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Patching and local-global principles for gerbes over semi-global fields with an application to homogeneous spaces

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Doctor of Philosophy

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An abstract of
A dissertation submitted to the Faculty of the
James T. Laney School of Graduate Studies of Emory University
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Abstract

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By Bastian Haase

Starting in 2007, Harbater and Hartmann introduced a new patching setup for semi-global fields, establishing a patching framework for vector spaces, central simple algebras, quadratic forms and other algebraic structures. In subsequent work with Krashen, the patching framework was refined and extended to torsors and certain Galois cohomology groups. After describing this framework, we will discuss an extension of the patching equivalence to bitorsors and gerbes. Building up on these results, we then proceed to derive a characterisation of a local-global principle for gerbes and bitorsors in terms of factorization. These results can be expressed in the form of a Mayer-Vietoris sequence in non-abelian hypercohomology with values in the crossed-module $G \to \mathrm{Aut}(G)$. After proving the local-global principle for certain bitorsors and gerbes using the characterization mentioned above, we conclude with an application on rational points for homogeneous spaces via a study of the associated quotient stack.

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This thesis is dedicated to Julie.

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Chapter 1

Introduction

The main topic of this thesis concerns patching and local-global principles for bitorsors and gerbes over semi-global fields with an application towards homogeneous spaces.

Local-global principles for varieties over global fields have been (and still are) a popular topic in algebraic geometry. Starting with the famous Hasse principle, there have been a plethora of results proving local-global principles for certain varieties. Additionally, the Brauer-Manin obstruction has been introduced to study the failure of the local-global principle to great success. In particular, Sansuc ([San81]) was able to completely answer when the local-global principle holds for G-torsors under connected linear algebraic groups. This was then extended to homogeneous spaces, most notably by Borovoi ([Bor92b], [Bor93], [Bor95], [BK97]).

In 2007, Harbater and Hartmann introduced a novel patching framework for function fields of arithmetic curves, so called *semi-global fields* (compare [HH10]). They proved that patching holds for vector spaces, central simple algebras and quadratic forms. Their patching framework also introduced an analogue of the local fields associated with global fields. Hence, it makes sense to ask which varieties over semi-global fields satisfy the local-global principle with respect to this patching framework.

Harbater, Hartmann and Krashen extended the patching framework to torsors and Galois cohomology (compare [HHK15a] and [HHK14]). By use of their patching result, they were also able to deduce a characterization of when the local-global principle holds for G-torsors in terms of a factorization condition on G.

They also proved that patching holds for certain homogeneous spaces, which in turn implies new and improved bounds on the u-invariant and the period index problem for semi-global fields ([HHK09]).

In this thesis, we want to extend patching to bitorsors and gerbes. We will then study when the local-global principle holds for those objects. This will be achieved via studying characterizations of local-global principles for bitorsors and gerbes that follows from the patching result.

We will then use this results to turn our attention to homogeneous spaces. By studying their quotient stack (which is a gerbe), we can use our results on gerbes to prove local-global principles for certain homogeneous spaces.

1.1 Overview

In Chapter 2, we will start by reviewing background material concerning Grothendieck topologies, descent and stacks.

Chapter 3 discusses basic definitions of the theory of algebraic groups, leading to the classification of split semisimple groups. We continue with a discussion of étale cohomology and torsors. Lastly, we discuss why torsors are classified by the first étale cohomology groups.

Next, in chapter 4, we introduce hypercohomology groups with values in crossed modules following Borovoi's cocyclic description. Apart from the main definitions, we also briefly discuss their relation to bitorsors and gerbes.

Our last background chapter, Chapter 5, introduces the main setup considered in this thesis: The patching setup of Harbater, Hartmann and Krashen of semi-global fields. After discussing the abstract setup, we discuss two particular patching setups in detail. We then focus on patching and local-global principles for torsors, which leads us naturally to the topic of factorization of linear algebraic groups. Lastly, we discuss separable factorization and patching for Galois cohomology.

The remaining chapters cover the novel material of the thesis.

In Chapter 6, we prove that patching for torsors implies patching for bitorsors. Let G, H be linear algebraic groups and let P be a (G, H)-bitorsors. Breen ([Bre90]) has given an equivalent description of P in terms of the left G-torsor P and a G-equivariant morphism $P \to \mathrm{Isom}(H, G)$. This description allows us to deduce patching for bitorsors to patching for torsors.

This result can be interpreted as a Mayer-Vietoris type sequence in hypercohomology with values in the crossed module $G \to \mathrm{Aut}(G)$. This sequence in turn allows us to characterize the local-global principle for G-bitorsors in terms of factorization of the center of G.

We conclude the chapter with a factorization result for neutral bitorsors the we will later relate to a local-global principle for gerbes.

Chapter 7 discuss patching for gerbes. We first review the definition of gerbes and bands and prove a result for patching the second non-abelian cohomology group. We then proceed by discussing a semi-cocyclic description of gerbes in terms of bitorsors by Breen ([Bre90]). This description allows us to reduce gerbe patching to bitorsor patching, under one technical condition.

With this patching result, we can extend our Mayer-Vietoris sequence of the previous chapter and characterize gerbe patching in terms of factorization for bitorsors.

While the last two chapters were describing results over an arbitrary inverse factorization system, we discuss their concrete realization to the HHK patching setup in Chapter 8. We also discuss various factorization results for linear algebraic groups such as $\mathrm{SL}_1(D)$ or split semisimple adjoint groups of type B_n. This in turn implies local-global principles for gerbes banded by those groups using the characterization of the previous chapter.

In the last chapter, Chapter 9, we study local-global principles for homogeneous spaces in the patching setup. Let X be a homogeneous space under a linear algebraic group H. Our main tool is to consider the quotient stack $[X/H]$ which is in fact a gerbe, whose band is induced by the geometric stabilizers of X. Using our local-global results for gerbes, we can deduce local-global principles for X.

Chapter 2

Grothendieck topologies, descent and stacks

In the following chapter, will recall standard results in étale descent theory and then review notions of Grothendieck topologies and stacks. This chapter is mainly expository and we will omit some proofs/details.

Our main references for the material presented here are [Ols16] and [LMB00].

2.1 Grothendieck Topologies

Let $\mathcal{C}$ be a category. A functor $F\colon \mathcal{C}^{op} \to \mathrm{Sets}$ is also called a presheaf. In order to define a sheaf over $\mathcal{C}$, we need the notion of coverings. This is precisely what a Grothendieck topology on a category defines.

Definition 1. A Grothendieck topology $\mathcal{T}$ on $\mathcal{C}$ is given by a set $\mathrm{Cov}(\mathcal{C})$ of families of morphisms $\{U_i \to U\}_{i\in I}$ such that

- $\mathrm{Cov}(\mathcal{C})$ contains isomorphisms: if $\phi\colon V_i \to U$ is an isomorphism, then $\phi\colon V \to U$ is in $\mathrm{Cov}(\mathcal{C})$.

- $\mathrm{Cov}(\mathcal{C})$ is closed under composition: if $\{\phi_i\colon V_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and, for all

$i \in I$ we have a covering $\{\alpha_{ij} \colon V_{ij} \to V_i\}_{j \in I_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{\phi_i \circ \alpha_{ij} \colon V_{ij} \to U\}_{i,j} \in \mathrm{Cov}(\mathcal{C})$.

- Cov($\mathcal{C}$) is closed under base change: if $\{\phi_i \colon V_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ and $f \colon W \to U$ is any morphism in $\mathcal{C}$, then $V_i \times_U W$ exists for all $i \in I$ and $\{V_i \times_U W \to W\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$.

An element $\{V_i \to U\} \in \mathrm{Cov}(\mathcal{C})$ is called a covering. A site $(\mathcal{C}, \mathcal{T})$ consists of a category $\mathcal{C}$ equipped with a Grothendieck topology.

Examples 1.

- Let X be a topological space and let $\mathcal{C}_X$ denote the associated category of open subsets: objects in $\mathcal{C}_X$ are open subsets of X and morphisms are given by inclusions. We can endow $\mathcal{C}_X$ with a Grothendieck topology induced by the topology on X: a family of morphisms $\{V_i \to U\}$ is a covering if and only if $U = \bigcup_i V_i$. In the special case where X is a scheme, we call $\mathcal{C}_X$ the small Zariski site of X.

