Distribution Agreement

Haoyang Cui Nov 20, 2024

On the Fairness of Rent Division Among Roommates

By

Haoyang Cui

Dr. John Patty
Adviser

Quantitative Theory and Methods

Dr. John Patty

Adviser

Dr. Elizabeth Newman

Committee Member

Dr. Ruoxuan Xiong

Committee Member

2024

On the Fairness of Rent Division Among Roommates

By

Haoyang Cui

Dr. John Patty
Adviser

An abstract of
a thesis submitted to the Faculty of Emory College of Arts and Sciences
of Emory University in partial fulfillment
of the requirements of the degree of
Bachelor of Science with Honors

Quantitative Theory and Methods

2024

Abstract

On the Fairness of Rent Division Among Roommates
By Haoyang Cui

In this study, we aim to find a method for assigning rooms and dividing rent among roommates who wish to rent a house together. We assume there are n rooms to be allocated to the same number of roommates with quasi-linear utility functions. Our primary focus is on identifying different procedural algorithms that can generate an allocation meeting specific fairness criteria, including envy-freeness, equitability, individual rationality, and efficiency, among others. We demonstrated the incompatibility between envy-freeness and equitability in certain scenarios by proving a necessary and sufficient condition for the existence of an equitable and envy-free allocation. We also analyzed some main trade-offs in a rent division problem and the lack of incentive compatibility in our model. Besides deriving conclusions based on the valuation matrix, we developed a graph representation of the model that visualizes the envy network and guarantees the functionality of graph-based algorithms. By referencing previous studies in the field of fair division, we designed a procedure that generates an allocation that is individually rational, utilitarian, envy-free, and whenever possible, equitable.

On the Fairness of Rent Division Among Roommates

By

Haoyang Cui

Dr. John Patty
Adviser

A thesis submitted to the Faculty of Emory College of Arts and Sciences
of Emory University in partial fulfillment
of the requirements of the degree of
Bachelor of Science with Honors

Quantitative Theory and Methods

2024

Acknowledgments

My thanks to Dr. Patty, my thesis advisor, who has provided me with indispensable guidance and help in completing this thesis. Also, thanks to Dr. Newman and Dr. Xiong for supporting me and being members of my thesis committee.

In addition, thanks to all my friends who attended my thesis defense to support me in person: Chang (Christina) Xu, Jinghan (Summer) Sun, Baixiao (Shawn) Chen, Yaxuan (Lily) Liu, Jiahui (Helen) Jin, Wenye Song, Shuqian (Emily) Ni, Lei (Tobby) Yang, Haoyuan (Bobby) Lin, and Zheyu (Marco) Ren.

Special thanks to Marco, who has been offering me emotional support throughout the past three semesters.

Table of Contents

List of Tables

List of Figures

1. Introduction

Consider a group of individuals who wish to rent a house together and become roommates. They need to assign each person a room and divide the rent in a way that everyone agrees to be fair. In this paper, we study the approaches to fairly assign the rooms and divide the rent among these roommates. In our case, an allocation is defined as a one-to-one matching between individuals and available rooms with an assigned price for each room under the assumption that there are the same number of rooms as roommates. We are concerned about the following criteria for fairness for an allocation:

(i) Individual Rationality: An allocation is individually rational if all individuals agree that the rooms they are assigned to are worth no less than the prices they pay.

(ii) Equitability: An allocation is equitable if all individuals have the same utility.

(iii) Envy-freeness: An allocation is envy-free if no individual wishes to exchange room with anybody else under the assigned prices.

(iv) Utility Maximization: An allocation is utilitarian if the utility sum of all roommates is maximized.

We made several key assumptions concerning individuals' utility and their preferences over the rooms. We assume the individuals have cardinal valuations, which means they can provide monetary bids to signal how much they value the rooms. We also assume that they have quasi-linear utility functions, meaning an individual who values a room as q and pays a price of p would have a utility of $q - p$. Finally, we add a condition that for any individual, the sum of her valuations of all rooms must exceed the cost of the entire house. These three key assumptions guarantee the existence of an allocation that is individually rational, utilitarian, and envy-free or equitable.

Past studies in the field of fair allocation mainly focus on achieving envy-free results because of the inherent complexity, wide applicability, and incentive compatibility associated with envy-freeness. The concept of envy-freeness was introduced by Foley (1967) and further formalized and popularized by Brams and Tylor (1996). Based on the study of Foley (1967), Varian (1974) formulates a fair allocation as one that is both envy-free and Pareto-efficient and shows the existence of fair allocations under the key assumption that the goods to be allocated are infinitely divisible. The problems for indivisible goods were first handled by Maskin (1987) and Svensson (1983), both of whom point out that an envy-free allocation does not always exist in the cases of indivisible goods. They formulate independently two models in which there are some indivisible goods to be allocated to the same number of individuals and prove the existence of fair allocations under different assumptions: either there exists sufficient money to be allocated to some individuals as compensation or the preference relations satisfy certain constraints. Based on the results presented by Maskin (1987), Alkan et al. (1991) propose a more generalized model in which any number of people and objects are allowed and the objected and money to be allocated can be undesirable (e.g., assignments that must be completed or costs that must be shared). They assume quasi-linear utility functions of the individuals and use the duality theorem to prove the existence of fair allocations while proposing a value-Rawlsian allocation that maximizes the utility of the individual who is the worst off.

Notably, the rent division problem we consider is essentially an allocation problem for discrete goods in which there is an equal number of agents and goods and a fixed cost to be divided. An ordinal version of this problem has been directly addressed by Su (1999), who assumes that the roommates can tell which room they prefer for a given price vector but do not have the exact monetary valuations for the rooms. Su (1999) named this problem "Rental Harmony" and applied *Sperner's Lemma* to find an envy-free allocation with non-negative rents for all roommates. As previously mentioned, we assume that the roommates have cardinal instead of ordinal valuations, which is less general but enables more in-depth discussions concerning equitability and optimality.

There have been several studies that introduce algorithms for finding fair allocations of indivisible objects, assuming cardinal valuations and quasi-linear utility functions. Aragones (1995) presents a cubic-time algorithm that produces the same envy-free and value-Rawlsian allocation given by Alkan et al. (1991), but this algorithm is not procedural in that Aragones only provides a general description rather than specific procedures for implementation. Klijn (2000) and Haake et al. (2002) introduce two procedural algorithms for solving the fair allocation problem of indivisible objects. Both algorithms are polynomial-time and graph-based and they are directly applicable to rent division problems that focus on envy-freeness. On the other hand, Abdulkadiroğlu et al. (2004) developed a market-like auction mechanism for solving the room assignment and rent division problem, which is polynomial-time and ensures non-negative rents whenever possible conditional on envy-freeness. Additionally, Gal et al. (2016) propose a linear programming (LP) framework that achieves envy-freeness in the rent division problem while optimizing certain criteria of social justice (e.g., maximin solution). Their framework has been utilized by *Spliddit*, an online platform that offers free solutions for fair division, since April 2015.

Our procedure for fair rent division is primarily based upon the study of Haake et al. (2002) because the algorithm they present is the most manageable and intuitive among all that guarantee envy-freeness. Compared to the algorithm of Haake et al. (2002), our procedure makes the following improvements:

(i) The procedure incorporates the Hungarian method in identifying a utilitarian (i.e., max-utility) allocation efficiently.

(ii) The procedure generates an equitable allocation whenever possible conditional on envy-freeness.

To begin with, we define a rent division problem and introduce our rent division model. The introduction will include definitions of key concepts (e.g., an assignment, a price vector, and an allocation) and specifications of model assumptions. We also provide mathematical definitions of several properties related to fair allocations, including individual rationality, envy-freeness, equitability, efficiency, and utility maximization.

To continue, we introduce a procedure for finding an equitable allocation for an arbitrary number of roommates, followed by an extra step that guarantees the eventual allocation to be efficient and utilitarian using the Hungarian method.

The core section of this paper focuses on envy-freeness. We start this section by offering procedures for finding an envy-free allocation for the $n = 2$ and $n = 3$ cases, such as *the Selfridge-Conway procedure*. We also demonstrate the incompatibility between equitability and envy-freeness for $n \geq 3$ by proving a necessary and sufficient condition for the existence of an envy-free and equitable allocation. We continue to propose a graph representation of the model that visualizes the envy network for a given allocation with a complete weighted directed graph, followed by some explanations concerning the properties of the graph. We then introduce the graph-based algorithm proposed by Haake et al. (2002) for finding an envy-free room allocation for an arbitrary number of roommates, followed by an extension of their procedure, which includes extra steps to guarantee equitability whenever possible and incorporate the Hungarian algorithm in identifying a utilitarian assignment.

In the last section, we discuss truthfulness and non-negativity. We start by looking at the model from the game theory perspective and analyze why the roommates might intend to conceal their true utilities. We also analyze a special example in which our procedure generates an allocation where some roommates must pay negative rent to ensure envy-freeness.

2. Model and Fairness Criteria

2.1 Model Specification

Consider a finite set of individuals $I = \{1, 2, 3, \ldots, n\}$ renting a house with a finite set of rooms $R = \{1, 2, 3, \ldots, n\}$ together. Following Abdulkadiroğlu et al. (2004), we define $V = [v_{ir_i}]_{i \in I, r_i \in R}$ as a **valuation matrix** in which $v_{ir_i} \in \mathbb{R}_+$ represents how much individual i values room r_i. Following Aragones (1995), we define an **assignment** as a bijection $\sigma: I \to R$ that matches each individual $i \in I$ with a specific room $r_i \in R$. Following Gal et al. (2016), we represent the rent division with a **price vector** $P \in \mathbb{R}^n$ such that $\sum_{r_i \in R} p_{r_i} = C$, where $p_{r_i} \in \mathbb{R}$ is the price assigned to room r_i and $C \in \mathbb{R}_+$ is the total rent of the house. Following Klijn (2000) and Gal et al. (2016), we make two key model assumptions:

Assumption 1. **(**Gal et al., 2016) *Every individual agrees that the sum of the values of all rooms is no less than the rent of the house. Thereby for any* $i \in I$, $\sum_{r_i \in R} v_{ir_i} \geq C$.

