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An abstract of

a thesis submitted to the Faculty of Emory College of Arts and Sciences

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Abstract

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In this paper, I want to present to the readers some basic knowledge about circle packing in the setting of Euclidean plane. Circle packing was introduced by William Thurston [8] in his lecture notes. I will establish the background on the discrete analytic function, which maps carriers of circle packings to carriers of circle packings and preserve the orientation and tangency. Last, I will present the proof of the Thurston's Conjecture on circle packings, which was proved by Burton Rodin and Dennis Sullivan [5], and is now called the Rodin-Sullivan Theorem.

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1. Introduction

In this thesis, we will focus on proving the Rodin-Sullivan Theorem in the Euclidean plane. We will begin with an introduction to circle packings, whichwill help the reader build up the background knowledge. Then we will review definitions and theorems of conformal and quasiconformal mapping. Before we prove the Rodin-Sullivan Theorem, we need to prove some useful lemmas. After assembling all the necessary tools, we will prove the main theorem.

2. Introduction to Circle Packing

First, I introduce some basic concepts about circle packing in the Euclidean-geometric setting. Throughout the paper, we treat $\mathbb{R}^2$ as the complex plane $\mathbb{C}$, with complex arithmetic, Euclidean and spherical metrics.

Before I list any definitions, I will provide some picture examples of circle packing to illustrate this mathematical term intuitively. Those examples include not only the circle packing in the plane, but also in the hyperbolic plane and the sphere.

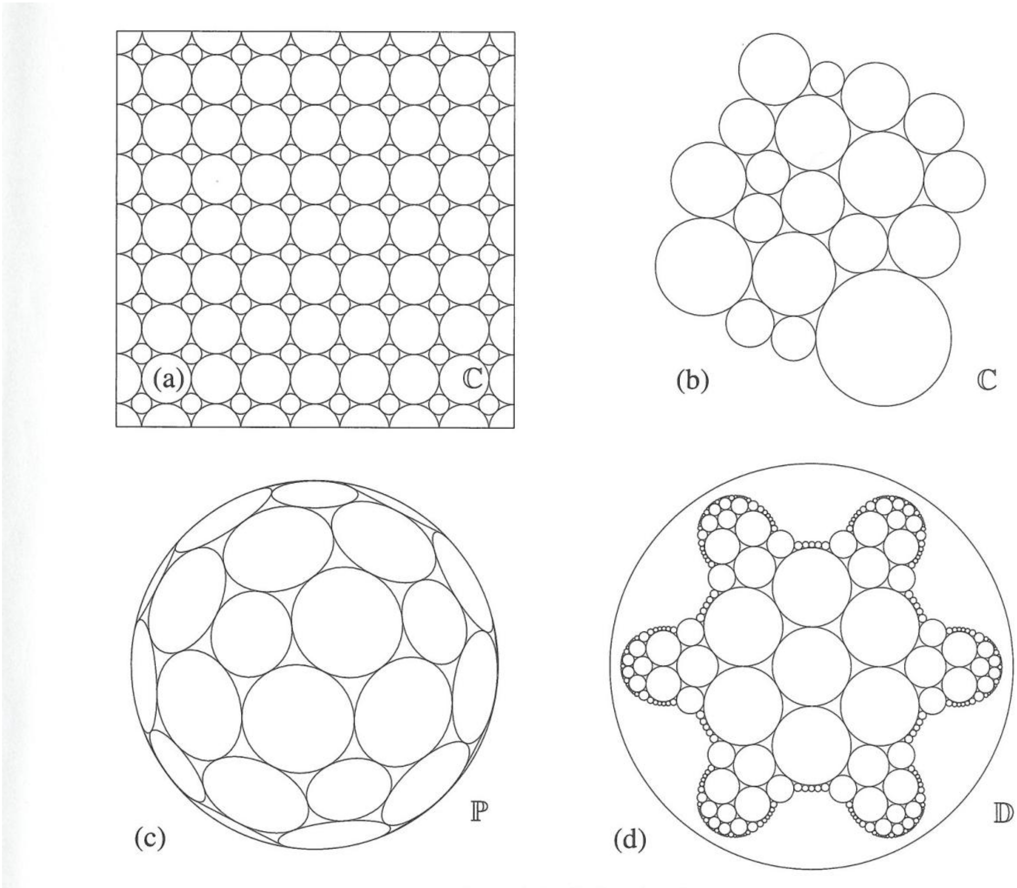

Figure 1: Circle packing in different geometric settings. [7]

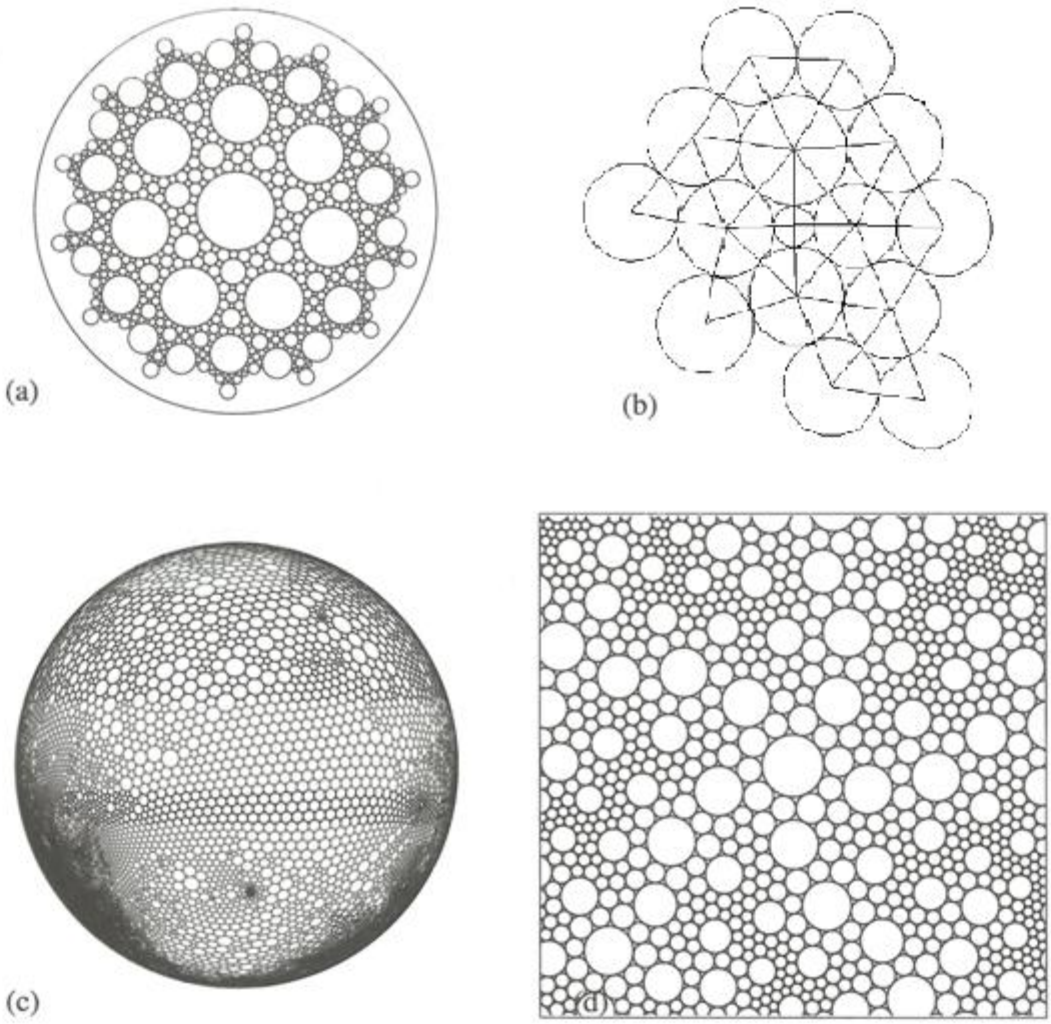

Figure 2: Circle packing in different geometric settings. [7]

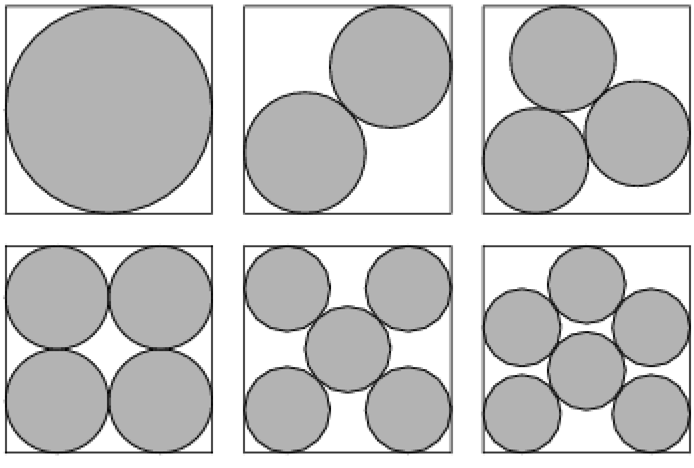

Figure 3: Circle packings for a square.