- Let X be a scheme and consider the category Sch/X whose objects are morphisms $U \to X$ and whose morphisms are commutative triangles

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X. & \end{tikzcd}$$

(We will often write U for the object $U \to X$ when the base morphism is clear from context) We endow Sch/X with a Grothendieck topology by saying that a family of morphisms $\{\phi_i \colon V_i \to U\}$ is in $\mathrm{Cov}((\mathrm{Sch/X})_{\acute{e}t})$ if and only if each ϕ_i is an open immersion and $U = \bigcup_i \phi_i(V_i)$. The resulting site, which we will denote by $(\mathrm{Sch/X})_{Zar}$, is called the big Zariski site of X.

- Let X be a scheme and consider the category $X_{ét}$ whose objects are étale morphisms $U \to X$ and whose morphisms are commutative triangles

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & U \arrow[dl] \\ & X. & \end{tikzcd}$$

  (Note that this forces $U' \to U$ to also be étale.) We endow $X_{ét}$ with a Grothendieck topology by saying that a family of morphisms $\{\phi_i \colon V_i \to U\}$ is in $\mathrm{Cov}(X_{ét})$ if and only if $U = \bigcup_i \phi_i(V_i)$. The resulting site, which we will also denote by $X_{ét}$, is called the small étale site of X. As every open immersion is étale, it follows that the étale topology is finer than the Zariski topology.

- Let X be a scheme and consider the category Sch/X. We endow Sch/X with a Grothendieck topology by saying that a family of morphisms $\{\phi_i \colon V_i \to U\}$ is in $\mathrm{Cov}((\mathrm{Sch/X})_{ét})$ if and only if each ϕ_i is étale and $U = \bigcup_i \phi_i(V_i)$. The resulting site, which we will denote by $(\mathrm{Sch/X})_{ét}$, is called the big étale site of X.

- Repeating the last two examples with faithfully flat finitely presented morphisms yields the fppf topology: the small fppf site of X (denoted by X_{fppf}) and the big fppf site of X, denoted by $(\mathrm{Sch/X})_{fppf}$. As every étale morphism is in particular faithfully flat and finitely presented, we observe that that the fppf topology is finer than the étale topology.

With the notion of a covering, we can define sheaves on a site to be presheaves that satisfy a descent axiom with respect to the Grothendieck topology.

Definition 2. Let $(\mathcal{C}, \mathcal{T})$ be a site and let $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ be a presheaf in sets. We say that F is a sheaf if, for all coverings $\{U_i \to U\}_{i \in I}$, the following sequence is exact:

$$F(U) \longrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{(i,j) \in I^2} F(U_i \times_U U_j)$$

is exact.

A morphism of sheaves $F \to F'$ is just a natural transformation of the underlying functors.

The descent results in Section 2.3 yield an important class of sheaves.

Example 1. Let X be a scheme and let $Y \to X \in (\mathrm{Sch/X})_{\acute{e}t}$. Then, the functor

$$h_Y \colon (\mathrm{Sch/X})_{\acute{e}t}^{\mathrm{op}} \longrightarrow \mathrm{Sets}$$
$$U \to X \mapsto \mathrm{Hom}(U, Y)$$

is a sheaf by Lemma 2.3.1. A sheaf F on $(\mathrm{Sch/X})_{\acute{e}t}$ isomorphic to h_Y for some $Y \in (\mathrm{Sch/X})_{\acute{e}t}$ is called representable.

2.2 Categories fibered in groupoids

Categories fibered in groupoids are the category-theoretic notion we will use to define stacks, just like presheaves are the category-theoretic notion used to define sheaves. As the notion of a presheaf is often first introduced by looking at the presheaf of regular functions, we will also motivate the definition of a category fibered in groupoids by looking at an example.

Let G be an algebraic group over a scheme S and let BG denote the classifying stack of G-torsors. We will later see that this is in fact an (algebraic) stack (and even a gerbe). Let us first define it as a category:

1. Objects: An object in BG is a tuple $(X \to S, P)$, where $X \to S$ is an object in Sch/S and P is a left $G|_X$-torsor over X in the étale topology. We will often write (X, P) to simplify notation.

2. Morphisms: A morphism $(X, P) \to (X', P')$ consists of a morphism $f \colon X \to X'$ in Sch/X and an isomorphism $\alpha \colon f^*P' \to P$ of $G|_X$-torsors.

Note that there is a natural functor $BG \to \mathrm{Sch/S}$ which leads us to our first definition.

Definition 3. let $\mathcal{C}$ be a category. A category over $\mathcal{C}$ is a functor $\mathcal{D} \to \mathcal{C}$. We will often say that $\mathcal{D}$ is a category over $\mathcal{C}$ if the functor $\mathcal{D} \to \mathcal{C}$ is clear from context.

Recall that we have the notion of pullbacks in BG: Given an object $(X, P) \in BG$ and a morphism $f\colon Y \to X$, we get another object $(Y, f^*P) \in BG$. Note that f^*P is not unique, which leads us to axiomatizing the existence of pullbacks through the notion of cartesian arrows.

Definition 4. Let $p\colon \mathcal{D} \to \mathcal{C}$ be a category over $\mathcal{C}$. We say that a morphism $\eta \to \zeta \in \mathcal{D}$ is cartesian if any commutative diagram

$$\begin{tikzcd}
\tau \arrow[r, dotted, "?"] \arrow[rr, bend left, "\alpha"] \arrow[d, "p"] & \eta \arrow[r] \arrow[d, "p"] & \zeta \arrow[d, "p"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C
\end{tikzcd}$$

where $p(\alpha) = g \circ f$ can be uniquely completed by a morphism $\alpha'\colon \tau \to \eta$ satisfying $p(\alpha') = f$.

We say that $\mathcal{D} \to \mathcal{C}$ is a category fibered over $\mathcal{C}$ if for all $\zeta \in \mathcal{D}$ and any morphism $f\colon B \to p(\zeta) \in \mathcal{C}$, there exists a cartesian arrow $\beta\colon \eta \to \zeta$ with $p(\beta) = f$. This is called a pullback of η along f.

As the natural map $(Y, f^*P) \to (X, P)$ is cartesian, it follows that $BG \to \mathrm{Sch/X}$ is a category fibered over Sch/X.

Fixing a base category $\mathcal{C}$, let us define the (2)-category of fibered categories over $\mathcal{C}$, denoted by $\mathrm{FIB}(\mathcal{C})$:

1. Objects: An object is given by fibered categories over $\mathcal{C}$.

2. 1-Morphisms: A 1-morphism between $p\colon \mathcal{D} \to \mathcal{C}$ and $p'\colon \mathcal{D}' \to \mathcal{C}$ is given by a commutative (**not** 2-commutative) diagram of functors:

$$\begin{tikzcd} \mathcal{D} \arrow[rr] \arrow[dr, "p"] & & \mathcal{D}' \arrow[dl, "p'"] \\ & \mathcal{C}. & \end{tikzcd}$$

3. 2-Morphisms: A 2-morphism between two 1-morphisms $\alpha, \beta\colon (\mathcal{D}, p) \to (\mathcal{D}', p')$ is given by a natural transformation $\alpha \Rightarrow \beta$ such that for all $\eta \in \mathcal{D}$, the map $\alpha(\eta) \to \beta(\eta)$ maps to the identity under p'.

An equivalence of fibered categories over $\mathcal{C}$ is defined to be a quasi-isomorphism in FIB($\mathcal{C}$)

We will now define fibers of a fibered category $p\colon \mathcal{D} \to \mathcal{C}$. This will both explain the name *fibered category* as well as shed light on the mysterious condition on 2-morphisms.

Given $U \in \mathcal{C}$, the fiber of $p\colon \mathcal{D} \to \mathcal{C}$ over U, denoted by $\mathcal{D}(U)$, is the following category:

1. Objects: An object is an object $\eta \in \mathcal{D}$ such that $p(\eta) = U$.

2. Morphisms: A morphism $\eta \to \eta'$ is given by a morphism $\alpha\colon \eta \to \eta' \in \mathcal{D}$ such that $p(\alpha) = \mathrm{id}_U$ holds.

Note that for a map $U \to V \in \mathcal{C}$, we can define a (unique up to unique isomorphism) pullback map $\mathcal{D}(V) \to \mathcal{C}(U)$.

Given a scheme $X \to S$ over S the fiber $BG(X)$ is just the category of $G|_X$-torsors over X with morphisms given by $G|_X$-invariant morphisms. This is where we can see that the requirement $p(\alpha) = \mathrm{id}_U$ is natural: Otherwise, we would allow morphisms $P \to P'$ in $BG(X)$ that are not compatible with the structure morphisms $P \to X$ and $P' \to X$.