Assumption 2. **(**Klijn, 2000) *The utility of any* $i \in I$ *assigned to any room* $r_i \in R$ *is denoted as*

$U_i(r_i, p_{r_i}) = v_{ir_i} - p_{r_i}$, *where* $U_i: R \times \mathbb{R} \to \mathbb{R}$ *is a quasi-linear utility function.*

Assumption 1 ensures that all roommates agree the house is worth at least as much as the total rent, while Assumption 2 states that all roommates have quasi-linear utility functions. We formally define a rent division problem as follows:

Rent Division Problem: Given an ordered 4-tuple $S = (C, I, R, V)$ where $C \in \mathbb{R}_+$ is the house rent, I is the set of individuals, R the set of rooms, and V the valuation matrix, find an ordered pair $A = (\sigma, P)$ where σ is an **assignment** and P a **price vector** that fulfills certain given constraints (i.e., certain fairness criteria).

We refer to the ordered 4-tuple $S = (C, I, R, V)$ as a **roommate system** and the ordered pair $A = (\sigma, P)$ as an **allocation**. Note that besides $U_i(r_i, p_{r_i})$, an alternative representation of the utility of an individual $i \in I$ under an allocation $A = (\sigma, P)$ is $U_i(A)$.

2.2 Different Criteria for Fairness

We continue to introduce some common criteria for a fair allocation. The first is individual rationality, which states that no individual should pay more than her valuation for an assigned room under a given allocation:

Definition 1. *Given a roommate system* $S = (C, I, R, V)$, *an allocation* $A = (\sigma, P)$ *is* ***individually rational*** *if for any* $i \in I$ *with an assigned room* $r_i = \sigma(i)$, $v_{ir_i} \geq p_{r_i}$.

Note that individual rationality can be phrased equivalently as no individuals should endure a negative utility under the current allocation, because for any $i \in I$ and $r_i \in R$, the condition $v_{ir_i} \geq p_{r_i}$ is equivalent to $U_i(r_i, p_{r_i}) = v_{ir_i} - p_{r_i} \geq 0$.

Next, we define equitability, which states that all individuals should have the same utility under a given allocation:

Definition 2. *Given a roommate system* $S = (C, I, R, V)$, *an allocation* $A = (\sigma, P)$ *is* ***equitable*** *if for any* $i, j \in I$ *with* $r_i = \sigma(i)$ *and* $r_j = \sigma(j)$, $U_i(r_i, p_{r_i}) = U_j\left(r_j, p_{r_j}\right)$.

Then, we follow the definition introduced by Foley (1967) to define envy-freeness:

Definition 3. Foley (1967) *Given a roommate system* $S = (C, I, R, V)$, *an allocation* $A = (\sigma, P)$ *is* ***envy-free*** *if for any* $i, j \in I$ *with* $r_i = \sigma(i)$ *and* $r_j = \sigma(j)$, $U_i(r_i, p_{r_i}) \geq U_i\left(r_j, p_{r_j}\right)$.

Envy-freeness ensures that under a given allocation, no individual wishes to exchange room with anybody else.

After that, we define efficiency (i.e., Pareto efficiency), which states that given the current allocation, it is impossible to increase the utilities of some individuals without decreasing the utilities of others:

Definition 4. *Given a roommate system* $S = (C, I, R, V)$, *an allocation* $A = (\sigma, P)$ *is* ***efficient*** *if there exists no other allocation* $A' = (\sigma', P')$ *in which there exists some* $i \in I$ *with* $r_i = \sigma(i)$ *under* A *and* $r_i{}' = \sigma'(i)$ *under* A' *such that* $U_i(r_i{}', p_{r_i{}'}) > U_i(r_i, p_{r_i})$, *while for all* $j \in I, j \neq i$ *with* $r_j = \sigma(j)$ *under* A *and* $r_j{}' = \sigma'(j)$ *under* A', $U_j\left(r_j{}', p_{r_j{}'}\right) \geq U_j\left(r_j, p_{r_j}\right)$.

Notably, the efficiency of an allocation does not depend on the price vector. Given a room assignment, it is impossible to decrease the prices of some rooms without increasing the prices of others, as the prices of all rooms must sum to a constant that equals the house rent. Thereby, it is always impossible to increase the utilities of some roommates without decreasing the utilities of others under a given assignment, and whether an allocation is efficient depends only on the room assignment, not on the price vector.

Finally, we define utility maximization, which states that the utility sum of all individuals is maximized under the current allocation:

Definition 5. *Given a roommate system* $S = (C, I, R, V)$*, an allocation* $A = (\sigma, P)$ *is* ***utilitarian*** *if for any allocation* $A' = (\sigma', P')$*, where any* $i \in I$ *is assigned* $r_i = \sigma(i)$ *under A and* $r_i' = \sigma'(i)$ *under* A'*,* $\sum_{i\in I} U_i(r_i', p_{r_i'}) \leq \sum_{i\in I} U_i(r_i, p_{r_i})$.

We end this section by proving the following useful lemma:

Lemma 1. (Haake et al., 2002) *A utilitarian allocation must have an underlying room assignment that maximizes the sum of the valuations of all individuals.*

Proof. Let $\mathcal{A}$ represent the set of all allocations, $\mathcal{Q}$ represent the set of all assignments and $\mathcal{P}$ represent the set of all price vectors. We can verify:

$$\begin{aligned} arg\ max_{A\in\mathcal{A}} \sum_{i\in I} U_i(A) &= arg\ max_{\sigma\in\mathcal{Q},P\in\mathcal{P}} \sum_{i\in I} U_i(r_i, p_{r_i}) \\ &= arg\ max_{\sigma\in\mathcal{Q},P\in\mathcal{P}} \sum_{i\in I} (v_{ir_i} - p_{r_i}) \\ &= arg\ max_{\sigma\in\mathcal{Q}} \sum_{i\in I} v_{ir_i} - arg\ max_{P\in\mathcal{P}} \sum_{i\in I} p_{r_i} \\ &= arg\ max_{\sigma\in\mathcal{Q}} \sum_{i\in I} v_{ir_i} - arg\ max_{P\in\mathcal{P}}\ C \\ &= arg\ max_{\sigma\in\mathcal{Q}} \sum_{i\in I} v_{ir_i} \end{aligned}$$

According to Lemma 1, whether an allocation is utilitarian is independent of the price vector and depends solely on the underlying assignment. A similar argument applies to efficiency, as has been justified under Definition 4. From here on, we refer to an assignment that maximizes the valuation sum of all roommates as a **utilitarian assignment**. Additionally, as proven by Svensson (1983), an envy-free allocation necessarily has an underlying utilitarian assignment, which we will prove again in section 4.

3. Equitability and Utility Maximization

3.1 Equitable Allocation for Arbitrary n

In section 3.1, we focus on the procedure for finding an equitable allocation. Intuitively, for any given room assignment (i.e., a one-to-one matching between roommates and rooms), we can always find a unique price vector that generates an equitable allocation. Inspired by the *Adjusted Winner (AW) Procedure* developed by Brams and Tylor (1996), we present a procedure that guarantees equitability, which works as follows:

Procedure 3.1

Step 1. Given an assignment σ, calculate the sum of valuations of all roommates $\sum_{i \in I} v_{i\sigma(i)}$.

Step 2. Subtract the house rent C from the valuation sum $\sum_{i \in I} v_{i\sigma(i)}$ to obtain the utility sum, then divide the sum by n to obtain the average utility $\frac{\sum_{i \in I} v_{i\sigma(i)} - C}{n}$.

Step 3. For any individual $j \in I$, ask her to pay her valuation of the assigned room minus the average utility, which is $v_{j\sigma(j)} - \frac{\sum_{i \in I} v_{i\sigma(i)} - C}{n}$.

The procedure above guarantees equitability because it divides total utility equally among all roommates so that they have the same utility of $\frac{\sum_{i \in I} v_{i\sigma(i)} - C}{n}$. Whether such a common utility is positive depends on the room assignment given. More specifically, the utility $\frac{\sum_{i \in I} v_{i\sigma(i)} - C}{n}$ is positive if the sum of valuations of all roommates $\sum_{i \in I} v_{i\sigma(i)}$ is greater than the house rent C under room assignment σ. Notably, the price vector that guarantees such an equitable allocation always exists and must be unique because given any room assignment σ, we can find a unique price vector $P = (v_{\sigma^{-1}(1)1} - \frac{\sum_{i \in I} v_{i\sigma(i)} - C}{n}, v_{\sigma^{-1}(2)2} - \frac{\sum_{i \in I} v_{i\sigma(i)} - C}{n}, \ldots, v_{\sigma^{-1}(n)n} - \frac{\sum_{i \in I} v_{i\sigma(i)} - C}{n}) \in \mathbb{R}^n$ that generates the common utility of $\frac{\sum_{i \in I} v_{i\sigma(i)} - C}{n}$. Since a unique equitable allocation exists for every given room assignment, the procedure for finding an equitable allocation depends only on the price vector and not on the underlying room assignment.

3.2 Utilitarian Room Assignment for Arbitrary n

In section 3.2, we focus on finding utilitarian allocations. According to Lemma 1, whether an allocation is utilitarian depends on whether its underlying room assignment maximizes the total valuation of all roommates and is independent of the price vector. Finding a utilitarian allocation is thereby equivalent to finding a valuation-maximizing room assignment (i.e., a utilitarian assignment), and we need a method for finding a utilitarian assignment given the valuation matrix.