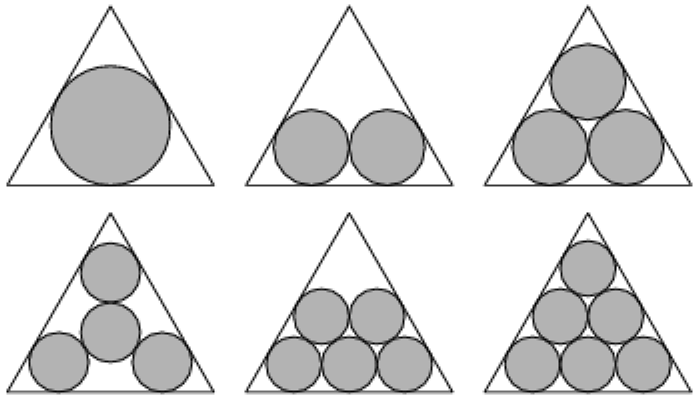

Figure 4: Circle packings for a triangle.

A circle packing consists of patterns of tangent circles. All the tangencies are external and no two circles share any points except the point of tangency. In the following figure, figure 5, we observe that tangent circles come in mutually tangent *triples*. A finite sequence of circles from a circle packing is called a *chain* if each circle except the last is tangent to its successor. The chain is a *cycle* if the first and last circles are tangent. Then we can form a *flower*, with a *central circle*, surrounded by a chain of successively tangent circles, called *petal* circles. The number of the petal circles is defined as the *degree* of the central circle. Last, the entire circle packing consists of interlinked flowers. We usually denote a circle packing with *P*.

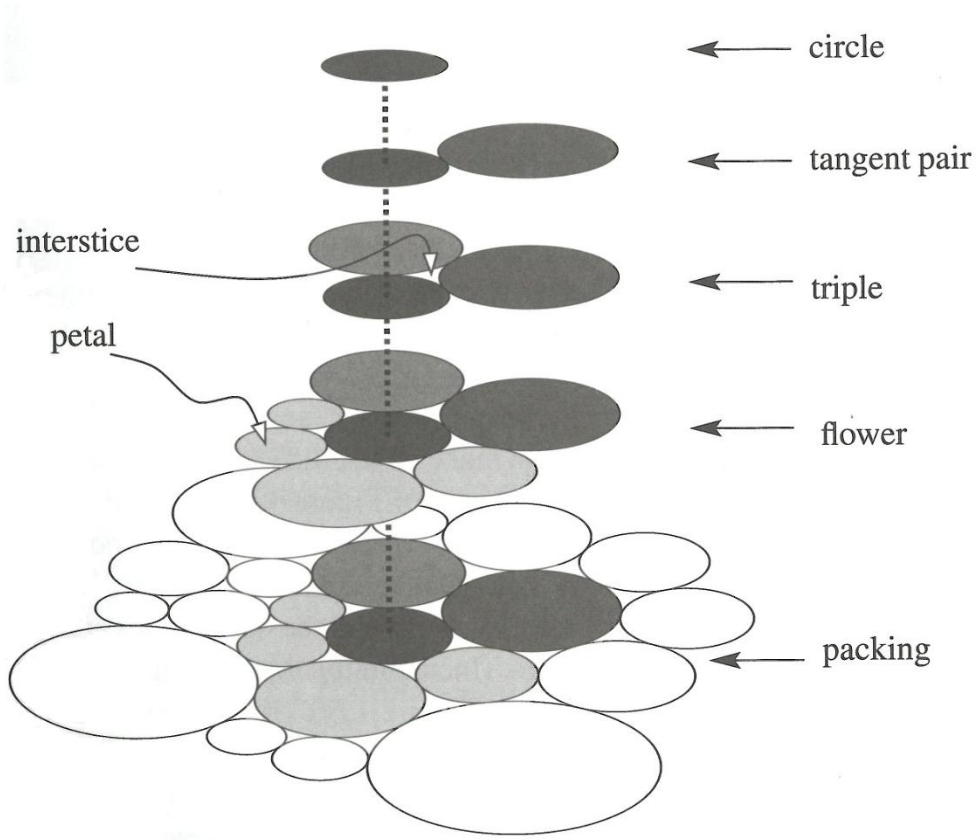

Figure 5: a description of components of circle packings. [7]

Before we study the structure of the circle packings, we need to understand the topological term, *simplicial complex*, for denoting the structures.

Definition 1.1 A simplicial complex K consists of a set $\{v\}$ of vertices and a set $\{s\}$ of finite nonempty subsets $\{v\}$ called simplexes such that

(a) Any set consisting of exactly one vertex is a simplex.

(b) Any nonempty subset of a simplex is a simplex.

A simplex s containing exact $q+1$ vertices is called a q-simplex. We also say that the dimension of s is q. We also use the term complex as a shorthand of simplicial 2-complex in this paper.

In figure 6, we obtain a structure called *carrier.* A *carrier* is the geometric complex formed by connecting the center of the tangent circle with the geodesic segment to form *edges* and *faces*.

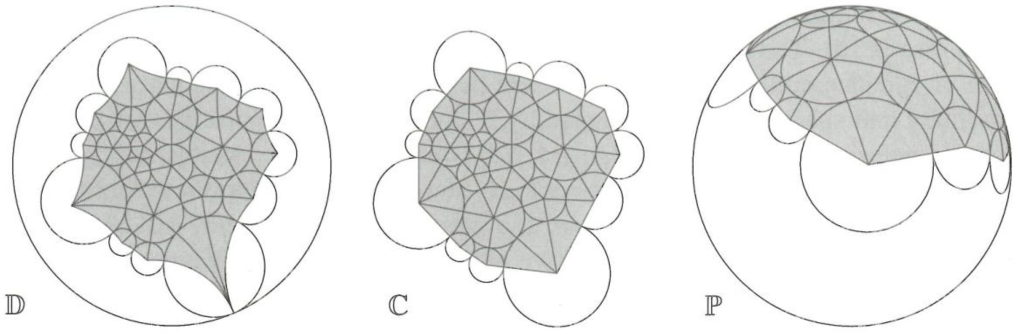

Figure 6: carrier of packings in different geometric surfaces. [7]

One of the most familiar packings of the complex plane is the *Regular Hexagonal Packing,* which consists of circles with same radius and each interior circle has a degree of six, as in the background of figure 7.