As a G-equivariant morphism of torsors is an isomorphism, we observe that the categories $BG(X)$ are in fact a groupoids, i.e. categories where all morphisms are isomorphisms.

Definition 5. A fibered category $\mathcal{D} \to \mathcal{C}$ is a category fibered in groupoids over $\mathcal{C}$ if, for all $U \in \mathcal{C}$, the fiber $\mathcal{D}(U)$ is a groupoid.

A stack will later be defined to be a category fibered in groupoids satisfying a descent criterion.

Just as the Yoneda Lemma is crucial to embed the category of schemes into the category of algebraic spaces, the 2-Yoneda Lemma is crucial to embed the category of schemes into the category of stacks.

Given an element $U \in \mathcal{C}$, let $\mathcal{C}/U$ denote the localization of $\mathcal{C}$ at U, i.e. the category of morphisms $V \to U \in \mathcal{C}$. This category admits a natural functor $\mathcal{C}/U \to \mathcal{C}$ making it a category fibered in groupoids (in fact in setoids!).

Let $\mathcal{D} \to \mathcal{C}$ be another category fibered over $\mathcal{C}$. There is a natural functor

$$\begin{aligned} \mu\colon\ \mathrm{Hom}_{\mathrm{FIB}(\mathcal{C})}(\mathcal{C}/U, \mathcal{D}) &\longrightarrow \mathcal{D}(U) \\ g &\mapsto g(\mathrm{id}_U). \end{aligned}$$

The 2-Yoneda Lemma asserts that this functor is an equivalence.

Lemma 2.2.1 (2-Yoneda Lemma)**.** *The functor μ is an equivalence of categories.*

Proof. Compare [Ols16, Proposition 3.2.2]. □

In particular, for a scheme $X \to S$ over S, the category of morphisms Sch/X $\to$ BG is equivalent to the category of $G|_X$-torsors $BG(X)$.

Another special case is that the set of morphisms $X \to Y \in$ Sch/S is in a natural bijection with the set of equivalences of fibered categories over Sch/S between Sch/X and Sch/Y.

This results thus allows us to embed the category of schemes over S into the category of fibered categories over Sch/S via identifying X with Sch/X, i.e. the functor

$$\mathrm{Sch/S} \to \mathrm{FIB(Sch/S)}$$

sending $X \to S$ to $\mathrm{Sch/X} \to \mathrm{Sch/S}$ is fully faithful. We will often abuse notation and write for instance $X \to BG$ as opposed to $\mathrm{Sch/X} \to BG$.

2.3 Descent

While the last chapter described the category-theoretic part of stacks, this chapter focuses on the geometric aspect. Apart from the main references for the appendix ([Ols16] and [LMB00]), a paper of Vistoli ([Vis89], especially the appendix) is an excellent reference for this chapter.

The main result we want to cover is the following theorem.

Theorem 2.3.1. *Let X be a scheme and let $Y \in \mathrm{Sch/X}$. Then, the presheaf h_Y is a sheaf in the fppf topology.*

In more concrete terms, the above result says that we can glue morphisms in the fppf topology. As the fppf topology is finer than the étale topology, we immediately obtain the following corollary.

Corollary 2.3.2. *Let X be a scheme and let $Y \in \mathrm{Sch/X}$. Then, the presheaf h_Y is a sheaf in the étale topology.*

The proof of theorem 2.3.1 will consist of multiple intermediate results and reductions. Roughly, we will first prove the affine case and then reduce the general case to the affine case. So, our first goal is to prove that we can patch morphisms between affine schemes in the fppf topology.

Proposition 2.3.3. *Let $Y \to Y'$ be a faithfully flat morphism of affine schemes in* $\mathrm{Sch/X}$. *Let U be an affine scheme in* $\mathrm{Sch/X}$. *Then, the sequence*

$$h_U(Y') \longrightarrow h_U(Y) \rightrightarrows h_U(Y \times_Y Y')$$

is exact.

Let us translate this statement into a statement in commutative algebra. Let $Y = \operatorname{Spec}(B)$, $Y' = \operatorname{Spec}(A)$ and let $R = \operatorname{Spec}(U)$. The morphism $Y \to Y'$ corresponds to a homomorphism $f\colon A \to B$ of commutative rings with multiplicative identity. The morphisms $Y' \times_Y Y' \to Y'$ correspond to the two projections $B \to B \otimes_A B$ which we will denote by pr_1 and pr_2 respectively. The statement of Proposition 2.3.3 translates to proving that

$$\operatorname{Hom}(R, A) \longrightarrow \operatorname{Hom}(R, B) \rightrightarrows \operatorname{Hom}(R, B \otimes_A B)$$

is exact (Here, $\operatorname{Hom}(R, A)$ denotes the set of homomorphisms of rings with multiplicative identity). We need the following proposition to prove this.

Proposition 2.3.4. *Let $A \to B$ be a faithfully flat ring map. The sequence of A-modules*

$$A \longrightarrow B \rightrightarrows B \otimes_A B$$

is exact.

Proof. As $A \to B$ is faithfully flat, it is equivalent to prove that the sequence

$$B \xrightarrow{f} B \otimes_A B \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} B \otimes_A B \otimes_A B$$

is exact. The first map is given by $b \mapsto 1 \otimes b$, whereas the second and third map are given by $b_1 \otimes b_2 \mapsto b_1 \otimes 1 \otimes b_2$ and $b_1 \otimes b_2 \mapsto 1 \otimes b_1 \otimes b_2$. Note that a f is a section to the map $m\colon B \otimes_A B \to B$ given by $b_1 \otimes b_2 \mapsto b_1 b_2$ and is thus injective.

Assume now that $b_1 \otimes b_2$ is in the equalizer of pr_1 and pr_2. Then, consider $m'\colon B \otimes_A B \otimes_A B \to B \otimes_A B$ defined via $b_1 \otimes b_2 \otimes b_3 \mapsto b_1 \otimes b_2 b_3$. Note that $m' \circ \mathrm{pr}_1 = \mathrm{id}$ and

$m' \circ \mathrm{pr}_2 = f \circ m$. Hence,

$$b_1 \otimes b_2 = (m' \circ \mathrm{pr}_1)(b_1 \otimes b_2) = (m' \circ \mathrm{pr}_2)(b_1 \otimes b_2) = (f \circ m)(b_1 \otimes b_2)$$

holds. Thus, the image of f equals the equalizer of pr_1 and pr_2. □

Proof of Proposition 2.3.3. As $\mathrm{Hom}(R, \circ)$ is left exact, Proposition 2.3.4 yields that $\mathrm{Hom}(R, A) \to \mathrm{Hom}(R, B)$ is injective. Fix $f \colon R \to B$ and assume that $\mathrm{pr}_1 \circ f = \mathrm{pr}_2 \circ f$. Thus, for any $r \in R$, we have $1 \otimes_A f(r) = f(r) \otimes_A 1$. Hence, $f(r) \in A$ so that f is induced by a morphism $R \to A$. □

Proposition 2.3.5. *Let* $F \colon \mathrm{Sch/X} \to \mathrm{Sets}$ *be a sheaf in the big Zariski topology of* X*. Then, the following are equivalent*

1. *F is a sheaf in the big fppf topology of X*

2. *For all fppf covers of the form $U \to V$ in* $\mathrm{Sch/X}$*, the sequence*

$$F(V) \longrightarrow F(U) \rightrightarrows F(U \times_V U)$$

is exact.

3. *F is a sheaf in the big Zariski topology and for all fppf covers of the form $U \to V$ in* $\mathrm{Sch/X}$ *with U and V affine, the sequence*

$$F(V) \longrightarrow F(U) \rightrightarrows F(U \times_V U)$$

is exact.

Sketch: Clearly, it is enough to show that 3) implies 1). So, assume that condition 3) holds and let $\{U_i \to V\}$ be a fppf cover. By replacing the collection of morphisms $U_i \to V$ with the single morphism $\bigsqcup_i U_i \to V$ it is easy to see that we may assume that the cover is given by a single morphism $U \to V$. (This is essentially proves that the first two conditions are equivalent.

Let us check that the map $F(V) \to F(U)$ is injective. Cover $V = \bigcup_i V_i$ by open affines and let $U = \bigcup_{i,j} U_{ij}$ be an affine covering of U, so that $\{U_{ij} \to U_i\}$ is an fppf cover. Consider the commutative diagram

$$\begin{tikzcd} F(V) \arrow[r] \arrow[d] & F(U) \arrow[d] \\ \prod_i F(V_i) \arrow[r] & \prod_{i,j} F(U_{ij}) \end{tikzcd}$$

and note that the bottom row is injective as F by assumption. As F is a Zariski sheaf, the horizontal arrows are also injective and thus $F(V) \to F(U)$ is also injective.

