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A Connection Between K3 Surfaces and Conway Moonshine Module

by

Qiyu Zhang

John Duncan
Adviser

Mathematics

John Duncan

Adviser

David Zureick-Brown

Committee Member

Ajit Srivastava

Committee Member

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An abstract of
a thesis submitted to the Faculty of Emory College of Arts and Sciences
of Emory University in partial fulfillment
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Abstract

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In this paper, we survey an interesting duality between K3 surfaces and the Conway moonshine modules via Jacobi forms. K3 surfaces are studied extensively in the context of string compactification since they are manageable, yet non-trivial kind of Calabi-Yau manifold. An important topological invariant discovered by Witten in [1] is the elliptic genus. The elliptic genus of a K3 surface is a weak modular form of weight zero and index 1. In this text we show that a certain graded trace associated to the Conway moonshine module coincides with the K3 elliptic genus.

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1 Introduction

In this paper, we survey an interesting duality between K3 surfaces and the Conway moonshine modules via Jacobi forms. K3 surfaces are studied extensively in the context of string compactification since they are manageable, yet non-trivial kind of Calabi-Yau manifold. An important topological invariant discovered by Witten in [1] is the elliptic genus. The elliptic genus of a K3 surface is a weak modular form of weight zero and index 1. In this text we show that a certain graded trace associated to the Conway moonshine module coincides with the K3 elliptic genus.

2 Background

In this section, we review results on modular forms, K3 surfaces, and vertex operator algebras that will be useful in Section 3.

2.1 Modular Forms

Modular forms are ubiquitous in mathematics. In this section, we look at some general properties of certain modular forms. Our main reference for this section will be [9].

Definition 2.1. *Let $\mathbb{H}$ be the upper half complex plane, and let $SL(2,\mathbb{Z})$ be the group of matrices*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

with integer entries such that $ad-bc=1$. A modular form of weight k $f(\tau)$ is a holomorphic function on $\mathbb{H}$ such that

$$f(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k f(\tau). \tag{1}$$

and $f(\tau)$ is bounded as $Im(\tau)\to\infty$ where $\tau\in\mathbb{H}, k\in\mathbb{Z}$.

By definition, $f(\tau)$ is invariant under the transformation $\tau \to \tau + 1$ and thus can be expanded into a Fourier series:

$$f(\tau) = \sum_{n=-\infty}^{\infty} a(n)q^n, q := e^{2\pi i\tau}. \tag{2}$$

If the modular form vanishes at infinity, i.e. $a(0) = 0$, then it is called a cusp form. If we weaken the growth condition to $O(q^{-N})$ instead of $O(1)$ for some $N \geq 0$, then it is called a weakly holomorphic modular form. The vector space over $\mathbb{C}$ of holomorphic modular forms of weight k is referred to as M_k, the space of weakly holomorphic modular forms of weight k is referred to as $M_k^!$, and the space of cusp forms is referred to as S_k.
Some important modular forms of $SL(2,\mathbb{Z})$ include Eisenstein series, the discriminant function, and Dedekind eta-function.
The Einsenstein series $E_k \in M_k(k \geq 4)$ For instance,

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty} \frac{n^3q^n}{1-q^n} = 1 + 240q + 2160q^2... \tag{3}$$

$$E_6(\tau) = 1 - 504\sum_{n=1}^{\infty} \frac{n^5q^n}{1-q^n} = 1 - 504q - 16332q^2... \tag{4}$$

The discriminant function $\Delta(\tau)$ is a modular form of weight 12 given by

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = q - 24q^2 + 252q^3... \tag{5}$$

, and the Dedekind eta function is defined as $\Delta^{1/24}$ with weight $\frac{1}{2}$:

$$\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{6}$$

A modular form of level N is defined as a modular form for $\Gamma_0(N)$

$$\Gamma_0(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) | c = 0 mod N\} \tag{7}$$

which has the same definition as modular form of level 0, replacing $SL_2(\mathbb{Z})$ with $\Gamma_0(N)$. But they have a different growth condition: $f(\frac{a\tau+b}{c\tau+d})(c\tau+d)^{-1}$ is bounded when $Im(\tau)$ goes to ∞, $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, and f is a cusp form if $f(\frac{a\tau+b}{c\tau+d})(c\tau+d)^{-1}$ goes to zero as $Im(\tau)$ goes to ∞. And the functions $\Lambda_N(\tau)$ are a family of modular forms of weight 2 with level N is defined as:

$$\Lambda_N(\tau) = \frac{N}{2\pi i}\frac{d}{d\tau}log(\frac{\eta(N\tau)}{\eta\tau}) = \frac{N}{24}(NE_2(N\tau) - E_2(\tau)) \tag{8}$$

2.2 Jacobi Forms

Jacobi forms are roughly a multivariate version of modular forms which is modular in τ and elliptic in z.

Definition 2.2. *Consider a holomorphic funtion $\phi(\tau, z)$ from $\mathbb{H} \times \mathbb{C}$ to $\mathbb{C}$. The function is a Jacobi form of weight k and index m if*

$$\phi(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k e^{\frac{2\pi i m c z^2}{c\tau + d}} \phi(\tau, z) \tag{9}$$

for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

and transforms as

$$\phi(\tau, z + \lambda z + \mu) = e^{-2\pi i m(\lambda^2 \tau + 2\lambda z)} \phi(\tau, z) \tag{10}$$

for any $\lambda, \mu \in \mathbb{Z}$ with a growth condition described by the following Fourier coefficients. Then we define the growth condition. From the definition, we deduce that $\phi(\tau + 1, z) = \phi(\tau, z)$, and $\phi(\tau, z + 1) = \phi(\tau, z)$, therefore ϕ has the Fourier expansion:

$$\phi(\tau, z) = \sum_{n,r} c(n, r) q^n y^r \tag{11}$$

where $q := e^{2\pi i \tau}$ and $y := e^{2\pi i z}$. The periodicity condition can be recast as the condition that:

$$c(n, r) = C(4nm - r^2, r) \tag{12}$$

where $C(\Delta, r)$ depends only on r mod $2m$. If $c(n, r) = 0$ for $\Delta \geq 0$, then the the function is a holomorphic Jacobi form.
If $c(n, r) = 0$ for $\Delta > 0$, then the the function is a Jacobi cusp form.
If $c(n, r) = 0$ unless $n \geq 0$, then the the function is a weak holomorphic Jacobi form.

The Jacobi forms we are interested in are theta functions and the generator of weak Jacobi forms of weight zero and index one $\phi_{0,1}$.