In fact, we can apply the Hungarian method, which solves balanced assignment problems efficiently. Since we have an $n \times n$ valuation matrix and aim to find a one-to-one matching between roommates and rooms, our utilitarian assignment problem is clearly balanced. However, since finding a utilitarian room assignment is a maximization problem while the Hungarian method solves minimization problems, we need to transform the valuation matrix by replacing every element with its difference from the largest element in the matrix. After that, we can directly apply the Hungarian method to the transformed matrix to find a utilitarian room assignment.

Details on how the Hungarian method works are omitted and left for the readers to consider if interested. Note that there might be multiple utilitarian room assignments depending on the valuation matrix given, while the Hungarian method finds only one of them.

Finally, there's an equivalence between efficiency and utility maximization as shown in the lemma that follows:

Lemma 2. *An allocation is efficient if and only if it is utilitarian.*

Proof. Note that it is impossible to increase the utilities of some individuals without decreasing the utilities of others under an allocation that is already maximizing the utility sum. Thereby the fact that a utilitarian allocation is necessarily efficient is self-evident.

To prove that an efficient allocation is necessarily utilitarian, we use proof by contrapositive and assume there exists an allocation $A = (\sigma, P)$ that is not utilitarian. We also assume that there exists a utilitarian allocation $A' = (\sigma', P')$. Recall that we have justified in section 3.1 that we can find a unique price vector that guarantees equitability for any given room assignment. We have also justified in section 2.2 that whether an allocation is utilitarian or efficient is independent of the price vector. Therefore, we can assume without losing generality that A and A' are equitable allocations for price vectors P and P' respectively, so that all roommates have equal utilities under either A or A'. We assumed that A is not utilitarian while A' is utilitarian, so the utility sum of all roommates under A' must be greater than the utility sum under A. It follows that the equal utility shared by all roommates under allocation A' must be larger than the equal utility they share under allocation A. It is therefore possible to increase the utilities of all roommates if we switch from A to A', which shows that A is not efficient. Therefore, an allocation that is not utilitarian is necessarily not efficient, and it follows that an efficient allocation must be utilitarian.

According to Lemma 2, efficiency and utility maximization are equivalent in our rent division model. We will thus only discuss utility maximization in the following sections for the simplicity of its definition and proof.

3.3 Equitable and Utilitarian Allocation for Arbitrary n

In section 3.3, we combine our findings in sections 3.1 and 3.2 to present a procedure that yields an allocation that is both equitable and utilitarian. We have demonstrated that a unique equitable allocation exists for any given room assignment, and whether an allocation is utilitarian depends only on the room assignment and not on the price vector. Therefore, we can address equitability and utility maximization independently by finding a utilitarian room assignment first, and then finding the price vector that guarantees equitability. The complete procedure contains two more initial steps compared to Procedure 3.1:

Procedure 3.3

Step 1. Transform the valuation matrix by replacing every element with its difference from the largest element in the matrix.

Step 2. Apply the Hungarian method to the transformed matrix to find a utilitarian room assignment σ.

Step 3. Under the utilitarian room assignment σ, calculate the sum of valuations of all roommates $\sum_{i\in I} v_{i\sigma(i)}$.

Step 4. Subtract the house rent C from the valuation sum $\sum_{i\in I} v_{i\sigma(i)}$ to obtain the utility sum, then divide the sum by n to obtain the average utility $\frac{\sum_{i\in I} v_{i\sigma(i)}-C}{n}$.

Step 5. For any individual j, ask her to pay her valuation of the assigned room minus the average utility, which is $v_{j\sigma(j)} - \frac{\sum_{i\in I} v_{i\sigma(i)}-C}{n}$.

Steps 1 and 2 guarantee a utilitarian room assignment, while steps 3, 4, and 5 guarantee an equitable allocation based on a utilitarian room assignment. The resulting allocation is thereby both equitable and utilitarian.

The procedure also guarantees individual rationality (all roommates have non-negative utilities), which can be proved as follows:

Proof. According to assumption 1, for any individual $j \in I$, we have $\sum_{r_j \in R} v_{jr_j} \geq C$. As a result, the sum of the entries in every row of the valuation matrix is a constant that is no less than the house rent C. Under a utilitarian room assignment σ, the valuation sum $\sum_{i \in I} v_{i\sigma(i)}$ is maximized, and it follows that $\sum_{i \in I} v_{i\sigma(i)} \geq \sum_{r_j \in R} v_{jr_j} \geq C$. Thus, we have $\frac{\sum_{i \in I} v_{i\sigma(i)} - C}{n} \geq 0$, so the common utility shared by all roommates is non-negative and individual rationality is satisfied.

Therefore, we have justified that Procedure 3.3 is individually rational, equitable, and utilitarian.

We proceed to give an example to illustrate Procedure 3.3:

Example 1. An Illustration of Procedure 3.3:

Suppose there are four roommates: Amy, Betty, Charlie, and Danny, renting a house with 4 rooms together, whose total rent is $\$1000$. The following is their valuation matrix:

Table 1. The Valuation Matrix

	Room 1	Room 2	Room 3	Room 4
Amy	$200	$400	$350	$150
Betty	$400	$250	$300	$200

Charlie	$200	$450	$250	$250
Danny	$300	$300	$200	$200

Note that in the valuation matrix, the minimum row sum is $1000, thereby assumption 1 is satisfied. Following step 1 of Procedure 3.3, we replace every element with its difference from the largest element in the valuation matrix:

Table 2. The Transformed Valuation Matrix

	Room 1	Room 2	Room 3	Room 4
Amy	$250	$50	$100	$300
Betty	$50	$200	$150	$250
Charlie	$250	$0	$200	$200
Danny	$150	$150	$250	$250

Next, following step 2, we apply the Hungarian method to the transformed valuation matrix to find a utilitarian assignment.

The Hungarian method yields the utilitarian assignment {(Amy, Room 3), (Betty, Room 1), (Charlie, Room 2), (Danny, Room 4)}.

We mark this assignment in the original valuation matrix using boxes:

Table 3. The Utilitarian Assignment

	Room 1	Room 2	Room 3	Room 4
Amy	$200	$400	[$350]	$150

Betty	$400	$250	$300	$200
Charlie	$200	$450	$250	$250
Danny	$300	$300	$200	$200

Then, following steps 3 and 4, we calculate the sum of valuations of all roommates under the utilitarian assignment $\sum_{i \in I} v_{i\sigma(i)} = \$400 + \$450 + \$350 + \$200 = \$1400$, subtract from it the house rent $C = \$1000$, then divide by $n = 4$ to obtain the common utility, which yields $\frac{\sum_{i \in I} v_{i\sigma(i)} - C}{n} = \frac{\$1400 - \$1000}{4} = \$100$.

Finally, we ask the four roommates to pay their valuations of the assigned room minus the average utility of $\$100$, which yields the price vector $P = (\$300, \$350, \$250, \$100)$ corresponding to the prices of rooms 1, 2, 3, and 4.

We have therefore found the allocation $A = (\{(\text{Betty, Room 1}), (\text{Charlie, Room 2}), (\text{Amy, Room 3}), (\text{Danny, Room 4})\}, (\$300, \$350, \$250, \$100))$ as required. This allocation is equitable and individually rational as it guarantees every roommate an equal utility of $\$100$. It is also utilitarian as it yields a maximum group utility of $\$400$.

4 Envy-freeness

4.1 Envy-free Allocations for $n = 2$ and $n = 3$.

We start section 4 by introducing the procedures for finding envy-free allocations for the toy cases where $n = 2$ and $n = 3$.

For $n = 2$, we can ensure envy-freeness through a basic *divide-and-choose* method: randomly choose one roommate to divide the house rent between the two rooms so that she is indifferent between living in either room, then let the other roommate choose which room she prefers at the given prices.

The roommate who divides the prices does not envy the other roommate because she is indifferent between living in either room at the prices she sets; the roommate who chooses the room also does not envy the other roommate because she gets to choose the room that she prefers at the set prices. The method thereby guarantees envy-freeness.

For $n = 3$, we can apply a modified version of *the Selfridge-Conway procedure* (Brams and Tylor, 1996) to find an envy-free allocation. An advantage of this procedure is that it generates an envy-free allocation without requiring the roommates to disclose their exact valuations of the rooms. The procedure works as follows:

The Selfridge-Conway Procedure

Step 1. Randomly choose a roommate (marked as A) among the three and let her divide the rent in a way that she is indifferent about living in any of the three rooms.

Step 2. Randomly choose another roommate (marked as B), let her determine the room she likes the most (marked as $\mathrm{R1}$) and the 2nd most (marked as $\mathrm{R2}$) at the given prices, then let her increase the price of $\mathrm{R1}$ until she is indifferent between choosing $\mathrm{R1}$ or $\mathrm{R2}$ (mark the room other than $\mathrm{R1}$ and $\mathrm{R2}$ as $\mathrm{R3}$).

Step 3. Let the last roommate (marked as C) choose a room at the set prices.

Step 4. Let B choose between the two rooms left at the set prices, with the limitation that if C didn't choose $\mathrm{R1}$, B now must choose it.

Step 5. Let A get the room left at the set prices.

Step 6. Calculate the total price of the three rooms, subtract the house rent from it, and mark the difference by D, then decrease the price of each room by $D/3$.

The procedure will result in an envy-free allocation since:

1. C envies nobody since she gets her most desired room at the set prices as the first person to choose.
2. B envies nobody since she will get either R1 or R2, between which she is indifferent at the set prices, and she does not prefer R3 to either R2 or room R1.
3. A envies nobody since she is indifferent between getting R2 or R3, and she will never get R1.