To prove the gluing axiom, let us first reduce to the case where U is affine. For general V, pick a cover $V_i \to V$ of affine open subsets V_i and let U_i be the preimage of V_i in U. Consider the commutative diagram

$$\begin{tikzcd} F(V) \arrow[r] \arrow[d] & F(U) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F(U \times_V U) \arrow[d] \\ \prod_i F(V_i) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_i F(U_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & F(U_i \times_{V_i} U_i) \\ \prod_{i,j} F(V_i \cap V_i) \arrow[r] & \prod_{i,j} F(U_i \cap U_j) & \end{tikzcd}$$

By use of a diagram chase, one can show that it is enough to prove that

$$F(V_i) \to F(U_i) \rightrightarrows F(U_i \times_{V_i} U_i)$$

is exact for all i.

We will leave another reduction to the case where U is quasi-compact to the reader. Assuming this, let $U = \bigcup_{i=1}^n U_i$ be a finite affine cover of U. Then, $\bigsqcup_i U_i \to U$ is an fppf-cover and $\bigsqcup_i U_i$ is affine. Hence, in the commutative diagram

$$\begin{tikzcd}
F(V) \arrow[r] \arrow[d, equal] & F(U) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & F(V \times_U V) \arrow[d, hook] \\
F(V) \arrow[r] & F(\bigsqcup_i U_i) \arrow[r, shift left] \arrow[r, shift right] & F(\bigsqcup_i U_i \times_V \bigsqcup_i U_i)
\end{tikzcd}$$

the middle vertical arrow is injective. By assumption, the bottom row is exact. Hence, the upper row is also exact in the middle. □

Proof of Theorem 2.3.1. Note that we may assume without loss of generality that $X = \mathrm{Spec}(\mathbb{Z})$. If Y is affine, then the statement follows from Proposition 2.3.3 and Proposition 2.3.5.

For the general case, by Proposition 2.3.5, it is enough to prove exactness of the sequence

$$h_Y(V) \to h_Y(U) \rightrightarrows h_Y(U \times_V U)$$

where $U \to V$ is an fppf cover of affine schemes. To prove injectivity, let $f, g \in h_Y(V)$ such that their image in $h_Y(U)$ agree. This means that f, g are morphisms $V \to Y$ that, when precomposed with the cover $U \to V$, yield the same morphism $U \to Y$. This implies that f, g agree set-theoretically, as $U \to V$ is by definition surjective. Let $Y' \subset Y$ be an open affine subscheme of Y. Then, on $f^{-1}(Y') = g^{-1}(Y')$, f and g agree by the affine case. Covering Y with open affines finishes the proof of injectivity.

To prove exactness in the middle, let $f\colon U \to Y$ be an element in the equalizer $h_Y(U) \rightrightarrows h_y(U \times_V U)$. We omit the verification that this implies in particular, that the compositions

$$|U \times_V U| \rightrightarrows |U| \xrightarrow{|f|} |V|$$

are the same, where $|U|$ denotes the underlying set of the scheme U. Hence, by the universal property, we see that there is a set-theoretical map $h\colon |V| \to |Y|$ such that

$$
\begin{tikzcd}
|U| \arrow[r, "|f|"] \arrow[dr] & |Y| \\
& |V| \arrow[u, "h"]
\end{tikzcd}
$$

commutes. It remains to promote h to a scheme-theoretic map. This can be done on affine patches; the uniqueness of a lift follows from the injectivity of the first morphism, it guarantees that we can glue along the patches to get a map schemes $h\colon V \to Y$. By construction, this is a preimage of f. □

2.4 Stacks

Just like a sheaf is a presheaf satisfying a gluing/descent condition, a stack is a category fibered in groupoids satisfying a gluing/descent condition. To make this precise, we will first define a descent functor for categories fibered in groupoids over sites. We will then state the definition of a stack and reconsider our running example BG.

Let $\mathcal{D} \to \mathcal{C}$ be a category fibered in groupoids where $\mathcal{C}$ is a site. Fix now a collection of morphisms $\{Y_i \to Y\}_{i\in I}$ in $\mathcal{C}$. We mostly care about the case where this collection is a cover, but this is not relevant yet. Let us define a descent category of $\mathcal{D}$ with respect to $\{Y_i \to Y\}$ which we will denote by $\mathcal{D}(\{Y_i \to Y\}_{i\in I})$:

1. Objects: An object is given by a tuple $(\{\eta_i\}_{i\in I}, \{\sigma_{ij}\}_{i,j\in I})$ where $\eta_i \in \mathcal{D}(Y_i)$ and $\sigma_{ij}\colon \eta_i|_{Y_i\times_Y Y_j} \to \eta_j|_{Y_i\times_Y Y_j}$ such that the following diagram is commutative:

$$
\begin{tikzcd}
\mathrm{pr}_{12}^*\,\mathrm{pr}_1^*\,\eta_i \arrow[r, "\mathrm{pr}_{12}^*\,\sigma_{ij}"] \arrow[d, equal] & \mathrm{pr}_{12}^*\,\mathrm{pr}_2^*\,\eta_j \arrow[r, "\simeq"] & \mathrm{pr}_{23}^*\,\mathrm{pr}_1^*\,\eta_j \arrow[d, "\mathrm{pr}_{23}^*\,\sigma_{jk}"] \\
\mathrm{pr}_{13}^*\,\mathrm{pr}_1^*\,\eta_i \arrow[r, "\mathrm{pr}_{13}^*\,\sigma_{ik}"] & \mathrm{pr}_{13}^*\,\mathrm{pr}_2^*\,\eta_k \arrow[r, "\simeq"] & \mathrm{pr}_{23}^*\,\mathrm{pr}_2^*\,\eta_k
\end{tikzcd}
$$

(which we should think of as the usual cocycle condition $\sigma_{ij}\sigma_{jk} = \sigma_{ik}$). An object is also called a gluing datum.

2. Morphisms: A morphism $(\{\eta_i\}_{i\in I}, \{\sigma_{ij}\}_{i,j\in I}) \to (\{\eta'_i\}_{i\in I}, \{\sigma'_{ij}\}_{i,j\in I})$ is given by a collection of morphisms $\eta_i \to \eta'_i \in \mathcal{D}(Y_i)$ that satisfy the obvious compatibility condition with σ_{ij} and σ'_{ij}.

Note that there is a natural functor

$$\mathcal{D}(Y) \to \mathcal{D}(\{Y_i \to Y\}_{i\in I})$$

sending an object η to the canonical descent datum $(\{\eta|_{Y_i}\}_{i\in I}, \{\sigma^{\text{can}}_{ij}\}_{i,j\in I})$. Here σ^{can}_{ij} are the canonical isomorphisms induced by the universal property of pullbacks.

Let us consider our running example $BG \to$ Sch/S, where we equip Sch/S with the étale topology. Let $\{Y_i \to Y\}_{i\in I}$ be an étale covering in Sch/S. An object in $BG(\{Y_i \to Y\}_{i\in I})$ is given by a collection of $G|_{Y_i}$-torsors P_i over Y_i together with isomorphisms $\sigma_{ij}\colon P_i|_{Y_i\times_Y Y_j} \to P_j|_{Y_i\times_Y Y_j}$ satisfying the gluing condition. Compare [Ols16, Section 4.5] for a detailed proof that this induces a $G|_Y$-torsor P over Y together with isomorphisms $P|_{Y_i} \to P_i$ that are compatible with σ_{ij}. This follows readily from the fact that we can glue sheaves as this implies that we can glue the G-action (as this action can be expressed in terms of morphisms of sheaves) and the fact that being simply transitive is a local property in the étale topology.

In other words, the natural functor $BG(Y) \to BG(\{Y_i \to Y\}_{i\in I})$ is essentially surjective. Furthermore, we have also seen that we can glue morphisms in the étale topology - this just means that the functor is fully faithful. We define a stack to be a category fibered in groupoids where *descent works*; i.e. the above functor is an equivalence for all coverings.

Definition 6. Let $\mathcal{D} \to \mathcal{C}$ be a category fibered in groupoids. We say that $\mathcal{D} \to \mathcal{C}$ is a stack if, for all coverings $\{Y_i \to Y\}_{i\in I}$ the natural functor

$$\mathcal{D}(Y) \to \mathcal{D}(\{Y_i \to Y\}_{i\in I})$$

is an equivalence.