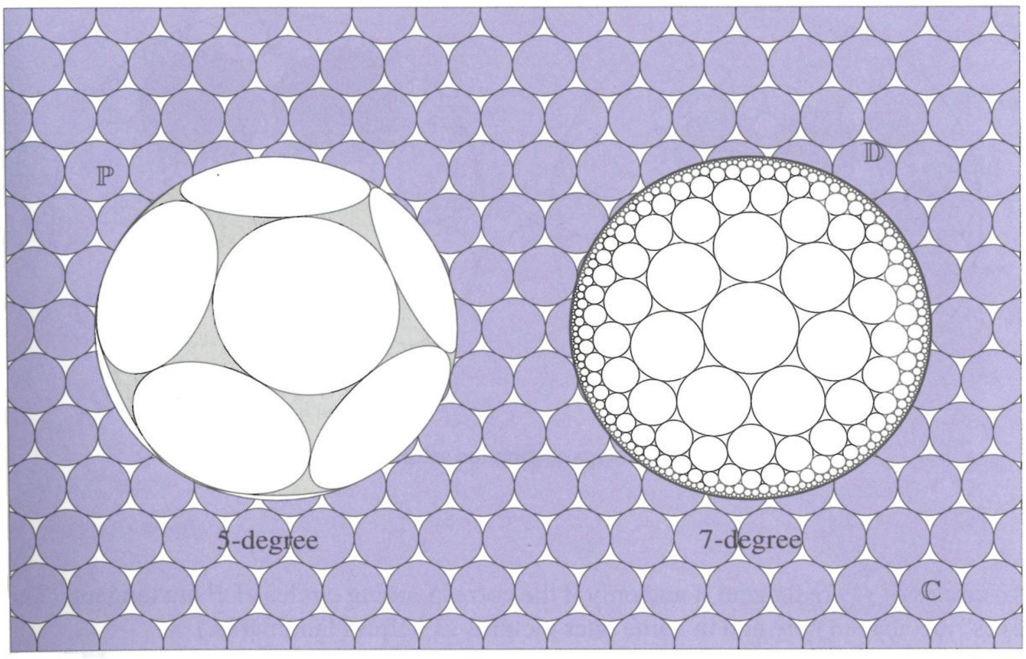

Figure 7: circle packings of degree 5, 6, and 7. [7]

We notice that a circle packing consists of its combinatorics information, the tangencies of circles, and its geometric information, the radii of the circles in the packing. We first focus on the combinatoric properties of packing.

Before we can study the combinatorics of circle packing, we need to find a way to represent the combinatorics of a circle packing P. We use the construction of the abstract vertices, edges and faces of the carrier. We call this structure the *complex* for P, written as K. K is a simplicial complex, an abstract object without any metric or geometry. Then we attach and label R to K which contains a set of positive numbers associated with the radius of each circle in the packing. With tangential of the circles and their radii, a circle packing is determined in certain domain.

From the observation of the figures above, we find that carrier consists of a group of triangles and with the vertices and edges of those triangles, we can obtain the abstract combinatoric information of the circle packing, so we introduce the concept of triangulation of the a surface.

Definition 1.2 A triangulation T of a surface S is a locally finite decomposition of S into a collection of topological closed triangles, $T = \{t_j\}$, so that any two either are disjoint, intersect in a single vertex, or intersect in a single complete edge. (Locally finite means that every point of S has a neighborhood that intersects at most finitely many triangles of T; the collection T itself may be finite or countably infinite)

Every circle packing has its associated carrier, and therefore its own triangulation.

The structure for the packing and the triangulation of the domain help us to study the relationship of topology and structure between different domains and packings. So our next step is to define these terms precisely.

Definition 1.3 The prescribed pattern for a circle will be encoded as an abstract simplicial 2-complex (or complex for short) which is a triangulation of an oriented topological surface. The notation K will be used both for the complex and for its realization as a topological surface.

Definition 1.4 Given a complex K with vertices $\{\mathrm{v}_1, \mathrm{v}_2, \dots\}$, a label R for K in a particular geometry consists of a set $\mathrm{R} = \{\mathrm{r}_1, \mathrm{r}_2, \dots\}$ of real numbers that qualify as radii. We refer to $K(R)$ as a labeled complex.

Definition 1.5 A collection $P = \{c_v\}$ of circles in $\mathcal{G}$ is said to be a circle packing for a complex K or K is the complex of P if

(1) P has a circle c_v associated with each vertex *v of K*,

(2) two circles c_v, $c_u$are externally tangent whenever $\langle \mathrm{u}, \mathrm{v} \rangle$ is an edge of K, and

(3) three circles c_v, c_u, c_w form a positively oriented triple in $\mathcal{G}$ whenever $\langle u, v, w \rangle$ forms a positively oriented face of K

One important fact about triangulation we will use to prove the main theorem is Andreev's Theorem. It help us to prove the existence of the circle packing in the co-domain.

Theorem 1.1 [1] (Andreev's Theorem)

Any triangulation of the sphere is isomorphic to the triangulation associated to some circle packing. The isomorphism can be required to preserve the orientation of the sphere and then this circle packing is unique up to Möbius transformations.

Next, we want to study more about a single flower, which is essential to the circle packing. We need to introduce the definition of angle sum and packing label.

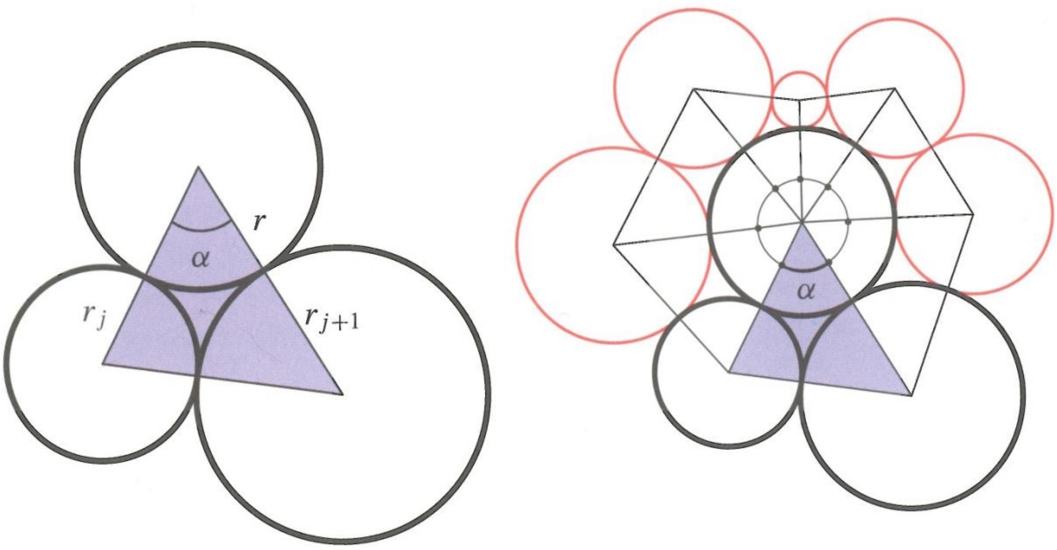

Figure 8: angle of petal circle for circle packing. [7]

Definition 1.6 Let R be a label for K. Then the angle sum map $\theta_R: K^{(0)} \longrightarrow [0,\infty)$ assigns to each vertex $v \in K$ the sum of the angles at v in the faces of K the sum of the angles at v in the faces of K to which v belonges:

$$\theta_R(v) = \sum_{\langle u,v,w \rangle} \alpha(R(v); R(u), R(w)).$$

In the Euclidean case,

$$\alpha(\mathrm{x}; \mathrm{y}, \mathrm{z}) = \arccos\left(\frac{(\mathrm{x}+\mathrm{y})^2 + (\mathrm{x}+\mathrm{z})^2 - (\mathrm{y}+\mathrm{z})^2}{2(\mathrm{x}+\mathrm{y})(\mathrm{x}+\mathrm{z})}\right).$$

Definition 1.7 A label R is termed a packing label for K is for every interior vertex v of K, then the angle sum of the vertex would be such that $\theta_R(v) = 2\pi$. If, we say that R has an unbranch point at v.

In order to form a circle packing, the angle sum of the petal circle in a flower should equal to 2π or multiple of 2π for branched packing; otherwise, the petal circle will not form a closed chain which surrounds the center circle.

After we grasp the basic understanding of circle packing, I want to introduce some further properties.

Theorem 1.2 [7]

Let K be a simply connected complex with label R. Let $\mathbb{G}$ be one of $\mathbb{P}$ (the hyperbolic plane), $\mathbb{C}$ (the complex plane), or $\mathbb{D}$ (the unit disk) , depending on the geometry of R. Then there exists a circle packing P in $\mathbb{G}$ with $P \longleftrightarrow K(R)$ if and only if R is a packing label. The circle packing P is unique up to isometries of $\mathbb{G}$.

Once we establish the structure and notation of the circle packing, we then want to introduce the mappings between circle packing. Therefore, we have to introduce the concept of discrete analytic function.