Note that since increasing/decreasing the prices paid by all roommates by the same amount will not affect the envy of any roommates, step 6 is presented here only to make sure the total price paid equals the house rent. The procedure thereby generates an envy-free allocation. A more rigorous proof concerning the envy-freeness guaranteed by *the Selfridge–Conway procedure* can be found in the appendix of *Fair Division: From Cake-Cutting to Dispute Resolution* by Brams and Tylor (1996).

Besides envy-freeness, the two procedures we presented for $n = 2$ and $n = 3$ both guarantee individual rationality and utility maximization. In fact, we will show in section 4.2 that in our rent division model, an envy-free allocation is necessarily individually rational and utilitarian.

4.2 The Incompatibility Between Equitability and Envy-freeness

In this section, we aim to prove the incompatibility between equitability and envy-freeness by providing a necessary and sufficient condition for the existence of an equitable and envy-free allocation. We start by proving two important lemmas:

Lemma 3. *An envy-free allocation is necessarily individually rational.*

Proof. We use proof by contradiction. Assume that given a roommate system $S = (C, I, R, V)$ with n roommates, we have an envy-free allocation $A = (\sigma, P)$ that is not individually rational. It follows that we can find some specific $i \in I$ with the assigned room $r_i = \sigma(i)$ such that $v_{ir_i} < P_{r_i}$. It follows that the utility of i is $U_i(r_i, P_{r_i}) = v_{ir_i} - P_{r_i} < 0$. Since the allocation is envy-free, for any individual $j \in I$ with the assigned room $r_j = \sigma(j)$, we must have $U_i\left(r_j, P_{r_j}\right) \leq U_i(r_i, P_{r_i}) < 0$, thereby $U_i\left(r_j, P_{r_j}\right) = v_{ir_j} - P_{r_j} < 0$ and $v_{ir_j} < P_{r_j}$. It follows then $\sum_{r_j \in R} v_{ir_j} < \sum_{r_j \in R} P_{r_j} = C$, which contradicts assumption 1 that for any $i \in I$, $\sum_{r_i \in R} v_{ir_i} \geq C$. Therefore, an envy-free allocation is necessarily individually rational.

Lemma 4. Svensson (1983) *An envy-free allocation necessarily has an underlying utilitarian assignment.*

Proof. We use proof by contrapositive and assume that given a roommate system $S = (C, I, R, V)$ with n roommates, we have a utilitarian assignment σ' and an allocation $A = (\sigma, P)$ where σ is not a utilitarian assignment. It follows then there must exist some individual $i \in I$ with assigned rooms $r_i = \sigma(i)$ under σ and $r_i' = \sigma'(i)$ under σ' such that $U_i(r_i', p_{r_i'}) > U_i(r_i, p_{r_i})$. Otherwise, we would have $\sum_{i \in I} U_i(r_i', p_{r_i'}) \leq \sum_{i \in I} U_i(r_i, p_{r_i})$ so that either σ' is not utilitarian or σ is utilitarian, which contradicts the assumption that σ' is a utilitarian assignment and σ is not a utilitarian assignment. Given that $U_i(r_i', p_{r_i'}) > U_i(r_i, p_{r_i})$ for some specific $i \in I$, roommate i must be envious under allocation A since by switching from room r_i to r_i'

under the given price vector, i can obtain a larger utility. Therefore, A is not an envy-free allocation, and it follows that an allocation with an underlying assignment that is not utilitarian must not be envy-free. We have thereby proved that an envy-free allocation necessarily has an underlying utilitarian assignment.

Having proved Lemma 3 and Lemma 4, we proceed to prove the following theorem, which gives a necessary and sufficient condition for the existence of an equitable and envy-free allocation.

Theorem 1. *Given a roommate system* $S = (I, R, V, C)$, *an equitable and envy-free allocation exists if and only if there is an assignment* σ *such that for any individuals* $i, j \in I$ *with assigned rooms* $r_i = \sigma(i)$ *and* $r_j = \sigma(j)$, $v_{ir_i} \geq v_{jr_i}$.

Proof. We start by proving the "if" statement. Assume that we have an assignment σ such that for any individuals $i, j \in I$ with the assigned rooms $r_i = \sigma(i)$ and $r_j = \sigma(j)$, $v_{ir_i} \geq v_{jr_i}$. We can apply Procedure 3.1 to assignment σ to find a price vector P such that the allocation $A = (\sigma, P)$ is a unique equitable allocation given assignment σ. Since A is equitable, we have $U_i(r_i, p_{r_i}) = U_j(r_j, p_{r_j})$ so that $v_{ir_i} - p_{r_i} = v_{jr_j} - p_{r_j}$. We also assumed $v_{ir_i} \geq v_{jr_i}$, so that $v_{ir_i} - p_{r_i} \geq v_{jr_i} - p_{r_i}$. Therefore, we have $v_{jr_j} - p_{r_j} = v_{ir_i} - p_{r_i} \geq v_{jr_i} - p_{r_i}$, and it follows that $U_j(r_j, p_{r_j}) = v_{jr_j} - p_{r_j} \geq v_{jr_i} - p_{r_i} = U_j(r_i, p_{r_i})$, which proves that j does not envy i. Since both i and j are arbitrarily chosen, all roommates must be unenvious under allocation A, so A is both equitable and envy-free. Thereby an equitable and envy-free allocation must exist under our assumption.

We proceed to prove the "only if" statement. Assume that there exists an equitable and envy-free allocation $A = (\sigma, P)$. It follows from the equitability that for any roommates $i, j \in I$ with assigned rooms $r_i = \sigma(i)$ and $r_j = \sigma(j)$, we have $U_i(r_i, p_{r_i}) = U_j(r_j, p_{r_j})$ so that $v_{ir_i} - p_{r_i} = v_{jr_j} - p_{r_j}$. It also follows from the envy-freeness that $U_j(r_j, p_{r_j}) \geq U_j(r_i, p_{r_i})$ so that $v_{jr_j} - p_{r_j} \geq v_{jr_i} - p_{r_i}$. Therefore, we have $v_{ir_i} - p_{r_i} = v_{jr_j} - p_{r_j} \geq v_{jr_i} - p_{r_i}$, so that $v_{ir_i} \geq v_{jr_i}$. We have thus proved that if there exists an equitable and envy-free allocation, then for any roommates $i, j \in I$ with assigned rooms $r_i = \sigma(i)$ and $r_j = \sigma(j)$, we must have $v_{ir_i} \geq v_{jr_i}$.

This theorem can be verbally interpreted as follows: an equitable and envy-free allocation exists if and only if there exists an assignment in which every room is assigned to the individual that values it the most. We define such an assignment as a **dominant assignment**:

Definition 6. *An assignment σ is a dominant assignment if for any individuals $i, j \in I$ with assigned rooms $r_i = \sigma(i)$ and $r_j = \sigma(j)$, $v_{ir_i} \geq v_{jr_i}$.*

Theorem 1 demonstrates the incompatibility between equitability and envy-freeness, stating that in the absence of a dominant assignment, an equitable and envy-free allocation does not exist. It is worth noting that since a dominant assignment derives an envy-free allocation according to the proof of Theorem 1, any dominant assignment must be utilitarian based on Lemma 4.

We continue to prove a corollary concerning dominant assignments:

Corollary 1. *If dominant assignments exist in a roommate system, then all utilitarian assignments are necessarily dominant assignments.*

Proof. Assume we have a dominant assignment σ and some utilitarian assignment σ'. Since σ is a dominant assignment, it follows that for any $i \in I$ with the assigned rooms $r_i = \sigma(i)$ and $r_i' = \sigma'(i)$, we have $v_{ir_i} \geq v_{ir_i'}$. Since σ' is a utilitarian assignment, it follows from Lemma 1 that $\sum_{i \in I} v_{ir_i} \leq \sum_{i \in I} v_{ir_i'}$. Note that we have $v_{ir_i} \geq v_{ir_i'}$ for any $i \in I$ and $\sum_{i \in I} v_{ir_i} \leq \sum_{i \in I} v_{ir_i'}$, therefore $v_{ir_i} = v_{ir_i'}$ for all $i \in I$, and σ' is a dominant assignment. We have therefore proved that if dominant assignments exist, then all utilitarian assignments are necessarily dominant assignments.

Note that according to Corollary 1, either dominant assignments do not exist or all utilitarian assignments are dominant assignments. We therefore have the following efficient procedure to determine whether an equitable and envy-free allocation exists and to identify if it does:

Procedure 4.2

Step 1. Given a roommate system, we apply the Hungarian method to find a utilitarian assignment σ.

Step 2. Determine whether σ is a dominant assignment by checking whether $v_{ir_i} \geq v_{jr_i}$ for all individuals $i, j \in I$ with assigned rooms $r_i = \sigma(i)$ and $r_j = \sigma(j)$.

Step 3. If σ is not a dominant assignment, then an equitable and envy-free allocation does not exist. If σ is a dominant assignment, then we apply Procedure 3.1 to σ, which is guaranteed to yield an equitable and envy-free allocation based on the justification in the proof for Theorem 1.

We end this section by providing two examples, one in which a dominant assignment exists and the other in which a dominant assignment does not exist, to demonstrate how Procedure 4.2 works:

Example 2. A case in which a dominant assignment exists:

Suppose there are four roommates: Amy, Betty, Charlie, and Danny, renting a house with 4 rooms together, whose total rent is $\$1000$. The following is their valuation matrix:

Table 4. The Valuation Matrix

	Room 1	Room 2	Room 3	Room 4
Amy	$200	$400	$350	$150
Betty	$400	$250	$300	$200
Charlie	$200	$450	$250	$250
Danny	$300	$300	$200	$300

The Hungarian method yields the following utilitarian assignment:

Table 5. The Utilitarian Assignment

	Room 1	Room 2	Room 3	Room 4
Amy	$200	$400	[$350]	$150
Betty	[$400]	$250	$300	$200
Charlie	$200	[$450]	$250	$250
Danny	$300	$300	$200	[$300]

Note that the utilitarian assignment in Table 5 is clearly a dominant assignment because every boxed element in Table 5 is the largest in its column. Therefore, an equitable and envy-free allocation exists under this dominant assignment, and we can find it by applying Procedure 3.1.