By our discussion above, $BG \to \mathrm{Sch/S}$ is a stack. A morphism of stacks is just a 1-morphism of categories fibered in groupoids. Let $f, g : \mathcal{X} \to \mathcal{Y}$ be two morphisms of stacks. A morphism $f \Rightarrow g$ is just a natural transformation of functors.

We denote by STACKS the 2-category of stacks where 1-morphisms are given by morphisms of stacks and 2-morphisms are given by natural transformations.

Chapter 3

Algebraic groups

In this chapter, we recall the basic notions of algebraic groups and discuss their classification. We will then proceed and study specific groups a little closer, as we will need this when discussing bitorsor factorization in Section 8.2. We will end the section by a review of (étale) cohomology of algebraic groups (or, more generally, étale group sheaves) and torsors.

The main references for this chapter are [KMTR98], [Bor91], [Ser88] and [Sko01].

3.1 Introduction

In this section, we will give basic definitions and constructions without proofs. Let us fix a perfect ground field F and let G be a scheme over F. A collection of morphisms $(e\colon F \to G, m\colon G \times_F G \to G, i\colon G \to G)$ equips G with a group structure if the following conditions are satisfied:

- Associativity: The diagram

$$\begin{tikzcd} G \times_F G \times_F G \arrow[r, "\mu \times \mathrm{id}_G"] \arrow[d, "\mathrm{id}_G \times \mu"] & G \times_F G \arrow[d, "\mu"] \\ G \times_F G \arrow[r, "\mu"] & G \end{tikzcd}$$

commutes.

- Neutral element: The diagrams

$$\begin{tikzcd} \mathrm{Spec}(F) \times_F G \arrow[rr, "e\times \mathrm{id}_G"] \arrow[dr, "\mathrm{pr}_2"] & & G \times_F G \arrow[dl, "\mu"] \\ & G & \end{tikzcd}$$

and

$$\begin{tikzcd} G \times_F \mathrm{Spec}(F) \arrow[rr, "\mathrm{id}_G \times e"] \arrow[dr, "\mathrm{pr}_2"] & & G \times_F G \arrow[dl, "\mu"] \\ & G & \end{tikzcd}$$

commute where $\mathrm{pr}_2 \colon \mathrm{Spec}(F) \times_F G \to G$ denotes the projection on the second factor.

- Inverse Element: Let $\delta \colon G \to G \times_F G$ denote the diagonal map defined via $g \mapsto (g, g)$. The diagrams

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d] & G \times_F G \arrow[dr, "i\times \mathrm{id}_G"] & \\ \mathrm{Spec}(F) \arrow[r, "e"] & G & G \times_F G \arrow[l, "\mu"'] \end{tikzcd}$$

and

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d] & G \times_F G \arrow[dr, "\mathrm{id}_G \times i"] & \\ \mathrm{Spec}(F) \arrow[r, "e"] & G & G \times_F G \arrow[l, "\mu"'] \end{tikzcd}$$

commute.

An algebraic group over F is given by a tuple (G, e, m, i) such that G is a variety over F and (e, m, i) define a group structure on G. We will often omit (e, m, i) and simply speak of G as an algebraic group when there is no risk of confusion.

Examples 2.

- The additive group $\mathbb{G}_a$: The underlying scheme is given by $\mathrm{Spec}(F[T])$; the group structure is given by addition.

- The multiplicative group $\mathbb{G}_m$: The underlying scheme is given by $\mathrm{Spec}(F[T, T^{-1}])$; the group structure is given by multiplication.

- More generally, the group of invertible matrices GL_n: The underlying scheme is given by $\mathrm{Spec}(F[x_{ij}][1/\det])$ where $0 \leq i, j \leq n$ and $\det = \det((x_{ij}))$. Its group structure is induced by matrix multiplication.

Let us now collect some definitions and constructions concerning algebraic groups:

- The fiber product of two algebraic groups is again an algebraic group.

- Let (G, e, m, i) and (H, e', m', i') be algebraic groups over F. A homomorphism of algebraic groups is given by a morphism of schemes $G \to H$ that is compatible with the maps $e, e', m,', i$ and i'. An example of a morphism of algebraic groups is given by the determinant map $\det\colon \mathrm{GL}_n \to \mathbb{G}_m$.

  We denote by $\mathrm{Hom}_F(G, H)$ the sheaf of group homomorphisms which maps any F-scheme to the set of homomorphisms. In particular, we write $\mathrm{Aut}_F(G)$ for the group sheaf of group automorphisms of G. There is a natural map $G \to \mathrm{Aut}_F(G)$ given by conjugation.

- Let $\mathrm{swap}\colon G \times_F G \to G \times_F G$ denote the morphism given by $(a, b) \mapsto (b, a)$. We say that G is abelian if $m \circ \mathrm{swap} = m$, i.e. if for any scheme $S \to F$ and any $a, b \in G(S)$, we have $m(S)(a, b) = m(S)(b, a)$.

- We say that an algebraic group is

  - smooth,

- connected,
- affine

if the underlying scheme has said property.

- Let G be an abstract group. Consider the scheme $G' = \coprod_{g\in G} \mathrm{Spec}(F)$. We define multiplication by sending the component corresponding to (g, g') to the component corresponding to gg'. The inversion map sends the component corresponding to g to the component corresponding g^{-1}. Lastly, the morphism e is given by mapping $\mathrm{Spec}(F)$ on the component corresponding to the neutral element is G. We say that G', equipped with this group structure, is the constant group scheme associated to G. We will often identify G with G'. Note that G' represents the sheafification of the functor

$$\mathrm{Sch/F} \to \mathrm{Groups}$$
$$S \mapsto G.$$

- A subgroup H of an algebraic group G is a subvariety of G such that the restriction of the group structure of G defines a group structure of H. An example is the subgroup SL_n of GL_n of invertible matrices of determinant 1. It is the kernel of the group homomorphism $\mathrm{GL}_n \xrightarrow{\det} \mathbb{G}_m$.

  Another important example is the group of n-th roots of unity μ_n. It is the kernel of the map $\mathbb{G}_m \xrightarrow{()^n} \mathbb{G}_m$

  A subgroup H of G is normal if H is stable under conjugation by G, i.e. if the conjugation map $G \to \mathrm{Aut}(G)$ induces a a map $G \to \mathrm{Aut}(H)$.

- A linear algebraic group is a closed, smooth subgroup of GL_n. In particular, a linear algebraic group is affine. In fact, any affine group is isomorphic to a linear algebraic group.

- An algebraic group G is diagonalizable if it is a closed subgroup of $(\mathbb{G}_m)^n$ for some n. It is of multiplicative type if $G|_{F^{\text{sep}}}$ is diagonalizable.

- An algebraic group G is simple if it is connected and does not admit any connected, normal subgroups. An example of a simple group is given by PSL_n.

- An algebraic group G is solvable if it admits a subnormal series

  $$G = G_0 \supset G_1 \supset \ldots \supset G_t = e$$

  of normal subgroups G_i such that G_i/G_{i+1} is abelian.

  An algebraic group G is nilpotent if in addition G_i/G_{i+1} is contained in the center of G/G_{i+1}.

  The group of upper triangular matrices is solvable, whereas the group of strictly upper triangular matrices is nilpotent.

- The radical of an algebraic group is the identity component (i.e. the connected component of the identity) of its maximal normal solvable subgroup. For example the radical of GL_n is given by the subgroup of scalar matrices.

  An algebraic group is semisimple if its radical is trivial. An example of a semisimple group is given by SL_n, as the identity matrix is the only scalar matrix with determinant 1.

- A linear algebraic group is unipotent if it is a closed subgroup of the group of upper triangular matrices with diagonal entries equal to 1.

  The unipotent radical of G is the maximal unipotent subgroup of the radical of G (which is automatically normal).

  An algebraic group is reductive if its unipotent radical is trivial. Examples of reductive groups are given by SL_n and the group of special orthogonal matrices,

SO_n.

- An isogeny of algebraic groups G, G' is a an epimorphism $G \to G'$ with finite kernel. We say that G and G' are isogenous if an isogeny between them exits.

  A multiplicative isogeny is an isogeny $G \to G'$ where the kernel is of multiplicative type.

- Let G be connected and semisimple. We say that G is simply connected if any multiplicative isogeny $G' \to G$ with G' connected is an isomorphism.

- Given an algebraic group G, there is a subgroup G^{ad} of $\mathrm{Aut}(G)$ such the sequence

$$1 \to Z(G) \to G \to G^{\mathrm{ad}} \to 1$$

  is exact. We say that G^{ad} is the adjoint group of G. We say that G is adjoint if it is isomorphic to G^{ad} via the morphism of the above sequence.