Definition 2.3. *The four Jacobi theta functions are Jacobi forms of weight $1/2$ and index $1/2$ defined as:*

$$\vartheta_1(\tau, z) := -\mathbf{i} \sum_{n \in \mathbb{Z}} (-1)^n y^{n+1/2} q^{(n+1/2)^2/2},$$

(13)

$$\vartheta_2(\tau, z) := \sum_{n \in \mathbb{Z}} y^{n+1/2} q^{(n+1/2)^2/2}, \tag{14}$$

$$\vartheta_3(\tau,z) := \sum_{n\in\mathbb{Z}} y^n q^{n^2/2}, \tag{15}$$

$$\vartheta_4(\tau,z) := \sum_{n\in\mathbb{Z}} (-1)^n y^n q^{n^2/2}, \tag{16}$$

To prove these have the desired weight and index, we need theories of multiplier systems. However, we will not get in to the details here. And these functions can be rearranged into product formulas:

$$\vartheta_1(\tau,z) = -\mathbf{i} q^{1/8} y^{1/2} \left(1-y^{-1}\right) \prod_{n>0} \left(1-y^{-1}q^n\right)\left(1-yq^n\right)\left(1-q^n\right), \tag{17}$$

$$\vartheta_2(\tau,z) = q^{1/8} y^{1/2} \left(1+y^{-1}\right) \prod_{n>0} \left(1+y^{-1}q^n\right)\left(1+yq^n\right)\left(1-q^n\right), \tag{18}$$

$$\vartheta_3(\tau,z) = \prod_{n>0} \left(1+y^{-1}q^{n-1/2}\right)\left(1+yq^{n-1/2}\right)\left(1-q^n\right), \tag{19}$$

$$\vartheta_4(\tau,z) = \prod_{n>0} \left(1-y^{-1}q^{n-1/2}\right)\left(1-yq^{n-1/2}\right)\left(1-q^n\right), \tag{20}$$

Furthermore, we define the weak Jacobi form of weight zero and index one and the weak Jacobi form of weight -2 and index one as

$$\phi_{0,1}(\tau,z) := 4\left(\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right) \tag{21}$$

$$\phi_{-2,1}(\tau,z) := -\frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^6} \tag{22}$$

We refer to [10] for the proof of the following lemma that we will use later:

Lemma 2.1. *The following statements are true:*

$$\frac{1}{12}\phi_{0,1}(\tau,z) + 2\Lambda_2(\tau)\phi_{-2,1}(\tau,z) = \frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} \tag{23}$$

$$\frac{1}{12}\phi_{0,1}(\tau,z) - \Lambda_2(\tau/2+1/2)\phi_{-2,1}(\tau,z) = \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} \tag{24}$$

$$\frac{1}{12}\phi_{0,1}(\tau,z) - \Lambda_2(\tau/2)\phi_{-2,1}(\tau,z) = \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2} \tag{25}$$

2.3 K3 Surfaces

In this section, we state some basic facts about K3 surfaces, the basic notions can be found in [2][3]. First, we define the notion of the canonical bundle.

Definition 2.4. *For any non-singular algebraic variety V of dimension n, the canonical bundle is defined to be the $n-th$ exterior power of the cotangeant bundle Ω on V. For complex manifolds, the definition coincides with the determinant bundle of holomorphic n-forms.*

A (complex analytic) K3 surface is essentially defined to be a compact 2-dimensional complex manifold with a nowhere vanishing two-form which is not a complex torus.

Definition 2.5. *A K3 surface is a compact two-dimensional complex manifold S with trivial canonical bundle $\omega_S \simeq \mathcal{O}_S$ and $H^1(S, \mathcal{O}_S) = 0$.*

An example of the K3 surfaces is Fermat quartic surface. It is defined as the locus of the following polynomial of degree four in complex projective 3-space.

$$x^4 + y^4 + z^4 + w^4 = 0$$

An important fact about K3 surfaces is that all K3 surfaces are diffeomorphic. Therefore, the topological invariants such as the Euler characteristic, betti numbers etc. will be identical for all K3 surfaces. We start off our study with the Hodge diamond of K3, which refer to the dimensions of the Dolbeaut cohomology groups. Dolbeaut cohomology groups are isomorphic to the coherent sheaf cohomology groups, with respect to the sheaf of holomorphic p-forms.

$$H^{p,q}(S) \simeq H^q(S, \Omega^p) \tag{26}$$

For a complex projective variety X, i.e. a closed submanifold of $\mathbb{CP}^n$, we have the Hodge decomposition:

$$H^n(S, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(S) \tag{27}$$

Therefore, the Hodge diamond which is a diamond that juxtaposes the dimension of $h^{p,q} := dim(H^{p,q})$ encodes the information of the cohomology groups of smooth complex algebraic varieties.The Hodge diamond of a complex surface is as follows:

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

For K3 surfaces, it computes to be,

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

A proof can be found in [4]. As we can see the only non-trivial cohomology group is the second cohomology group. The study of moduli spaces states that the structure of cohomology group determines the isomorphism class of K3 surfaces, however, we will not get into the details in this paper. Equipped with the Hodge diamond, we can compute the Euler characteristic of K3 surfaces,

$$\chi(S) = \sum_i (-1)^i h^i(S, \mathcal{O}_S) = 1 + 0 + 22 + 0 + 1 = 24. \tag{28}$$

2.4 Elliptic Genus

Witten defined an important topological invariant called elliptic genus in [1]. We are going to show that the elliptic genus for K3 surfaces is $2\phi_{0,1}$, the weak Jacobi form of weight zero and index one shown in equation (21). First of all, we define characteristic classes that characterize vector bundles. Specifically, we define Chern classes, the Chern character and the Todd class. We refer to [5] for basic notions about characteristic classes.

Definition 2.6. *Given a complex hermitian vector bundle V of rank n on a smooth manifold M, a representative of each Chern class $c_k(V)$ comes from the coefficients of the characteristic polynomial of the curvature form Ω of V*

$$det(\frac{it\Omega}{2\pi} + I) = \sum_k c_k(V) t^k \tag{29}$$

The right hand side of equation(29) is also called a Chern polynomial. The determinant is taken over the rings of $n * n$ matrix with entries of polynomials in t, and the coefficients are in the even differentiable forms of M. The expression of Chern class thus expands as:

$$\sum_k c_k(V) t^k = [I + i\frac{tr(\Omega)}{2\pi} t + \frac{tr(\Omega^2) - tr(\Omega)^2}{8\pi^2} t^2 + ...] \tag{30}$$

Now we introduce the Chern characters and the Todd classes:

Definition 2.7. *For a line bundle L, the Chern character is defined as:*

$$ch(L) = exp((c_1(L))) = \sum_{n=0}^{\infty} \frac{c_1(L)^n}{n!} \tag{31}$$

If a complex vector bundle can be written as direct sum of line bundles,

$$V = L(1) \oplus L(2) \oplus L(n) \tag{32}$$

, the Chern character is defined additively:

$$ch(E) = e^{c_1(L(1))} + e^{c_1(L(2))}.. \tag{33}$$

and thus Chern characters respect the tensor products and direct sums of vector bundles:

$$ch(V \otimes W) = ch(V)ch(W) \tag{34}$$

$$ch(V \oplus W) = ch(V) + ch(W) \tag{35}$$

Using the fact that $c_i(V)$ are elementary symmetric polynomials in formal variables x_i and

$$V = \sum_{i=0}^{n} c_i(V) = \prod_{i=1}^{n} c(L_i) = \prod_{i=1}^{n}(1 + x_i) \tag{36}$$

