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Audrey Lynne Malagon Date

Killing Forms of Lie Algebras

By

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Doctor of Philosophy

Mathematics

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An abstract of
A dissertation submitted to the Faculty of the Graduate School of Emory University
in partial fulfillment of the requirements of the degree of
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Abstract

Killing Forms of Lie Algebras
By Audrey Lynne Malagon

One approach to the problem of classifying Lie Algebras is to find invariants. One such invariant is the Killing form. In this dissertation, I give a formula for computing the Killing form of any semisimple isotropic Lie algebra defined over an arbitrary field of characteristic zero, based on the Killing form of a subalgebra containing its anisotropic kernel. I then explicitly compute the Killing form for several Lie algebras of exceptional type and give a general formula for the Killing form of all Lie algebras of inner type E_6, including the anisotropic ones.

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Chapter 1

Introduction

Lie algebras have been of interest since Sophus Lie began studying them in the mid-1800s. In attempting to classify Lie algebras, several invariants have been studied. One invariant is their root system, described in detail for algebraic groups in [28]. Another such invariant is their Killing form, a symmetric bilinear form given by the trace of the adjoint representation. Today Lie algebras are classified into classical and exceptional types. While much is known about the classical types, several properties of the exceptional type algebras have remained open problems. Nathan Jacobson computes the Killing form for certain exceptional Lie algebras in [18]. Jean-Pierre Serre has also given a formulas for Lie algebras of type F_4 and G_2 [10, p.67]. While Jacobson's methods cover the exceptional Lie algebras over $\mathbb{R}$, his methods are quite complicated and omit many cases over arbitrary fields. By studying Lie algebras via their root systems, we have a more streamlined method for computing Killing forms which allows us to find Killing forms for many exceptional Lie algebras over arbitrary fields of characteristic zero. Computing Killing forms of these exceptional Lie algebras also involves studying central simple algebras defined by Tits in [29] and Galois cohomology and cohomological invariants, including the work of Jean-Pierre Serre and Markus Rost.

Chapter 2

Background Information I

2.1 Quadratic Forms

Quadratic forms can be defined over any field of characteristic not 2, but we will work strictly in characteristic zero. A **quadratic form** of dimension n over a field F is a homogenous polynomial of degree 2 in n variables:

$$f(X) = \sum_{i,j=1}^{n} a_{ij} X_i X_j$$

We typically write a quadratic form with symmetric coefficients $a'_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$. By doing so, we can associate to $f(X)$ a symmetric matrix of the coefficients (a'_{ij}). This matrix is called the **Gram matrix** for $f(X)$. A quadratic form $f(X)$ is nonsingular if its Gram matrix is nonsingular. Two quadratic forms are said to be equivalent if their Gram matrices are congruent. The Gram matrix also allows us to define the determinant of a quadratic form. Specifically $\det q$ is the determinant of the Gram matrix of q. We will often make use of the signed determinant, or **discriminant** of q. This is defined as

$$\operatorname{disc} q := (-1)^{\frac{n(n-1)}{2}} \det q$$

for an n-dimensional form q.

Clearly a quadratic form $f(X)$ defines a quadratic map $q : F^n \to F$ or from any n-dimensional vector space V over F. Such a vector space V will be called a **quadratic space**. The map q has the property that $q(ax) = a^2x$

for any $a \in F, x \in V$, and $q(x)$ defines a symmetric bilinear form B on $V \times V$ by

$$B(x, y) = \frac{q(x+y) - q(x) - q(y)}{2}$$

Notice that $B(x, x) = q(x)$. The Gram matrix can also be defined as the matrix $(B(x_i, x_j))$. When the Gram matrix is nonsingular, we know that $B(x, y) = 0 \text{for all} y \in V$ implies that $x = 0$, and the quadratic form is regular. From this point on we will write q to denote a quadratic form. The one dimensional quadratic form dX^2 will be written $\langle d \rangle$

One can easily define orthogonal sums of quadratic forms and quadratic spaces. If V_1, V_2 are each quadratic spaces with associated quadratic forms q_1, q_2 (and symmetric bilinear forms B_1, B_2) we define a quadratic form $q : V_1 \oplus V_2 \to F$ and symmetric bilinear form B by

$$q(v_1, v_2) = B((v_1, v_2), (v_1, v_2)) = B_1(v_1, v_1) + B_2(v_2, v_2) = q_1(v_1) + q_2(v_2)$$

We will use the notation $q = q_1 \perp q_2$ to denote the orthogonal sum of two quadratic forms and the notation mq to denote the orthogonal sum of m copies of a quadratic form q.

We say that a quadratic form q represents an element $d \in F$ if there exists a vector $v \in V$ such that $q(v) = d$. The elements represented by q are defined only up to square classes since $q(av) = a^2 d$. The Representation Criterion [21, I.2.3] states that if $q : V \to F$ represents d, then V decomposes as the orthogonal sum

$$V = \langle d \rangle \oplus V'$$

Using this criterion, we obtain a diagonalization of any quadratic form q.

Proposition 2.1.1. *([21, 1.2.4]) Any quadratic form q over F of dimension n can be written as*

$$q = \langle a_1, a_2, \cdots, a_n \rangle$$

for $a_i \in F/F^2$.

From this point, we will use the notation $\langle a_1, \cdots, a_n\rangle$ to denote a quadratic form. Notice that for a diagonal form the determinant, which is determined only up to squares, is

$$\det q = a_1 a_2 \cdots a_n$$

The orthogonal sum of two diagonal forms is again a diagonal form:

$$\langle a_1, a_2, \cdots, a_n\rangle \perp \langle b_1, b_2, \cdots, b_n\rangle = \langle a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\rangle$$

In addition to the orthogonal sum of quadratic forms, we can also define a multiplication of forms $q_1 \otimes q_2$. For diagonal forms

$$\langle a_1, a_2, \cdots, a_n\rangle \otimes \langle b_1, b_2, \cdots, b_n\rangle$$
$$= \langle a_1 b_1, \cdots, a_n, a_2 b_1, \cdots a_2 b_n, \cdots, \cdots, a_n b_1, \cdots a_n b_n\rangle$$

Notice that the product of a one-dimensional form $\langle a\rangle$ with q simply scales each diagonal entry of q by a. We use the notation $\langle a\rangle q$ to denote the product $\langle a\rangle \otimes q$.

A quadratic form is said to be **isotropic** if there exists $v \in V$ such that $q(v) = 0$. Otherwise, q is **anisotropic**. An important two dimensional isotropic form will be the **hyperbolic plane**. This is the form $\langle 1, -1\rangle$ and is denoted $\mathcal{H}$. In fact, any isotropic form must contain a copy of $\mathcal{H}$ as an orthogonal summand [21, I.3.4]. It is a theorem of Witt [21, I.4.1] that any nonsingular quadratic form decomposes as

$$q = q_{an} \oplus m\mathcal{H}$$

where q_{an} is an anisotropic form. Witt's Cancellation theorem [21, I.4.2] tells us that two forms q_1, q_2 are equivalent (isometric) if and only if the have the same dimension and their anisotropic parts are equivalent. For this reason, we are often only concerned with the anisotropic part of a quadratic form. The **Witt Ring** $W(F)$ is the set of isometry classes of anisotropic forms.

If two quadratic forms are Witt equivalent, they differ only by hyperbolic planes. When we give the formula for Killing forms in the later sections, however, we will prove give the Killing forms up to isomorphism, not simply Witt equivalence.

One particular class of quadratic forms which we will use in future chapters is that of **Pfister forms**. A Pfister form is a product of binary forms $\langle 1, a\rangle$. We use the notation below for an n-fold Pfister form.

$$\langle\langle a_1, a_2, \cdots, a_n\rangle\rangle = \langle 1, -a_1\rangle \otimes \langle 1, -a_2\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$$

Notice that if even one of the summands above is a hyperbolic plane, the entire Pfister form is hyperbolic, so any istropic Pfister form must be hyperbolic. Another useful property of Pfister forms is that they always represent 1, so any Pfister form q can be written as $1 \perp q_0$, and we say q_0 is the **pure** part of q. In the Witt ring $W(F)$, the ideal I^nF is the ideal of all even-dimensional forms. This ideal is generated by the n-fold Pfister forms [21, X.1.2].

2.2 Central Simple Algebras and The Brauer Group

A **central simple algebra** over a field F is an algebra A which is simple, i.e. has no two-sided non-trivial ideals, and whose center is the field F. We say A is **division** if each non-zero element of A has an inverse. Wedderburn's theorem [12, 2.1.3] tells us that any central simple algebra over F is isomorphic to the matrix algebra of a uniquely determined division algebra. That is

$$A \cong M_r(D)$$

for some integer r and division algebra D which is uniquely determined up to isomorphism. If $D = F$, we say that the algebra A is split and call F a splitting field of A. It is clear that over an algebraic closure F_{alg} of F.

$$A \otimes F_{alg} \cong M_n(F_{alg})$$

so every central simple algebra over F splits over F_{alg}. The **dimension of** A is then n^2 and the **degree** of A is n.

We can define an equivalence relation on central simple algebras over F. We first note that the tensor product of two central simple algebras is again central simple [12, 2.4.4]. Two central simple algebras A, B over F are **Brauer equivalent** if

$$A \otimes_F M_n(F) \cong B \otimes_F M_{n'}(F)$$

for some n, n'. The **Brauer group** $B(F)$ of F is the set of equivalence classes of central simple algebras over F given by this relation. For a central simple algebra A write $[A]$ for its Brauer class. Then $B(F)$ is an abelian group under the tensor product whose identity element is $[M_n(F)]$. The inverse of $[A]$ in the Brauer group is A^{op}. This is the algebra A with multiplication given by $a * b = ba$ where ba is the usual multiplication in A. Furthermore, each Brauer class contains a unique division algebra.

2.3 Clifford Algebras and Merkurjev's Theorem

We are often interested in invariants of quadratic forms. One invariant that will be of particular use to us is the even Clifford invariant of a quadratic form. We first define the Clifford algebra of a quadratic form. Let q be a quadratic form on a vector space V. Let $T(V)$ be the tensor algebra of V and let $I(q)$ be the ideal generated by $\{v \otimes v - q(v) \mid v \in V\}$. The the **Clifford algebra** of V (and of q) is $C(q) = T(V)/I(q)$. The Clifford algebra of an n-dimensional quadratic form has dimension $\dim C(q) = 2^n$ and decomposes into an odd and even part.

$$C(q) = C_0(q) \oplus C_1(q)$$

via the natural $\mathbb{Z}/2\mathbb{Z}$ gradation on $T(V)$. For even dimensional forms, we have some very nice properties of the Clifford algebra. In this case, $C(V)$ is a central simple algebra over F, and there is a map

$$c : W(F) \to B(F)$$

sending $q \to [C(q)]$. The Brauer class $C(q)$ is the **Clifford invariant** of q for q of even dimension. (In the case that q has odd dimension, we define this map to take q to the even Clifford algebra $C_0(q)$ which is a central simple algebra over F.) Furthermore for $\dim q$ even, [21, V.2.5] tells us that for $C(q) \cong M_t(D)$,

$$C_0(q) \cong M_{t/2}(D) \times M_{t/2}(D)$$

Merkurjev's theorem allows us to tell when an even dimensional form is Pfister based on its Clifford invariant.

Theorem 2.3.1 (Merkurjev)**.** *[21, p.138] A form q is in I^3F if and only if* $\dim q = 2m$, *$det(q) = (-1)^m$, and $c(q) = 1$.*

Furthermore, any 8-dimensional form in I^3 must be a scalar multiple of a Pfister form [21, X.5.6]

2.4 Cohomological Invariants

One of the main results of this dissertation involves relating the Killing form, a quadratic form invariant of Lie algebras to the Rost invariant, a cohomological invariant. In this section we describe the cohomological invariants of Lie algebras that will be used in later chapters.

We say that an algebra B' defined over F is a **twisted form** of an algebra B defined over F if $B' \otimes F_{sep} \cong B \otimes F_{sep}$. There is a natural automorphism between the set of isomorphism classes of twisted forms of an algebra B and $H^1(F, \mathrm{Aut}(B))$ where $\mathrm{Aut}(B)$ denotes algebra automorphisms of A [12, 2.2.3]. This allows us to classify algebras over F which become isomorphic to B over F_{sep} using elements of the cohomology group $H^1(F, Aut(B))$. Given a cocycle $b_\sigma \in H^1(F, Aut(B))$, we define the twisted Galois action σ' on B by

$$\sigma'(b) = b_\sigma \sigma(b)$$

where $\sigma(b)$ is the usual Galois action defined on B and the multiplication on the right is the usual multiplication in B. Then the elements of $B' \otimes K$ fixed under the twisted Galois action of Γ give a twisted form of B [12, 2.3.3].

In particular we have the following classification of n-dimensional quadratic forms over F (up to isometry), where $O(q)$ is the orthogonal group of q [20, 29.28].

$$\{n\text{-dimensional quadratic forms } q \text{ over } F\} \leftrightarrow H^1(F, O(q))$$

The Brauer group gives us a nice description of $H^2(F, A)$. If $A = F^*_{sep}$, then $H^2(F, A) = \mathrm{Br}(F)$ [12, 4.4.7].

We can also define cohomological invariants of algebraic structures. We begin by describing the **Arason invariant** e_3 of a 3-fold Pfister form. Let F be our base field of characteristic not 2, and let $q = \langle\langle a_1, a_2, a_3 \rangle\rangle$ be a 3-fold Pfister form over F. The Arason invariant is a group homomorphism

$$e_3 : I^3F \to H^3(F, \mathbb{Z}/2\mathbb{Z})$$

sending

$$\langle\langle a_1, a_2, a_3 \rangle\rangle \to (a_1) \cdot (a_2) \cdot (a_3)$$

Since I^3 is generated by 3-fold Pfister forms, this completely determines the group homomorphism.