- A Borel subgroup of G is a maximal closed, connected and solvable algebraic subgroup.

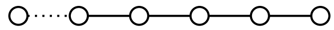
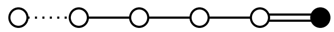
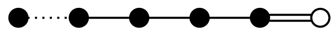
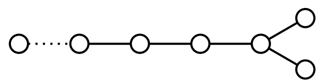
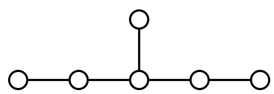
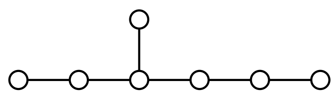
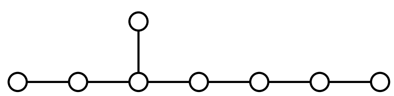
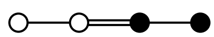
- A torus is an algebraic group G such that $G|_{F^{\mathrm{sep}}} \simeq \mathbb{G}_m^n|_{F^{\mathrm{sep}}}$ for some n. A torus is called split if $G \simeq \mathbb{G}_m^n$. An example of a non-split torus is given by SO_2.

3.2 Classification of split semisimple algebraic groups

Let G be a semisimple algebraic group. A subtorus of G is maximal if it is not contained in any larger subtorus of G. We say that G is split if it contains a split maximal torus. Clearly, if F is separably closed, any semisimple group is split.

For a split semisimple group G, fix a split maximal torus T. Recall that, through the restriction of the adjoint representation of G to T, we obtain a set of non-zero

Table 3.1: Classification of Dynkin diagrams

Type	**Dynkin Diagram** Ψ	$\mathrm{Aut}(\Psi)$
A_n		$\{e\}$ if $n = 1$, $\mathbb{Z}/2\mathbb{Z}$ otherwise
B_n		$\{e\}$
C_n		$\{e\}$
D_n, $n \geq 3$		S_3 if $n = 4$, $\mathbb{Z}/2\mathbb{Z}$ otherwise
E_6		$\mathbb{Z}/2\mathbb{Z}$
E_7		$\{e\}$
E_8		$\{e\}$
F_4		
G_2		$\{e\}$

weights $\Phi(G)$ in $T^* \otimes_{\mathbb{Z}} \mathbb{R}$ that form a root system. This system is, up to isomorphism, independent of the choice of T. We call $\Phi(G)$ the root system of G.

As the root system is an invariant of G, we can study the classification of root systems to study the classification of split semisimple algebraic groups. It turns out that the root system is a *rich* invariant in the sense that it yields a powerful classification for split semisimple groups.

Compare Table 3.1 for a list of all Dynkin diagrams. For later use, we will also note the automorphism group of the Dynkin diagram.

We say that a split semisimple group G is of type X, if the associated Dynkin diagram of G is of type X.

Examples 3. • Let V be a $n+1$ dimensional vector space over F. Then, the group $\mathrm{SL}(V)$ is is of type A_n.

- Let (V, q) be a hyperbolic quadratic form of dimension $2n$. Consider the group orthognal group of (V, q), denoted by $\mathrm{O}(V, q)$ and defined on points via

$$\mathrm{O}(V, q) = \{A \in \mathrm{GL}(V)(R) \mid q_R(Av) = q_r(v) \text{ for } v \in V\}.$$

If $\mathrm{char}(F) \neq 2$, consider the group

$$\mathrm{SO}(V, q) = \ker(\mathrm{O}(V, q) \xrightarrow{\det} \mathbb{G}_m).$$

Then, $\mathrm{SO}(V, q)$ is of type D_n.

Our main application of this classification is the information we get about outer automorphisms of algebraic groups.

Let G be a split semisimple group with split maximal torus T. Fix a simple root system Π in $\Phi(G)$. For any scheme S over F, any element $g \in G(S)$ induces an automorphism $G|_S \to G|_S$ via conjugation. This defines a morphism of group sheaves $\mathrm{Int}\colon G \to \mathrm{Aut}(G)$. In particular, we have a map $G(F) \to \mathrm{Aut}(G)(F)$.

Given an automorphism $\psi \in \mathrm{Aut}(G)(F)$, there is $g \in G(F)$ such that $\psi \circ \mathrm{Int}(g)(\Pi) = \Pi$. Hence, ψ induces an automorphism of $\mathrm{Dyn}(\Phi)$ and we have a group homomorphism $\mathrm{Aut}(G)(F) \to \mathrm{Aut}(\mathrm{Dyn}(\Phi))$, compare [KMTR98, VI.25.B].

For split semisimple adjoint groups, there is a strong relation between outer automorphisms and the automorphism group of the Dynkin diagram induced by the maps described above.

Proposition 3.2.1 ([KMTR98, Proposition 25.15]). *If G is a split semisimple adjoint group, then the sequence*

$$1 \to G(F) \to \mathrm{Aut}(G)(F) \to \mathrm{Aut}(\mathrm{Dyn}(\Phi)) \to 1$$

is exact.

If $\mathrm{Aut}(\mathrm{Dyn}(\Phi))$ is trivial, then the last theorem tells us in particular, that there are no outer automorphisms. This is true even if G is not split, as it becomes split after base change to F^{sep}.

Corollary 3.2.2. *If G is semisimple adjoint of type $A_1, B_n, C_n, E_7, E_8, F_4$ or G_2, then G does not admit outer automorphisms.*

If G is not adjoint, a variant of the last proposition remains true. Recall that $\overline{G} = G/C$ denotes the associated adjoint group of G, where C is the kernel of ad_G.

Theorem 3.2.3 ([KMTR98, Theorem 25.16])**.** *Let G be a split semisimple group. Then, the sequence*

$$1 \to \overline{G}(F) \to \mathrm{Aut}(G)(F) \to \mathrm{Aut}(\mathrm{Dyn}(\Phi))$$

is exact. The last map is surjective and the sequence splits if G is adjoint or simply connected.

If G is simply connceted, the last theorem implies that the group $\mathrm{Aut}(G)$ coincides with $\mathrm{Aut}(\overline{G})$.

Corollary 3.2.4. *Let G be a split semisimple simply connected group. Then, the natural map $\mathrm{Aut}(G) \to \mathrm{Aut}(\overline{G})$ is an isomorphism.*

Proof. In [KMTR98, Corollary 25.17] this is proven for $\mathrm{Aut}(G)(F)$ and $\mathrm{Aut}(\overline{G})(F)$, however this statement implies our statement as all assumption are stable under separable field extensions. □

3.3 Examples

In this section, we will take a closer look at the groups $\mathrm{SO}(q)$ and $\mathrm{SL}_1(D)$ associated to a quadratic form q or a division algebra D. We will use the results discussed here later in Section 8.2, when we discuss bitorsor factorization for these groups.

Let us start with $\mathrm{SO}(q)$ and assume $\mathrm{char}(F) = 2$. We have already defined this group in Example 3. Consider the quadratic form $q' = \sum_{i=1}^{n} x_i^2$. Then, it is an easy computation to check that

$$\mathrm{O}(q')(R) = \left\{A \in \mathrm{GL}_n(R) \mid A^t A = I_n\right\}$$

holds. This recovers the standard notion of orthogonal matrix. Clearly, this also implies

$$\mathrm{SO}(q')(R) = \left\{A \in \mathrm{GL}_n(R) \mid A^t A = I_n;\ \det(A) = 1\right\}.$$

As any two quadratic forms are isomorphic over F^{sep}, it follows that any $\mathrm{SO}(q')$ is a form of $\mathrm{SO}(q)$.

For $\mathrm{SL}_1(D)$, let us fix a central simple algebra D over F. Let $\mathrm{GL}_1(D)$ denote the algebraic group of units in D, i.e. $\mathrm{GL}_1(D)(R) = (D \otimes_F R)^*$ for an F-algebra R. The reduced norm map induces a morphism of algebraic groups

$$\mathrm{Nrd}\colon\ \mathrm{GL}_1(D) \to \mathbb{G}_m.$$

We define $\mathrm{SL}_1(D)$ to the kernel of this map. Hence, for an F-algebra R, we have

$$\mathrm{SL}_1(D)(R) = \{a \in (D \otimes_F R)^* \mid \mathrm{Nrd}(a) = 1\}.$$

Note that if $D = \mathrm{M}_n(F)$, then $\mathrm{GL}_1(D) = \mathrm{GL}_n$ and $\mathrm{SL}_1(D) = \mathrm{SL}_n$.