Using theories of symmetric polynomials, we arrive at the following expansion[5]:

$$ch(V) = rk(V)+c_1(V)+1/2(c_1(V)^2-2c_2(V))+1/6(c_1(V)^3-2c_1(V)c_2(V)+3c_3(V)) \tag{37}$$

By the Riemann-Roch theorem, for S a K3 surface, $c_1(S) := c_1(TS)$

$$\chi(S) = 1/12(c_1(S)^2 + c_2(S)) = 24 \tag{38}$$

and we know $c_1(S) = -c_1(K_S) = 0$, and thus

$$c_2(S) = 24 \tag{39}$$

Definition 2.8. *The Todd class is defined as:*

$$\prod_{i=1}^{n} \frac{a_i}{1 - e^{-a_i}} \tag{40}$$

where $a_i := c_1(L_i)$ is defined as the Chern roots. We can expand Todd class explicitly into:

$$td(V) = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1 c_2}{24}... \tag{41}$$

Definition 2.9. *The elliptic genus of a complex manifold M is defined as*

$$Z_M = \int_M ch(\mathbb{E}_{q,y})td(M) \tag{42}$$

where $ch(\mathbb{E}_{q,y})$ is a the Chern character of the formal power series whose coefficients are vector bundles

$$\mathbb{E}_{q,y} = y^{\frac{d}{2}} \bigotimes_{n=1}^{\infty}\Big(\bigwedge_{-y^{-1}q^{n-1}} TM \otimes \bigwedge_{-yq^n} T^*M \otimes \bigvee_{q^n} TM \otimes \bigvee_{q^n} T^*M\Big) \tag{43}$$

where

$$\bigwedge_t(V) = \sum_{k\geq 0} \wedge^k(V)t^k$$
$$\bigvee_t(V) = \sum_{k\geq 0} \vee^k(V)t^k \tag{44}$$

It has been shown that[7] since K3 surface S has $c_1 = 0$, $Z_M(\tau, z)$ is a weak Jacobi form of weight zero and index $dim(S)/2 = 1$. It is also known that Z_M specializes to Euler characteristic $\chi(S) = 24$ when $z = 0$[8]and that the space of weak Jacobi forms of weight zero and index one is one dimensional. Therefore, since $\phi(\tau, 0) = 12$,

$$Z_M(\tau, 0) = 2\phi_{0,1} \tag{45}$$

The main goal of this paper is to show that the K3 elliptic genus equals a graded trace of Conway moonshine module.

2.5 Vertex Operator Algebra

In this section we review vertex operator algebras.
Below, we denote $R[[z^{\pm 1}]]$ as the formal power series in z with coefficients in the ring R, $R[[z]]$ denotes formal Taylor series with only non-negative coefficients being non-zero, $R[z]$ denotes the polynomial ring and $R((z)) = R[[z]][z^{-1}]$ denotes the Laurent series in z with coefficients in R. We are going to briefly introduce the notion of vertex operator algebra. For further reading, we refer to the text[11].

Definition 2.10. *A super vertex algebra is a super vector space $V = V_{\bar{0}} \bigoplus V_{\bar{1}}$ equipped with a vacuum $\mathbf{1} \in V_{\bar{0}}$, a linear operator*

$$V \to End(V)[[z^{\pm 1}]] \tag{46}$$

$$a \to Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \tag{47}$$

satisfying the following axioms for $a, b, c \in V$:
(1) $Y(a, z)b \in V((z))$ and, if $a \in V_{\bar{0}}$ (respectively, $a \in V_{\bar{1}}$), then $a_{(n)}$ is an even (respectively, odd) operator for all n:
(2) $Y(\mathbf{1}, Z) = Id_V$ and $Y(a, z)\mathbf{1} \in a + zV((z))$
(3) $[T, Y(a, z)] = \partial_z Y(a, z), T\mathbf{1} = 0$, and T is an even operator.
(4) If $a \in V_{p(a)}$ and $b \in V_{p(b)}$ are $\mathbb{Z}_2$ homogeneous, there exists an element:

$$f \in V[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}]$$

depending on a, b, and c, such that

$$Y(a, z)Y(b, w)c, (-1)^{p(a)p(b)}Y(b, w)Y(a, z)c$$

, and

$$Y(Y(a,z-w)b,w)c$$

are the expansions of f in $V((z))((w))$, $V((w))((z))$, and $V((w))((z-w))$, respectively.

Next we define the notion of a module over a super vertex algebra.

Definition 2.11. *A module over a super vertex algebra V is a super vector space $M = M_{\bar{0}} \bigoplus M_{\bar{1}}$ equipped with a linear map*

$$V \to End(M)[[z^{\pm 1}]] \tag{48}$$

$$a \to Y_M(a,z) = \sum_{n\in\mathbb{Z}} a_{(n),M} z^{-n-1} \tag{49}$$

satisfying the following axioms for $a, b \in V$, and $u \in M$:
(1) $Y_M(a,z)u \in M((z))$ and, if $a \in V_{\bar{0}}$ (respectively, $a \in V_{\bar{1}}$), then $a_{(n),M}$ is an even (respectively, odd) operator for all n:
(2) $Y_M(\boldsymbol{1}, Z) = Id_M$ and,
(3) If $a \in V_{p(a)}$ and $b \in V_{p(b)}$ are $\mathbb{Z}_2$ homogeneous, there exists an element:

$$f \in M[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$$

depending on a, b, and u, such that

$$Y_M(a,z)Y_M(b,w)u, (-1)^{p(a)p(b)}Y_M(b,w)Y_M(a,z)u$$

, and

$$Y_M(Y(a,z-w)b,w)u$$

are the expansions of f in $M((z))((w))$, $M((w))((z))$, and $M((w))((z-w))$, respectively.

Then we can define modules that are twisted by an automorphism of the vertex algebra. Now we consider the special case of $\theta := Id_{V_{\bar{0}}} \bigoplus -Id_{V_{\bar{1}}}$, a canonically twisted module over V is defined as follows.