Definition 1.8 A discrete analytic function f from $\mathcal{P}_\mathrm{K}$ to P is a continuous orientation-preserving simplicial mapping $f\colon \mathrm{carr}(\mathcal{P}_\mathrm{K}) \to \mathrm{carr}(P)$. In particular, it maps the center of each circle of $\mathcal{P}_\mathrm{K}$ to the center of the corresponding circle of P, each edge of $\mathrm{carr}(\mathcal{P}_\mathrm{K})$ one-to-one onto the corresponding edge of $\mathrm{carr}(P)$, each face of $\mathrm{carr}(\mathcal{P}_\mathrm{K})$ one-to-one in an orientation-preserving way onto the corresponding face of $\mathrm{carr}(P)$. The associate ration function $f^{\#}$ is defined on the vertices of carrier of Q, $carr(Q)$: if C_v is a circle of Q and c_v is the corresponding circle of P, then at the center Z_v of C_v,

$$f^{\#}(Z_v) = \frac{radius(c_v)}{radius(C_v)}.$$

3. Conformal and Quasiconformal Mappings

I assume the reader has some fundamental knowledge about the complex numbers and complex functions. However, I find a brief introduction on conformal and quansiconformal mappings theories would be very helpful to understand this paper.

The basic objects we are focusing on are complex functions $f\colon \mathrm{G} \to \mathbb{C}$.on the domain $\mathrm{G} \subset \mathbb{C}$. A function is also called a *mapping*.

A mapping $f\colon \mathrm{G} \to \mathbb{C}$ is said to be *analytic* at $\mathrm{a} \in \mathrm{G}$ if it has a complex derivative defined as

$$f^{'}(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a};$$

f is analytic on G if F is analytic at each point of G.

Geometrically, when $f^{'}(a)$ is nonzero, the mapping preserves the angle (the magnitude and orientation) between any two smooth curves that intersect at z_0.

Definition 2.1 An analytic function is said to be a *conformal mapping* at points where its derivative does not vanish. Regions $\mathrm{G}, \widetilde{\mathrm{G}}$ are conformal-equivalent if there is a univalent analytic bijection $F\colon \mathrm{G} \to \widetilde{\mathrm{G}}$.

Conformal Mapping is one of the central things we need to understand in order to read the proof of the theorem. It is also the mapping we are trying to approximate with the circle packings and discrete analytic functions. In the next few paragraphs, we will study some basic theorem about the analytic function.

Definition 2.2 If γ is a closed C^1(continuous first derivative) curve in $\mathbb{C}$ and $z_0 \notin \{\gamma\}$ then

$$n(\gamma; a) = \frac{1}{2\pi i}\int_{\gamma} (z-a)^{-1}\, dz$$

is called the *index of* γ with respects to the point. It is also called the winding number of γ around a.

Theorem 2.1 [3] (Cauchy Theorem)

Let G be an open subset of the plane and $f: \mathrm{G} \to \mathbb{C}$ an analytic function. If γ is a closed rectifiable curve in G, such that the $n(\gamma; w) = 0$ for all w in $\mathbb{C} - \mathrm{G}$, then for a in $\mathrm{G} - \{\gamma\}$,

$$n(\gamma; w)f(a) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{z-a}dz.$$

Power series

Let $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ has a radius of $\mathrm{R} > 0$, then:

(a) For each $k \geq 1$ the series

$$\sum_{n=0}^{\infty}(n)(n-1)\ldots(n-k+1)a_n(z-a)^{n-k}$$

has radius of convergence R;

(b) The function is infinitely differentiable on $B(a, R)$ and furthermore, $f^{(k)}(z)$ is given be the formula in part (a), for all $k \geq 1$ and $|\mathrm{z} - \mathrm{a}| < R$;

(c) For $n \geq 0$,

$$a_n = \frac{1}{n!} f^{(n)}(a).$$

If a function is analytic at a point a, we can find a Taylor's series like above that converges to it at point a.

Quasiconformal Mappings

Let $w = f(z)$, where $z = x + iy$ and $w = u + iv$ be a C^1 homeomorphism from one region G to another. At a point $a \in G$, it induces a linear mapping of the differentials

$$du = u_x dx + u_y dy \quad (1)$$

$$dv = v_x dx + v_y dy \quad (2)$$

which we also can write in the complex form

$$dw = f_z dz + f_{\bar{z}} d\bar{z} \quad (3)$$

with

$$f_z = \frac{1}{2}(f_x - if_y), \;\; f_{\bar{z}} = \frac{1}{2}(f_x + if_y). \quad (4)$$

Using this complex notation, we notice that

$$f_z = \frac{1}{2}\left(u_x + v_y\right) + \frac{i}{2}\left(v_x - u_y\right) \quad (5)$$

$$f_{\bar{z}} = \frac{1}{2}\left(u_x - v_y\right) + \frac{\mathrm{i}}{2}\left(v_x + u_y\right). \quad (6)$$

This result gives

$$|f_z|^2 - |f_{\bar{z}}|^2 = u_x v_y - u_y v_x = J \quad (7)$$

which is the determent of the Jacobian. The Jacobian is positive for orientation preserving and negative for sense reversing mappings. For our interest, we shall only consider the sense preserving case. Then $|f_{\bar{z}}| < |f_z|$.

It follows the equation (3)

$$(|f_z| - |f_{\bar{z}}|)|dz| \leq |dw| \leq (|f_z| + |f_{\bar{z}}|)|dz| \qquad (8)$$

where both limits can be attained. Then we conclude that the ratio of the major to the minor axis is

$$D_f = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \geq 1. \qquad (9)$$

This is called the *dilatation* at the point a. Sometime it is more convenient to consider

$$d_f = \frac{|f_{\bar{z}}|}{|f_z|} < 1 \qquad (10)$$

which is related to D_f by

$$D_f = \frac{1 + d_f}{1 - d_f}; \quad d_f = \frac{D_f - 1}{D_f + 1}. \qquad (11)$$

Definition 2.3 The mapping f is said to be *quasiconformal* if D_f is bounded. It is *K-quasiconformal* if $D_f \leq K$.

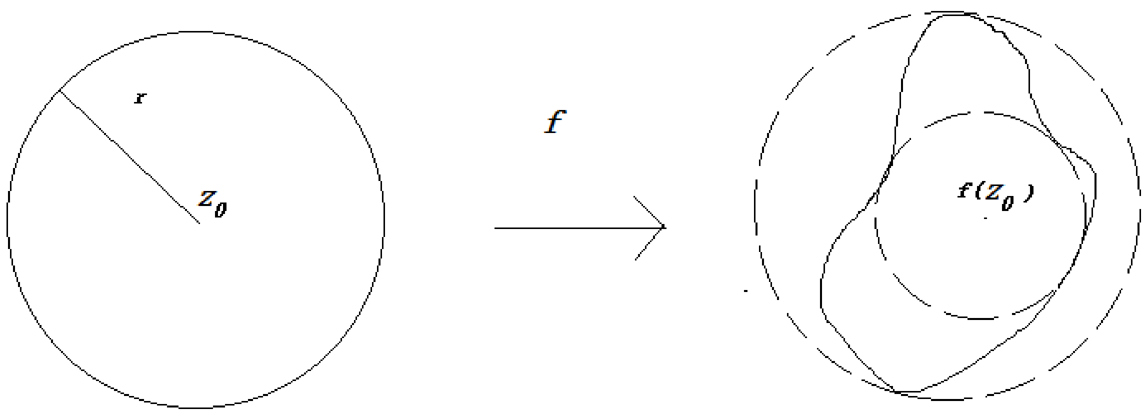

Figure 9: image of a circle under a quasiconformal mapping.

Definition 2.4 For a Euclidean rectangle $\mathfrak{R}$ with two designated “ends”, the modulus is $\mathrm{Mod}(\mathfrak{R}) = \mathrm{height/length}$ (the heights of two ends divided by their distance apart). If A is a round annulus, meaning $A = \{r < |z| < R\}$, for some $0 < r < R < \infty$, then its modulus is $\mathrm{Mod(A)} = 2\pi/\log\mathrm{(R/r)}$.

In order to understand the following theorems about quasiconformal mapping, we introduce the *extended complex plane*, $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. The Euclidean plane will be called the *finite plane*. Besides the Euclidean metric, I also use the spherical metric in the extended complex plane. Two finite points z_1 and z_2 have the spherical distance

$$k(z_1, z_2) = arctan\left|\frac{z_1 - z_2}{1 + \overline{z_1} z_2}\right|,$$

where $0 \le k(z_1, z_2) \le \frac{\pi}{2}$.