We will also make use of the Rost invariant of a Lie algebra. The Rost invariant is, strictly speaking, only defined for simply connected semisimple algebraic groups, but we will make use of a generalization of the invariant by Garibaldi and Gille [9]. The **Rost Invariant** of a simply connected semisimple group G is a map

$$R_G := H^1(*, G) \to H^3(*, \mathbb{Q}/\mathbb{Z}(2)) = H^3(*, \mu_\delta^{\otimes 2})$$

where δ depends on the group G. We will be particularly interested in the case ${}^1E_6$ where $\delta = 6$. Let G be any group of type ${}^1E_6$. Garibaldi and Gille's invariant $r(G)$ is defined as follows. First, take $\overline{G}$ to be the adjoint group associated to G and let $\widetilde{G}$ be the simply connected cover of G. By [9, Prop 5.2], there exists a unique invariant

$$r_{\overline{G}} := H^1(*, G) \to H^3(*, \mu_2^{\otimes 2})$$

such that

$$r_{\overline{G}} = \pi \circ R_{\widetilde{G}}$$

with $\pi = H^3(*, \mu_6^{\otimes_2}) \to H^3(*, \mu_2^{\otimes 2})$ the natural projection. Then

$$r(G) := r_{\overline{G}}(\theta_\eta^{-1}(0))$$

where θ_η is the twisting homomorphism taking $\overline{G}$ to the quasi-split group of type E_6. (For any seimsimple group G there is a unique inner form of the quasi-split group of that type equal to ${}_\eta G$ for a uniquely determined $\eta \in H^1(F, \overline{G})[20, 31.5, 31.6].$)

Chapter 3

Background Information II

3.1 Introduction to Lie Algebras

Lie Algebras were introduced by Sophus Lie in the mid-1800s. Formally, a Lie algebra L is a vector space over a field F with an additional multiplication called the bracket that satisfies the following identities:

1. The bracket multiplication is bilinear.

2. $[xx] = 0$

3. $[x[yz]] + [y[zx]] + [z[xy]] = 0$ (Jacobi identity)

In characteristic not 2, the bracket is anti-commutative.

Any vector space with a trivial bracket multiplication of course defines a Lie algebra, known as an abelian Lie algebra. Another simple example of a Lie algebra is the cross product of vectors in $\mathbb{R}^3$. It is well known that the cross product is anti-commutative and non-associative, and it is straightforward to check that it satisfies the Jacobi identity. We may also define a bracket multiplication on $M_n(F)$ in the following manner

$$[AB] = AB - BA$$

where AB and BA are the usual matrix multiplication. With this operation, $M_n(F)$ becomes a Lie algebra over F. The subspace of $M_n(F)$ consisting of trace zero matrices is known as the special linear Lie algebra $\mathfrak{sl}_n(F)$.

Every Lie algebra acts on itself via the **adjoint representation**, defined by the bracket operation, i.e.

$$\mathrm{ad}(x)(y) = [xy].$$

We will work exclusively with semisimple Lie algebras in this dissertation. An element in the Lie algebra is **semisimple** if the endomorphism $\mathrm{ad}(x)$ is semisimple (diagonalizable over an algebraic closure of the base field). Every semisimple Lie algebra contains a subalgebra consisting entirely of semisimple elements. Such a subalgebra is abelian and is called a **toral subalgebra**. These subalgebras act on the Lie algebra via the adjoint representation as well. In the case that all endomorphisms $\mathrm{ad}(h)$ for $h \in H$ are diagonalizable over the base field of the Lie algebra, we say H is **split** (or F**-split** to emphasize the fact that the endomorphisms are split over the base field F). A maximal toral subalgebra is known as a **Cartan subalgebra**. In the special linear Lie algebra $\mathfrak{sl}_n(F)$, the trace zero diagonal matrices form a split Cartan subalgebra. Over an algebraically closed field, any Cartan subalgebra is split, but since we are working with Lie algebras defined over arbitrary fields, we will not always have a split maximal toral subalgebra.

3.2 Root Systems and Dynkin Diagrams

We will begin by defining the roots and root spaces of a split Lie algebra. Since the Cartan subalgebra H is split, we have a basis that simultaneously diagonalizes all $\mathrm{ad}(h)$. We can therefore decompose the Lie algebra into generalized eigenspaces based on the action of H. That is, L can be written as a direct sum of subspaces

$$L_\alpha = \{x \in L \mid [hx] = \alpha(h)x \text{ for all } h \in H\}$$

where $\alpha \in H^*$. The α's can be thought of as generalized eigenvalues and the nonzero α's are called the **roots** of L. The collection of all non-zero roots is denoted Φ and the decomposition of L into L_α's is known as the **root space decomposition**.

Example 3.2.1. $\mathfrak{sl}_4(F)$ gives a nice example of the root space decomposition. The Cartan subalgebra here is diagonalizable over F. One basis is given below.

$$H = \langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \rangle$$

A typical element in H can be written as

$$h = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_2 - h_1 & 0 & 0 \\ 0 & 0 & h_3 - h_2 & 0 \\ 0 & 0 & 0 & -h_3 \end{pmatrix}$$

Then

$$\mathrm{ad}(h)(x) = [hx] = hx - xh$$

$$= \begin{pmatrix} 0 & (2h_1 - h_2)* & (h_1 + h_2 - h_3)* & (h_1 + h_3)* \\ -(2h_1 - h_2)* & 0 & (-h_1 + 2h_2 - h_3)* & (-h_1 + h_2 + h_3)* \\ -(h_1 + h_2 - h_3)* & -(-h_1 + 2h_2 - h_3)* & 0 & (-h_2 + 2h_3)* \\ -(h_1 + h_3)* & -(-h_1 + h_2 + h_3)* & -(-h_2 + 2h_3)* & 0 \end{pmatrix}$$

where * denotes the original entry in x. The root $h_1 + h_3 \in H^*$, for example, is the map that sends h to its first diagonal entry minus its fourth diagonal entry. Notice that every root above the diagonal can be written as a positive sum of

$$\alpha_1 = h \to 2h_1 - h_2$$

$$\alpha_2 = h \to -h_1 + 2h_2 - h_3$$

$$\alpha_3 = h \to -h_2 + 2h_3$$

and every root below the diagonal can be written as a negative sum of these three roots. The roots $\alpha_1, \alpha_2, \alpha_3$ are called **simple roots**. The collection of simple roots is denoted Δ. In terms of the simple roots the roots from the previous matrix are

$$\begin{pmatrix} 0 & \alpha_1 & \alpha_1+\alpha_2 & \alpha_1+\alpha_2+\alpha_3 \\ -\alpha_1 & 0 & \alpha_2 & \alpha_2+\alpha_3 \\ -(\alpha_1+\alpha_2) & -\alpha_2 & 0 & \alpha_3 \\ -(\alpha_1+\alpha_2+\alpha_3) & -(\alpha_2+\alpha_3) & -(\alpha_3) & 0 \end{pmatrix}$$

The roots above the diagonal can all be written as positive sums of the simple roots and are called **positive roots** and are denoted Φ^+. Those below the diagonal can be written as strictly negative sums of simple roots and are **negative roots**, Φ^-. The root in the upper right hand corner of the matrix, here $\alpha_1 + \alpha_2 + \alpha_3$, is known as the **highest root**.

One notices immediately the symmetry of the roots. In particular, roots occur in positive, negative pairs. If α is a root of L relative to H, then so is $-\alpha$. (But notice that no other multiple of α is a root). This is not mere coincidence for $\mathfrak{sl}_4(F)$ but is in fact true for all root systems. [15, 8.3] gives the following properties of roots.

1. Φ spans H^*
2. If $\alpha \in \Phi$, then $-\alpha \in \Phi$.

As in $\mathfrak{sl}_n(F)$, the zero root space is precisely the Cartan subalgebra H and all other root spaces are 1-dimensional.With $L_0 = H$, the root space decomposition of a Lie algebra with respect to a Cartan subalgebra H is

$$L = H \oplus_{\alpha\in\Phi^+} (L_\alpha \oplus L_{-\alpha}) \tag{3.1}$$

The roots of a Lie algebra form a finite dimensional subspace of Euclidean space known as a root system. There is a bijection between elements of H and roots which can be described as follows:

For any $\alpha \in \Phi$, let t_α be the unique element of H with the property that

$$\kappa(t_\alpha, h) = \alpha(h) \text{for all } h \in H.$$

Using this identification of roots with elements of H, we can define a symmetric bilinear form on the roots using the Killing form:

$$(\alpha, \beta) = \kappa(t_\alpha, t_\beta)$$

It will always be the case that for any $\alpha, \beta \in \Phi$, $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ [15, Theorem 8.5]. This integer is denoted $\langle \alpha, \beta \rangle$. For distinct positive roots $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 0, 1, 2, 3$ [15, 9.4]. The **Cartan matrix** of a split Lie algebra gives the relations $\langle \alpha_i, \alpha_j \rangle$ for the simple roots under a fixed ordering. By definition the Cartan matrix is the matrix $(a_{ij}) = \langle \alpha_i, \alpha_j \rangle$. Some examples are given below.

$$D_4 \quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$E_6 \quad \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$E_7 \quad \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Table 3.1: Cartan Matrices

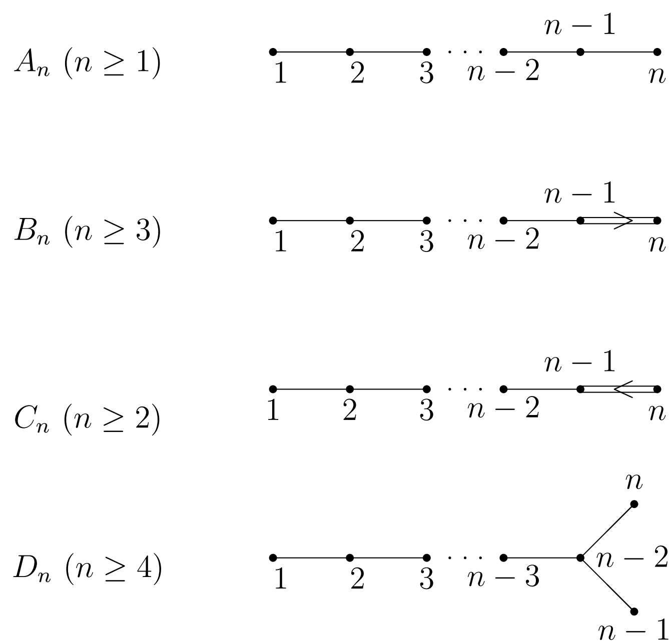


Table 3.2: Classical Dynkin diagrams

A Dynkin Diagram depicts the simple roots of a root system and their relations with respect to the $\langle , \rangle$. Each simple root is denoted by a vertex in the Dynkin diagram, and the vertices α_i, α_j are connected by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges. By [15, 9.4], any two vertices will be connected by at most 3 edges. A root system in which any adjacent vertices are connected by only one edge is called **simply laced**. In this case, all roots have the same length. When more than one root length occurs, an arrow is added to the Dynkin diagram pointing to the shorter of the two roots. An irreducible root system is one with a connected Dynkin Diagram.

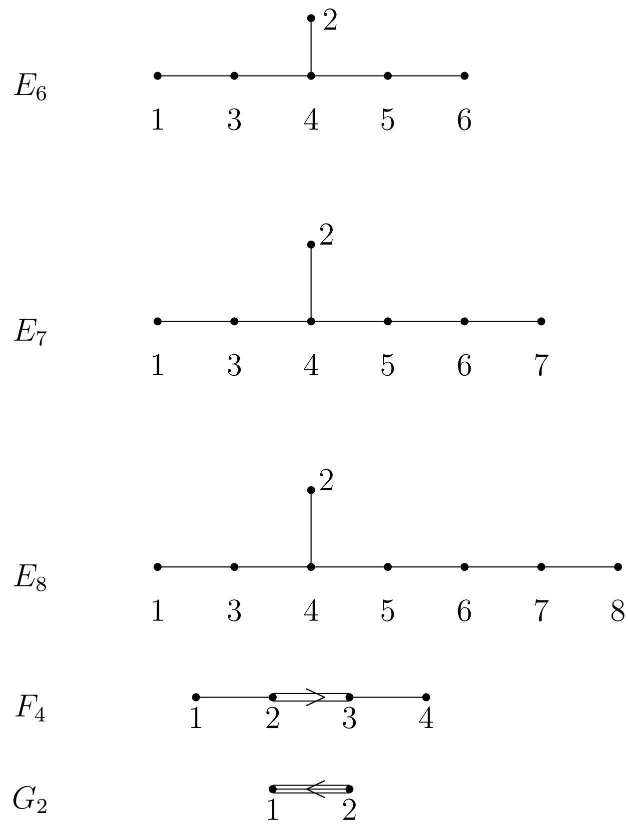


Table 3.3: Exceptional Dynkin diagrams

These allow us to completely classify Lie algebras over the complex numbers. However, over arbitrary fields, it is often the case that two distinct Lie algebras will have the same root system. It is for this reason that we are interested in other invariants, particularly the Killing form.

3.3 Killing Form

The Killing form is a quadratic form invariant that aids in the classification of Lie algebras. The Killing form κ of a Lie algebra L is the symmetric bilinear form given by the trace of the adjoint representation. That is for $x, y \in L$:

$$\kappa(x, y) = \mathrm{tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y)$$

For simplicity, the Killing form will often be denoted (x, y). The Killing form is associative with respect to the bracket operation:

$$([xy], z) = (x, [yz])$$

When L is semisimple, the Killing form is nondegenerate. In fact κ nondegenerate is a necessary and sufficient condition to have L semisimple ([15, Theorem 5.1]).

We can compute the Killing form directly for some Lie algebras.

Example 3.3.1 (Quaternion Algebras)**.** Let $Q = \frac{(a,b)}{F}$ be a quaternion algebra with basis $\{1, i, j, k\}$ and let $Q_0 = \{q \in Q \mid \mathrm{tr}(q) = 0\}$. Let $x = x_0 + x_1 i + x_2 j + x_3 k \in Q$. Then $tr(x) = x + \bar{x} = 2x_0$ so for $x \in Q_0$, $x_0 = 0$, and a basis for Q_0 is $\{i, j, k\}$. Let $x = x_1 i + x_2 j + x_3 k$ be in Q_0. Then the matrix for $\mathrm{ad}(x)$ is given by

$$\begin{bmatrix} \mathrm{ad}(x)(i) | & \mathrm{ad}(x)(j) | & \mathrm{ad}(x)(k) \end{bmatrix} = 2 \begin{bmatrix} 0 & bx_3 & -bx_2 \\ -ax_3 & 0 & ax_1 \\ -x_2 & -x_1 & 0 \end{bmatrix}$$

The form κ is given by $\kappa(x) = trace(\mathrm{ad}\, x\, \mathrm{ad}\, x)$

$$= tr(4 \begin{bmatrix} 0 & bx_3 & -bx_2 \\ -ax_3 & 0 & ax_1 \\ -x_2 & -x_1 & 0 \end{bmatrix}^2)$$

$$= tr(4 \begin{bmatrix} -x_3^2ab + x_2^2b & * & * \\ * & -x_3^2ab + x_1^2a & * \\ * & * & x_2^2b + x_1^2a \end{bmatrix})$$

$$= 4(2ax_1^2 + 2bx_2^2 - 2abx_3^2)$$

and so

$$\kappa = \langle 8a, 8b, -8ab \rangle$$

or up to squares

$$\kappa = \langle 2a, 2b - 2ab \rangle$$

Example 3.3.2 (Orthogonal Group of a Quadratic Form)**.** Let $q = \langle a_1, ..., a_n \rangle$ be a quadratic form with Gram matrix A. Let

$$\mathfrak{o}(q) = \{g \in \mathfrak{gl_n} \mid g^t A + Ag = 0\}.$$

We can calculate the Killing form of $\mathfrak{o}(q)$ as follows.