More specifically, we calculate the sum of valuations of all roommates under the utilitarian assignment $\sum_{i\in I} v_{i\sigma(i)} = \$400 + \$450 + \$350 + \$300 = \$1500$, subtract from it the house rent $C = \$1000$, then divide by 4 obtain the common utility, which yields $\frac{\sum_{i\in I} v_{i\sigma(i)} - C}{n} = \frac{\$1500 - \$1000}{4} =$

$\$125$.

Finally, we ask the four roommates to pay their valuations of the assigned room minus the average utility of $\$125$, which yields the price vector $P = (\$275, \$325, \$225, \$175)$ corresponding to the prices of rooms 1, 2, 3, and 4.

We have therefore found the allocation $A = (\{(\text{Betty, Room 1}), (\text{Charlie, Room 2}), (\text{Amy, Room 3}), (\text{Danny, Room 4})\}, (\$275, \$325, \$225, \$175))$ as required. This allocation is equitable and envy-free as it guarantees every roommate an equal utility of $\$125$ while ensuring no roommate wishes to exchange room with anybody else.

Example 3. A case in which a dominant assignment does not exist:

Suppose there are four roommates: Amy, Betty, Charlie, and Danny, renting a house with 4 rooms together, whose total rent is $\$1000$. Following is their valuation matrix:

Table 6. The Valuation Matrix

	Room 1	Room 2	Room 3	Room 4
Amy	$\$200$	$\$400$	$\$350$	$\$150$
Betty	$\$400$	$\$250$	$\$300$	$\$200$
Charlie	$\$200$	$\$450$	$\$250$	$\$250$
Danny	$\$300$	$\$300$	$\$200$	$\$200$

The Hungarian method yields the following utilitarian assignment:

Table 7. The Valuation Matrix

	Room 1	Room 2	Room 3	Room 4
Amy	$200	$400	$350	$150
Betty	$400	$250	$300	$200
Charlie	$200	$450	$250	$250
Danny	$300	$300	$200	$200

Note that in the utilitarian assignment, Danny is assigned to Room 4. However, Charlie has a larger valuation than Danny for Room 4 (i.e., $\$250 > \$200$). Therefore, a dominant assignment does not exist according to Corollary 1, and it follows from Theorem 1 that there exist no equitable and envy-free allocations.

4.3 Graph Representation of the Envy Network

This section presents a graph representation of the envy network given an allocation, which is necessary for our graph-based procedure for finding an envy-free allocation.

Alkan et al. (1991) showed the existence of envy-free allocations in a more general model compared to ours, where any number of people and objects are allowed; Aragones (1995) also showed in a similar setting that envy-free allocations of objects exist under any given utilitarian assignment of objects. Also, according to Lemma 4, an envy-free allocation must have an underlying utilitarian assignment. It follows that finding an envy-free allocation is equivalent to finding a price vector that guarantees envy-freeness under any given utilitarian room assignment.

To find such a price vector, we proceed to present a complete weighted directed graph to illustrate an envy network consisting of all roommates. Given a roommate system of n roommates with an allocation $A = (\sigma, P)$, we can construct a unique complete weighted directed graph $G = (V, E, W)$, where:

1. V is the set of n vertices where each vertex $v_i \in V$ represents a roommate $i \in I$.
2. E is the set of directed edges, where each edge $e_{ij} \in E$ represents an envy relation directed from roommate i toward roommate j (i.e., an edge e_{ij} pointing from v_i to v_j represents the relation "i envies j").
3. $W: E \to \mathbb{R}$ is a function that assigns a weight (i.e., a real number) to each edge, where each weight represents the amount of monetary compensation required to eliminate envy (i.e., a weight w_{ij} assigned to an edge e_{ij} means we need to compensate roommate i a minimum dollar amount of w_{ij} to eliminate her envy toward j).

Note that we allow negative weights to be assigned to edges. In this case, a negative weight w_{ij} assigned to an edge e_{ij} means we need to detract from roommate i a minimum dollar amount of $|w_{ij}|$ for her to become envious of roommate j. We assign a pair of edges pointing in the opposite direction between every two vertices to represent the envy relation between two roommates.

The envy network can be derived following the steps below:

Step 1. Given an allocation $A = (\sigma, P)$, for every $i \in I$ with assigned room $r_i = \sigma(i)$, calculate $U_i(r_j, P_{r_j})$ for all $j \in I$ with assigned room $r_j = \sigma(j)$.

Step 2. For each $i \in I$, calculate $w_{ij} = U_i(r_j, P_{r_j}) - U_i(r_i, P_{r_i})$ for all $j \in I$.

Step 3. Create a complete weighted directed graph, representing each individual $i \in I$ with a vertex v_i, drawing a pair of edges e_{ij} and e_{ji} pointing from v_i to v_j and from v_j to v_i, respectively, and assigning e_{ij} the weight w_{ij} and e_{ji} the weight w_{ji}.

To illustrate how to derive such an envy network given an allocation, we show the following example:

Example 4. Derivation of an Envy Network

Suppose we have a roommate system of three individuals: A, B, and C, with the total rent $C = \$600$ and the following valuation matrix.

Table 8. The Valuation Matrix

	Room 1	Room 2	Room 3
A	$200	$400	$350
B	$400	$250	$300
C	$200	$450	$250

Let's start with a random allocation $A = (\sigma, P)$, where $\sigma = \{(\text{A}, \text{Room 3}),\ (\text{B}, \text{Room 2}),$ $(\text{C}, \text{Room 1})\}$ and a price vector $P = (\$200, \$200, \$200)$:

Table 9. The Assignment

	Room 1	Room 2	Room 3
A	$200	$400	**$350**
B	$400	**$250**	$300
C	**$200**	$450	$250

Following step 1, we replace the valuation matrix with a utility matrix by subtracting from each column the price of the room corresponding to the column index:

Table 10. The Utility Matrix

	Room 1	Room 2	Room 3
A	$0	$200	$\boxed{\$150}$
B	$200	$\boxed{\$50}$	$100
C	$\boxed{\$0}$	$250	$50

Following step 2, subtract from each row the utility of the corresponding roommate (i.e., the value in this row that is boxed), and change the column indices from the rooms to the roommates that are assigned to the rooms to obtain the labeled adjacency matrix:

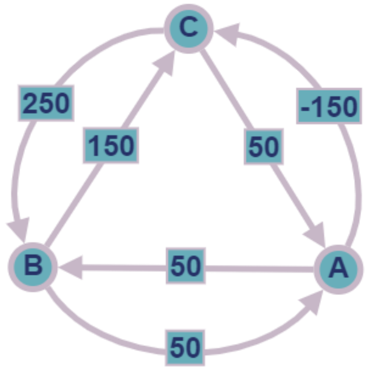
Table 11. The Labeled Adjacency Matrix

	C	B	A
A	$\$-150$	$50	$\boxed{\$0}$
B	$150	$\boxed{\$0}$	$50
C	$\boxed{\$0}$	$250	$50

Now that the value in the row of roommate i and column of roommate j of the adjacency matrix represents the minimum dollar amount needed to compensate i to eliminate her envy toward j and is thus the weight assigned to edge e_{ij}. Following step 3, we proceed to generate the envy network based on the adjacency matrix:

Figure 1. The Envy Network

	Room 1	Room 2	Room 3
Roommate	C	B	A
Price	\$200	\$200	\$200



Note that the table above the graph keeps track of the room assignment and the price of each room. A non-positive weight $w_{AC} = -150$ assigned to the edge e_{AC} suggests that A does not envy C under the current allocation. Since a positive weight suggests the presence of envy in the network, we can obtain an envy-free allocation if and only if all positive weights in the envy network are eliminated.

Having clarified how to construct an envy network given an allocation, we continue to analyze some of its properties. We start by proving the following lemma:

Lemma 5. *Given an envy network G under the allocation $A = (\sigma, P)$, for any two roommates i, $j \in I$ with the assigned rooms $r_i = \sigma(i)$ and $r_j = \sigma(j)$, moving i from r_i to r_j increases the utility of i by w_{ij}.*

Proof. Recall that the weight assigned to edge e_{ij} is calculated as $w_{ij} = U_i\left(r_j, P_{r_j}\right) - U_i(r_i, P_{r_i})$ for i, $j \in I$. Thereby, if we move roommate i from r_i to r_j, her utility will change from

$U_i(r_i, P_{r_i})$ to $U_i(r_j, P_{r_j})$ so she gains a utility of $U_i(r_j, P_{r_j}) - U_i(r_i, P_{r_i})$, which is exactly equal to the weight w_{ij}.

Lemma 5 can be used to prove the following important theorem:

Theorem 2. (Aragones, 1995) *Any allocation* $A = (\sigma, P)$ *where* σ *is a utilitarian assignment derives an envy network in which no cycle has a positive total weight.*

Proof. We use proof by contrapositive. Assume we have an allocation $A = (\sigma, P)$ and derive based on A an envy network $G = (V, E, W)$ that contains a cycle with a positive total weight. It follows that there exist some vertices $v_1, v_2, \ldots, v_k \in V$ such that $w_{12} + w_{23} + \ldots + w_{(k-1)k} + w_{k1} > 0$.