Definition 2.12. *A canonically twisted module over V is a super vector space over V , $M = M_{\bar{0}} \bigoplus M_{\bar{1}}$ equipped with a linear map:*

$$V \to End(M)[[z^{\pm 1/2}]] \tag{50}$$

$$a \to Y_{tw}(a, z^{1/2}) = \sum_{n\in\frac{1}{2}\mathbb{Z}} a_{(n)_{tw}} z^{-n-1} \tag{51}$$

equipped with twisted vertex operator $Y_{tw}(a, z^{1/2})$ such that satisfying the following axioms for $a, b, c \in V$:
(1) $Y_M(a, z^{1/2})u \in M((z^{1/2}))$ and, if $a \in V_{\bar{0}}$ (respectively, $a \in V_{\bar{1}}$), then $a_{(n)_{tw}}$ is an even (respectively, odd) operator for all n:
(2) $Y_{tw}(\boldsymbol{1}, z^{1/2}) = Id_M$ and,

(3) If $a \in V_{p(a)}$ and $b \in V_{p(b)}$, there exists an element:

$$f \in M[[z^{1/2}, w^{1/2}]][z^{-1/2}, w^{-1/2}, (z-w)^{-1}]$$

depending on a, b, and c, such that

$$Y_{tw}(a, z^{1/2})Y_{tw}(b, w^{1/2})u, (-1)^{p(a)p(b)}Y_{tw}(b, w^{1/2})Y_{tw}(a, z^{1/2})u$$

, and

$$Y_{tw}(Y(a, z-w)b, w^{1/2})u$$

are the expansions of f in $M((z^{1/2}))((w^{1/2}))$, $M((w^{1/2}))((z^{1/2}))$, and $M((w^{1/2}))((z-w))$, respectively.
(4)if $\theta(a) = (-1)^m a$, then $a_{(n),tw} = 0$ for $n \notin \mathbb{Z} + (m/2)$

We can also enrich the notion of super vertex algebra with representation of Virasoro algebra. The Virasoro algebra is a Lie algebra spanned by $L(m), m \in Z$ and central element $\mathbf{c}$, with Lie bracket:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\delta_{m+n,o}\mathbf{c} \tag{52}$$

Definition 2.13. *A super vertex operator algebra is a super vertex algebra containing a Virasoro element $\omega \in V_{\bar{0}}$ such that if $L(n) := \omega_{n+1}$ for $n \in \mathbb{Z}$, then*
(5) L(-1) = T
(6)

$$[L(m), L(n)] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,o}cId_v$$

(7)L(0) is a diagnolizable linear operator, with half-integer eigenvalues bounded below, with finite dimensional eigenspaces.
(8)the superspace structure on V is recovered from the eigendata of $L(0)$, i.e. $p(a) = 2n(mod2)$ for $L(0)v = nv$.

2.6 Clifford Construction of VOA

Next, we construct super vertex operator algebras. There exists a standard construction from Clifford modules. Now we review some basic facts about Clifford modules. Consider a complex vector space of even dimension $\mathfrak{a}$ with a nondegenerate, symmetric bilinear form $<,>$. We call a subspace $W < \mathfrak{a}$ isotropic if $< u, v >= 0, \forall u, v \in W$.We call the decomposition into maximally isotropic subspaces, $\mathfrak{a} = \mathfrak{a}^+ \oplus \mathfrak{a}^-$ a polarization of $\mathfrak{a}$. We define the Clifford algebra as the quotient algebra $T(\mathfrak{a})/I(\mathfrak{a})$ and $I(\mathfrak{a})$ is the two-sided ideal $< u \otimes u + < u, u > \mathbf{1} >$ where $\mathbf{1}$ denotes the unity element in the tensor algebra. Suppose there is a family of isomorphisms $\mathfrak{a} \to \mathfrak{a}(n + \frac{1}{2})$ for $n \in \mathbb{Z}$, and we define the vector space

$$\hat{\mathfrak{a}} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{a}(n + \frac{1}{2}) \tag{53}$$

We can extend the nondegenerate, symmetric bilinear form to $\hat{\mathfrak{a}}$ by putting $< u(r), v(s) >=< u, v > \delta_{r+s,0}$. Then, there exists a polarization of $\hat{\mathfrak{a}}$, i.e. a decomposition of $\hat{\mathfrak{a}}$ into maximal isotropic subspaces,$\hat{\mathfrak{a}} = \hat{\mathfrak{a}}^+ \oplus \hat{\mathfrak{a}}^-$, where

$$\hat{\mathfrak{a}}^+ = \bigoplus_{n\geq 0} \mathfrak{a}(n+\frac{1}{2}), \hat{\mathfrak{a}}^- = \bigoplus_{n<0} \mathfrak{a}(n+\frac{1}{2}). \tag{54}$$

Similar to the previous discussion, we can define a Clifford algebra $Cliff(\hat{\mathfrak{a}}) = T(\hat{\mathfrak{a}})/I(\hat{\mathfrak{a}})$. And we denote B^+ and B^- as subalgebras generated by $\hat{\mathfrak{a}}^+$ and $\hat{\mathfrak{a}}^-$, respectively. The map $-\mathbf{Id} : \hat{\mathfrak{a}} \to \hat{\mathfrak{a}}$ naturally extends to a linear map that satisfies $(-\mathbf{Id}(v))^2 = - < v, v > \mathbf{1}$ by the definition of Clifford algebra. Therefore, by the universal property of the Clifford algebra, there exists a unique involution $\theta : Cliff(\hat{\mathfrak{a}}) \to Cliff(\hat{\mathfrak{a}})$. Decompose $Cliff(\hat{\mathfrak{a}})$ according to the eigenspace of the involution, $Cliff(\hat{\mathfrak{a}}) = Cliff(\hat{\mathfrak{a}})^0 \oplus Cliff(\hat{\mathfrak{a}})^1$, where $Cliff(\hat{\mathfrak{a}})^j$ denotes the subalgebra with eigenvalue $(-1)^j$.
Now we examine the tensor product of B^--modules: $A(\mathfrak{a}) = Cliff(\hat{\mathfrak{a}}) \otimes_{B^+} \mathbb{C}\mathbf{v}$. The B^+ action of $\mathbb{C}\mathbf{v}$, a one-dimensional vector space is defined as follows: $\mathbf{1v} = \mathbf{v}$, and $u\mathbf{v} = 0$ for $u \in \hat{\mathfrak{a}}^+$. There exists a natural isomorphism:

$$A(\mathfrak{a}) \simeq \bigwedge(\hat{\mathfrak{a}}^-)\mathbf{v} \tag{55}$$

. Define a vertex algebra structure on $A(\mathfrak{a})$ by extending the vertex operator on $a(-\frac{1}{2})$:

$$Y(a(-1/2)\mathbf{v}, z) = \sum_{n\in\mathbb{Z}} a(n+1/2)z^{-n-1} \tag{56}$$

By reconstruction theorem [11], these vertex operators extend uniquely to a super vertex algebra structure on $A(\mathfrak{a})$.The super space structure comes from the parity decomposition of the exterior algebra, that is:

$$A(\mathfrak{a})^0 \simeq \bigwedge^{even}(\hat{\mathfrak{a}}^-)\mathbf{v}, A(\mathfrak{a})^1 \simeq \bigwedge^{odd}(\hat{\mathfrak{a}}^-)\mathbf{v} \tag{57}$$