Let M_{ij} be the $n \times n$ matrix with 1 in position ij, $-\frac{a_i}{a_j}$ in position ji and 0s elsewhere. Then the collection $\{M_{ij} \mid i < j\}$ forms an orthogonal basis for $\mathfrak{o}(q)$ with respect to κ.The matrix for $\text{ad}(M_{ij})(x)$ has $2(n-2)$ non-zero columns. In addition, the product of the mth row of this matrix with the mth column is either 0 or $-\frac{a_i}{a_j}$:

To compute the matrix for $\text{ad}(M_{ij})$ we use the following table for values of $\text{ad}(M_{ij})(x)$ when x is another basis vector ($k \neq i, j$, $l \neq i, j$)

x	M_{ij}	M_{ki}	$M_{ik(k<j)}$	$M_{ik(k>j)}$	$M_{kj(k<i)}$	$M_{kj(k>i)}$	M_{jk}	M_{kl}
$\text{ad}(M_{ij})(x)$	0	$-M_{kj}$	$\frac{a_i}{a_k} M_{kj}$	$\frac{-a_i}{a_j} M_{jk}$	$\frac{a_i}{a_j} M_{ki}$	$\frac{-a_k}{a_j} M_{ik}$	M_{ik}	0

Since $\text{ad}(M_{ij})(M_{kl})$ is zero unless k or l equals i or j (but $kl \neq ij$), the matrix for $\text{ad}(M_{ij})$ then has only

$$\binom{n}{2} - \binom{n-2}{2} - 1 = 2(n-2)$$

nonzero columns.

We now compute the product of a row with column of the same index.

1. Row M_{ik} times Column M_{ik}, $k < i$:
   Only column M_{kj} gives a non-zero entry in row M_{ik}; this value is $\frac{-a_k}{a_j}$. Column M_{ik} has a nonzero entry only in row M_{kj}, this value is $\frac{a_i}{a_k}$. So in $\mathrm{ad}(M_{ij})^2$, the product of row M_{ik} and column M_{ik} is $\frac{-a_i}{a_j}$.

2. Row M_{ik} times Column M_{ik} $k > i$:
   Only column M_{jk} gives a non-zero entry in row M_{ik}; this value is 1. Column M_{ik} has a nonzero entry only in row M_{jk}, this value is $\frac{a_i}{a_j}$. So in $\mathrm{ad}(M_{ij})^2$, the product of row M_{ik} and column M_{ik} is $\frac{-a_i}{a_j}$.

3. Row M_{ki} times Column M_{ki} $k < i$:
   Only column M_{kj} gives a non-zero entry in row M_{ki}; this value is $\frac{a_i}{a_j}$. Column M_{ki} has a nonzero entry only in row M_{kj}, this value is -1. So in $\mathrm{ad}(M_{ij})^2$, the product of row M_{ki} and column M_{ki} is $\frac{-a_i}{a_j}$.

4. Row M_{jk} times Column M_{jk} $k > j$:
   Only column M_{ik} gives a non-zero entry in row M_{jk}; this value is $\frac{-a_i}{a_j}$. Column M_{jk} has a nonzero entry only in row M_{ik}, this value is 1. So in $\mathrm{ad}(M_{ij})^2$, the product of row M_{jk} and column M_{jk} is $\frac{-a_i}{a_j}$.

5. Row M_{kj} times Column M_{kj} $k < i$:
   Only column M_{ki} gives a non-zero entry in row M_{kj}; this value is -1. Column M_{kj} has a nonzero entry only in row M_{ki}, this value is $\frac{a_i}{a_j}$. So in $\mathrm{ad}(M_{ij})^2$, the product of row M_{kj} and column M_{kj} is $\frac{-a_i}{a_j}$.

6. Row M_{kj} times Column M_{kj} $k > i$:
   Only column M_{ik} gives a non-zero entry in row M_{kj}; this value is $\frac{a_i}{a_k}$. Column M_{kj} has a nonzero entry only in row M_{ik}, this value is $\frac{-a_k}{a_j}$. So in $\mathrm{ad}(M_{ij})^2$, the product of row M_{kj} and column M_{kj} is $\frac{-a_i}{a_j}$.

7. All other products of row M_{kl} with column M_{kl} are zero since these columns are all zero.

Since $\{M_{ij}\}$ form an orthogonal basis, we only need to compute $\kappa(M_{ij}) = trace((\text{ad}(M_{ij}))^2)$. We saw that the diagonal entries in $\text{ad}(M_{ij})^2$ are either 0 or $\frac{-a_i}{a_j}$ and there are precisely $2(n-2)$ non-zero entries on the diagonal. Thus

$$\kappa(M_{ij}) = trace((\text{ad}(M_ij))^2) = -2(n-2)\frac{a_i}{a_j}.$$

And therefore

$$\begin{aligned}\kappa &= \langle -2(n-2)\rangle\langle\frac{a_1}{a_2}, ..., \frac{a_i}{a_j}, ...\frac{a_{n-1}}{a_n}\rangle \\ &= \langle -2(n-2)\rangle\langle a_1a_2, ...a_ia_j, ...a_{n-1}a_n\rangle \\ &= \langle -2(n-2)\rangle\lambda^2 q.\end{aligned}$$

3.4 Chevalley Basis and Killing Form

In the previous section we computed the Killing form by finding an orthogonal basis that simplified the computation. For any split Lie algebra, there is a canonical basis arising from a decomposition of the Lie algebra with respect to its split maximal toral subalgebra. This basis is known as a Chevalley basis.

Given a basis Δ of Φ, we define

$$\check{\Delta} = \{\check{\alpha}_i = \frac{2\alpha_i}{(\alpha_i, \alpha_i)} \mid \alpha_i \in \Delta\}$$

Elements in the root system $\check{\Phi}$ with basis $\check{\Delta}$ are called **coroots**, and we have a map $\check{\Phi} \to H$. For each $\check{\alpha} \in \check{\Phi}$, let h_α be the unique element of H satisfying $\check{\alpha}(h) = \kappa(h_\alpha, h)$ for all $h \in H$. This gives a bijection between the coroot lattice $\check{\Lambda}_r$ and the $\mathbb{Z}$-form of H. The image of $\check{\Delta}$ under this map gives a basis for H:

$$\{h_{\alpha_i} \mid \alpha_i \in \Delta\}$$

Notice that $h_\alpha = \frac{2t_\alpha}{(\alpha,\alpha)}$ where t_α corresponds to the root α in under the analogous map from $\Phi \to H$ defined in section 3.2.

We construct a Chevalley basis is the following manner. First, recall that the zero root space L_0 is precisely the Cartan H and we have given a basis for H above. Recall also that each root space L_α is one dimensional. By [15, Proposition 25.2], it is possible to choose generators x_α of L_α such that $[x_\alpha x_{-\alpha}] = h_\alpha$. Then

$$\{h_{\alpha_i} \mid \alpha_i \in \Delta\} \cup \{x_\alpha \mid \alpha \in \Phi^+\} \cup \{x_{-\alpha} \mid \alpha \in \Phi^+\}$$

form a basis for L known as the **Chevalley basis**. The following properties hold for a Chevalley basis [15, 25.2](Chevalley).

1. $[x_\alpha x_{-\alpha}] = h_\alpha$
2. $[h_{\alpha_i} h_{\alpha_j}] = 0$
3. $[h_{\alpha_i} x_\alpha] = \langle \alpha, \alpha_i \rangle x_\alpha$
4. For α, $\beta \in \Phi$ such that $\alpha + \beta \in \Phi$, $[x_\alpha x_\beta] = c_{\alpha\beta} x_{\alpha+\beta}$ for a scalar $c_{\alpha\beta}$.
5. The elements $x_\alpha, x_{-\alpha}, h_\alpha$ generate a subalgebra isomorphic to $\mathfrak{sl}_2$.
6. Every h_α is in the $\mathbb{Z}$ span of the h_{α_i} ($\alpha_i \in \Delta$).

We can use the decomposition given in 3.1 together with the properties of a Chevalley basis to simplify the computation of the Killing form. Let

$$L = H \oplus \bigoplus_{\alpha \in \Phi^+} (L_\alpha \oplus L_{-\alpha})$$

be the root space decomposition of L.

Lemma 3.4.1. *If α, $\beta \in H^*$ with $\alpha + \beta \neq 0$, then L_α is perpendicular to L_β relative to the Killing form of L.*

Proof. Since $\alpha + \beta \neq 0$, we must have $h \in H$ such that $(\alpha+\beta)(h) \neq 0$. Let $v_1 \in L_\alpha$ and $v_2 \in L_\beta$. The Killing form is associative so we have

$$\begin{aligned} \alpha(h)\kappa(v_1, v_2) &= \kappa([hv_1], v_2) = -\kappa([v_1 h], v_2) \\ &= -\kappa(v_1, [hv_2]) = -\beta(h)\kappa(v_1, v_2) \\ (\alpha(h) + \beta(h))\kappa(v_1, v_2) &= 0 \end{aligned}$$

Since $(\alpha+\beta)(h) = \alpha(h) + \beta(h) \neq 0$, we must have

$$\kappa(v_1, v_2) = 0$$

□

This means that in the root space decomposition, the sums outside the parenthesis are perpendicular with respect to the Killing form. The next lemma describes the Killing form on each orthogonal summand using a Chevalley basis.

Lemma 3.4.2. *Let L be a simple Lie algebra with Chevalley basis $\{h_{\alpha_i} \mid \alpha_i \in \Delta\} \cup \{x_\alpha \mid \alpha \in \Phi^+\} \cup \{x_{-\alpha} \mid \alpha \in \Phi^+\}$. The Killing form κ is hyperbolic on $\sum_{\alpha\in\Phi^+}(L_\alpha \oplus L_{-\alpha})$ and on H, κ is given by the Weyl-invariant inner product on the dual root space. Specifically*

$$\kappa(h_\alpha, h_\beta) = 2m^*(L)(\check{\alpha}, \check{\beta})$$

where $(\check{\alpha}, \check{\beta})$ is the Weyl-invariant inner product with $(\check{\alpha}, \check{\alpha}) = 2$ for a long root α and $m^*(L)$ is the dual Coxeter number of the algebra.

Proof. Let x_α, y_α be in L_α. Since $\alpha + \alpha \neq 0$, by Lemma 3.4.1 $\kappa(x_\alpha, y_\alpha) = 0$, and L_α is totally isotropic with respect to κ. Since $\dim(L_\alpha) = \frac{1}{2}\dim(L_\alpha \oplus L_{-\alpha})$ and $L_\alpha \oplus L_{-\alpha}$ is non-degenerate, κ restricts to be hyperbolic on $L_\alpha \oplus L_{-\alpha}$ ([21, I.3.4(1)]). Let $\check{\alpha}, \check{\beta} \in \check{\Phi}$. Define a symmetric bilinear form f on $\check{\Phi}$ by

$$f(\check{\alpha}, \check{\beta}) = \kappa(h_\alpha, h_\beta)$$

This form is Weyl-invariant. (By [15, Lemma 9.2], $\langle \alpha, \beta \rangle$ is Weyl-invariant, and since the Weyl group action preserves root length, this implies the inner product on the root space (α, β) is Weyl-invariant. But $\sigma_{\check{\alpha}}(\check{\beta}) = (\sigma_\alpha \beta)\check{}$, which implies that the form f on the dual root system is also Weyl-invariant.) If Φ is an irreducible root system, then Φ is an irreducible representation of the Weyl group. If not, Φ decomposes uniquely as a direct sum of irreducible root systems ([4, Prop 6, VI.1.2]) so it suffices to work in the irreducible case. Then Schur's lemma states that there is at most one Weyl-invariant symmetric form on Φ up to scalars. Take the scalar multiple of f that gives $(\check{\alpha}, \check{\alpha}) = 2$ for a long root α so that the form may be computed from the literature (see for example [4]). By [26, p.14] for a long root α

$$\kappa(h_\alpha, h_\alpha) = 4m^*(L)$$

Therefore

$$\kappa(h_\alpha, h_\alpha) = 2m^*(L)(\check{\alpha}, \check{\alpha}) \qquad \square$$

Note: In the case of a simply laced root system, $\check{\alpha} = \alpha$ and $\langle \alpha, \beta \rangle = (\alpha, \beta)$ so the Killing form on H is given by the Cartan matrix.

Example 3.4.3. In the case of $\mathfrak{sl}_4(F)$, we have $\kappa|_H$ given by

$$= \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

By decomposing the Lie algebra into root spaces, we simplified the calculation of the Killing form to the calculation on the Cartan subalgebra. This process is quite useful, but is only applicable for split Lie algebras. In Chapter 4, we will examine a similar decomposition that simplifies the computation of the Killing form for isotropic Lie algebras.

Chapter 4

Isotropic Lie Algebras

Much of the work on classification of Lie algebras has been done over algebraically closed fields. In this section, we examine Lie algebras defined over arbitrary fields which may not have a split Cartan subalgebra. Calculating the Killing form without a split Cartan subalgebra proves to be more difficult. We will often make use of the theory of algebraic groups, which applies to these Lie algebras (see [2, I.3], [16, III.9]).

4.1 Tits Indices

When studying Lie algebras over arbitrary fields, it is preferable to use Tits indices instead of Dynkin diagrams. A Tits index includes the Dynkin diagram of the root system together with information about split toral subalgebras and the action of the Galois group of F on the Lie algebra. Tits indices were described and classified by J. Tits in [28] for algebraic groups.