Suppose roommate i is assigned to room $r_i = \sigma(i)$ for any $i \in \{1,2,\ldots k\}$. Then, according to Lemma 5, if we move roommate 1 from r_1 to r_2, roommate 2 from r_2 to r_3, …, roommate $k-1$ from r_{k-1} to r_k, and roommate k from r_k to r_1, we can increase the utility of roommate 1 by w_{12}, the utility of roommate 2 by w_{23}, …, the utility of roommate $k-1$ by $w_{(k-1)k}$, and the utility of roommate k by w_{k1}. This will increase the total utility by an amount $w_{12} + w_{23} + \ldots + w_{(k-1)k} + w_{k1} > 0$. Therefore, σ is not a utilitarian assignment since the total utility under σ can be further increased. We have thus proved that an envy network derived from an allocation with a utilitarian assignment does not contain a cycle with a positive total weight.

Aragones (1995) managed to prove a stronger version of this theorem, but in the context of our study, Theorem 2 itself is sufficient.

We proceed to examine the effect that a change in room price has on the envy network. From here on, we forsake the assumption that the sum of the prices of all rooms must equal the house rent and define a price vector $P \in \mathbb{R}^n$ as feasible if $\sum_{r_i \in R} p_{r_i} = C$. In other words, we allow the existence of infeasible price vectors in which $\sum_{r_i \in R} p_{r_i} \neq C$.

We start by proving the following property of an envy network, which illustrates how a change in the price paid by a roommate affects the weights of the edges in an envy network:

Property 1. *Given an envy network $G = (V, E, W)$, if we increase the price paid by some roommate $i \in I$ by an amount $c \in \mathbb{R}$, then for vertex $v_i \in V$, the weights of all its out-edges will increase by c, and the weights of all its in-edges will decrease by c.*

Proof. Given an envy network $G = (V, E, W)$, suppose we increase the price paid by some roommate $i \in I$ by an amount $c \in \mathbb{R}$. Recall that the weight of any out-edge of vertex v_i is expressed as $w_{ij} = U_i(r_j, P_{r_j}) - U_i(r_i, P_{r_i})$ where $j \neq i, j \in I$ is some other roommate. Note that $w_{ij} = U_i(r_j, P_{r_j}) - U_i(r_i, P_{r_i}) = v_{ir_j} - P_{r_j} - v_{ir_i} + P_{r_i}$, so when P_{r_i} is increased by c, w_{ij} is also increased by c.

Also, recall that the weight of any in-edge of vertex v_i is expressed as $w_{ji} = U_j(r_i, P_{r_i}) - U_j(r_j, P_{r_j})$ where $j \neq i \in I$ is some other roommate. Note that $w_{ji} = U_j(r_i, P_{r_i}) - U_j(r_j, P_{r_j}) = v_{jr_i} - P_{r_i} - v_{jr_j} + P_{r_j}$, so when P_{r_i} is increased by c, w_{ji} is decreased by c.

We have thus proved Property 1.

Following Property 1, we can prove another property of an envy network:

Property 2. *Given an envy network G, the total weight of any cycle in G will stay constant regardless of the changes in prices paid by any roommates.*

Proof. Given an envy network $G = (V, E, W)$, suppose we increase the price paid by some roommate $i \in I$ by an amount $c \in \mathbb{R}$. Based on Property 1, for vertex $v_i \in V$, the weights of all its out-edges will increase by c, and the weights of all its in-edges will decrease by c.

Consider any cycle in G: if the cycle does not contain vertex v_i, then the change in the price paid by i will not affect the total weight of this cycle; if the cycle contains vertex v_i, then it must contain the same number of in-edges as the number of out-edges of v_i, and since the weights of all in-edges are decreased by c while the weights of all out-edges are increased by c, the total weight of this cycle will not change either.

We have therefore proved that the total weight of any cycle in G will not change as a result of the changes in prices paid by any roommates.

Recall that our aim is to eliminate all positive weights in the envy network to obtain an envy-free allocation. According to Property 2, changes in prices paid by any roommates do not change the total weight of any cycle. It follows that if there exists some cycle in an envy network G with a positive total weight, then we can never eliminate all positive weights by changing the prices because the weights in the cycle must sum to a positive constant. Therefore, given a cycle with a positive total weight, it is impossible to obtain an envy-free allocation through price changes.

Fortunately, according to Theorem 2, an envy network derived based on a utilitarian assignment must contain no cycle with a positive total weight. It follows that based on the existence of envy-free allocations under any utilitarian assignments, an envy network derived based on a utilitarian assignment can always generate an envy-free allocation through some changes in room prices. More specifically, given a utilitarian assignment, we can always construct an envy network based on any initial price vector and eliminate all positive weights through some price operations (i.e., increasing/decreasing the prices paid by some roommates). In section 4.4, we will introduce an algorithm proposed by Haake et al. (2002) that generates an envy-free allocation through price operations starting from any utilitarian room assignment.

4.4 Procedure for Finding an Envy-free Allocation

We start this section by introducing an adjusted version of *the compensation procedure* developed by Haake et al. (2002), which is a graph-based procedural algorithm for finding envy-free allocations. This algorithm yields an envy-free allocation for indivisible goods through monetary compensation and can be applied to our model after some minor adjustments since both of us make the same fundamental assumptions as many other related studies (e.g., quasi-linear utility function). For this algorithm to function, we need to first identify a utilitarian assignment and ask all roommates to pay their valuations. The adjusted algorithm works as follows:

The Compensation Procedure

Step 1. Given a roommate system $S = (C, I, R, V)$, find an allocation with a utilitarian assignment and a price vector where all roommates pay their valuations of the assigned rooms. Note that the initial price vector may not be feasible (i.e., the sum of the room prices can be greater than the house rent C).

Step 2. Construct an envy network based on the utilitarian allocation. There always exists at least one roommate who envies nobody (i.e., a vertex that has no out-edges with positive weights), as stated by Haake et al. (2002) in Theorem 1 of their paper.

Step 3. For each roommate (i.e., vertex), mark an out-edge with the maximum positive weight as the *max-envy edge*. Note that a roommate (i.e., vertex) does not have a max-envy edge if all her out-edges have non-positive weights.

Step 4. Compensate each roommate (i.e., vertex) with a max-envy edge pointing to a roommate who envies nobody (i.e., a vertex that has no out-edges with positive weights) by decreasing the price she pays by the weight of her max-envy edge.

Step 5. Examine whether the current allocation is envy-free by checking if all max-envy edges in the envy network have been removed. Note that this is equivalent to checking if all edges have non-positive weights. If not, go to step 3 and repeat. Haake et al. (2002) state in Theorem 2 of their paper that at most $n-1$ repetitions are needed to remove all edges with positive weights.

Step 6. After eliminating all edges with positive weights, calculate the sum of the room prices and denote it as Q, then decrease the price of each room by $\frac{Q-C}{n}$ to ensure the total amount of money paid by the roommates equals the rent C while maintaining envy-freeness. Haake et al. (2002) state in Theorem 4 of their paper that $Q-C$ is guaranteed to be positive.

This algorithm generates an envy-free allocation because it removes all the positive weights in an envy network within $n-1$ repetitions. Based on Lemma 3, the allocation generated using this algorithm is also individually rational. In their paper, Haake et al. (2002) proved some important properties of this algorithm that ensure its functionality given an initial utilitarian assignment. Details concerning the proof of these properties can be found in section 4 of their paper and are thus omitted here and left for the readers to discover if interested. We continue to provide an example to demonstrate how this algorithm functions:

Example 5. Demonstration of the Compensation Procedure

Suppose we have a roommate system of four individuals: A, B, C, and D with the total rent $C = \$\,1000$ and the following valuation matrix:

Table 12. The Valuation Matrix

	Room 1	Room 2	Room 3	Room 4
A	\$550	\$350	\$450	\$350
B	\$550	\$450	\$400	\$400
C	\$400	\$400	\$350	\$350
D	\$500	\$300	\$400	\$350

Applying the Hungarian method yields the following utilitarian assignment:

Table 13. The Utilitarian Assignment

	Room 1	Room 2	Room 3	Room 4
A	\$550	\$350	$\boxed{\$450}$	\$350
B	\$550	\$450	\$400	$\boxed{\$400}$
C	\$400	$\boxed{\$400}$	\$350	\$350
D	$\boxed{\$500}$	\$300	\$400	\$350

Let each roommate pay her valuation of the assigned room so that we have the price vector $P = (\$500, \$400, \$450, \$400)$. We replace the valuation matrix with a utility matrix by subtracting from each column the price of the room corresponding to the column index:

Table 14. The Utility Matrix

	Room 1	Room 2	Room 3	Room 4
A	$\$50$	$\$-50$	$\boxed{\$0}$	$\$-50$
B	$\$50$	$\$50$	$\$-50$	$\boxed{\$0}$
C	$\$-100$	$\boxed{\$0}$	$\$-100$	$\$-50$
D	$\boxed{\$0}$	$\$-100$	$\$-50$	$\$-50$

Then, subtract from each row the utility of the corresponding roommate (i.e., the value in this row that is boxed), and change the column indices from the rooms to the roommates that are assigned to the rooms to obtain the labeled adjacency matrix. Note that since all boxed values are 0, the labeled adjacency matrix looks exactly the same as the utility matrix:

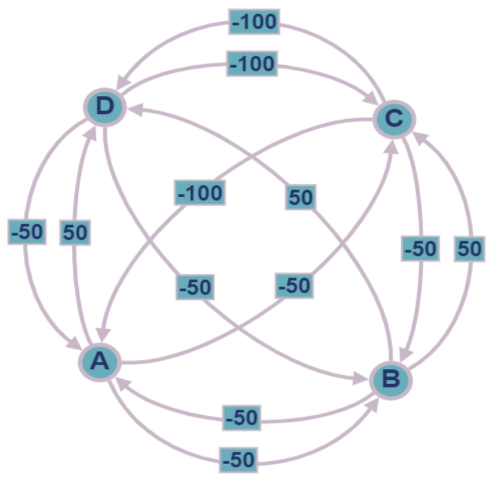
Table 15. The Labeled Adjacency Matrix

	D	C	A	B
A	$\$50$	$\$-50$	$\boxed{\$0}$	$\$-50$
B	$\$50$	$\$50$	$\$-50$	$\boxed{\$0}$
C	$\$-100$	$\boxed{\$0}$	$\$-100$	$\$-50$
D	$\boxed{\$0}$	$\$-100$	$\$-50$	$\$-50$