Choosing an orthonormal basis e_i for $\mathfrak{a}$, the Virasoro element

$$\omega = \frac{-1}{4}\sum_{i=1}^{dim\mathfrak{a}} e_i(\frac{-3}{2})e_i(\frac{-1}{2})\mathbf{v} \tag{58}$$

gives the super vertex operator algebra structure of $A(\mathfrak{a})$ with central charge $c = \frac{1}{2}dim(\mathfrak{a})$.
A similar construction gives us the twisted $A(\mathfrak{a})$-module $A(\mathfrak{a})_{tw}$. Given vector space a, we have isomorphisms $\mathfrak{a} \to \mathfrak{a}(n)$ for $n \in \mathbb{Z}$.
Define:

$$\hat{\mathfrak{a}} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{a}(n) \tag{59}$$

Picking the bilinear form and polarization as before. Given a polarization $\mathfrak{a} = \mathfrak{a}^+ \oplus \mathfrak{a}^-$, there is:

$$\hat{\mathfrak{a}}_{tw}^{+} = \mathfrak{a}(0)^{+} \oplus \bigoplus_{n\geq 0} \mathfrak{a}(n), \hat{\mathfrak{a}}_{tw}^{-} = \mathfrak{a}(0)^{-} \oplus \bigoplus_{n<0} \mathfrak{a}(n) \tag{60}$$

And similarly,

$$A(\mathfrak{a})_{tw} \simeq \bigwedge(\hat{\mathfrak{a}}_{tw}^{-})\mathbf{v}_{tw} \tag{61}$$

The twisted vertex operator

$$Y_{tw}(\omega, z^{-1/2}) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2} \tag{62}$$

equips $A(\mathfrak{a})_{tw}$ with representation of Virasoro algebra. By the axioms, $L(0)$ acts diagonizablly and the eigenvalues of the action can be computed to be contained in $\mathbb{Z} + \frac{1}{16}dim(\mathfrak{a})$.

We also notice that there is an embedding of $Cliff(\mathfrak{a})$ in $Cliff(\hat{\mathfrak{a}}_{tw})$ as the subalgebra generated by $\mathfrak{a}(0)$. Therefore, there is a $Cliff(\mathfrak{a})$ action on $A(\mathfrak{a})_{tw}$. And the $Cliff(\mathfrak{a})$ sub-module generated by $\mathbf{v}_{tw}$ is the unique nontrivial irreducible representation of $Cliff(\mathfrak{a})$ in $A(\mathfrak{a})_{tw}$. We denote this subspace of $A(\mathfrak{a})_{tw}$ by CM. Restricting the isomorphism (61), we have:

$$A(\mathfrak{a})_{tw} \simeq \bigwedge(\bigoplus_{n<0} \mathfrak{a}(n)) \otimes CM, CM \simeq \bigwedge(\mathfrak{a}(0)^{-})\mathbf{v}_{tw} \tag{63}$$

2.7 Spin Group

Spin group $Spin(n)$ is defined as the simple-connected double cover of the Lie Group $SO(n)$, i.e. there exists a short exact sequence of Lie groups:

$$1 \to \mathbb{Z}_2 \to Spin(n) \to SO(n) \to 1 \tag{64}$$

First we construct it as a subgroup of the group of invertible elements in the Clifford algebra $Cliff(\mathfrak{a})$. Define the main automorphism α of $Cliff(\mathfrak{a})$ as an automorphism such that $\alpha(u_1....u_k) = u_k....u_1$ where $u_i \in \mathfrak{a}$. The spin group is defined as the set of even invertible elements in $Cliff(\mathfrak{a})$ such that $\alpha(x)x = 1$, i.e. the invertible element x of $Cliff(\mathfrak{a})$ such that $xux^{-1} \in \mathfrak{a}$ whenever $u \in \mathfrak{a}$.

It is possible to construct elements of $Spin(\mathfrak{a})$ explicitly. Elements of the form $\frac{1}{2}(uv - vu) \in Cliff(\mathfrak{a})$, spans a simple Lie algebra of type D_c, where $c = \frac{1}{2}dim(\mathfrak{a})$, and the exponential map generates the Lie group $Span(exp(\frac{1}{2}(uv - vu))$. Choose $a^+, a^- \in \mathfrak{a}$ such that a^+ and a^- are isotropic and $< a^+, a^- >= 1$. The expression $X := \frac{\mathbf{i}}{2}(a^-a^+ + a^+a^-)$ satisfies $X = -\mathbf{1}$. Therefore, the Taylor expansion gives us $e^{\theta X} = cos(\theta)\mathbf{1} + sin(\theta)X$.

Define the map x that sends $u \in \mathfrak{a}$ to xux^{-1} that belongs to $SO(\mathfrak{a})$. Then the assignment $x \to x(-)$ is a group homomorphism from $Spin(\mathfrak{a})$ to $SO(\mathfrak{a})$ with kernel ± 1. We denote $\hat{g} \in Spin(\mathfrak{a})$ as a lift of $g \in SO(\mathfrak{a})$ if $\hat{g}(-) = g$.

Following the definition, X acts on $a^{\pm}$ as follows:

$$Xa^{\pm} = \pm\mathbf{i}a^{\pm} = -a^{\pm}X \tag{65}$$

, so
$$e^{\theta X}(a^{\pm}) = e^{\theta X} a^{\pm} e^{-\theta X} = e^{\pm 2\theta \mathbf{i}} a^{\pm} \tag{66}$$

Therefore, $e^{\theta \mathbf{X}}$ is a lift of the orthogonal transformation that acts on $a^{\pm}$ by multiplication by $e^{\pm 2\theta \mathbf{i}}$ and acts as identity on other basis elements.
For future reference, by definition $X\mathbf{v}_{tw} = \mathbf{i}\mathbf{v}_{tw}$, so the action of $e^{aX} \in Spin(\mathfrak{a})$ on $\mathbf{v}_{tw}$ is given by: $e^{aX}\mathbf{v}_{tw} = e^{a\mathbf{i}}\mathbf{v}_{tw}$.
There is a natural action of $Spin(\mathfrak{a})$ on $A(\mathfrak{a})$. If $a \in A(\mathfrak{a})$ has the form $a = u_1(-n_1+1/2)u_2(-n_2+1/2)..$, the spin group element $x \in Spin(\mathfrak{a})$ acts as

$$xa = x(u_1)(-n_1+1/2)x(u_2)(-n_2+1/2)... \tag{67}$$

Due to the isomorphism in equation (63), we can identify an element of $A(\mathfrak{a})_{tw}$ as $u_1(-n_1)u_2(-n_2)... \otimes y$ where $u_i \in \mathfrak{a}$ and $y \in CM$. Thus the natural action is as follows,

$$x(u_1(-n_1)u_2(-n_2)... \otimes y) = x(u_1)(-n_1)x(u_2)(-n_2)... \otimes xy \tag{68}$$

Observe that if $x \in Spin(\mathfrak{a})$ is a lift of $-Id(\mathfrak{a})$, then $\mathbf{v}_{tw} \in CM$ satisfies $x\mathbf{v}_{tw} = \pm\mathbf{i}^c\mathbf{v}_{tw}$. Thus there exists a polarization that distinguished one unique lift $\mathfrak{z}$ of $-Id_{\mathfrak{a}}$ such that $\mathfrak{z}\mathbf{v}_{tw} = \mathbf{i}^c\mathbf{v}_{tw}$.Therefore, $\mathfrak{z}$ acts with order two on a 24 dimensional vector space, so its eigendata recovers the superspace structure of $A(\mathfrak{a})$.
Next, we define the Conway group Co_0. We first learn some properties of lattices.
Lattices are can be understood as generalization of the center points of a sphere packing, which are generated by finitely many number of points. The main reference on this topic is [12]