To understand the Tits index, it is important to first understand the Weyl group of a root system. Let E denote the rational vector space spanned by Φ. The Weyl group is the subgroup of $GL(E)$ generated by Weyl reflections w_α for $\alpha \in \Phi$ where for $\beta \in \Phi$

$$w_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$$

Notice that w_α fixes all roots orthogonal to α under the usual inner product on the root system and sends α to $-\alpha$. In particular, the image of a basis Δ under the Weyl group will again be a basis Δ', and there is a unique element

$w : \Delta \to \Delta'$ [17, VIII.1, Theorem 2]. Tits describes an action on the root system that combines the usual Galois action on L with the action of the Weyl group on roots.

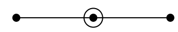
Let L be a Lie algebra defined over a field F, and let Γ be the Galois group of F_{sep}/F. There is a natural action of Γ on H^* since over F_{sep}, H is split. The image of Δ under the action of an element $\sigma \in \Gamma$ will be another basis for L, and since the Weyl group permutes basis, we have a unique element w in the Weyl group such that $w \circ \sigma(\Delta) = \Delta$. The $*$-action of Γ is the composition of the usual action with this element of the Weyl group. That is for $\sigma \in \Gamma$

$$\sigma^* := w \circ \sigma$$

The resulting action will be a graph automorphism of the Dynkin diagram. The Tits index is drawn from the Dynkin diagram by placing vertices of the Dynkin diagram which are in the same orbit close together in the Tits index. A Lie algebra is called *inner* if the *-action is trivial and *outer* if there are non-trivial orbits.

In addition to depicting the *-action of Γ on Δ, the Tits index also gives information about split toral subalgebras and the anisotropic kernel. The **(semisimple) anisotropic kernel** of a Lie algebra is roughly the part containing no split toral subalgebra. Precisely, it is the derived group of the centralizer of a maximal F-split toral subalgebra. A Tits index in which all vertices are circled indicates that the Lie algebra is split, or in the outer case, quasi-split.

Example 4.1.1. One common example of an isotropic Lie algebra is $\mathfrak{sl}_2(Q)$ for Q a quaternion algebra defined over a field F. It has Tits index below for Q division.



This is a Lie algebra of inner type with trivial $*$-action. Here the anisotropic kernel is $A = \mathfrak{sl}_1(Q) \times \mathfrak{sl}_1(Q)$, which sits inside $\mathfrak{sl}_2(Q)$ as

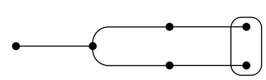
$$= \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

and an F-split toral subalgebra is

$$S = F \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The uncircled vertices correspond to the anisotropic kernel, and the circled vertex in the center tells us there is a one-dimensional F-split toral subalgebra.

Example 4.1.2. The Tits index of a Lie algebra of type ${}^2E_6$ with anisotropic kernel of type ${}^2D_4$ is given below. The *-action permutes vertices $1, 6$ and vertices $3, 5$.



This is a Lie algebra of outer type. Its anisotropic kernel has simple roots corresponding to the uncircled vertices of the Tits index. This will be an important example in the computation of the Killing form of Lie algebras of outer type E_6.

4.2 Tits Indices and F-Split Tori

It is important to note that the $*$-action of Γ on the Tits index (and hence on L) is not the same as the usual Galois action on L. In fact, elements fixed under the $*$-action may not be fixed under the usual Galois action and vice versa. We will often want to know the fixed elements under the usual Galois action since, for example, the elements of a (twisted) Cartan subalgebra fixed by Γ form an F-split toral subalgebra.

We begin by defining the weight lattice and co-weight lattice of a root system Φ. A **weight** λ is an element of the Euclidean space containing Φ such that

$$(\lambda, \check{\alpha}) \in \mathbb{Z}$$

for all $\alpha \in \Phi$. The **weight lattice** of Φ is written Λ. A weight is **dominant**if

$$(\lambda, \check{\alpha}) \geq 0$$

The root lattice Λ_r generated by Φ is contained in Λ since $(\beta, \check{\alpha}) \in \mathbb{Z}$ [15, Theorem 8.5]. We define the **fundamental dominant weights** as the dual basis to $\check{\Delta} = \{\check{\alpha}_i \mid i \in I\}$. These are the weights

$$\{\lambda_j \mid j \in I, (\lambda_j, \check{\alpha}_i) = \delta_{ij}\}$$

where δ_{ij} is the Kronecker delta, and they form a basis for the weight lattice Λ [15, 13.1]. Let $\check{\Lambda}$ be the dual lattice to Λ_r. Elements of $\check{\Lambda}$ are called **co-weights**. Notice that the coroot lattice $\check{\Lambda}_r \subset \check{\Lambda}$. Define the **fundamental dominant co-weights** as the dual basis to Δ. These are the co-weights

$$\{\check{\lambda}_j \mid j \in I, (\check{\lambda}_j, \alpha_i) = \delta_{ij}\}$$

which form a basis for the co-weight lattice. Since $\check{\Lambda}_r \subset \check{\Lambda}$, each co-root $\check{\alpha}$ can be written as an integer combination of fundamental co-weights, and each fundamental co-weight can be written as a rational combination of simple co-roots. Let $c = |\Lambda/\Lambda_r|$. Then each $c\check{\lambda}_j$ can be written as an integer combination of co-roots. Suppose $c\check{\lambda}_j = \sum_{\check{\alpha}\in\check{\Delta}} a_i\check{\alpha}_i$. Then using the map $\check{\Phi} \leftrightarrow H$ sending $\check{\alpha} \to h_\alpha$, let

$$h_{c\check{\lambda}_j} = \sum a_i h_{\alpha_i}$$

Borel and Tits [3, Corollary 6.9] describe precisely which elements of $\check{\Lambda}$ are fixed under the Galois action and therefore which elements of H are fixed under the Galois action. This allows us to construct a basis for F-split toral subalgebras from the Tits index.

Let $\{\alpha_{i_1}, ..., \alpha_{i_r}\}$ be a circled orbit in the Tits index (so that the α_{i_j} are not in the anisotropic kernel). Associate to this orbit the element $c\check{\lambda}_{i_1} + \cdots + c\check{\lambda}_{i_r} \in \check{\Lambda}_r$ where $c = |\Lambda/\Lambda_r|$. Then the subspace of the co-root lattice fixed under the usual Galois action is generated by elements of the form

$$\{c\check{\lambda}_{i_1} + \cdots + c\check{\lambda}_{i_r} \mid \ \{\alpha_{i_1}, ..., \alpha_{i_r}\} \text{ is a Tits orbit in } \Delta \setminus \Delta_0\}$$

(over $\mathbb{Q}$) [3, Corollary 6.9].

In the case that the $*$-action on the Tits index is trivial (so all orbits contain only one vertex),

$$\check{\Lambda}^\Gamma \otimes \mathbb{Q} = \langle c\check{\lambda}_i \mid \alpha_i \in \Delta \setminus \Delta_0 \rangle$$

and an F-split toral subalgebra has basis given by elements of the form

$$\{h_{c\check{\lambda}_i} \mid \text{vertex } i \text{ is circled in the Tits index}\} \tag{4.1}$$

In particular, the dimension of a maximal F-split toral subalgebra is equal to the number of circled vertices in the Tits index.

Suppose now that L is type 2. The $*$ action gives a map

$$* : \Gamma \to \mathrm{Aut}\Delta$$

$K = (F_{sep})^{\ker(*)}$ is a finite extension of F of degree $|\mathrm{im}(*)| = 2$ for which the Tits Index of $L \otimes K$ is of inner type. Letting $K = F(\sqrt{a})$ and using [25, p.279], a basis for the maximal F-split toral subalgebra in L is given by

$$\{h_{c\check{\lambda}_i} + h_{c\check{\lambda}_j}, \sqrt{a}h_{c\check{\lambda}_i} - \sqrt{a}h_{c\check{\lambda}_j} \mid \{\alpha_i, \alpha_j\} \text{ a circled Tits orbit}\} \tag{4.2}$$

In the case that all roots have the same length, weights and co-weights are equal since we adopt the bilinear form with $(\alpha, \alpha) = 2$. Formulas for the fundamental dominant weights written as sums of the simple roots can be found in [15, Sec. 13, Table 1].

In Example 4.1.2, the subspace of Λ^Γ fixed by the usual Galois action is generated by $3\lambda_1 + 3\lambda_6$ and Γ permutes $3\lambda_1$ and $3\lambda_6$. A basis for the F-split toral subalgebra is $\{h_{3\lambda_1} + h_{3\lambda_6}, \sqrt{a}h_{3\lambda_1} - \sqrt{a}h_{3\lambda_6}\}$.

4.3 Tits Algebras

Associated to each orbit in the Tits index is a central simple algebra known as a Tits Algebra. These were defined by Tits in [29] and are also described in [23]. These central simple algebra invariants give us information about the Lie algebra which will be necessary for the results on Killing forms in Chapters 7.3 and 7.3. Tits defines these algebras by giving a bijection between dominant weights fixed under the $*$-action and algebra representations of L. An algebra representation of a Lie algebra L defined over F is simply a map $\rho : L \to GL_1(A)$ for a central simple algebra A over F (see [23]).

It is well known that there exists a bijection between dominant weights of a split Lie algebra and irreducible representations of the Lie algebra ([29, Lemma 2.2]). Let

$$\beta : \Lambda_+ \to \text{irreducible representations of } L.$$

Tits extends this notion to non-split algebras by restricting β to just the elements of Λ_+ fixed under the $*$-action of Γ. In this case by [29, Theorem 3.3], we have a bijection

$$\Lambda_+^\Gamma \leftrightarrow \{\text{irreducible algebra representations of } L\}$$

Let $\lambda \in \Lambda_+^\Gamma$ be a dominant weight associated to an orbit in the Tits index, and let $\rho : G \to GL_1(A)$ be the algebra representation assigned to λ under this bijection. Then A defines the Tits algebra associated to that orbit. We denote this by $A(\lambda)$. In the case that L is split or quasi-split, we will always have $A(\lambda)$ split [29, 3.3]. In addition we have the following properties of Tits algebras.

Let $A(\lambda)$, $A(\mu)$ be the Tits algebras associated to $\lambda, \mu \in \Lambda_+^\Gamma$ and let $[A]$ denote the Brauer class. By [23, Proposition 7.4] we have

1. If λ is in the root lattice of L, then the $A(\lambda)$ is split.

2. $[A(\lambda + \mu)] = [A(\lambda)] + [A(\mu)]$

Notice that these properties allow us to define a homomorphsim

$$\alpha : (\Lambda/\Lambda_r)^\Gamma \to \mathrm{Br}(F).$$

We will define the Tits algebra only up to Brauer equivalence. So for any $\lambda \in \Lambda_+$ associated to an orbit in the Tits index, $\alpha(\lambda)$ will be the **Tits algebra** associated to λ.

In the case of an inner type Lie algebra, associated to vertex $i$is the Tits algebra $\alpha(\lambda_i)$ for the fundamental dominant weight λ_i. We will denote this algebra by $A(i)$. In the type 2 case, let B be the Tits algebra associated to the orbit containing vertices i, j and let K be the quadratic extension of F. which splits the quasi-split Lie algebra. Then B is a central simple algebra over K and

$$B \otimes_F K = (B \otimes_K K) \otimes_F K = B \otimes_K (K \otimes_F K) = B \otimes_K (K \times {}^\iota K)$$

$$B \otimes_K K \times B \otimes_K {}^\iota K = B \times {}^\iota B = A(i) \times A(j)$$

4.4 Weight Space Decomposition

For this section S is a (not necessarily maximal) F-split toral subalgebra, which sits inside a Cartan subalgebra H. It acts on the Lie algebra via the adjoint representation, and we can decompose the Lie algebra into the eigenspaces induced by this action just as we did for the action of H. When we decompose with respect to a non-maximal split toral subalgebra, we call the generalized eigenvalues **weights** and refer to this as the **weight space decomposition** of L with respect to S. A weight space L_μ is defined below.

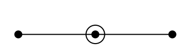
$$L_\mu := \{x \in L \mid [hx] = \mu(h)x \text{ for all } h \in S\}$$

Just as we had a highest root, we will also have a **highest weight**. Using the weight spaces, L decomposes as

$$L = L_0 \oplus \bigoplus_{\mu \in S^*, \mu \neq 0} (L_\mu \oplus L_{-\mu})$$

A weight space decomposition is similar to a root space decomposition in that the nonzero weight spaces occur in positive and negative pairs. If μ is a weight, $-\mu$ is also a weight, and furthermore L_μ and $L_{-\mu}$ have the same dimension. It is not true, however, that L_μ must be one dimensional. In addition, the zero weight space is not the same as the zero root space. It will contain the Cartan subalgebra H, but it is often larger than H alone.

Example 4.4.1. We return to the example of $\mathfrak{sl}_2(Q)$ for Q a quaternion algebra defined over a field F



Recall an F-split toral subalgebra is

$$S = F \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Decomposing with respect to this toral subalgebra in $\mathfrak{sl}_2(Q)$ gives

$$[hx] = \begin{pmatrix} 0 & 2* \\ -2* & 0 \end{pmatrix}$$

and our weights are $0, 2, -2$.

The weight space decomposition is useful for computing the Killing form because the properties of weights mirror the advantageous properties of roots described in Chapter 2.4. Specifically, we restate Lemma 3.4.1 and part of Lemma 3.4.2 in terms of weights. The proofs are analogous.

Lemma 4.4.2. *If μ, λ are weights of L such that $\mu + \lambda \neq 0$, then L_μ is perpendicular to L_λ relative to the Killing form of L.*

Lemma 4.4.3. *For any μ a weight of L, $L_\mu \oplus L_{-\mu}$ is hyperbolic with respect to κ.*

Notice that we cannot restate all of Lemma 3.4.2 in terms of the weight space decomposition because the weight space decomposition does not give us a Chevalley basis. That basis is unique to split Lie algebras. In this case, we must work harder to compute the Killing form on the zero weight space, which is larger than simply H. The next lemma describes the zero weight space.

Lemma 4.4.4. *Let S be an F-split toral subalgebra of L and let A be the derived subalgebra of $Z_L(S)$. Let H be a (not necessarily split) Cartan subalgebra of L. Then in the weight space decomposition of L with respect to S*

$$L_0 = Z_H(A) \oplus A$$

where $\oplus$ is an orthogonal sum with respect to the Killing form. Furthermore, A is semisimple and A contains the anisotropic kernel of L.

