2}{\int_{Y_1} \phi_1^2} \geq \left(\frac{m_1 \sin(\sigma \ell_0/2)}{M_1} \right)^2 \frac{|S''|}{|Y_2''|}. \tag{3.16}$$

Combining (3.14) and (3.16) gives

$$\frac{\sum_S \phi_1^2}{\int_{Y_1} \phi_1^2} \geq \left(\frac{m_1 \sin(\sigma \ell_0/2)}{M_1} \right)^2 h(\Gamma). \tag{3.17}$$

To summarize, Case 1 is ruled out by (3.9) and Case 2 can be reduced to Case 3 with a reduction in the weighted Cheeger ratio. Hence the bound (3.17) applies to $h_{\phi_1}(\Gamma)$.

$\square$

REMARK 3.6. We write down the explicit formula of $C(L, \ell_0)$ in Theorem 3.1. Note that $h(\Gamma) \geq \frac{2}{L}$ and $\sigma \geq \frac{\pi}{2L}$. Therefore, by Proposition 3.5, we obtain an lower bound of $h_{\phi_1}(\Gamma)$ in terms of L and ℓ_0.

$$\begin{aligned} h_{\phi_1}(\Gamma) &\geq \left(\frac{m_1 \sin(\sigma \ell_0/2)}{M_1} \right)^2 \frac{2}{L} \\ &\geq \sin^2(\frac{\pi \ell_0}{4L}) \left(\frac{b}{\sqrt{1+b^2}} \right)^{2q} \frac{2}{L}. \end{aligned}$$

By Theorem 1.2, we obtain the following inequality

$$\begin{aligned}
\lambda_2 - \lambda_1 &\geq \frac{1}{4} h_{\phi_1}(\Gamma)^2 \\
&\geq \frac{1}{4} \sin^4\left(\frac{\pi \ell_0}{4L}\right) \left(\frac{b}{\sqrt{1+b^2}} \right)^{4q} \frac{4}{L^2} \\
&= \frac{1}{L^2} \sin^4\left(\frac{\pi \ell_0}{4L}\right) \left(\frac{b}{\sqrt{1+b^2}} \right)^{4q}.
\end{aligned}$$

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