Definition 2.14. *A lattice in $\mathbb{R}^n$ is a discrete abelian group that is a free $\mathbb{Z}$-module of finite rank n, $L \simeq \mathbb{Z}^n$, equipped with symmetric bilinear form $< .,. >$ that takes values in $\mathbb{Z}$.*

Given a field k of characteristic zero, the bilinear form $< .,. >: L \otimes_{\mathbb{Z}} L \to L$ extends uniquely to the n-dimensional vector space over k: $L \otimes_{\mathbb{Z}} k$. The signature of the lattice is a pair (r,s) where r refers to the maximal dimension of a positive-definite subspace of $L \otimes_{\mathbb{Z}} \mathbb{R}$, and s refers to the maximal dimension of the negative-definite subspace of $L \otimes_{\mathbb{Z}} \mathbb{R}$. Call n the rank of L, if $n = r+s$, then we call L non-degenerate. If $s = 0$, we call L positive-definite. If $r = 0$, we call L negative-definite.
Then we define the dual of L as follows:

$$L^* := \{\gamma \in L \otimes_{\mathbb{Z}} \mathbb{Q} | \langle \lambda, \gamma \rangle \in \mathbb{Z}, \text{ for all } \lambda \in L\}. \tag{69}$$

We call L self-dual if $L^* = L$.
Given $\lambda \in L$, we call $< \lambda, \lambda >$ the square-length of λ. A lattice L is called even if all its square-lengths are even integers. The set of elements in an even lattice L with square-length ± 2 is called a root system. Some examples are the $D4$ lattice and the Leech lattice.

Definition 2.15. *A D4 lattice is a lattice whose vertices lie in $\mathbb{Z}_4$: (i,j,k,l) such that the sum of coordinate components is even, i.e. $i + j + k + l \in 2\mathbb{Z}$.*

Definition 2.16. *The Leech lattice Λ is the unique self-dual positive definite even lattice of rank 24 with no roots, and Conway group is defined as $Co_0 := Aut(\Lambda)$.*

The uniqueness of Conway group was proven by Conway.[13] Conway group has order 8,315,553,613,086,720,000. However, it is not simple, its quotient by its center $Co_1 := Co_0/\pm\{Id\}$ is the largest Conway sporadic simple group. A choice of identification $\mathfrak{a} = \Lambda \otimes \mathbb{C}$ gives an embedding of Co_0 in $SO(\mathfrak{a}$. Given a choice of such identification, we write G for the subgroup of $SO(\mathfrak{a})$ isomorphic to Co_0. We call χ_g the character of the corresponding representation of G.

$$\chi_g := tr_{\mathfrak{a}} g. \tag{70}$$

Given a subgroup $H < SO(\mathfrak{a})$, we call $\hat{H} < Spin(\mathfrak{a})$ is a lift of H if the natural map between $Spin(\mathfrak{a})$ and $SO(\mathfrak{a})$ restricts to an isomorphism between H and $\hat{H}$.

Theorem 2.2. *Let $G < SO(\mathfrak{a})$, and suppose G is isomorphic to Co_0, then there is a unique lift of G to $Spin(\mathfrak{a})$.*

For the proof, we refer to [12]. The theorem guarantees that there is a unique action of the copy of Conway group G on $A(\mathfrak{a})$ and $A(\mathfrak{a})_{tw}$.

3 Main Result

3.1 Conway Moonshine Module

In this section, we are proving that the trace of the Conway moonshine module is the elliptic genus. In order to compute the trace, we first construct the Conway moonshine module. The Conway moonshine module is constructed via the construction of a distinguished super vertex operator algebra $V^{s\natural}$ and its unique canonically twisted module $V^{s\natural}_{tw}$. There is an action of Conway group on both $V^{s\natural}$ and $V^{s\natural}_{tw}$ and in this section we show that they indeed carry the super vertex operator algebra structures. We recall that given a 24-dimensional vector space $\mathfrak{a}$, there is a polarization $\mathfrak{a} = \mathfrak{a}^- \oplus \mathfrak{a}^+$. Given such a polarization, there is an associated lift of $-Id_{\mathfrak{a}}$, $\mathfrak{z}$ so that $\mathfrak{z}\mathbf{v}_{tw} = \mathbf{v}_{tw}$. The decomposition of $A(\mathfrak{a})_{tw}$ is given by the eigenspace of $\mathfrak{z}$, where $\mathfrak{z}A(\mathfrak{a})^j_{tw} = (-1)^j A(\mathfrak{a})^j_{tw}$:

$$A(\mathfrak{a})_{tw} = A(\mathfrak{a})^0_{tw} \oplus A(\mathfrak{a})^1_{tw}. \tag{71}$$

Now, $A(\mathfrak{a})^0$ is a vertex operator algebra and $A(\mathfrak{a})_{tw}$ is a $A(\mathfrak{a})^0$-module.Therefore, we have the construction of $A(a)^0$-modules.

$$V^{s\natural} = A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^1_{tw} \tag{72}$$

$$V^{s\natural}_{tw} = A(\mathfrak{a})^0_{tw} \oplus A(\mathfrak{a})^1 \tag{73}$$

The following theorem gives these modules super vertex operator algebra and canonically twisted module structures, respectively. A proof can be found in [12].

Theorem 3.1. *The $A(\mathfrak{a})^0$-module structure on $V^{s\natural}$ extends uniquely to a super vertex operator structure on $V^{s\natural}$, and the $A(\mathfrak{a})^0$-module on $V^{s\natural}_{tw}$ extends uniquely to a canonically twisted $V^{s\natural}$-module structure*

Now we note that there is an action of Conway group on $V^{s\natural}$ and $V^{s\natural}_{tw}$. As we see in the spin group section, there are natural actions of $Spin(\mathfrak{a})$ on $V^{s\natural}$ and $V^{s\natural}_{tw}$ since there are natural actions of them on $A(\mathfrak{a})^j$ and $A(\mathfrak{a})^j_{tw}$. This action repsects the super vertex algebra structure and the canonically twisted module structure in the last theorem. Now, given the identification $\mathfrak{a} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$, we have a copy of Conway group $Aut(\Lambda)$ in $SO(\mathfrak{a})$, and thus lifted to $Spin(\mathfrak{a})$. Since there is a unique natural isomorphism induced by the lifting map as shown in Theorem 2.2 , the $Spin(\mathfrak{a})$ action restricts to the CO_0 action.
We can consider the graded trace of actions associated of $Spin(\mathfrak{a}$ since it preserves the $L(0)$-grading. First we show that the supertrace of $x \in Spin(\mathfrak{a})$ is

$$str_{A(\mathfrak{a})}(xq^{L(0)-c/24}) = tr_{A(\mathfrak{a})}(x\mathfrak{z}q^{L(0)-c/24}) = \frac{\eta_{\bar{x}}(\tau/2)}{\eta_{\bar{x}}(\tau)} \tag{74}$$