Proof. It is clear that $L_0 = Z_L(S)$. Since L is semisimple, L_0 is the centralizer of a toral subalgebra, so L_0 is reductive [16, Corollary A 26.2] and by [17, Theorem 11] decomposes as

$$L_0 = Z(L_0) \oplus [L_0 L_0]$$

By definition we have $A = [L_0 L_0]$. Let us now examine $Z(L_0)$. Let $x \in L_0$. Clearly $[xZ(L_0)] = 0$ so $x \in Z(L_0)$ if and only if $[xA] = 0$. By [17, Theorem 11] $Z(L_0)$ is a toral subalgebra and is therefore contained in a maximal toral subalgebra. Since all maximal toral subalgebras are conjugate, $Z(L_0)$ is contained in all maximal toral subalgebras, and in particular $Z(L_0)$ is contained in H. Therefore $Z(L_0)$ is precisely the elements of H which centralize A.

$$Z(L_0) = Z_H(A)$$

If S is a maximal F-split toral subalgebra, A is precisely the semisimple anisotropic kernel of L with simple roots corresponding to the non-circled vertices in the Tits index ([28]). If S is contained in a maximal F-split toral subalgebra, then A will contain the semisimple anisotropic kernel. $\square$

To summarize, any isotropic Lie algebra with split toral subalgebra S and subalgebra A containing the anisotropic kernel of L decomposes as

$$L = A \oplus Z_H(A) \oplus \bigoplus_{\mu \in S^*, \mu \neq 0} (L_\mu \oplus L_{-\mu})$$

where the sums oustide the parenthesis are orthogonal with respect to κ

This decomposition simplifies the computation of the Killing form. As in the split case, we are reduced to computing on the zero weight space only, however here we must compute the Killing form on two parts - a subalgebra A

$^1E_6$

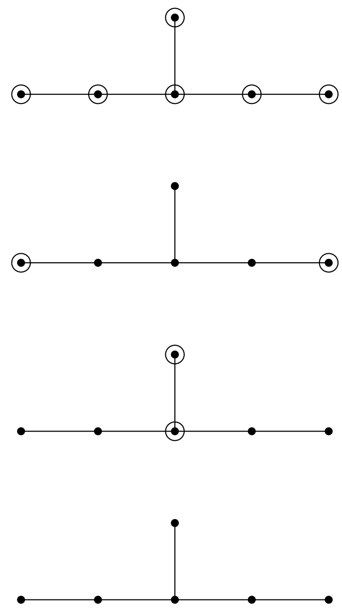


Table 4.1: $^1E_6$ Tits Indices

containing the semisimple anisotropic kernel and on a split toral subalgebra $Z_H(A)$. The goal will be to select S in such a manner that the Killing form on A is well known or easy to compute.

${}^2E_6$

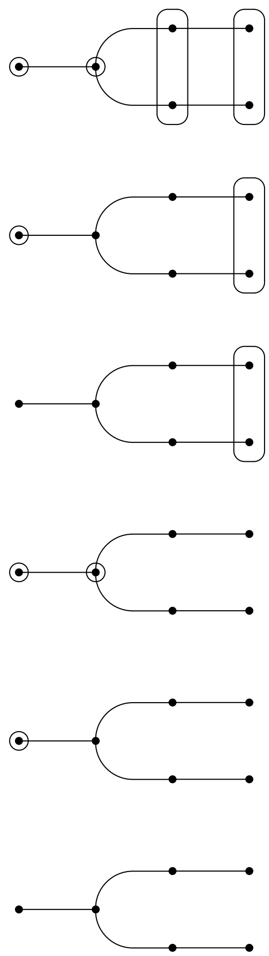


Table 4.2: ${}^2E_6$ Tits Indices

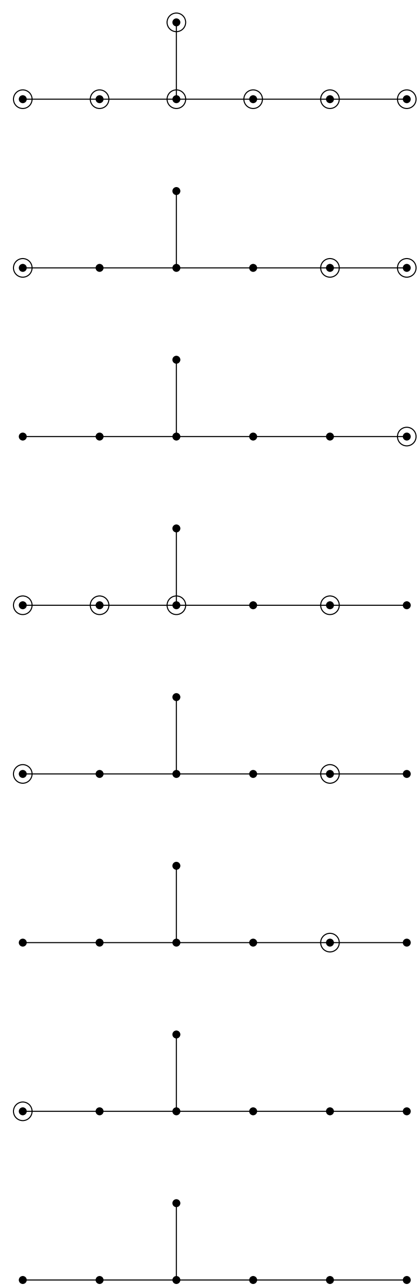


Table 4.3: E_7 Tits Indices

Chapter 5

Previous Results on Killing Forms of Lie Algebras

As we showed in Chapter 3, the Killing form of any split Lie algebra is well known thanks to the work of Killing and Cartan subalgebra. In this case, one needs only to know the dimension of the Lie algebra and its Cartan matrix, information that is readily available in the literature. In the non-split case, however, the Killing form is not as straightforward to compute especially in the case of exceptional Lie algebras. N. Jacobson has done work in the area of Killing forms of exceptional Lie algebras [18], and additionally Serre has given formulas for Lie algebras of type F_4 and G_2 [10, p.67].

We begin by examining Jacobson's results from [18, Chapter 11] for Lie algebras of type E_6. Following his notation, let $\mathcal{J} = \mathcal{H}(\mathfrak{C}_3, \gamma)$ be the set of 3×3 matrices over a Cayley algebra $\mathfrak{C}$ which are γ-hermitian for the diagonal matrix $\gamma = \mathrm{diag}[\gamma_1, \gamma_2, \gamma_3]$. Let $\mathcal{J}'$ be the set of trace 0 elements of $\mathcal{J}$ and let $R(\mathcal{J}') = \{R_a \mid a \in \mathcal{J}'\})$ the collection of right multiplication maps. Then $\mathcal{J}$ is a reduced simple Jordan algebra, and $\mathrm{Der}\mathcal{J} = \mathrm{Inder}\mathcal{J} = \mathcal{F}_4$ is a Lie algebra of type F_4. Furthermore $\mathcal{L}' = R(\mathcal{J}') \oplus \mathcal{F}_4$ is a Lie algebra of type E_6. Jacobson gives the following formula for the Killing form of such an E_6 [18, p.113]. Here $R_a \in R(\mathcal{J}')$ and $D = D_0 + [R_{e_1} R_{b_{12}}] + [R_{e_2} R_{c_{23}}] + [R_{e_3} R_{d_{31}}]$ where $D_0 \in \mathcal{D}_0 \subset \mathcal{F}_4$, the subalgebra which annihilates the diagonal idempotents. The $e_i, b_{12}, c_{23}, d_{31}$ come from the basis given by the Pierce decomposition of $\mathcal{J} = Fe_1 \oplus Fe_2 \oplus Fe_3 \oplus \mathcal{J}_{12} \oplus \mathcal{J}_{13} \oplus \mathcal{J}_{23}$ relative to $e_i = e_{ii}$ of the usual matrix basis for $\mathcal{J}$. The value $n(a)$ is the norm of the element $a \in \mathfrak{C}$. $T(,)$ is the trace form on $R(\mathcal{J}')$ and κ_{D_4} is the Killing form on $\mathcal{D}_0$ of type D_4. Then for $R_a \in R(\mathcal{J}'), D \in \mathcal{F}_4$, κ is given by

$$\kappa(R_a + D, R_a + D) = 12T(a,a) + 2\kappa_{D_4}(D_0, D_0) - \\ 6(\gamma_2^{-1}\gamma_1 n(b) + \gamma_3^{-1}\gamma_2 n(c) + \gamma_1^{-1}\gamma_3 n(d))$$

The above formula also applies to certain Lie algebra of type 2. Let $\alpha \in F$ such that α is not a square. Then the subalgebra of $\mathfrak{C} \otimes F(\sqrt{\alpha})$ of elements of the form $\sqrt{\alpha}R_a + D$ is a Lie algebra of type ${}^2E_6$ [18, p.114] and its Killing form is the same as above replacing $12T(a,a)$ by $12\alpha T(a,a)$.

Jacobson also gives a formula for the Tits Lie algebra $\mathcal{T} = \mathfrak{C} \otimes \mathcal{A}_1 \oplus \mathcal{D}$ of type E_7 where $\mathfrak{C}$ is as before, $\mathcal{A}_1$ a 3-dimensional Lie algebra of type A_1 and $\mathcal{D} = \mathrm{Inder}\mathfrak{C}$. Here $x, x' \in \mathfrak{C}, a, a' \in \mathcal{A}_1, D, D' \in \mathcal{D}$.

$$\kappa(x \otimes a + D, x \otimes a + D') = \frac{1}{2}T(x,x')\kappa_{A_1}(a,a') + \kappa_{\mathcal{D}}(D, D')$$

These formulas rely heavily on knowledge of the underlying Jordan and Cayley algebras. The formula we give for type ${}^1E_6$ is much more straightforward and can be computed from the Tits index of E_6 together with the Rost invariant. Jacobson also gives some results for certain other exceptional Lie algebras (including F_4), but the formulas are again complicated and we leave those to the reader to explore (see [18, p. 118-121]).

J.P. Serre has also given formulas for the Killing form of Lie algebras of type F_4 and G_2 over arbitrary fields of characteristic not 2 or 3. Here we let A be an octonion algebra over F and G_2 the Lie algebra of its derivations or we take A to be an Albert algebra with Lie algebra of derivations F_4. In either case, Serre has defined cohomological invariants $q_3(A)$, $q_5(A)$ in $H^3(F, \mathbb{Z}/2)$, $H^5(F, \mathbb{Z}/2)$ respectively. (See [22] for precise definitions of these invariants.) He then gives the Killing form in terms of these invariants [10, p.67].

$$\kappa_{G_2} = \langle -1, -3 \rangle (q_3 - 1)$$

$$\kappa_{F_4} = \langle -2 \rangle (q_5 - q_3) + \langle -1, -1, -1, -1 \rangle (q_3 - 1)$$

Chapter 6

Killing Forms of Isotropic Lie Algebras

In this chapter, we give a method for computing the Killing form of any semisimple isotropic Lie algebra based on the Killing form of a subalgebra containing the anisotropic kernel. The main result is given below.

6.1 Main Result

Theorem 6.1.1. *Let L be a semisimple isotropic Lie algebra of dimension n defined over a field of characteristic zero with simple roots Δ (all of the same length) and Cartan subalgebra H. Let A be a subalgebra of dimension n' containing the anisotropic kernel of L with simple roots $\Delta' \subset \Delta$. If $A = \oplus_{i=1}^{l} A_i$ with each A_i simple, then the Killing form κ on L is given by*

$$\kappa = \langle \frac{m(L)}{m(A_1)} \rangle \kappa_1 \perp ... \perp \langle \frac{m(L)}{m(A_l)} \rangle \kappa_l \perp \kappa|_{Z_H(A)} \perp \frac{n - n' - |\Delta \setminus \Delta'|}{2} \mathcal{H}$$

where m is the Coxeter number of the algebra.

Proof. Since A contains the anisotropic kernel of L, we know that $Z_H(A)$ is F-split. Decomposing with respect to $Z_H(A)$ gives

$$L = A \oplus Z_H(A) \oplus \bigoplus_{\mu \in S^*, \mu \neq 0} (L_\mu \oplus L_{-\mu})$$

where the sums outside the parenthesis are orthogonal with respect to κ (Lemma 3.4.1). Furthermore, each A_i is orthogonal with respect to the Killing form so $\kappa|_A$ is the sum of the $\kappa|_{A_i}$. Let κ_i denote the Killing form of the subalgebra A_i. We first note that $\kappa|_{A_i}$ is A_i-invariant by the properties of trace. Since κ_i is also A_i-invariant, and the nonzero A_i-invariant bilinear form on A_i is unique up to a scalar multiple ([13, Theorem 5.1.21]), we must have

$$\kappa|_{A_i} = \langle c \rangle \kappa_i$$

for some scalar c. Now let $\alpha \in \Delta' \subset \Delta$ so $h_\alpha \in A_i$. Then by Lemma 3.4.2

$$\kappa(h_\alpha, h_\alpha) = 4m(L)$$

$$\kappa_i(h_\alpha, h_\alpha) = 4m(A_i)$$

forcing $c = \frac{m(L)}{m(A_i)}$.That is,

$$\kappa|_A = \frac{m(L)}{m(A_1)}\kappa_1 \perp \dots \perp \frac{m(L)}{m(A_l)}\kappa_l$$

On the subspaces $L_{\mu_i} \oplus L_{-\mu_i}$, we know that κ restricts to be hyperbolic (Lemma 4.4.3) and the dimension of the nonzero (hyperbolic) weight space is $\dim L - \dim L_0 = n - (n' + |\Delta \setminus \Delta'|)$. This gives the result.

□

Note: In the case that L has roots of different lengths, the above theorem holds by simply replacing the Coxeter number with the dual Coxeter number, except in the following case. Suppose there exists $\alpha \in \Delta' \subset \Delta$ such that α is short in A_i but long in L. Then according to [26, p.14-15]

$$\kappa'(h_\alpha, h_\alpha) = 4m^*(A_i)$$

$$\kappa(h_\alpha, h_\alpha) = 4cm^*(L)$$

where c is the square of the ratio of the root lengths and m^* is the dual Coxeter number of the algebra. Then the scalar is $\frac{cm^*(L)}{m^*(A_i)}$.

The only remaining subspace of the decomposition for which we have not computed the Killing form is $Z_H(A)$. The next section discusses computing the Killing form on a toral subalgebra by calculating dimensions of appropriate irreducible representations.