For $z_2 = \infty$,

$$k(z_1, \infty) = arctan\left|\frac{1}{z_1}\right|.$$

We now will introduce the theorem that will be used directly or indirectly in the proof of the main theorem.

Theorem 2.2 [7] Suppose $f: G \to G'$ is *K-quasiconformal.* If G and G' are both rectangles, the map f maps the designated ends on to those of the other, or if both G and G' are round annuli, the modulus is quasi-invariant:

$$\frac{1}{\mathrm{K}}\mathrm{Mod(G)} \le \mathrm{mod(G)} \le \mathrm{KMod(G)}.$$

The existence bound on the modulus for a domain comes from the bound of the distortion of the quasiconformal mappings.

Theorem 2.3 [7] There exist no quasiconformal mapping from $\mathbb{C}$ to the unit disc.

This theorem is parallel to the version in conformal mapping such that there is no conformal mapping from $\mathbb{C}$ to the unit disc. Next, we want to focus on the sequence of functions. The properties of sequences of functions are crucial to our proof of the main theorem.

Definition 2.5 A family F of mappings of a domain G in to the plane is called equicontinuous at the point $z_0 \in G$ if to every $\varepsilon > 0$ there corresponds a neighborhood U of z_0 such that

$$\sup_{f \in F, z \in U} |f(z) - f(z_0)| < \varepsilon.$$

The family F is equicontinuous in a set $E \in G$ is it is equicontinuous at every point of E.

Equicontinuous families have very important properties which we will study more. These families are also related to the normal families of quasiconformal mappings.

Definition 2.6 Let F be a family of continuous mappings of a domain G. We say that F is *normal* if every sequence of elements of F contains a subsequence which converges uniformly in every compact subset of G. A normal family which contains all its limit functions is called closed.

Lemma 2.1 [2]

Let f_n, $n = 1,2,3 \ldots$, be a sequence of mappings, equicontinuous in a domain G, and E an everywhere dense subset of G. If the sequence f_n converges at every point of E then it is uniformaly convergent in every compact subset.

Proof:

Let $\varepsilon > 0$ and z_0 be a point of G. Since the sequence f_n is equicontinuous at the point z_0, then we can find a neighborhood of z_0, $B \subset G$ where $k(f_n(z), f_n(z_0)) < \frac{\varepsilon}{5}$ for all mappings f_n. Therefore, by the hypothesis, there is a number $N(B)$ such that $k(f_n(w), f_m(w)) < \frac{\varepsilon}{5}$ at a point $w \in E \cap B$. Whenever $m, n \geq N(B)$. Then we have

$$k(f_n(z), f_m(z))$$

$$\leq k(f_n(z), f_n(z_0)) + k(\, f_n(z_0),\ f_n(w),) + k(f_n(w), f_m(w))$$

$$+ k(f_m(w), f_m(z_0)) + k(f_m(z_0), f_m(z)) < \varepsilon.$$

For $z \in B$ and $m, n \geq N(B)$.

Every compact set $K \subset G$ can be covered by finitely many neighborhoods B_i of the same type mentioned above. If N represents the largest numbers $N(B_i)$, then $k(f_n(z), f_m(z)) < \varepsilon$ for all $z \in K$ and $m, n \geq N$. The sequence f_n is therefore converge uniformly in K. ■

Theorem 2.4 [2]

A family F of mapping which is equicontinuous in a domain G is a normal family.

Proof:

From the Lemma 2.1, it suffice to show that every sequence of f_n of mappings of G has a subsequence converging in a set E, which is everywhere dense in G. We can choose E to be a countable set $\{a_n\}, \mathrm{n} = 1,2,3 \ldots$

Since the extended complex plane is compact, the sequence $w_n(a_1)$ has an accumulation point. Therefore, the sequence w_n has a subsequence w_{1n} such that converges at the point a_1. The sequence w_{1n} has a further subsequence which converges at a_2. Repeating this process for k times and we obtain the sequence w_{kn} which converges at $a_1, a_2, a_3, \ldots, a_k$. The diagonal sequence w_{nn} then will converge at all points in E. Then by the previous lemma, w_n have a subsequence that converge uniformly on every compact subset of G and therefore is normal. ■

The following two theorems are very crucial to our proof, the proofs for both are quite long and difficult to read, while has little to do for us understand the circle packings, and therefore, I choose not to write them

Theorem 2.5 [2][7]

Let $\{f_n\}$ be a sequence of *K-quasiconformal* mappings defined in G and assume that the sequence converges to a limit function *f*. Then *f*. is either a constant, a mapping of G to two points, or a *K-quasiconformal* mapping of G.

Theorem 2.5 tell us that there are three possible choices for the limit function for a sequence of *K-quasiconformal* mappings if it converges. It also implies that if the images of the domain under a sequence of *K-quasiconformal* mappings converge to a set with 3 points or more, the limit function has to be *K-quasiconformal.*

Theorem 2.6 [2][7] (Caratheodory Kernel Theorem)

Assume G is a plane region with at least two boundary points and suppose z_0 is a point of G. Let $\{f_j\}$ be a sequence of K-quasiconformal mappings $f_j\colon G_j \to \mathbb{C}$ for open set $G_j \to G$ and suppose that f_j converge to a homeomorphism $f\colon G \to \mathbb{C}$, with $w_0 = lim_{j\to\infty} f_j(z_0)$. Let Γ be the kernel of the ranges, defined by

$$\Gamma = \bigcup_{n=1}^{\infty}\bigcap_{j=n}^{\infty} f_j(G_j)$$

Then, f is a *K-quasiconformal* mappings from G to Σ, where Σ is the connected component of Γ containing w_0.

The Caratheodory Kernel Theorem tells us that if the limit function of a sequence converges to a K-quasiconformal mapping, then its image will be one of the connected component of the kernel of the range. If we can show the kernel of the range is connected, then we know the limit function is one-to-one and onto.

Theorem 2.7 [7]

Simplicial homeomorphisms are K-quasiconformal for K depending only on the shapes of the triangles involved.

4. Lemmas for the Rodin-Sullivan Theorem

The Ring Lemma [7]

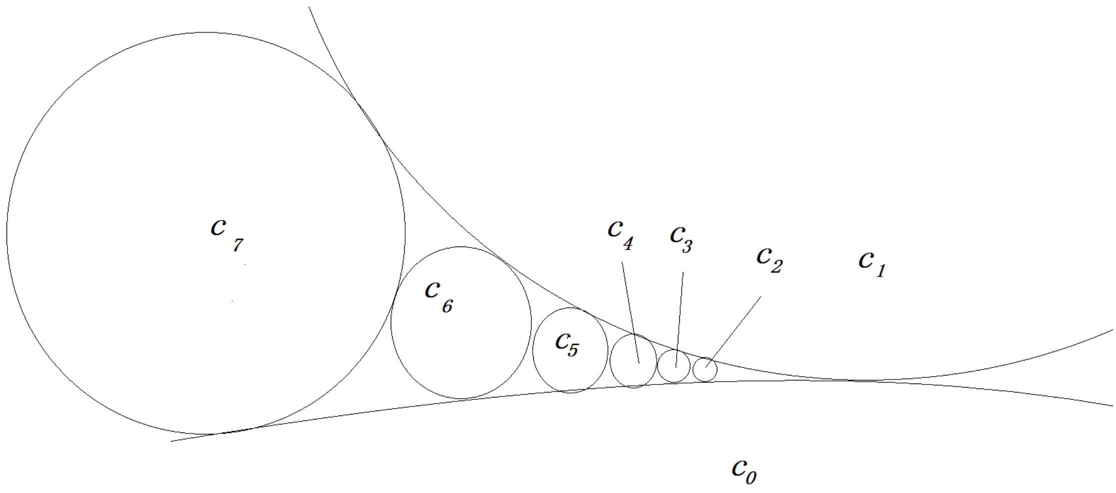

Figure 10: petal circles surrounding the center circle c_0.

For each integer $k \geq 3$ there exists a constant $c(k) > 0$ such that if F is any univalent *k-flower* of circles in the Euclidean or hyperbolic plane having a central circle of radius of r_0, then the radius of *r* of each petal satisfies $r \geq c(k) \cdot r_0$.