,where for $g \in SO(\mathfrak{a})$

$$\eta_g(\tau) = q\prod_{i=1}^{24}\prod_{n>0}(1-\epsilon_i q^n) \tag{75}$$

, where $q = e^{2\pi i\tau}$ and ϵ_i is the eigenvalue of the action of g on $\mathfrak{a}$. First, we introduce the grading by $J(0)$ on $A(\mathfrak{a})$ and $A(\mathfrak{a})_{tw}$. For V a vertex operator algebra, given an element $\jmath \in V$, and that $L(0)\jmath = \jmath$, and

$$[J(m), J(n)] = k\delta_{m+n,0}Id_V, k \in \mathbb{C} \tag{76}$$

, then $J(0)$ is called a $U(1)$ element of level k, and by definition, the grading of $L(0)$ is preserved. Therefore, if $J(0)$ is diagonalizable, then there exists a bigrading

$$V_{n,r} := \left\{v \in V| \left(L(0) - \tfrac{\mathbf{c}}{24}\right)v = nv,\ J(0)v = rv\right\} \tag{77}$$

Similarly, for a canonically twisted module V_{tw}

$$(V_{tw})_{n,r} := \left\{v \in V_{tw}| \left(L(0) - \tfrac{\mathbf{c}}{24}\right)v = nv,\ J(0)v = rv\right\} \tag{78}$$

. Consider the isotropic element $a_1^{\pm}, a_2^{\pm}$ such that $< a_i^{\pm}, a_j^{\mp} >= \delta_{i,j}$, we have the following lemma. A proof can be found in [10].

Lemma 3.2. *For*

$$\jmath = \frac{1}{2}\sum_{i=1}^{2} a_i^-(-1/2)a_i^+(-1/2)\mathbf{v} \in A(\mathfrak{a}), \tag{79}$$

it is a $U(1)$ element of level 2, and $J(0)\mathbf{v} = 0$, and $J(0)\mathbf{v_{tw}} = \mathbf{v_{tw}}$

3.2 A Graded Trace of Conway Moonshine Module

Now we proceed to compute the following trace:

$$\phi_g := -\mathrm{tr}_{V^{s\natural}_{\mathrm{tw}}} \mathfrak{z}\widehat{g} y^{J(0)} q^{L(0)-c/24} \tag{80}$$

for $g \in CO_0$. The main theorem of this paper is going to show that the trace ϕ_e is the K3 elliptic genus $Z_S = 2\phi_{0,1}(\tau, z)$. First we ought to show that it can be written as a certain combination of theta functions and η_gs.

Theorem 3.3.

$$\phi_e = -\frac{1}{2}\left(\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\frac{\eta_e(\tau/2)}{\eta_e(\tau)} - \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2}\frac{\eta_{-e}(\tau/2)}{\eta_{-g}(\tau)}\right)$$

- 1 $\frac{}{2(\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} C_{-e}\eta_{-e}(\tau))}$

Proof: To compute

$$tr_{A(\mathfrak{a})} q^{L(0)-c/24} y^{J(0)} = tr_{\wedge(\mathfrak{a}^-)\mathbf{v}} q^{L(0)-c/24} y^{J(0)} \tag{81}$$

We make use of the definition of $\hat{\mathfrak{a}}$

$$tr_{\wedge(\mathfrak{a}^-)\mathbf{v}} q^{L(0)-c/24} y^{J(0)} = tr_{\wedge(\mathfrak{a}(-1/2)\oplus\mathfrak{a}(-3/2)...)\mathbf{v}} q^{L(0)-c/24} y^{J(0)} \tag{82}$$

We make use of the fact that $\bigwedge(V \oplus W) = \bigwedge(V) \otimes \bigwedge(W)$

$$tr_{\wedge(\mathfrak{a}(-1/2)\oplus\mathfrak{a}(-3/2)...)\mathbf{v}} q^{L(0)-c/24} y^{J(0)} = tr_{\wedge(\mathfrak{a}(-1/2))\mathbf{v}\otimes\wedge(\mathfrak{a}(-3/2))\mathbf{v}\otimes...} q^{L(0)-c/24} y^{J(0)} \tag{83}$$

Since trace is additive and multiplicative

$$tr_{\wedge(\mathfrak{a}(-1/2))\mathbf{v}\otimes\wedge(\mathfrak{a}(-3/2))\mathbf{v}\otimes...} q^{L(0)-c/24} y^{J(0)} = tr_{\wedge(\mathfrak{a}(-1/2))\mathbf{v}} q^{L(0)-c/24} y^{J(0)} tr_{\wedge(\mathfrak{a}(-3/2))\mathbf{v}} q^{L(0)-c/24} y^{J(0)} ... \tag{84}$$

We expand the vector space in question into basis elements since we know how these operators act on $\mathbf{v}$ *and* $e_i(-n_i + 1/2)$*, make use of equation (19), we got*

$$= tr_{\wedge(\bigoplus_{i=1}^{24} <e_i(-1/2)>)\mathbf{v}} q^{L(0)-c/24} y^{J(0)} tr_{\wedge(\bigoplus_{i=1}^{24} <e_i(-3/2)>)\mathbf{v}} q^{L(0)-c/24} y^{J(0)} ... \tag{85}$$

$$= q^{-1/2} \prod_{n\in\mathbb{Z}^+} (1 + y^{-1}q^{n-1/2})^2 (1 + yq^{n-1/2})^2 (1 + q^{n-1/2})^{20}$$

$$= \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2}\frac{\eta_{-e}(\tau/2)}{\eta_{-e}(\tau)}$$

Along the same line of logic, further decompose eigenspace into eigenspace of $J(0)$*, we have*

$$tr_{A(\mathfrak{a})\mathfrak{z}} q^{L(0)-c/24} y^{J(0)} = q^{-1/2} \prod_{n>0} (1 - yq^{n-1/2})^2 (1 - y^{-1}q^{n-1/2})^2 (1 - q^{n-1/2})^{20} \tag{86}$$

$$tr_{A(\mathfrak{a})^1} q^{L(0)-c/24} y^{J(0)} = \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2} \frac{\eta_e(\tau/2)}{\eta_e(\tau)} \tag{87}$$

Therefore,

$$tr_{A(\mathfrak{a})^1} \mathfrak{z} q^{L(0)-c/24} y^{J(0)} = \frac{1}{2}\left(\frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} \frac{\eta_{-e}(\tau/2)}{\eta_{-e}(\tau)} - \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2} \frac{\eta_e(\tau/2)}{\eta_e(\tau)}\right) \tag{88}$$