6.2 Calculating the Killing form on $Z_H(A)$

In this section we discuss a streamlined method for computing the Killing form on a split toral subalgebra, specifically the centralizer of a subalgebra containing the anisotropic kernel. We will assume our root system is simply laced so that coroots equal roots and coweights equal weights. Since A contains the anisotropic kernel of L, $Z_H(A)$ is split or quasi-trivial. We use the results of Section 4.2 to see the basis of a maximal F-split toral subalgebra containing $Z_H(A)$. Since the Killing form on a toral subalgebra of outer type can be computed in terms of basis elements for the inner type toral subalgebra, we will work only in the case that L is of inner type and Φ is simply connected.

Proposition 6.2.1. *Let A have basis $\Delta' \subset \Delta$. Then a basis for $Z_H(A)$ is*

$$\{h_{c\lambda_j} \mid \alpha_j \in \Delta \setminus \Delta'\}$$

Proof. A basis for $Z_H(A)$ is contained in the basis 4.1. Let $h_{c\lambda_j}$ be a basis element from 4.1. Then $h_{c\lambda_j} \in Z_H(A)$ if and only if

$$[h_{c\lambda_j} x_{\alpha_k}] = \langle \alpha_k, c\lambda_j \rangle = 0$$

for all $\alpha_k \in \Delta'$. For each such α_k define

$$\phi_k : \Lambda_r \to \mathbb{Z} : \alpha_k \to \langle \alpha_k, \lambda \rangle$$

Then $h_\lambda \in Z_H(A)$ if and only if $\lambda \in \bigcap_k \ker \phi_k$. Clearly each basis element given above is in $\bigcap_k \ker \phi_k$, and from Section 4.2 we know the rank of $Z_H(A)$ is $|\Delta \setminus \Delta'|$. $\square$

We can compute the Killing form on $Z_H(A)$ using this basis. Let $h_{c\lambda_j} = \sum_{\alpha_i \in \Delta} a_i h_{\alpha_i}$ First note that

$$[h_{c\lambda_j} x_\alpha] = \alpha(h_{c\lambda_j}) x_\alpha = \langle \alpha, c\lambda_j \rangle c a_j x_\alpha$$

so $\mathrm{ad}(h_{c\lambda_j})$ is a diagonal matrix with entries c(coefficient of α_j). Using the symmetry of positive and negative roots we have

$$\mathrm{tr}(\mathrm{ad}(h_{c\lambda_i})\,\mathrm{ad}(h_{c\lambda_j})) = 2c^2 \sum_{\alpha\in\Phi+} (\text{coefficient of } \alpha_i \text{ in } \alpha)(\text{coefficient of } \alpha_j \text{ in } \alpha) \tag{6.1}$$

or in the case $i = j$

$$\mathrm{tr}(\mathrm{ad}(h_{c\lambda_i})\,\mathrm{ad}(h_{c\lambda_i})) = 2c^2 \sum_{\alpha\in\Phi+} (\text{coefficient of } \alpha_i \text{ in } \alpha)^2 \tag{6.2}$$

To compute $\kappa(h_{c\lambda_i}, h_{c\lambda_j})$, we need only those roots in Φ^+ with nonzero α_i and α_j coefficients. To count these roots we introduce the notation of [1].

Fix a subset $\Delta_J \subset \Delta$. The **level** of a positive root $\beta = \sum_{i\in I} c_i\alpha_i$ with respect to Δ_J is the sum of the coefficients of the $\alpha_i \in \Delta \setminus \Delta_J$. The **shape** of β is $\sum_{i\in I\setminus J} c_i\alpha_i$. The goal is to grade the Lie algebra according to the level and shape of its roots. Then the Killing form of an element $h_{n\lambda_i}$ can be given in terms of the dimension of an irreducible representation of a subalgebra.

Let $M_{I\setminus J}(l)$ denote the product of all root spaces $L_{-\beta}$ with $\mathrm{level}(\beta) = l$. By [1, Theorem 2],

$$M_{I\setminus J}(l) = \prod_{S \text{ a shape of level } l} V_S$$

where each V_S is an irreducible representation of the Lie algebra L_J generated by basis Δ_J.

Furthermore, the highest weight of V_S is the negative of the root with shape S and minimal height (in L) ([1]). In the case $\Delta \setminus \Delta_J = \alpha_i$, the only possible shape for a fixed level l is $l\alpha_i$ so $M_i(l)$ is a standard cyclic representation of L_J with highest weight $-l\alpha_i - \beta_l$, denoted $V_{L_J}(-l\alpha_i - \beta_l)$. The dimension of this representation can be used to compute $\kappa(h_{n\lambda_i}, h_{n\lambda_i})$.

Proposition 6.2.2.

$$\kappa(h_{n\lambda_i}, h_{n\lambda_i}) = 2n^2 \sum_{l>0} l^2 \dim V_{L_J}(-l\alpha_i - \beta_l)$$

Proof.

$$\begin{aligned}\operatorname{tr}(\operatorname{ad}(h_{n\lambda_i})\operatorname{ad}(h_{n\lambda_i})) &= 2n^2 \sum_{\beta\in\Phi^+} (\text{coefficient of } \alpha_i \text{ in } \beta)^2 \\ &= 2n^2 \sum_{l>0} l^2 \dim_{L_J} M_{\{i\}}(l).\end{aligned}$$ $\square$

The same method can be used in some cases to compute $\kappa(h_{n\lambda_i}, h_{n\lambda_j})$ for $i \neq j$. In this case, take $\Delta_J = \Delta \setminus \{\alpha_i, \alpha_j\}$. A root β contributes to the above sum if only if both its α_i and α_j coefficients are nonzero. In the E_6 case with $L' = D_4$, this condition states $level(\beta) = 2$ with respect to $\Delta \setminus \Delta_J = \{\alpha_1, \alpha_6\}$.

The notation makes this process appear more complicated than necessary. A few illustrative examples in the next section should clarify that this is really a straightforward process.

Chapter 7

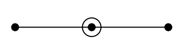
Classical Results with New Method

In this chapter, we compute the Killing form of some classical isotropic Lie algebras using the new methods described in the previous sections. While these Killing forms were previously known, this more streamlined method produces clean results in a straightfoward manner.

For these examples, we use λ_i to denote a fundamental dominant weight of the larger Lie algebra L and ω_i to denote a fundamental dominant weight of the subalgebra L_J with simple roots Δ_J.

7.1 Type A_n

Example 7.1.1 **($\mathfrak{sl}_2(Q)$).** Recall the example of an isotropic Lie algebra of type A_3 given earlier. Here $L = \mathfrak{sl}_2(Q)$ for $Q = (a, b)$ a quaternion algebra defined over F.



Let A be the anisotropic kernel of L of type $A_1 \times A_1$. We compute the Killing form of $A = \mathfrak{sl}(Q) \times \mathfrak{sl}(Q)$ by computing the Killing form on each (isomorphic) $\mathfrak{sl}(Q)$. Since each A_1 is anisotropic it is $\mathfrak{sl}(Q)$ where Q is a quaternion algebra ([28]). These are precisely the trace 0 elements of Q and we computed the Killing form on this Lie Algebra in Chapter 1 to be $\langle 2a, 2b, 2ab\rangle$ where $Q = (a, b)$. Thus, the Killing form of $\mathfrak{sl}(Q)$ is

$$\langle -2\rangle q_0$$

where $1 \perp q_0$ is the norm form of Q. Since the ratio of the Coxeter number of $\mathfrak{sl}_2(Q)$ (type A_3) and $\mathfrak{sl}(Q)$ (type A_1) is 2, we have

$$\kappa|_A = \langle -4 \rangle 2q_0$$

Here $Z_H(A)$ is generated by $h_{4\lambda_2}$, the fundamental dominant weight corresponding to the circled vertex. The Killing form on $Z_H(A)$ can be computed by finding the dimension of $V(-\alpha_2)$ in the anisotropic kernel.

$$\kappa(h_{4\lambda_2}, h_{4\lambda_2}) = 2(4^2) \dim_{A_1 \times A_1} V(-\alpha_2)$$

Since the fundamental weights also form a basis for the root system, we can rewrite α_2 in terms of the fundamental weights of $\mathfrak{sl}_2(Q)$.

$$\kappa(h_{4\lambda_2}, h_{4\lambda_2}) = 2(4^2) \dim_{A_1 \times A_1} V(\lambda_1 - 2\lambda_2 + \lambda_3)$$

As a weight of each A_1, $\lambda_1 - 2\lambda_2 + \lambda_3$ is just ω_1, the fundamental dominant weight of A_1 so

$$\begin{aligned}
\kappa(h_{4\lambda_2}, h_{4\lambda_2}) &= 2(4^2)(\dim_{A_1}(\omega_1) \times \dim_{A_1}(\omega_1)) \\
&= 2(4^2)(4) \\
&= 2^7 \\
&\equiv 2 \bmod F^{*2}
\end{aligned}$$

By Theorem 6.1.1 we have

$$\kappa = \langle -4 \rangle 2q_0 \perp \langle 2 \rangle \perp 4\mathcal{H}$$

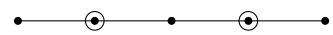
Note

$$\mathrm{tr}_{M_2(Q)}|_{\mathfrak{sl}_2(Q)} = \langle -2 \rangle 2q_0 \perp \langle 1 \rangle \perp 4\mathcal{H}$$

which agrees with [5, Chapter VIII,Ex. 12].

Knowing the Killing form of $\mathfrak{sl}(Q)$ of anisotropic type A_1 allows us to compute the Killing forms of many type A_n Lie algebras as the following example illustrates.

Example 7.1.2 ($\mathfrak{sl}_3(Q)$)**.** Consider the Lie Algebra of type A_5 with Dynkin diagram below. Here we have a slightly larger F-split toral subalgebra.



Here the anisotropic kernel is of type $A_1 \times A_1 \times A_1$. In this case, $Z_H(A)$ has basis $\{h_{6\lambda_2}, h_{6\lambda_4}\}$.

$$\begin{aligned}
\kappa(h_{6\lambda_2}, h_{6\lambda_2}) &= 2(36)\dim_{A_1\times A_3} V(-\alpha_2)\\
&= 2(36)\dim_{A_1\times A_3} V(\lambda_1 - 2\lambda_2 + \lambda_3)\\
&= 2(36)\dim_{A_1} V(\omega_1) \times \dim_{A_3} V(\omega_1)\\
&= 2(36)2 \times 4\\
&= 2^6 * 3^2
\end{aligned}$$

$$\begin{aligned}
\kappa(h_{6\lambda_4}, h_{6\lambda_4}) &= 2(36)\dim_{A_1\times A_3} V(-\alpha_4)\\
&= 2(36)\dim_{A_3\times A_1} V(\lambda_3 - 2\lambda_4 + \lambda_5)\\
&= 2(36)\dim_{A_3} V(\omega_3) \times \dim_{A_1} V(\omega_1)\\
&= 2(36)4 \times 2\\
&= 2^6 * 3^2
\end{aligned}$$

$$\begin{aligned}
\kappa(h_{6\lambda_2}, h_{6\lambda_4}) &= 2(36)\dim_{A_1\times A_1\times A_1} V(-\alpha_2 - \alpha_3 - \alpha_4)\\
&= 2(36)\dim_{A_1\times A_1\times A_1} V(\lambda_1 - \lambda_2 - \lambda_4 + \lambda_5)\\
&= 2(36)\dim_{A_1} V(\omega_1) \times \dim_{A_1} V(\omega_1)\\
&= 2(36)2 \times 2\\
&= 2^5 * 3^2
\end{aligned}$$

The Gram matrix is then

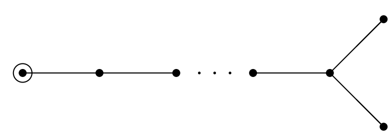
$$2^5 * 3^2 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 2^5 * 3^2 \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \text{(up to squares)}$$

The ratio of the Coxeter numbers here is 3. This gives the Killing form on $\mathfrak{sl}_3(Q)$ as

$$\kappa = \langle -6 \rangle 3q_0 \perp \langle 6, 2 \rangle \perp 12\mathcal{H}$$

7.2 Type D_n

Example 7.2.1 ($\mathfrak{so}(q)$)**.** Consider the Killing form on $\mathfrak{so}(q)$ with Tits index below, where q is a $2n$-dimensional isotropic quadratic form.



Since q is isotropic we can write $q = q_0 \oplus \mathcal{H}$. Letting $J = I \setminus \{1\}$ (so $A = \mathfrak{so}(q_0)$ of type D_{n-1}), Theorem 3.1 gives the Killing form on $\mathfrak{so}(q)$ as

$$\kappa_{\mathfrak{so}(q)} = \langle \frac{2n-2}{2n-4} \rangle \kappa_{\mathfrak{so}(q_0)} \perp \kappa|_{Z_H(A)} \perp (2n-2)\mathcal{H}$$

Since $Z_H(A)$ is generated by $h_{2\lambda_1}$, Proposition 6.2.2 gives the Killing form on $Z_H(A)$ as

$$\begin{aligned} \langle 8 \dim V_{D_{n-1}}(-\alpha_1) \rangle &= \langle 8 \dim V_{D_{n-1}}(\omega_1) \rangle \\ &= \langle 8(2)(n-1) \rangle \cong \langle n-1 \rangle \end{aligned}$$

We know from Example 3.3.2 that

$$\kappa_{\mathfrak{so}(q_0)} = \langle -2(2n-4) \rangle \lambda^2 q_0$$

so we have

$$\begin{aligned}\kappa_{\mathfrak{so}(q)} &\cong \langle \frac{2n-2}{2n-4}(-2)(2n-4)\rangle \lambda^2 q_0 \perp \langle n-1\rangle \perp (2n-2)\mathcal{H}\\ &= \langle -2(2n-2)\rangle \lambda^2 q_0 \perp \langle n-1\rangle \perp (2n-2)\mathcal{H}\\ &\cong \langle -(n-1)\rangle \lambda^2 q_0 \perp \langle n-1\rangle \perp (2n-2)\mathcal{H}\end{aligned}$$

Calculating $\kappa_{\mathfrak{so}(q)}$ as in Example 3.3.2

$$\kappa_{\mathfrak{so}(q)} = \langle -2(2n-2)\rangle \lambda^2 q$$

But since $q = \langle a_1, ...a_{2n-2}, 1, -1\rangle$, $\lambda^2 q = \lambda^2 q_0 \perp (2n-2)\mathcal{H} \perp \langle -1\rangle$ and we have

$$\begin{aligned}\kappa_{\mathfrak{so}(q)} &= \langle -2(2n-2)\rangle (\lambda^2 q_0 \perp (2n-2)\mathcal{H} \perp \langle -1\rangle)\\ &\cong \langle -2(2n-2)\rangle \lambda^2 q_0 \perp (2n-2)\mathcal{H} \perp \langle 2(2n-2)\rangle\\ &\cong \langle -(n-1)\rangle \lambda^2 q_0 \perp \langle n-1\rangle \perp (2n-2)\mathcal{H}\end{aligned}$$

We can see in this example that the two methods agree.