Proof:

Suppose that $\{c_0; c_1, c_2, \ldots, c_k\}$ is a closed univalent *k-flower*, with some $k \geq 3$. Without loss of generality, we can assume the radius of the central circle c_0 is 1. There exists at least one petal c_1, such that the radius of c_1 is larger or equal to the radius of the petal circles as if all the petal circles are of the same size. We now consider c_2. The smaller the radius of c_2 is, the deeper c_2 lies in the crevasse of formed by c_0 and c_1. If c_2 lies deep inside the crevasse, the radius of the next petal, c_3, is also very small. The univalence implies that c_3 must not overlap with c_1 or c_2.

If we repeat this reasoning, we find that if the radius of c_2 is too small, then the chain of k petals could not reach outside the crevasse or form a closed flower around c_0. Therefore, there exists a lower bound for the radius of c_2. Now we repeat this reasoning for $c_3, c_4, \ldots, c_k$, we conclude that there is a positive lower bound $c(k)$ for all the radii of the petals around c_0. Hereby, we establish the existence.

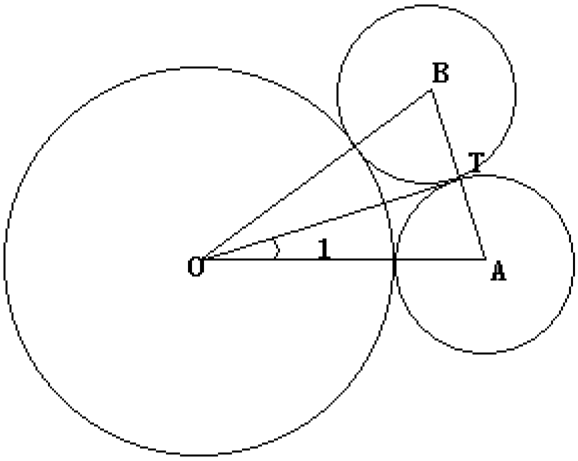

Figure 11: A triple consist two petal circles with the same size and the center circle.

To make the statement more rigorous, we first need to find the radius of c_1. As in figure 10, we have a triple form by the central circle c_0 and two of its neighboring petals c_1 and c_2. c_0 is centered at the origin, *O*, with assumed radius 1. Two circles c_1 and c_2 are of the shape as if all of the k petal circles are of the same size, with radius *r*. We denote the centers of c_1 and c_2 to be points *A* and *B*, respectively, and their tangent point to be point *T*. Let angle $\angle TOA$ be $\angle 1$. Then *OBA* form an isosceles triangle with $OA = OB$. Since $BT = AT$, line segment $OT \perp AB$. Therefore, we have:

$$r = AT = (1 + r) \cdot \sin(\angle 1);$$

$$r = (1 + r) \cdot sin\left(\frac{2\pi}{2k}\right);$$

$$r = \frac{sin\left(\frac{\pi}{k}\right)}{1 - sin\left(\frac{\pi}{k}\right)}.$$

We conclude that $r_1 \geq \frac{sin\left(\frac{\pi}{k}\right)}{1-sin\left(\frac{\pi}{k}\right)} > 0\ for\ k \geq 3.$

Let us think of the extreme case such that the last circle in the petal just escapes the crevasse formed by c_0 and c_1. Then the circle c_{k-1} is tangent to both c_0 and c_1 and there is a line tangent to c_0, c_1 and c_{k-1}. Then the half plane is the largest circle possible could be for c_k, but it still does not close the petal. Therefore, c_2 should have a radius larger than the smallest possible in this case. Then we can find the radius of c_{k-1}, which tangent to all the circles in the triple c_0, c_1 and c_k with Descartes' Theorem [7] (Descartes Circle Theorem)

$$k_{c_{k-1}} = k_{c_0} + k_{c_1} + k_{c_k} + 2\sqrt{k_{c_0}k_{c_1} + k_{c_0}k_{c_k} + k_{c_k}k_{c_1}},$$

where k_{c_i} is the curvature of the circle c_i for integer $0 \leq i \leq k$. and $k_{c_i} = \frac{1}{r_{c_i}}$.

Since k_{c_0} and k_{c_1} are positive, then $k_{c_{k-1}}$ is a positive number.

Then we can trace back down to the radius of the circle c_2. We then observe that the curvature of the j^{th} circle in the petal is given by the equation:

$$k_{c_j} = k_{c_0} + k_{c_1} + k_{c_{j+1}} + 2\sqrt{k_{c_0}k_{c_1} + k_{c_0}k_{c_{j+1}} + k_{c_{j+1}}k_{c_1}}, for\ integer\ 2 \leq j \leq k - 1.$$

Since each time we plug in a positive finite number as the curvature of $k_{c_{j+1}}$ and iterate it for a finite times, $(k-2)$ times, to obtain the curvature of c_2. Therefore, k_{c_2} is a finite positive number bounded above and its reciprocal, which is the radius of c_2, denoted as r_2^*, is bounded below.

If c_2 has any radius smaller than r_2^*, the corresponding chain of circles will be more constrain by the crevasse formed by c_0 and c_1 and certainly fail to reach out the crevasse or close around c_0. ■

Furthermore, we actually can compute the value for each $c(k)$. The values of $c(k)$ are given by the following formula:

$$c(k) = \frac{1}{\left[\left(\frac{5-2\sqrt{5}}{5}\right)\left(\frac{3+\sqrt{5}}{2}\right)^{\mathrm{k}} + \left(\frac{5+2\sqrt{5}}{5}\right)\left(\frac{3-\sqrt{5}}{2}\right)^{\mathrm{k}} - 1\right]}, \mathrm{k} \geq 3.$$

Moreover, these constants are all reciprocal integers,

$$c(3) = 1, \qquad c(4) = \frac{1}{4}, \qquad c(5) = \frac{1}{12}, \qquad c(6) = \frac{1}{33}, \ldots$$

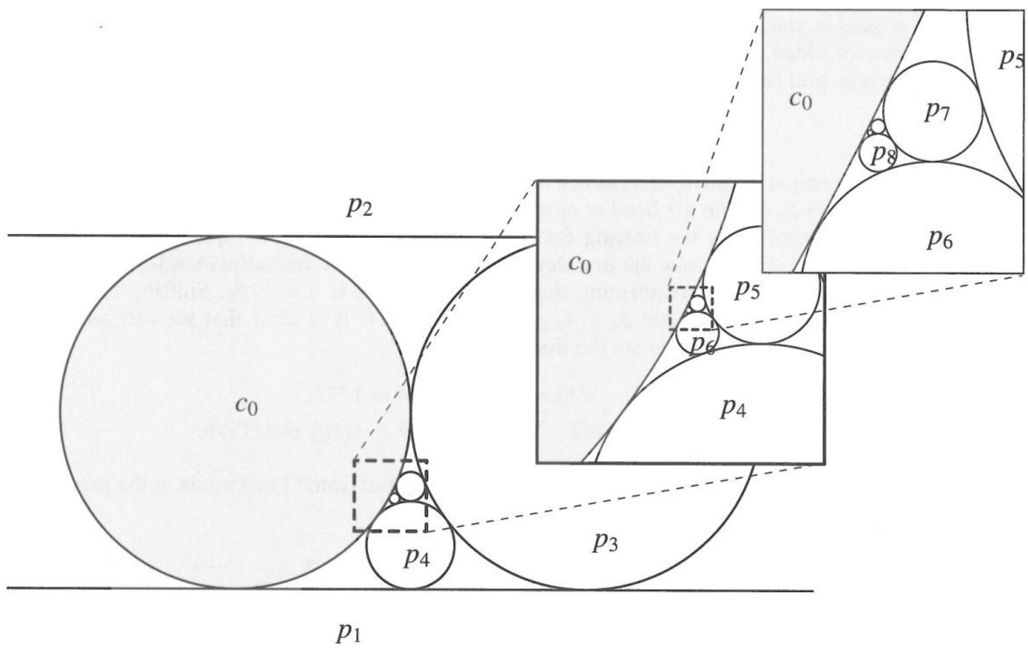

Figure 12: fit in the smallest circle possible to preserve the tangency of the petal circle.[7]

In order to find the $c(k)$ precisely, we can construct the flower as in the figure 10, such that both p_1 and p_2 have the radius of infinity and therefore are straight lines. Then we try to fit in a circle that is tangent to both previous circles and find the one with the smallest radius. We can use the Descartes Circle Theorem to find the radius and its ratio to the radius of c_0.