Now, we consider the twisted case, we note that $L(0)\mathbf{v}_{tw} = 3/2\mathbf{v_{tw}}$, $J(0)\mathbf{v}_{tw} = \mathbf{v}_{tw}$

$$tr_{A(\mathfrak{a})_{tw}} q^{L(0)-c/24} y^{J(0)} = tr_{\bigwedge(\mathfrak{a}_{tw}^-)\mathbf{v}_{tw}} q^{L(0)-c/24} y^{J(0)} \tag{89}$$

$$= tr_{\bigwedge(\mathfrak{a}(0)^- \oplus \mathfrak{a}(-1) \oplus \mathfrak{a}(-2)\ldots)\mathbf{v}_{tw}} q^{L(0)-c/24} y^{J(0)} \tag{90}$$

Follow similar logic, with $L(0)a(n) = na(n)$, $J(0)a_i^\pm = \pm a_i, i > 10$, $J(0)a_i^\pm = 0a_i, i \leq 10$, we get,

$$\begin{aligned} tr_{A(\mathfrak{a})_{tw}} q^{L(0)-c/24} y^{J(0)} &= qy \prod_{n>0} (1+y^{-1}q^{n-1})^2 (1+yq^n)^2 \prod_{n>0} (1+q^{n-1})^{10} (1+q^n)^{10} \\ &= \frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} C_{-e} \eta_{-e}(\tau) \end{aligned} \tag{91}$$

where, $C_{-e}(\tau) = 2^{12}$ Similarly,

$$tr_{A(\mathfrak{a})_{tw}} \mathfrak{z} q^{L(0)-c/24} y^{J(0)} = qy \prod_{n>0} (1-y^{-1}q^{n-1})^2 (1-yq^n)^2 \prod_{n>0} (1-q^{n-1})^{10} (1-q^n)^{10} = 0 \tag{92}$$

Therefore,

$$tr_{A(\mathfrak{a})_{tw}^0} q^{L(0)-c/24} y^{J(0)} = -1/2 \frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} C_{-e} \eta_{-e}(\tau) \tag{93}$$

Combining this with the trace of the $A(\mathfrak{a})^1$, we have proven the theorem.

Now the final step is to show that the expression indeed coincides with K3 elliptic genus.

Theorem 3.4.

$$\phi_e(\tau,z) = 2\phi_{0,1}(\tau,z) \tag{94}$$

Proof: Consider the following construction:

$$F_e(\tau) = 1/2\Lambda_2(\tau/2) \frac{\eta_e(\tau/2)}{\eta_e(\tau)} - 1/2\Lambda_2(\tau/2+1/2) \frac{\eta_{-e}(\tau/2)}{\eta_{-e}(\tau)} - \Lambda_2(\tau) C_{-e} \eta_{-e}(\tau) \tag{95}$$

To prove the theorem, we first need to show that:

$$\phi_e(\tau,z) = 2\phi_{0,1}(\tau,z) + F_e(\tau)\phi_{-2,1}(\tau,z) \tag{96}$$

Plug in expression for F, we rearrange the right hand side to the following:

$$
\begin{aligned}
2\phi_{0,1}(\tau,z)+\frac{1}{2}\Lambda_2(\tau/2)\frac{\eta_e(\tau/2)}{\eta_e(\tau)}\phi_{-2,1}(\tau,z)\\
-\frac{1}{2}\Lambda_2(\tau/2+1/2)\frac{\eta_{-e}(\tau/2)}{\eta_{-e}(\tau)}\phi_{-2,1}(\tau,z)\\
-\Lambda_2(\tau)C_{-e}\eta_{-e}(\tau)\phi_{-2,1}(\tau,z)
\end{aligned}
$$

Now, we make use of the observation that:

$$48-\frac{\eta_{-e}(\tau/2)}{\eta_{-e}(\tau)}+\frac{\eta_e(\tau/2)}{\eta_e(\tau)}+C_{-e}\eta_{-e}(\tau)=0 \tag{97}$$

Therefore, we subtract the previous expression by $1/24*0*\phi_e$, with zero as above we got right hand side of the expression as:

$$
\begin{aligned}
\frac{1}{24}\phi_{0,1}(\tau,z)-\frac{1}{2}\Lambda_2(\tau/2)\phi_{-2,1}(\tau,z))\frac{\eta_e(\tau/2)}{\eta_e(\tau)}+\\
\left(\frac{1}{24}\phi_{0,1}(\tau,z)-\frac{1}{2}\Lambda_2(\tau/2+1/2)\phi_{-2,1}(\tau,z)\right)\frac{\eta_{-e}(\tau/2)}{\eta_{-e}(\tau)}\\
-(\frac{1}{24}\phi_{0,1}(\tau,z)+\Lambda_2(\tau)\phi_{-2,1}(\tau,z))C_{-e}\eta_{-e}(\tau)
\end{aligned}
$$

Apply lemma 2.1, we can prove that the expression does coincide with the expression of ϕ_e that we have, it remains to show that $F_e(\tau)=0$
In order to prove this fact, we want to write everything in terms of $\theta_3 := \theta_3(\tau,0)$ and $\theta_4 := \theta_4(\tau,0)$. In order to do that, we show some preliminary formulas about theta-functions:

$$\frac{\eta_{-e}(\tau/2)}{\eta_{-e}(\tau)}=\frac{\theta_4^2}{\eta(\tau)^{12}} \tag{98}$$

$$\frac{\eta_e(\tau/2)}{\eta_e(\tau)}=\frac{\theta_3^2}{\eta(\tau)^{12}} \tag{99}$$

$$2^{12}\eta_{-e}(\tau)=\frac{\theta_2^{12}}{\eta(\tau)^{12}} \tag{100}$$

Also, since $\Lambda_2(\tau)$ is theta series of D_4 lattice and $\Lambda_2(\tau/2)$ is theta series of the dual D_4 lattice, we know that:

$$\Lambda_2(\tau/2)=\theta_3^4+\theta_2^4 \tag{101}$$

$$\Lambda_2(\tau/2+1/2)=\theta_4^4-\theta_2^4 \tag{102}$$

$$\Lambda_2(\tau/2)=\frac{1}{2}\theta_3^4+\frac{1}{2}\theta_2^4 \tag{103}$$

Therefore,making use of the identity that $\theta_2^4 = \theta_3^4 - \theta_4^4$*, we deduce,*

$$F_e(\tau) * 2\eta(\tau)^2 = \frac{1}{2}(\theta_3^4 + \theta_4^4) - 1/2(2\theta_4^4 - \theta_3^4) - 1/2(\theta_3^4 + \theta_4^4)(\theta_3^4 - \theta_4^4)^3 = 0 \quad (104)$$

q.e.d.

Therefore, we have shown that there is indeed a coincidence between Conway moonshine module and K3 surfaces, namely

$$Z_{K3} = -\mathrm{tr}_{V_{\mathrm{tw}}^{s\natural}} \mathfrak{z} y^{J(0)} q^{L(0)-c/24} \quad (105)$$

4 References

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