7.3 Real Lie Algebras

Before computing the Killing form on an isotropic real Lie algebra, we first describe the Killing form on a non-trivial orbit in the Tits index.

Lemma 7.3.1. *Let L be a Lie algebra of type 2 over F and let $\{\alpha_i, \alpha_j\}$ be an orbit in the Tits index of L not contained in Δ_0. The Killing form on the 2 dimensional toral subalgebra associated to this orbit is given by*

$$\langle 2(x+y), 2a(x-y)\rangle$$

where $x = \kappa(h_{c\lambda_i}, h_{c\lambda_i}) = \kappa(h_{c\lambda_j}, h_{c\lambda_j})$ and $y = \kappa(h_{c\lambda_i}, h_{c\lambda_j})$. The field $K = F(\sqrt{a})$ is the quadratic extension over which L is type 1. Furthermore $x > y > 0$.

Proof. By 4.2, an F-basis for the quasi-split toral subalgebra corresponding to the orbit $\{\alpha_i, \alpha_j\}$ is

$$\{h_{c\lambda_i} + h_{c\lambda_j}, \sqrt{a}h_{c\lambda_i} - \sqrt{a}h_{c\lambda_j}\}$$

We can compute the Killing form as follows.

$$(h_{c\lambda_i} + h_{c\lambda_j}, h_{c\lambda_i} + h_{c\lambda_j}) = (h_{c\lambda_i}, h_{c\lambda_i}) + 2(h_{c\lambda_i}, h_{c\lambda_j}) + (h_{c\lambda_j}, h_{c\lambda_j})$$

$$\begin{gathered}(\sqrt{a}h_{c\lambda_i} - \sqrt{a}h_{c\lambda_j}, \sqrt{a}h_{c\lambda_i} - \sqrt{a}h_{c\lambda_j}) = \\ a(h_{c\lambda_i}, h_{c\lambda_i}) - 2a(h_{c\lambda_i}, h_{c\lambda_j}) + a(h_{c\lambda_j}, h_{c\lambda_j})\end{gathered}$$

$$(h_{c\lambda_i} + h_{c\lambda_j}, \sqrt{a}h_{c\lambda_i} - \sqrt{a}h_{c\lambda_j}) = \sqrt{a}(h_{c\lambda_i}, h_{c\lambda_i}) - \sqrt{a}(h_{c\lambda_j}, h_{c\lambda_j})$$

Because of the symmetry of the root system $(h_{c\lambda_i}, h_{c\lambda_i}) = (h_{c\lambda_j}, h_{c\lambda_j})$. Let $x = (h_{c\lambda_j}, h_{c\lambda_j})$ and let $y = (h_{c\lambda_i}, h_{c\lambda_j})$ giving the Killing form as

$$\langle 2(x+y), 2a(x-y) \rangle$$

Since x, y are dimensions of irreducible representations $x, y > 0$. Furthermore roots with positive α_i and α_j coefficients contribute the sum in 6.1, while those with only positive α_i (or α_j) coefficient contribute only to the sum in 6.2. Since α_i, α_j are roots themselves, the latter sum will always be larger giving $x > y$. □

Example 7.3.2 (Real Lie Algebras)**.** We can use Theorem 6.1.1 to calculate the Killing form of any real Lie algebra given its Tits index. (Here $\kappa \in \mathbb{Z}$, the signature of the Killing form).

Let L be a Lie algebra over $\mathbb{R}$ with anisotropic kernel A. L decomposes as

$$L = Z_H(A) \oplus A \oplus \bigoplus_{\mu \in S^*, \mu \neq 0} (L_\mu \oplus L_{-\mu})$$

By Lemma 4.4.3, we know that $\kappa|_{(L_\mu \oplus L_{-\mu})}$ is hyperbolic so the signature here is 0. The subalgebra A is compact, so $\kappa_A = (\dim A)\langle -1\rangle$ [14, Proposition 6.6]. Since $m^*(L)$, $m^*(A)$ are positive, $\frac{m^*(L)}{m^*(A)} = 1 \in \mathbb{R}^*/\mathbb{R}^{*2}$ and $\kappa|_A = (\dim A)\langle -1\rangle$. This leaves only $\kappa|_{Z_H(A)}$. Since A is the anisotropic kernel of L, $Z_H(A)$ is split or quasi-split.

Suppose $Z_H(A)$ contains non-trivial orbits so that L is type 2. By Lemma 7.3.1, the Killing form on a non-trivial orbit is $\langle 2(x+y), -2(x-y)\rangle$ where $x > y > 0$. Up to squares this is hyperbolic, so its signature is also zero. The trivial orbits of $Z_H(A)$ are split. By Lemma 3.4.2, the non-hyperbolic part of the Killing form on a trivial orbit is a positive multiple of the Weyl-invariant bilinear form on coroots, which is positive definite [15, 8.5]. This gives the Killing form on a trivial orbit as $\langle 1\rangle$. Since each orbit in $Z_H(A)$ is orthogonal with respect to κ we have $\kappa|_{Z_H(A)} = n\langle 1\rangle$ where n is the number of circled single vertices in the Tits index. This gives the Killing form on L:

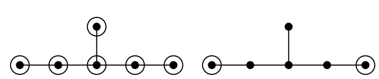
$$\kappa = (\text{\# of circled single vertices in Tits index}) - (\text{dim of the anisotropic kernel})$$

Chapter 8

Results for Lie algebras of type ${}^1E_6$

One of the main goals for this new method of calculating Killing form of isotropic Lie algebras is to be able to give a formula for the Killing form of the lesser understood exceptional Lie algebras. This method allows us to build up a Killing form for an exceptional Lie algebra based on the Killing form of a classical subalgebra. In the E_6 case, we utilize our knowledge of its D_4 subalgebra. Understanding D_4 together with the methods developed in the previous chapter allows us to give an explicit formula for any Lie algebra of inner type E_6 based on its Rost invariant. The critical case here is the E_6 Lie algebra whose anisotropic kernel is of type D_4. We first show that we can achieve this case (or the split case) over an odd degree extension for any ${}^1E_6$ Lie algebra.

Lemma 8.0.3. *For any Lie algebra L of type ${}^1E_6$ (inner type) over F, there exists an odd degree extension K/F such that the Tits index of $L \otimes_F K$ is one of the following:*



Proof. It suffices to prove this in the case L is an algebraic group of type ${}^1E_6$. We first show that there exists an odd-degree extension of F for which the Tits algebras of L are trivial. Recall that L has Tits algebras $A(2) = A(4) =$

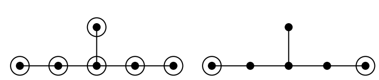
F and $A(1) = A(5) = A(3)^{op} = A(6)^{op}$. The Tits algebra $A(1)$ has order dividing 27 in $\mathrm{Br}(F)$ ([29, 6.4.1])and so there exists an odd degree extension K that splits $A(1)$.

Over K then, $L \cong \mathrm{Inv}(A)$, the group of isometries of the cubic norm form of an Albert algebra A ([7, Theorem 1.4]). We will show that there exists an odd degree extension of K for which A is reduced and that over this extension, the Tits index of L has vertex 1 circled. By Tits' classification ([28]), the Tits index will be one of the two given.

By [20, 40.8], A is reduced if and only if the invariant $g_3(A) \in H^3(*, \mathbb{Z}/3\mathbb{Z})$ is zero. But $g_3(A)$ is a symbol in $H^3(*, \mathbb{Z}/3\mathbb{Z})$ ([27, p.303]), so writing $g_3(A) = (a) \cdot (b) \cdot (c)$, it is enough to show that $\mathrm{res}(a) \in H^1(*, \mathbb{Z}/3\mathbb{Z}) = \mathrm{Hom}(*, \mathbb{Z}/3\mathbb{Z})$ is zero over an odd-degree extension of K. Let K_a be the Galois extension of K corresponding to $\ker(a)$. Then $[K_a : K] = |\operatorname{im}(a)|$ divides 3 and clearly $\mathrm{res}(a) = 0$ in $H^1(K_a, \mathbb{Z}/3\mathbb{Z})$. Therefore, over K_a, A is reduced.

We have a bijection between homogeneous varieties L-varieties over F and $*$-invariant subsets of the Dynkin diagram [3, 6.4,2]. Furthermore, such a variety has a point if and only if this subset consists of circled vertices [3, 6.3,1]. In this case, we have an explicit description of the variety associated to vertex 1 in [8, 7.10]. It is $\{Kv \mid v \in A \text{ with } v \neq 0 \text{ and } v^\sharp = 0\}$. Consider the diagonal matrix $(1, 0, 0) \in A$. Using the formula for $v^\sharp$ given in [19, p.385, (6)] we have $(1, 0, 0)^\sharp = 0$. Since this element is nonzero, it is in the variety associated to vertex 1 and hence vertex 1 is circled in the Tits index of $L \otimes K_a$. Since $[K_a : K]$ and $[K : F]$ are both odd, we have proven the proposition. $\square$

Lemma 8.0.4. *Let L be a Lie algebra of type ${}^1E_6$ with one of the two Tits indices below.*



Then the D_4 subalgebra is $\mathfrak{so}(q)$, where q is a 3-fold Pfister form and $e_3(q) = r(L)$, the mod 2 part of the Rost invariant of L as defined in [9].

Proof. Again we may prove this in the algebraic group case. Let L be an algebraic group wth Tits index above and let D_4 denote the D_4 subgroup with roots $\alpha_2, \alpha_3, \alpha_4, \alpha_5$. Let $A'(2)$ denote the Tits algebra associated to vertex 2 as a Tits algebra of D_4. In the first case, it is clear that $A'(2)$ is split. In the second, D_4 is the anisotropic kernel of L and so $A'(2)$ is the same as the Tits algebra $A(2)$ associated to vertex 2 in L [29, 5.5.2,5.5]. In L, $\lambda_2 \in \Lambda_r$ indicating that $A(2)$ is split over F and the irreducible 8-dimensional representation of D_4 with highest weight λ_2, $V(\lambda_2)$, is defined over F. By [13, 5.1.21, 5.1.24 and proof of 2.5.5], there exists a non-zero D_4-invariant symmetric bilinear form q on $V(\lambda_2)$ and a representation $\pi : D_4 \to so(q)$. Since D_4 is simple with $\dim D_4 = \dim so(q)$, π is an isomorphism. By [29, 6.2], $C_0(q) = A(3) \times A(5)$. Furthermore, $\lambda_5 \equiv -\lambda_3 \equiv \lambda_1$ modulo Λ_r and since $A(1)$ is split, $C_0(q) = M_8(F) \times M_8(F)$ and $C(q) = M_{16}(F)$ giving the Clifford invariant $c(q) = 1$ ([21, V.3.12]). Since $\dim q = 8$, $\mathrm{disc}(q) = (-1)^4$ and $c(q) = 1$, we have $q \in I^3F$ by Merkuryev's Theorem.

Let E_6 denote the split simply connected Lie algebra of type ${}^1E_6$. Then there is a unique $\eta \in H^1(F, E_6)$ such that E_6 twisted by η is L ([20, Proposition 31.5]). But L is also isomorphic to the algebra obtained by twisting E_6 by $\eta' \in H^1(F, so_8) \subset H^1(K, E_6)$ where so_8 twisted by η' is $so(q)$ ([28, Theorem 2.2]). Furthermore $\eta' \in H^1(F, spin_8)$ since q is Pfister ([20, 31.41]). Then the quadratic form q is uniquely associated to η by Arason-Pfister Haupstatz [21, X.5.1]. Consider the following sequence:

$$H^1(F, spin_8) \xrightarrow{\;i\;} H^1(F, E_6) \xrightarrow{\text{Rost}} H^3(F, \mu_6^{\otimes 2}) \xrightarrow{\;p\;} H^3(F, \mathbb{Z}/2\mathbb{Z})$$

The Rost multiplier of the inclusion $spin_8 \to E_6$ is 1 ([6, 2.2]). Therefore

$$\mathrm{Rost}_{E_6}(\eta) = \mathrm{Rost}_{spin_8}(\eta') = e_3(q) \in H^3(F, \mathbb{Z}/2\mathbb{Z}).$$

By [9, Prop. 5.2 and Def. 5.3], $\mathrm{Rost}_{E_6}(\eta)$ projected to $H^3(F, \mathbb{Z}/2\mathbb{Z})$ is precisely the r-invariant $r(L)$.

$\square$

Theorem 8.0.5. *The Killing form of a Lie Algebra of inner type E_6 is*

$$\kappa \cong \langle -24 \rangle 4q_0 \perp \langle 2, 6 \rangle \perp 24\mathcal{H}$$

where $e_3(1 \perp q_0)$ is $r(L)$, the mod 2 part of the Rost invariant of L as defined in [9].

Proof. Let L be a Lie algebra of type ${}^1E_6$ over F and let K/F be the odd degree extension described in Lemma 8.0.3. We will show that over K, L has a subalgebra of type $D_4 = \mathfrak{so}(q)$ where q is a Pfister form defined over F.

By Lemma 8.0.4, $L \otimes K$ has a D_4 subalgebra of the form $\mathfrak{so}(q)$ where q is a K-Pfister form and $e_3(q) \in H^3(K, \mathbb{Z}/2\mathbb{Z})$ is $r(L \otimes K) = \mathrm{res}(r(L \otimes F))$. Since $\mathrm{res}(r(L \otimes F))$ is a symbol in $H^3(K, \mathbb{Z}/2\mathbb{Z})$, $r(L \otimes F)$ is a symbol in $H^3(F, \mathbb{Z}/2\mathbb{Z})$ ([24, Proposition 2]). Therefore q is in fact a Pfister form over F.

We now compute the Killing form of $L \otimes K$ using Theorem 6.1.1. By Springer's Theorem, this is isomorphic to the Killing form of $L \otimes F$ since $[K : F]$ is odd. We have computed the Killing form of $\mathfrak{so}(q)$ in previous examples as $\kappa \cong \langle -2(2n-2) \rangle \lambda^2 q$. In the case q is 3-fold Pfister form, the Killing form of $\mathfrak{so}(q)$ is

$$\langle -3 \rangle 4q_0$$

where $q \cong 1 \perp q_0$.