3.4 Torsors and principal homogeneous spaces

In this section, we will review the notion of torsors and principal homogeneous spaces. Loosely speaking, given an algebraic group G, a G-torsor is a sheaf with a simply transitive G-action, whereas a principal homogeneous space is a scheme with a simply transitive G-action. Thus, any representable torsor is a principal homogeneous space. It turns out that in many interesting situations, e.g. if G is a linear algebraic group and the base is a field, these two notions coincide.

Definition 7. A left principal homogeneous space P over F under an F-group scheme G is an F-scheme P together with a left G-action $G \times_F P \to P$ such that the induced morphism

$$G \times_F P \to P \times_F P$$

given on points by $(g, t) \mapsto (g \cdot t, t)$ is an isomorphism.

A morphism of left principal G-homogeneous spaces P, P' over F is an F-morphism $P \to P'$ that is G-equivariant.

A trivial example of a principal homogeneous space is given by G itself, where the action is given by left translation. Let L/F be a Galois extension and let G be the constant group scheme associated to its Galois group. Then, $P = \mathrm{Spec}(L)$ is a principal homogeneous space under G via the usual Galois action.

Remark 1. When we speak of a principal homogeneous space, we mean left principal homogeneous space unless explicitly stated otherwise.

Let us now define the closely related notion of a torsor.

Definition 8. A left étale G-torsor T over F is an étale sheaf with a left action of the sheaf h_G such that:

1. For all K-schemes Y there is an étale cover $\{Y_i \to Y\}$ such that $T(Y_i) \neq \emptyset$.

2. The map $h_G \times T \to T \times T$ given by $(g, t) \mapsto (g \cdot t, t)$ is an isomorphism.

A morphism of G-torsors T, T' over K is a morphism of sheaves $T \to T'$ which is G-equivariant.

Remark 2. If G is affine over the base field, then the notions of (left) principal homogeneous space and (left) torsor coincide, i.e. the category of (left) principal G-homogeneous spaces over F is equivalent to the category of (left) G torsors over F (compare Proposition 4.5.6 in [Ols16]). This is also true if G is smooth. If one is willing to work with algebraic spaces instead of schemes, then this is true even if G is not affine or smooth. As we will focus on linear algebraic groups in this thesis, we do not have to worry about these subtleties.

Examples 4.

1. Let X and Y be two F-varieties such that $X|_{F^{\text{sep}}} \simeq Y|_{F^{\text{sep}}}$ holds, i.e. assume that X and Y are forms of each other. Let $\text{Aut}(X)$ denote the sheaf of automorphisms of X. Then, the sheaf of isomorphisms between X and Y, $\text{Isom}(X, Y)$ is a torsor under $\text{Aut}(X)$.

2. Let P be a left G-torsor. Define P^{op} to be the same sheaf as P but define a right action of G on P on sections as $p.g := g^{-1}.p$. This action makes P^{op} a right G-torsor. We call P^{op} the opposite torsors of P.

3.5 Étale Cohomology

Let S be a scheme and let Sch/S denote the category of schemes over S. In this section, we will (very) briefly recall the definition of étale cohomology groups of abelian étale sheaves in Sch/S. We will then show that G-torsors over S are classified by $\mathrm{H}^1(S, G)$ and use this to give a definition of non-abelian cohomology sets. We will conclude the section with the computation of certain cohomology groups that will be needed later on. For the general theory of étale cohomology, standard references are [Mil04] and [Tam94]. For the more specific results given in the last subsection, the main reference is [KMTR98].

Let $\mathrm{Sh}(S)$ denote the category of étale sheaves with values in abelian groups over the small étale site associated to Sch/S. As the category of abelian groups has enough injectives, $\mathrm{Sh}(S)$ also has enough injectives (compare [Tam94, Theorem I.3.2.2]). Hence, for any $F \in \mathrm{Sh}(S)$, we can find a resolution of étale sheaves

$$\ldots \to I_3 \xrightarrow{d_3} I_2 \xrightarrow{d_2} I_1 \xrightarrow{d_1} I_0 \xrightarrow{d_0} F \to 0$$

where $I_i \in \mathrm{Sh}(S)$ are injective. After applying the global section functor we obtain a complex

$$\ldots \to I_3(S) \xrightarrow{d_3} I_2(S) \xrightarrow{d_2} I_1(S) \xrightarrow{d_1} I_0(S) \xrightarrow{d_0} 0$$

of abelian groups. We define the i-th étale cohomology group of S with values in F, denoted by $\mathrm{H}^i(S,F)$ as the group

$$\mathrm{H}^i(S,F) = \ker(d_i)/\operatorname{Im}(d_{i+1}).$$

Let us recollect some properties, proofs of these can all be found in [Tam94, I.3.3]:

- For any sheaf F,

$$\mathrm{H}^0(S,F) = F(S).$$

- If I is an injective sheaf, then

$$\mathrm{H}^i(S,I) = 0$$

for all $i > 0$.

- Cohomology groups are functorial in F: A morphism of sheaves $G \to F$ induces morphisms

$$\mathrm{H}^i(S, G) \to \mathrm{H}^i(S, F)$$

for all $i \geq 0$.

- Short exact sequences yield long exact sequences: For every short exact sequence

$$1 \to F_1 \to F_0 \to F_2 \to 1$$

of étale sheaves, there are (unique) connecting maps δ_i

$$\delta_i \colon \mathrm{H}^i(S, F_2) \to \mathrm{H}^{i+1}(S, F_1)$$

such that we obtain a long exact sequence

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{H}^0(S, F_1) \arrow[r] & \mathrm{H}^0(S, F_0) \arrow[r] & \mathrm{H}^0(S, F_2) \arrow[dll, rounded corners] \\
 & \mathrm{H}^1(S, F_1) \arrow[r] & \mathrm{H}^1(S, F_0) \arrow[r] & \mathrm{H}^1(S, F_2) \arrow[dll, rounded corners] \\
 & \mathrm{H}^2(S, F_1) \arrow[r] & \mathrm{H}^2(S, F_0) \arrow[r] & \mathrm{H}^2(S, F_2) \arrow[dll, rounded corners] \\
 & \mathrm{H}^3(S, F_1) \arrow[r] & \dots &
\end{tikzcd}$$

Let us now specialize to the case $S = \mathrm{Spec}(F)$. In this case, it is well-known that étale and Galois cohomology agree (compare [Tam94, Corollary II.2.2]). In particular, the famous Hilbert 90 Theorem also holds in étale cohomology.

Theorem 3.5.1 (Hilbert 90)**.** *For any $n \geq 1$, we have*

$$\mathrm{H}^1(F, \mathbb{G}_m) = 0.$$

We can use this result to compute the first cohomology groups of μ_n. Starting with the short exact sequence

$$1 \to \mu_n \hookrightarrow \mathbb{G}_m \xrightarrow{()^n} \mathbb{G}_m \to 1,$$

we obtain the long exact sequence

$$1 \to \mu_n(F) \to F^* \xrightarrow{()^n} F^* \to \mathrm{H}^1(F, \mu_n) \to \mathrm{H}^1(F, \mathbb{G}_m) \to \mathrm{H}^1(F, \mathbb{G}_m).$$

Hence, Hilbert 90 immediately yields the following corollary.

Corollary 3.5.2. *There is a natural isomorphism*

$$\mathrm{H}^1(F, \mu_n) \simeq F^*/(F^*)^n.$$

3.5.1 Torsors and Cohomology

Fix a base scheme S and let G be an abelian group sheaf on the big (or small) étale site of S. The goal of this subsection is to prove the following theorem:

Theorem 3.5.3 ([Sta15, Tag 03AG])**.** *There is a canonical bijection*

$$\mathrm{H}^1(S, G) \simeq \{\textit{isomorphism classes of } G\textit{-torsors over } S\}.$$

Sketch. Let F denote a G-torsor. Let $\mathbb{Z}[F]$ denote the sheafification of the presheaf

$\mathbb{Z}(F)$ given by

$$U \mapsto \mathbb{Z}[F(U)].$$

Further, let $\overline{\mathbb{Z}}$ denote the constant group sheaf associated with $\mathbb{Z}$ and consider the map of sheaves $\sigma\colon \mathbb{Z}[F] \mapsto \mathbb{Z}$ induced by the map $\mathbb{Z}(F) \to \mathbb{Z}$ defined over $U \in \mathrm{Sch/S}$ via

$$\sum_i n_i s_i \mapsto \sum_i n_i.$$

for $s_i \in F(U)$.