Length-Area Lemma [7]

Let c be a circle in a circle packing in the unit disk. Let $S_1, S_2, \ldots, S_k$ be k disjoint chains which separate c from the origin and a point of the boundary of the disk. Denote the combinatorial lengths (simply the number of circles in a chain) of these chains by $\mathrm{n}_1, \mathrm{n}_2, \ldots, \mathrm{n}_\mathrm{k}$. Then,

$$\mathrm{radius(c)} \leq \frac{1}{\pi}\left(\mathrm{n}_1^{-1} + \mathrm{n}_2^{-1} + \cdots + \mathrm{n}_\mathrm{k}^{-1}\right)^{-1/2}.$$

Proof:

Suppose S_j consists of circles with Euclidean radii r_{ji}, $1 \leq \mathrm{i} \leq \mathrm{n_j}$. Therefore, by Schwarz inequality, we have

$$\left(\sum_i r_{ji}\right)^2 \leq n_j \sum_i r_{ji}^2.$$

Let $s_j = 2\sum_i r_{ji}$ be the geometric length of the chain S_j, then we have $\mathrm{s_j^2 n_j^{-1}} \leq 4\sum_i r_{ji}^2$, and therefore,

$$\sum_j s_j^2 n_j^{-1} \leq 4 \sum_{ji} r_{ji}^2 \leq 4.$$

The last inequality comes from the fact that the disjoint interior circles have a total area less then the unit circle. If we let $\mathrm{s} = \min\{\mathrm{s}_1, \mathrm{s}_2, \dots, \mathrm{s}_\mathrm{k}\}$, then

$$s^2 \sum_j n_j^{-1} \leq 4;$$

$$s \leq 2(n_1^{-1} + n_2^{-1} + \cdots + n_k^{-1})^{-1/2}.$$

Combining the fact that $\mathrm{s} \geq 2\pi \cdot \mathrm{radius(c)}$, we achieve the inequality stated in the theorem. ■

Hexagonal Packing Lemma [7]

There is a sequence $\{\mathrm{s_n}\}$, decreasing to zero, with the following property. Let c_1 be a circle in a univalent Euclidean circle packing P in the plane and suppose the first n generations of circles about c_1 are combinatorially equivalent to n generations of the regular hexagonal packing around one of its circles. Then for any circle $c \in P$ tangent to c_1,

$$\left|1 - \frac{\mathrm{radius(c)}}{\mathrm{radius(c_1)}}\right| \leq \mathrm{s_n}.$$

Proof:

Suppose that for $\mathrm{n} = 1, 2, 3, \dots$ we have a circle packing P_n which is combinatorically equivalent to the n-generation of circle packing centered around the c_1. Without loss of generality, we can assume the radius of c_1 is 1. Based on the ring lemma, we know that for each P_n, the radii of circles in generation k in the packing P_k, P_{k+1} ... are all uniformly bounded away from zero or infinite. Therefore the possible numbers for the radii of circles form a compact set. We can choose a subsequence of P_n such that the generation one circles converge geometrically. A further subsequence can be selected so that generation two circles and so on.

In this way we can obtain a limit infinite circle packing, which has the combinatorics of the regular hexagonal packing. Since the only packing with hexagonal pattern in the Euclidean plane is regular hexagonal packing, we could not choose a subsequence of P_n which converge to a limit such that six circles around c_1 have different radii. Since the radii of circles in each generation are bounded below by the constant from the Ring Lemma, we can find a decreasing sequence$\{s_n\}$, depending only on n, decreasing to zero such that

$$\left|1-\frac{\mathrm{radius(c)}}{\mathrm{radius(c_1)}}\right| < s_n$$

for each nature number n. ■

5. The proof of Rodin-Sullivan Theorem

Riemann Mapping Theorem

If G is a bounded and simply connected domain in the complex plane and let $a \in G$ the Riemann Mapping Theorem states that there exists a conformal mapping $f: \mathbb{D} \to G$ with

(a) $f(a) = 0$ and $f'(0) > 0$;

(b) f is one-to-one;

(c) $f(G) = \mathbb{D} = \{z: |z| < 1\}$.

Then, let us consider the circle packing for the domain G. If G is filled with a hexagonal packing P_ϵ of circles with radius $\epsilon > 0$ and defines the discrete analytic function $f_\epsilon: P_\epsilon \to Q_\epsilon$ where Q_ϵ is the packing associated with the unit disk. The circle packing Q_ϵ shares the same combinatorics with P_ϵ. For small ϵ, there will be distinct vertices $v_1, v_2 \in K_\epsilon$ whose flowers in P_ϵ contain z_0 and z_1, respectively, and one normalizes Q_ϵ to center $f_\epsilon(z_0)$ at the origin and $f_\epsilon(z_1)$ on the real positive axis. Each function $f_\epsilon: \mathbb{D} \to G$ is a discrete conformal mapping. In fact, each mapping is the simplical mappings between Euclidean carriers, $f_\epsilon: carr(Q_\epsilon) \to carr(P_\epsilon)$.

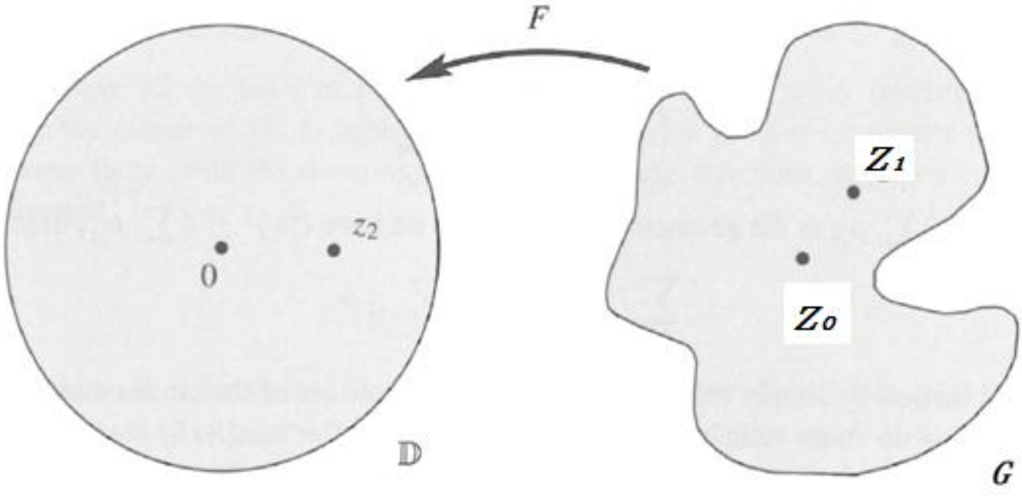

Figure 13: mapping the domain G on the unit disc. [7]

Rodin-Sullivan Theorem

Let G be a bounded and simply connected plane region with distinguished points z_0 and z_2. Assume that the classical conformal mapping $f: G \to \mathbb{D}$ and the discrete conformal mappings $f_\epsilon: G \to \mathbb{D}$ are defined and normalized as described above. Then the mappings f_ε converge uniformly on compact subsets of $\mathbb{D}$ to f as $\epsilon \to 0$.

The Proof of Rodin-Sullivan Theorem

Let G be a simply connected bounded region in the complex plane with two distinguished points z_0 and z_1. The map f_ϵ maps triangles in the $carr(H_\epsilon')$ to corresponding triangles in $f_\epsilon(\mathrm{t})$ in the $carr(H_\epsilon)$. Each packing H_ϵ is regular hexagonal, with circles sharing the same radius $\epsilon > 0$. It is clear that when $\epsilon \to 0$, the $\mathrm{carr}(H_\epsilon)$ exhaust the G. Given any compact set $E \subset G$, $\mathrm{E} \subset \mathrm{carr}(H_\epsilon)$ and the number of generations of circle between E and the boundary of G goes to infinity as

$\epsilon \to 0$ because the $dist(E, \partial G) > 0$.