Based on the adjacency matrix, we can derive the following envy network:

Figure 2. The Envy Network

	Room 1	Room 2	Room 3	Room 4
Roommate	D	C	A	B
Price	$500	$400	$450	$400



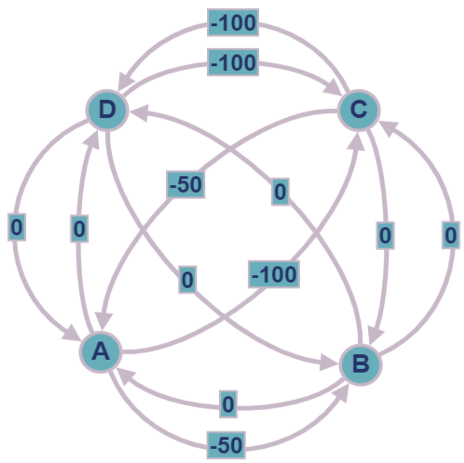
Based on the envy network, we notice that D envies nobody because all her out-edges have non-positive weights. Then we examine the max-envy edges of all roommates and check which of these max-envy edges point toward D.

All the out-edges of C have negative weights so C does not have a max-envy edge; A has a max-envy edge that points to D with a weight of 50; B has two out-edges with a weight of 50 that point to D and C respectively, so we mark the edge that points to D as the max-envy edge. It follows that we have identified two max-envy edges: e_{AD} and e_{BD}, so we compensate A and B the weights of their max-envy edges respectively. That is, we decrease the price that A pays by $50 and the price that B pays by $50.

After completing this round of compensation, we can refer to Property 1 to examine the effect that a change in prices has on the envy network and redraw the envy network.

Figure 3. The Updated Envy Network

	Room 1	Room 2	Room 3	Room 4
Roommate	D	C	A	B
Price	\$500	\$400	\$400	\$350



Note that after the first round of compensation, there are no more edges with positive weights in the envy network. Therefore, the algorithm terminates, and we have found an envy-free allocation. To ensure the feasibility of the price vector, we calculate the sum of all prices $Q = \$500 + \$400 + \$400 + \$350 = \$1650$, subtract from it the house rent $C = \$1000$, and divide the difference by $n = 4$ to obtain the average surplus $\frac{Q-C}{n} = \frac{\$1650-\$1000}{4}$ $= \$162.5$. Finally, we subtract the average surplus from the original price vector to obtain the feasible price vector $P = (\$337.5, \$237.5, \$237.5, \$187.5)$. We have therefore found an envy-free allocation: $A = (\{(\text{D, Room 1}),\ (\text{C, Room 2}),\ (\text{A, Room 3}),\ (\text{B, Room 4})\},$ $(\$337.5, \$237.5, \$237.5, \$187.5))$.

Notably, *the compensation procedure* does not explicitly state how to find a utilitarian assignment to start with since they do not assume their allocation problem to be balanced; it also does not consider equitability of the allocation. We can improve this algorithm by adding the following two extra steps:

(1). Identify a utilitarian assignment to start with using the Hungarian method.

(2). Examine whether the utilitarian assignment is a dominant assignment to check whether there exists an equitable and envy-free allocation and identify it if it exists.

Then, it follows that we can design the following procedure, which yields an allocation that is utilitarian, individually rational, envy-free, and whenever possible, equitable:

Procedure 4.4

Step 1. Given a roommate system $S = (I, R, V, C)$, apply the Hungarian method to find a utilitarian assignment σ.

Step 2. Check whether assignment σ is a dominant assignment.

Step 3. If σ is a dominant assignment, we apply Procedure 3.1 to σ, which yields an equitable and envy-free allocation; if σ is not a dominant assignment, we derive an envy network based on σ and apply *the compensation procedure*, which yields an envy-free allocation.

5 Discussion

5.1 A Discussion on Truthfulness

An assumption we have been making throughout our study is that individuals will truthfully report their valuations. In practice, however, individuals might be incentivized to misreport their valuations to obtain a larger utility. The following is an example:

Example 6. Truthfulness Analysis

Suppose we have a roommate system of three individuals: A, B, and C with the total rent $C = \$1002$ and the following valuation matrix:

Table 16. The Valuation Matrix

	Room 1	Room 2	Room 3
A	$1000	$1	$1
B	$1	$1000	$1
C	$1	$1	$1000

Following Procedure 4.4, we can find the equitable and envy-free allocation $A = (\{(\text{A}, \text{Room 1}), (\text{B}, \text{Room 2}), (\text{C}, \text{Room 3})\},\ (\$334, \$334, \$334\ ))$.

Notably, given the valuations reported by B and C, A has the incentive to report a lower valuation for Room 1. For instance, A can report the valuations $(\$502, \$250, \$250)$ for Rooms 1, 2, and 3, respectively, which results in the following valuation matrix:

Table 17. The Untruthfully Reported Valuation Matrix

	Room 1	Room 2	Room 3
A	$502	$250	$250
B	$1	$1000	$1
C	$1	$1	$1000

Following Procedure 4.4, we find the equitable and envy-free allocation $A' = (\{(\text{A}, \text{Room 1}), (\text{B}, \text{Room 2}), (\text{C}, \text{Room 3})\},\ (\$2, \$500, \$500\ ))$, in which A pays only $\$2$.

Roommate A therefore can gain a utility of $\$998$ by misreporting her valuation, while B and C both gain a utility of $\$500$ by reporting their true valuations. Similarly, given the truthful valuations reported by others, both B and C have the incentive to report lower valuations for their assigned rooms.

For this specific example, we can find a symmetric Nash Equilibrium under the dominant assignment, where each player reports $(\$334, \$334, \$334)$ as her valuations for Rooms 1, 2, and 3 respectively, which yields the following valuation matrix:

Table 18. The Reported Valuation Matrix Under Symmetric NE

	Room 1	Room 2	Room 3
A	\$334	\$334	\$334
B	\$334	\$334	\$334
C	\$334	\$334	\$334

Following Procedure 4.4, we can find the equitable and envy-free allocation $A'' = (\{(\mathrm{A}, \text{Room } 1), (\mathrm{B}, \text{Room } 2), (\mathrm{C}, \text{Room } 3)\}, (\$334, \$334, \$334))$, which happens to be the same as the allocation achieved under the truthful report of valuations.

Example 6 demonstrates how individuals may have the incentive to misreport their valuations to obtain a higher utility. In example 6, we can find a symmetric Nash Equilibrium that yields the same allocation as achieved under the truthful report of valuations. However, in most cases, such a symmetric Nash Equilibrium in which all individuals report the same valuations for all rooms does not exist.

When applying Procedure 4.4 to reality, we need to ensure that the valuations of all roommates are private information and that the procedure is only implemented once, so that no roommate can adjust her bids based on her knowledge of the valuations of others.

However, each individual can still make inferences concerning the valuations of others based on her own valuations. For instance, in example 6, knowing that her valuation for Room 1 is exceptionally high, roommate A can make a reasonable inference that others wouldn't value Room 1 as much as she does. Thus, A can still slightly lower her bids for Room 1 while ensuring that she gets assigned to Room 1 at a lower price.

5.2 A Discussion on Non-Negativity

We end our discussion by analyzing a special example in which some roommate pays a negative rent in the resulting envy-free allocation:

Example 7. An Inevitable Negative Rent

Suppose we have a roommate system of four individuals: A, B, C, and D, with the total rent $C = \$1000$ and the following valuation matrix:

Table 19. The Valuation Matrix

	Room 1	Room 2	Room 3	Room 4
A	\$1000	\$1	\$1	\$0
B	\$1	\$1000	\$1	\$0
C	\$1	\$1	\$1000	\$0
D	\$501	\$501	\$501	\$1

We can notice the existence of a dominant assignment:

Table 20. The Dominant Assignment

	Room 1	Room 2	Room 3	Room 4
A	$\boxed{\$1000}$	$1	$1	$0
B	$1	$\boxed{\$1000}$	$1	$0
C	$1	$1	$\boxed{\$1000}$	$0
D	$501	$501	$501	$\boxed{\$1}$

Then, following Procedure 4.4, we choose this dominant assignment and ask each roommate to pay her valuation minus the average surplus under the dominant assignment. The average surplus is calculated as $Q = (\$1000 + \$1000 + \$1000 + \$1 - \$1000) \div 4 = \$500.25$.

Then we compensate each roommate an amount of $\$500.25$ to obtain the price vector $P = (\$499.75, \$499.75, \$499.75, -\$499.25)$. We therefore have the allocation $A = (\{(\text{A}, \text{Room } 1), (\text{B}, \text{Room } 2), (\text{C}, \text{Room } 3), (\text{D}, \text{Room } 4)\}, (\$499.75, \$499.75, \$499.75, -\$499.25))$. This allocation must be both equitable and envy-free, as backed by Procedure 4.4. Notably, roommate D is paying negative rent, which means besides covering all the rent, roommates A, B, and C must pay an extra amount of money to compensate D. This makes sense as Room 4 is considered the most undesirable choice for all four roommates according to the valuation matrix. In such an extreme case where nobody values Room 4 more than a dollar, whoever ends up being assigned to Room 4 should be given some monetary compensation to guarantee envy-freeness and equitability.