Let $A = \mathfrak{so}(q)$, the subalgebra of type D_4 in this E_6. Then $Z_H(A)$ is generated by $h_{3\lambda_1}, h_{3\lambda_6}$ and we compute the Killing form using the methods described in the previous chapter. Again, we use λ_i to denote a fundamental dominant weight of $L \otimes K$ and ω_i to denote a fundamental dominant weight of $L_J \otimes K$. In these calculations we will use $J = \{1, 2, 3, 4, 5\}$, $\{2, 3, 4, 5, 6\}$, and $\{2, 3, 4, 5\}$ so $L_J \otimes K$ is of type D_5, D_5, and D_4, respectively.

$$\begin{aligned}\kappa(h_{3\lambda_1}, h_{3\lambda_1}) &= 18 \dim V_{D_5}(-\alpha_1) = 18 \dim V_{D_5}(-2\lambda_1 + \lambda_3) \\ &= 18 \dim V_{D_5}(-\omega_4) = 18 * 2^4 = 288\end{aligned}$$

$$\begin{aligned}\kappa(h_{3\lambda_6}, h_{3\lambda_6}) &= 18 \dim V_{D_5}(-\alpha_6) = 18 \dim V_{D_5}(\lambda_5 - 2\lambda_6) \\ &= 18 \dim V_{D_5}(\omega_5) = 18 * 2^4 = 288\end{aligned}$$

$$\begin{aligned}\kappa(h_{3\lambda_1}, h_{3\lambda_6}) &= 18 \dim V_{D_4}(-(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)) \\ &= 18 \dim V_{D_4}(-\lambda_1 + \lambda_2 - \lambda_6) = 18 \dim V_{D_4}(\omega_2) \\ &= 18 * 8 = 144\end{aligned}$$

This form diagonalizes to $\langle 2, 6 \rangle$. Using Theorem 6.1.1 and the fact that $\frac{m(E_6)}{m(D_4)} = \frac{12}{6} = 2$ gives the result. $\square$

The table below gives the results for each type of ${}^1E_6$ where $r(L) = 1 \perp q_0$. We need only know the Rost invariant to compute these forms. Notice how streamlined these formulas are compared to Jacobson's results in Chapter 5. In the split case $r(L) = 4\mathcal{H}$, so we can give the formula with no q_0. In the third case, Lemma 8.0.3 shows that the Killing form is isomorphic to the Killing form of the split Lie algebra.

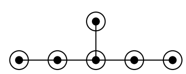
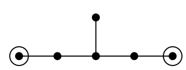
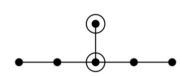
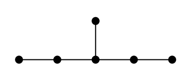
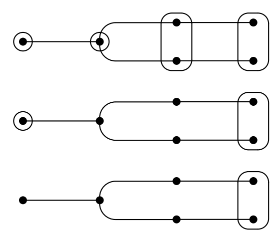
Tits Index	Killing form
	$\langle 1,1,1,1,2,6\rangle \perp 36\mathcal{H}$
	$\langle -6\rangle 4q_0 \perp \langle 2,6\rangle \perp 24\mathcal{H}$
	$\langle 1,1,1,1,2,6\rangle \perp 36\mathcal{H}$
	$\langle -6\rangle 4q_0 \perp \langle 2,6\rangle \perp 24\mathcal{H}$

Table 8.1: Results for ${}^1E_6$

Chapter 9

Results for Lie algebras of type $^2E_6$

Theorem 6.1.1 also yields results for some outer type E_6 Lie algebras. For this section L refers to a Lie algebra of outer type E_6 with one of the following Tits indices:



As in the inner type case, the subalgebra of type $^2D_4$ plays a crucial role. We begin by describing this subalgebra.

Lemma 9.0.6. *Let $\Gamma = Gal(F_{sep}/F)$ and let $* : \Gamma \to S_3$ be the map from Γ into the automorphism group of the Tits index of $^2D_4$ given by the $*$-action of Γ. Let $K = (F_{sep})^{\ker *}$. Then $K = F(\sqrt(a))$ for some $a \in \dot{F}$ and*

$$^2D_4 = \mathfrak{so}(1 \perp \langle a \rangle q_0)$$

where $1 \perp q_0$ is a Pfister form over K.

Proof. Over F, the fundamental weight λ_2 is in the root lattice of E_6 so $A(2)$ is split as a Tits algebra of E_6. When vertex 2 is circled in the Tits index of a Lie algebra of type $^2E_6$, $A(2)$ is split as a Tits algebra of D_4. When D_4 is the anisotropic kernel of E_6, the Tits algebras for the vertices in D_4 are the same as those for the vertices 2,3,4,5 in E_6 above [29, p.211]. This covers all Tits indices of type ${}^2E_6$ except the anisotropic case. Since A_2 is split in each of these cases, $V(\lambda_2)$, the 8-dimensional irreducible representation of D_4 with highest weight λ_2, is defined over the base field F. There exists a D_4 invariant symmetric bilinear form on $V(\lambda_2)$ by [13, 5.1.21, 5.1.24 and Proof of 2.5.5] so we have a representation $\pi : D_4 \to \mathfrak{so}(q)$ for some 8-dimensional quadratic form q. Since D_4 is simple, π is injective, and since $\dim D_4 = \dim \mathfrak{so}(q) = 28$, π is an isomorphism.

The image of the map $* : \Gamma \to S_3$ has order 2 since our D_4 is of type 2 [28]. Therefore $[K : F] = 2$ and so $K = F(\sqrt{a})$ for some $a \in \dot{F}$. Notice that $L \otimes K$ is of inner type. We now describe the Tits algebras of L over K. Let B be the Tits algebra associated to the orbit $\{\alpha_3, \alpha_5\}$ over F (using the number of the vertices in E_6). Then over K

$$B \otimes_F K = (B \otimes_K K) \otimes_F K = B \otimes_K (K \otimes_F K) = B \otimes_K (K \times {}^{\iota}K)$$

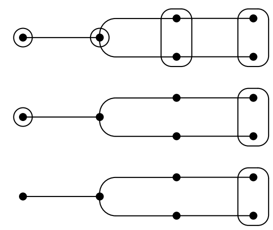
$$B \otimes_K K \times B \otimes_K {}^{\iota}K = B \times {}^{\iota}B = A(3) \times A(5)$$

where $A(3)$ and $A(5)$ are Tits algebras associated to vertices $3, 5$ in the inner type $D_4 \subset E_6$. The even Clifford algebra of $q \otimes K = A(3) \otimes A(5)$ [29, 6.2]. But we are in the inner case over K, so there exists an odd degree extension of K for which all Tits algebras are trivial. (See proof of Lemma 8.0.3.) Over this extension K' then $C_0(q \otimes K') = M_8(K') \otimes M_8(K')$ and $C(q \otimes K') = M_16(K')$ [21, V.2.5] so $c(q \otimes K') = 1$ and since also $\dim q = 8$, $\operatorname{disc} q = 1$, $q \otimes K'$ is a Pfister form. But then $q \otimes K$ is Pfister [24, Proposition 2] and

$$q \otimes F = \langle 1 \rangle \perp \langle a \rangle q_0$$

where $1 \perp q_0 = q \otimes K$ [21, VII.3.3]. □

Theorem 9.0.7. *Let L be a Lie algebra of outer type E_6 with one of the following Tits indices*



The Killing form of L is

$$\kappa \cong \langle -24\rangle(3q_0 \perp \langle a\rangle q_0) \perp \langle 6, 2a\rangle \perp 24\mathcal{H}$$

where $\langle 1 \perp q_0\rangle$ is $r(L \otimes K)$.

Proof. Let A be the subalgebra of type ${}^2D_4$. The ratio of the Coxeter numbers of L and A is 2 so Theorem 6.1.1 gives

$$\kappa = 2\kappa_A \oplus \kappa|_{Z_H(A)} \oplus 24\mathcal{H}$$

Since $A = \mathfrak{so}(1 \perp \langle a\rangle q_0)$, the Killing form of A is

$$\kappa_A = -2(6)\lambda^2 q = 3q_0 \perp \langle a\rangle q_0$$

The quasi-split toral subalgebra $Z_H(A)$ has F-basis $\{h_{3\lambda_1} + h_{3\lambda_6}, \sqrt{a}h_{3\lambda_1} - \sqrt{a}h_{3\lambda_6}\}$ [3, 6.9]. We calculate the Killing form using results from the ${}^1E_6$ case which are restated below.

$$\kappa(h_{3\lambda_1}, h_{3\lambda_1}) = 288$$
$$\kappa(h_{3\lambda_6}, h_{3\lambda_6}) = 288$$
$$\kappa(h_{3\lambda_1}, h_{3\lambda_6}) = 144$$

The Killing form for $Z_H(A)$ in the type 2 case is then:

$$\kappa(h_{3\lambda_1} + h_{3\lambda_6}, h_{3\lambda_1} + h_{3\lambda_6}) = 3(288)$$

$$\kappa(h_{3\lambda_1} + h_{3\lambda_6}, \sqrt{a}h_{3\lambda_1} - \sqrt{a}h_{3\lambda_6}) = 0$$

$$\kappa(\sqrt{a}h_{3\lambda_1} - \sqrt{a}h_{3\lambda_6}, \sqrt{a}h_{3\lambda_1} - \sqrt{a}h_{3\lambda_6}) = a(288)$$

This form diagonalizes to

$$\langle 6, 2a \rangle.$$

Combining these results gives the formula for outer type E_6. $\square$

In the quasi-split case, $1 \perp q_0 = 4\mathcal{H}$ so $3q_0 = \langle -1 \rangle \perp 3\mathcal{H}$ giving the Killing form on ${}^2D_4$ as

$$\langle -1 \rangle \langle 1, 1, 1, a \rangle \perp 12\mathcal{H}$$

The table below summarizes the Killing form for the other cases.

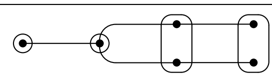
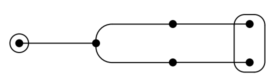
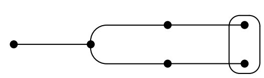
Tits Index	Killing form
	$\langle 6 \rangle \langle 1, 1, 1, a \rangle \perp \langle 6, 2a \rangle \perp 36\mathcal{H}$
	$\langle -6 \rangle (3q_0 \perp \langle a \rangle q_0) \perp \langle 6, 2a \rangle \perp 24\mathcal{H}$
	$\langle -6 \rangle (3q_0 \perp \langle a \rangle q_0) \perp \langle 6, 2a \rangle \perp 24\mathcal{H}$

Table 9.1: Results for ${}^2E_6$

Garibaldi and Petersson [11, 11.1] describe a bijection between pairs (C, K) of octonion F-algebras C and quadratic étale algebras K and simply connected groups of type E_6. In the above formulas $1 \perp q_0$ is the norm form of C and $K = F(\sqrt{a})$.

It should also be noted that Jacobson's methods also give the above results, although his formulas are much more compliacted (see [18, p.114]).

Chapter 10

Results for Lie algebras of type E_7

For Lie algebras of type E_7 we have only inner type. Theorems 6.1.1 and 8.0.5 give us immediate results for isotropic Lie algebras of type E_7 when the anisotropic kernel is contained in the subalgebra of type E_6. The Killing form for these Lie algebras is obtained by letting $A = E_6$ in Theorem 6.1.1 and using the results from Theorem 8.0.5 to give the Killing form on A.

Theorem 10.0.8. *Let L be an isotropic Lie algebra of type E_7 whose anisotropic kernel is contained in the subalgebra of type E_6 generated by the simple roots $\Delta \setminus \{\alpha_7\}$. Let q_0 denote the invariant $r(E_6)$. Then the Killing form on L is*

$$\kappa \cong \langle 2, 1, 1 \rangle \perp 4\langle -1 \rangle q_0 \perp 51\mathcal{H}$$

Proof. Let $A = E_6$. $Z_H(A)$ is a one dimensional F-split toral subalgebra corresponding to vertex 7. A basis for $Z_H(A)$ is just $h_{2\lambda_7}$ and the Killing form on $Z_H(A)$ can be computed as follows.

$$\begin{aligned}\kappa(h_{2\lambda_7}, h_{2\lambda_7}) &= 2(2^2)\dim V_{E_6}(-\alpha_7) = 8\dim V_{E_6}(\lambda_6 - 2\lambda_7)\\ &= 8\dim V_{E_6}(\omega_6) = 8 * 27 = 216\\ &\equiv 6 \text{mod } F^{*2}\end{aligned}$$

The Coxeter number of E_7 is 18 and the Coxeter number of E_6 is 12 so Theorem 6.1.1 and the computations above give

$$\kappa = \langle \frac{3}{2} \rangle (\langle -24 \rangle 4q_0 \perp \langle 2, 6 \rangle \perp 24\mathcal{H}) \perp \langle 6 \rangle \perp 27\mathcal{H}$$

Simplifying the form and reducing modulo squares gives

$$\kappa = 4\langle -1\rangle q_0 \perp \langle 3, 1, 6\rangle \perp 51\mathcal{H}$$

But

$$\langle 3, 6\rangle \cong \langle 1, 2\rangle$$

since they have the same discriminant and both represent 9 [21, I.5.1]. This gives

$$\kappa = \langle 2, 1, 1\rangle \perp 4\langle -1q_0\rangle \perp 51\mathcal{H} \qquad \square$$

The following table summarizes the results for E_7. Here $1 \perp q_0$ is the r-invariant of the E_6 subalgebra. In the split case $1 \perp q_0 = 4\mathcal{H}$ so $q_0 = \langle -1\rangle \perp 3\mathcal{H}$.

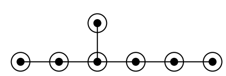
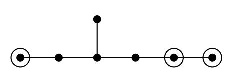
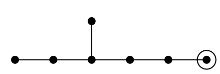
Tits Index	Killing form
	$\langle 2, 1, 1, 1, 1, 1, 1\rangle \perp 63\mathcal{H}$
	$\langle 2, 1, 1\rangle \perp 4\langle -1\rangle q_0 \perp 51\mathcal{H}$
	$\langle 2, 1, 1\rangle \perp 4\langle -1\rangle q_0 \perp 51\mathcal{H}$

Table 10.1: Results for E_7

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