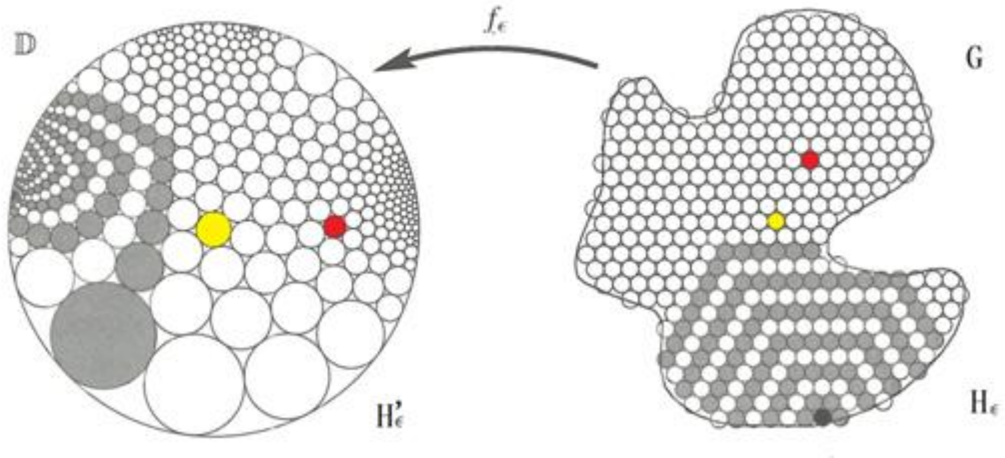

Figure 14: discrete analytic functions map circles packing of G to circle packings of the unit disc. [7]

Consider for small radius ϵ. Let c_0 be a circle whose flower contains z_0. Form all chains $C_0, C_1, \dots, C_k$ of circles from H_ϵ, starting with the circle c_0, such that the flowers of the circles in the chain are contained in G. The circles that appear in such chains are called inner circles. In other words, inner circles are the circles with completed flowers that are contained in H_ϵ. The set of inner circles is denoted as I_ϵ.

The circles in H_ϵ but not in I_ϵ are called *border circles*. Border circles are tangent to the inner circles. The set of border circles is denoted as B_ϵ. The set B_ϵ form a cycle called the border. The line segments joining the centers of the border circle is a Jordan curve surrounding the inner circles. A *Jordan curve* is a simple closed curve in the plane $\mathbb{R}^2$ is the image C of an injective continuous map of a circle into the plane, $\varphi: S^1 \to \mathbb{R}^2$ The set I_ϵ and B_ϵ together form the packing H_ϵ for G, with edges of its carrier being the edges of a triangulation T_ϵ.

T_ϵ can be completed to a topological triangulation T_ϵ^* of the sphere by adding a vertex at ∞ and disjoint Jordan arcs from ∞ to the center of the border circles. A Jordan arc in the plane is the image of an injective continuous map of a closed interval into the plane.

By Andreev's theorem, there exists a circle packing of the sphere with triangulation isomorphic to T_ϵ^* by an isomorphism that preserves the orientation of the sphere. This circle packing is unique up to Möbius transformations. We can normalize this packing such that the exterior of the unit disk is the disk centered at the vertex ∞ in T_ϵ^*. Then if $\mathrm{c} \in H_\epsilon$ and $f_\epsilon(c) = c'$ for $\mathrm{c}' \in \mathrm{H}_\epsilon'$, which is the circle packing of the unit disk with the triangulation as the part of T_ϵ^* on the unit disk. We can normalize the packing by Möbius transformations so that $\mathrm{c}_0' = f_\epsilon(c_0)$ is centered at the origin and $\mathrm{c}_1' = f_\epsilon(c_1)$ is a circle whose flower contains z_1, is centered on the positive real axis.

Let z be a point in the region G. If ϵ is sufficient small, then we can find z in one of the circles c in I_ϵ. As $\epsilon \to 0$, the circle c will be surrounded by more and more generations of cycles in I_ϵ. Since the circle packing H_ϵ' for the unit disk $\mathbb{D}$ have the same triangulation as H_ϵ in G, the image of c, $f_\epsilon(c) = c'$ will also be surrounded by the same number of generations of circles in H_ϵ' as c in H_ϵ. Then the Length-Area lemma gives that:

$$\left(\mathrm{radius}(\mathrm{c}')\right)^2 < \frac{1}{\frac{1}{\mathrm{n}_1} + \frac{1}{\mathrm{n}_2} + \cdots + \frac{1}{n_k}}.$$

where $\mathrm{n_i}$ is the combinatoric length of the cycle, in this hexagonal packing case $n_i = 6i$ for $i = 0, 1,2,3 \dots, k$. Also, each border circle is separated from the origin by chains with length $\leq 6, 12,18$ ...As the $\epsilon \to 0$, $k \to \infty$ and the radius of the for all

$c' = f_\epsilon(c), where\ c \in H_\epsilon$. c' goes to zero uniformly as $\epsilon \to 0$. Therefore, the circle packing H'_ϵ exhausts the unit disk and the discrete analytic function f_ϵ determines the approximate position of the image of z.

Next, we will show that the approximate map f_ϵ converges to a conformal map from the unit disk to G.

The circle packing H_ϵ for G is isomorphic to the circle packing H'_ϵ of the unit disk. Their associated triangulations are T_ϵ and T'_ϵ and their carriers are denoted as G_ϵ and D_ϵ, respectively. $f_\epsilon: G_\epsilon \to D_\epsilon$ is the simplicial mapping that maps vertices in T_ϵ to T'_ϵ with orientation preserved. As shown above, G_ϵ converges to G from its construction in the sense that G is the union of G_ϵ and any compact subset of G is contained in all G_ϵ with sufficiently small positive ϵ. In the same sense, D_ϵ converges to $\mathbb{D}$.

The angles of inner triangles in T'_ϵ are bounded away from zero independent of ϵ because the Ring Lemma provides that the ratio of the three sides of the triangles are bounded. The Ring Lemma also shows that the ratio of the radius of the border circle and the radii of its tangent circles are bounded, therefore their angles are bounded away from zero. Thus, the maps $f_\epsilon: G_\epsilon \to D_\epsilon$ are uniformly *K-quasiconformal* because they map equilateral triangles to triangles of uniformly bounded distortion.

Since mappings f_ϵ are *K-quasiconformal*, they are equicontinuous on any compact subset of G. If z and z′ are two different elements in G, then as $|z - z'| \to 0$, $|f_\epsilon(z) - f_\epsilon(z')| \to 0$ as $\epsilon \to 0$. Since f_ϵ are equicontinuous, they form a normal family. From the Caratheodory Domain Theorem, G_ϵ converges to G and D_ϵ converges $\mathbb{D}$ and the limit function f is *K-quasiconformal* of G onto $\mathbb{D}$ with $f(z_0) = 0$ and $f(z_1) > 0$. It is clear that f(G) $\subset \mathbb{D}$. Let $w_0 \in \mathbb{D}$ and Y be a subdomain of $\mathbb{D}$ with $0, w_0 \in Y$ and $D_\epsilon \supset Y$ for sufficient small $\epsilon > 0$. Consider the restrictions of f_ϵ^{-1} to Y and denote them by y_ϵ. Then choose $\epsilon(n) \to 0+$ such that $f_{\epsilon(n)} \to f$ and $y_{\epsilon(n)} \to y$ uniformly on compacta. Then $G \supset y(Y)$ because $g(0) = z_0$. It follows from $f_{\epsilon(n)}\left(y_{\epsilon(n)}(w_0)\right) = w_0$, and the uniform convergence of $f_{\epsilon(n)}$ near $y(w_0)$. That $f\big(y(w_0)\big) = w_0$. Therefore $\mathbb{D} = f(G)$. There for f is one-to-one.

The hexagonal Packing Lemma shows that each simplicial mapping f_ϵ restricted to a fixed compact subset of G maps equilateral triangles to triangles of T'_ϵ that are arbitrarily close to equilateral. Therefore, the limit function f has to be 1-quasiconformal and therefore conformal. Since $f(z_0) = 0$ and $\mathrm{f}(\mathrm{z}_1) > 0$, this limit function is the unique Riemann Mapping under the same normalization. ■

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