However, in example 7, roommates A, B, and C can choose to collude with each other by excluding D from the house and distributing the rent among the three of them. Excluding D will yield the following valuation matrix and dominant assignment:

Table 21. The Valuation Matrix and Dominant Assignment (excluding D)

	Room 1	Room 2	Room 3	Room 4
A	$\boxed{\$1000}$	$1	$1	$0
B	$1	$\boxed{\$1000}$	$1	$0
C	$1	$1	$\boxed{\$1000}$	$0

Similarly, following Procedure 4.4, we can find an envy-free and equitable allocation in which each of A, B, and C pays one-third of the rent and obtains a common utility of $U = \$1000 - \$333.33 = \$666.67$. Recall in example 7, A, B, and C all obtain a common utility of $\$500.25$, which is smaller than a common utility of $\$666.67$ under collusion. Therefore, A, B, and C all have the incentive to exclude D from the house to obtain a larger utility.

Additionally, even if we give up equitability, it is still impossible to find an envy-free allocation in which everybody pays a non-negative rent in example 7. This is known as the incompatibility between envy-freeness and non-negativity. Following is a proof for such incompatibility in example 7:

Proposition 1. *In example 7, there exists no envy-free allocation with a feasible non-negative price vector.*

Proof. We use proof by contradiction. Suppose there exists an envy-free allocation A with a non-negative price vector P. Since an envy-free allocation must have an underlying utilitarian assignment, allocation A must have the assignment $\sigma = \{(\text{A}, \text{Room 1}), (\text{B}, \text{Room 2}),$

$(\text{C}, \text{Room 3}),\ (\text{D}, \text{Room 4})\}$, since σ is the only utilitarian assignment.

Then, since P is non-negative, the utility of roommate D must not exceed 1 since one's utility cannot be greater than her valuation of the assigned room under a non-negative price (i.e., $U_D(Room\ 4, p_{Room\ 4}) = \$1 - p_{Room\ 4} \leq \$1\ \forall\ p_{Room\ 4} \in \mathbb{R} \geq 0$). Also, since D values each of Rooms 1, 2, and 3 at $\$501$ and is unenvious, the price of each of these three rooms must be no less than $\$501 - \$1 = \$500$. Otherwise, D must be envious: suppose $p_r < \$500$ for some $r \in \{\text{Room 1}, \text{Room 2}, \text{ Room 3}\}$, then $U_D(r, p_r) = \$501 - p_r > 1 \geq U_D(Room\ 4, p_{Room\ 4})$, and D will envy whoever lives in room r.

Therefore, the prices of Rooms 1,2, and 3 must all be greater than or equal to $\$500$, and the sum of the prices of all 4 rooms must be greater than \$1500 so that the price vector P is not feasible, contradicting the assumption that P is a feasible price vector. Therefore, in example 7, there exists no envy-free allocation with a feasible non-negative price vector.

In fact, Brams (2008) demonstrated in a similar setting that for $n \geq 4$, envy-free allocations with non-negative prices might not exist. An issue that arises from the incompatibility between envy-freeness and non-negativity is that we cannot always find a stable allocation. In example 7, if we want to ensure envy-freeness, then we cannot eliminate the incentives of A, B, and C to collude with each other and exclude D from the house resulting from the negative price that D pays; if we want to ensure non-negative prices, then we cannot eliminate the envy of D toward some other roommates. In either case, the allocation lacks stability, which makes it difficult to sustain the current allocation because some roommates will have the incentive to negotiate for changes in either prices or room assignments.

Procedure 4.4 guarantees envy-freeness but not non-negativity. An implicit assumption made in our model is that the roommates have agreed in advance that they will live together, leaving no potential to exclude any roommates from the house. Under such an assumption, negative prices paid by some roommates no longer imply a lack of stability in the allocation, because the possibility of collusion has been eliminated. Therefore, it becomes reasonable to make monetary compensation to roommates who are assigned to the least desirable rooms, if necessary, to guarantee envy-freeness.

In reality, our model is most appropriate for situations where there exist some external factors that contribute to an agreement among the roommates to live together before they decide on the exact room assignment and prices (e.g., roommates are very close friends, relatives, or coworkers). In cases where such an agreement does not exist in advance, a more complex model that considers the roommate matching process is needed. We argue that it is never applicable to simply abandon envy-freeness in exchange for non-negativity in a rent division problem because envy-freeness, as a core measure of fairness, is highly correlated with the long-term stability of an allocation.

6 Conclusion

Our study focuses on identifying procedures that can generate an allocation meeting specific criteria for fairness. We start by giving mathematically rigorous definitions for fairness criteria such as individual rationality, envy-freeness, and equitability. We demonstrate that a utilitarian allocation implies a utilitarian assignment and that an efficient allocation is equivalent to a utilitarian allocation. We proceed to propose a procedure that generates an equitable, utilitarian, and individually rational allocation based on the Hungarian method and justify that the price vector that guarantees equitability is unique.

Then, we present modified versions of *the divide-and-choose method* and *the Selfridge-Conway procedure*, which can generate an envy-free allocation for $n = 2$ and $n = 3$ respectively. We continue to demonstrate the incompatibility between equitability and envy-freeness for $n \geq 3$ by proving Theorem 1, which provides a necessary and sufficient condition for the existence of an envy-free and equitable allocation. Based on Theorem 1, we define a dominant assignment, prove that either all utilitarian assignments are dominant assignments or none of them is a dominant assignment, and derive an efficient procedure that determines whether an equitable and envy-free allocation exists and identifies it if it does. Apart from Theorem 1, we also prove that an envy-free allocation is necessarily individually rational and that an envy-free allocation necessarily has an underlying utilitarian assignment.

To guarantee the functionality of graph-based algorithms in generating an envy-free allocation, we develop a graph representation of the envy network. More specifically, given any allocation, we can derive an envy network consisting of all roommates represented as a complete weighted directed graph. We prove that if the given allocation is utilitarian, then the envy network contains no cycles with a positive total weight. We also illustrated two important properties of the envy network: 1. Increasing the price paid by a roommate by a specific amount will increase the weights of all out-edges and decrease the weights of all in-edges of the corresponding vertex by the same amount. 2. The total weight of any cycle will stay constant regardless of the changes in prices paid by any roommates.

Based on the envy network, we introduce an adjusted version of *the compensation procedure* developed by Haake et al. (2002) and demonstrate how it can be directly applied to generate an envy-free allocation for arbitrary n. By including two extra steps, we combine Procedure 3.3 with *the compensation procedure* into Procedure 4.4, which yields two improvements: 1. The procedure incorporates the Hungarian method in identifying a utilitarian assignment. 2. The procedure generates an equitable allocation whenever possible conditional on envy-freeness.

Finally, we discuss truthfulness and non-negativity. We demonstrate that roommates have the incentive to misreport their valuations for the rooms to obtain a larger utility, which shows that our model cannot guarantee roommates' truthfulness in reporting true valuations. Additionally, we demonstrate the incompatibility between non-negativity and envy-freeness, which shows that our procedure will inevitably assign negative rents to some roommates under specific circumstances. Intuitively, under the existence of such negative rents, some roommates will have the incentive to collude with one another and exclude the roommates who are paying negative rents from the house to obtain a larger utility. However, if we accept the implicit assumption in our model that there exist some external factors that contribute to an agreement among some roommates to live together before they decide on the exact room assignment and prices, then it becomes reasonable to make monetary compensation to roommates who are assigned to the least desirable rooms, if necessary, to guarantee envy-freeness.

References

Alkan, A., Demange, G., & Gale, D. (1991). Fair allocation of indivisible goods and criteria of Justice. *Econometrica*, *59*(4), 1023–1039. https://doi.org/10.2307/2938172

Abdulkadiroğlu, A., Sönmez, T., & Unver, M. Ü. (2004). Room assignment-rent division: A market approach. *Social Choice and Welfare*, *22*(3), 515–538. https://doi.org/10.1007/ s00355-003-0231-0

Aragones, E. (1995). A derivation of the money Rawlsian Solution. *Social Choice and Welfare*, *12*(3), 267–276. https://doi.org/10.1007/bf00179981

Brams, S. J., & Taylor, A. D. (1996). *Fair division: From cake-cutting to dispute resolution*. Cambridge University Press.

Brams, S. J. (2008). *Mathematics and democracy: Designing better voting and fair-division procedures*. Princeton University Press.

Foley, D. (1967). Resource Allocation and the Public Sector. *Yale Economic Essays*, 7, 45-98.

Gal, Y. (Kobi), Mash, M., Procaccia, A. D., & Zick, Y. (2016). Which is the fairest (rent division) of them all? *Proceedings of the 2016 ACM Conference on Economics and Computation*, *7*, 67–84. https://doi.org/10.1145/2940716.2940724

Haake, C.-J., Raith, M. G., & Su, F. E. (2002). Bidding for envy-freeness: A procedural approach to N-player fair-division problems. *Social Choice and Welfare*, *19*(4), 723–749. https://doi.org/10.1007/s003550100149

Klijn, F. (2000). An algorithm for envy-free allocations in an economy with indivisible objects and money. *Social Choice and Welfare*, *17*(2), 201–215. https://doi.org/10.1007/ s003550050015

Maskin, E. S. (1987). *On the fair allocation of indivisible goods*. Harvard Institute of Economic Research, Harvard University.

Su, F. E. (1999). Rental Harmony: Sperner's Lemma in fair division. *The American Mathematical Monthly*, *106*(10), 930–942. https://doi.org/10.2307/2589747

Svensson, L.-G. (1983). Large indivisibles: An analysis with respect to price equilibrium and fairness. *Econometrica*, *51*(4), 939–954. https://doi.org/10.2307/1912044

Varian, H. R. (1974). *Equity, envy, and efficiency*. Inst. of Technology.