The kernel of σ is locally generated by sections of the form $s - s'$, we can thus define a map $\ker(\sigma) \to G$ by mapping $s - s'$ to g such that $g.s = s'$.

We can thus form a morphism of exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \ker(\sigma) \arrow[r] \arrow[d] & \mathbb{Z}[F] \arrow[r, "\sigma"] \arrow[d] & \overline{\mathbb{Z}} \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\
1 \arrow[r] & G \arrow[r] & Q \arrow[r] & \overline{\mathbb{Z}} \arrow[r] & 1
\end{tikzcd}$$

where $Q = G \times^{\ker(\sigma)} \mathbb{Z}[F]$ is the pushout of $\mathbb{Z}[F]$ along $\ker(\sigma) \to G$.

The long exact sequence in cohomology associated to the lower exact sequence yields the connecting map $\mathrm{H}^0(S, \overline{\mathbb{Z}}) \to \mathrm{H}^1(S, G)$. Let ζ denote the image of 1 under this map. To F, we associate the class $\zeta \in \mathrm{H}^1(S, \overline{\mathbb{Z}})$.

Given a class $\zeta \in \mathrm{H}^1(S, G)$, let $G \hookrightarrow I$ be an embedding of G into an injective sheaf I. Let Q denote the quotient sheaf of I by G. We then have the short exact sequence

$$1 \to G \to I \to Q \to 1$$

which yields the long exact sequence

$$\ldots\ldots \to \mathrm{H}^0(S, I) \to \mathrm{H}^0(S, Q) \to \mathrm{H}^1(S, G) \to \mathrm{H}^1(S, I) \to \ldots$$

As I is injective, we have $\mathrm{H}^1(S, I) = 0$. Hence, the map $\mathrm{H}^0(S, Q) \to \mathrm{H}^1(S, G)$ is surjective. Let $f \in Q(S)$ denote a preimage of ζ. We can then define a subsheaf of I via

$$U \mapsto \{s \in I(U) \mid s \mapsto f|_U\}$$

which is a G-torsor via the natural action.

It is easy to check that this construction is, up to isomorphism, independent of the choice of f and that both constructions are inverse to each other. □

Remark 3. This statement is true in much greater generality on any site. However, as we have only introduced étale cohomology, we only state the étale version of the statement.

While we cannot define étale cohomology for non-abelian groups via the derived functor approach due to the lack of injective resolutions, torsors under non-abelian groups are defined naturally. This motivates the next definition.

Definition 9. Let G be a non-abelian algebraic group. We define

$$\mathrm{H}^1(F, G) = \{P \mid P \text{ is a } G\text{-torsor over } F\} / \simeq$$

We note without proof that even in the non-abelian case, a short exact sequence of group sheaves yields a (mildly) long exact sequence in cohomology. (See [Ser97] for details) Given a short exact sequence

$$1 \to G' \to G \to G'' \to 1$$

with G' is normal in G, we obtain a long exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{H}^0(F, G') \arrow[r] & \mathrm{H}^0(F, G) \arrow[r] & \mathrm{H}^0(F, G'') \arrow[dll, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztotarget.north) -- (\tikztotarget)}] \\ & \mathrm{H}^1(F, G') \arrow[r] & \mathrm{H}^1(F, G) \arrow[r] & \mathrm{H}^1(F, G'') \end{tikzcd}$$

3.5.2 Examples

Let us finish this chapter with some elementary calculations concerning the étale cohomology of $\mathrm{SL}_1, \mathrm{SL}_1(D)$ and $\mathrm{SO}(q)$ for a quadratic form q and a central simple algebra D over a field F of characteristic not 2.

Let us first state a generalized version of Hilbert 90.

Theorem 3.5.4 ([KMTR98, Theorem 29.2])**.** *Let D be a central simple algebra over F. Then,*

$$\mathrm{H}^1(F, \mathrm{GL}_1(D)) = 0.$$

Corollary 3.5.5. *For any $n \geq 1$, we have*

$$\mathrm{H}^1(F, \mathrm{SL}_n) = 0.$$

Proof. The short exact sequence

$$1 \to \mathrm{SL}_n \to \mathrm{GL}_n \xrightarrow{\det} \mathbb{G}_m \to 1$$

yields the long exact sequence

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{SL}_n(F) \arrow[r] & \mathrm{GL}_n(F) \arrow[r, "\det"] & F^* \arrow[dll, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& \mathrm{H}^1(F, \mathrm{SL}_n) \arrow[r] & \mathrm{H}^1(F, \mathrm{GL}_n) \arrow[r] & \mathrm{H}^1(F, \mathbb{G}_m)
\end{tikzcd}$$

By use of Hilbert 90, we obtain

$$\mathrm{GL}_n(F) \xrightarrow{\det} F^* \to \mathrm{H}^1(F, \mathrm{SL}_n) \to 0$$

The result follows if the determinant map is surjective. But, for any $a \in F^*$, the diagonal matrix A with $A_{11} = a$ and $A_{ii} = 1$ for $i > 1$ is a preimage. $\square$

Let us now turn to the case more general case of $\mathrm{SL}_1(D)$. The short exact sequence

$$1 \to \mathrm{SL}_1(D) \to \mathrm{GL}_1(D) \xrightarrow{\mathrm{Nrd}} \mathbb{G}_m \to 1$$

yields the long exact sequence

$$1 \to \mathrm{SL}_1(D)(F) \to \mathrm{GL}_1(D)(F) \xrightarrow{\det} F^* \to \mathrm{H}^1(F, \mathrm{SL}_1(D)) \to \mathrm{H}^1(F, \mathrm{GL}_1(D)) \to \mathrm{H}^1(F, \mathbb{G}_m).$$

As stated above, Hilbert 90 generalizes to algebraic groups of the form $\mathrm{GL}_1(D)$ to yield $\mathrm{H}^1(F, \mathrm{GL}_1(D)) = 0$. We have thus proven the next lemma.

Lemma 3.5.6. *There is a natural isomorphism*

$$\mathrm{H}^1(F, \mathrm{SL}_1(D)) = F^* / \mathrm{Nrd}(D).$$

Consider the short exact sequence defining $\mathrm{SO}(q)$:

$$1 \to \mathrm{SO}(q) \to \mathrm{O}(q) \xrightarrow{\det} \mu_2 \to 1.$$

The corresponding long exact sequence starts as

$$1 \to \mathrm{SO}(q)(F) \to \mathrm{O}(q)(F) \xrightarrow{\det} \{\pm 1\} \to \mathrm{H}^1(F, \mathrm{SO}(q)) \to \mathrm{H}^1(F, \mathrm{O}(Q)(F)) \to \mathrm{H}^1(F, \mu_2).$$

As $\mathrm{O}(q)(F) \xrightarrow{\det} \{\pm 1\}$ is surjective. This implies that $\mathrm{H}^1(F, \mathrm{SO}(q)) \to \mathrm{H}^1(F, \mathrm{O}(Q)(F))$ is injective at the base point. Using twisting, one can prove the following Lemma.

Lemma 3.5.7. *The natural morphism*

$$\mathrm{H}^1(F, \mathrm{SO}(q)) \to \mathrm{H}^1(F, \mathrm{O}(q))$$

is injective.

Chapter 4

Non-abelian hypercohomology

Non-abelian hypercohomology with values in crossed modules has been introduced and studied by Breen in [Bre90]. He defines non-abelian hypercohomology in any topos by methods from homotopy theory. Independently, Borovoi ([Bor92a]), based on work of Deligne and Dedecker, introduced a cocyclic approach to non-abelian hypercohomology generalizing the cocyclic approach of Galois cohomology. His approach is only defined over the étale site of a field.

For the purpose of this thesis, it is enough to consider Borovoi's approach. This has the advantage that we do not have to introduce any advanced notions from homotopy theory.

4.1 Definition

Throughout this section, let F be a field and Γ its absolute Galois group. Let F^{sep} denote the separable closure of F. Furthermore, let H, G be a algebraic groups over F. In this section, we will identify G with its set of points $G(F^{\mathrm{sep}})$ to simplify notation. Our hypercohomology groups will have coefficients in crossed modules. Hence, we will first discuss crossed modules.

Definition 10. A crossed module over F is a morphism $\rho\colon G \to H$ of algebraic groups over F together with a left action $\alpha\colon H \times G \to G$ such that

- ${}^{\rho(g')}g = g'g(g')^{-1}$ for $g', g \in G$ and
- $\rho({}^h g) = h\rho(g)h^{-1}$ for $g \in G$ and $h \in H$

holds.

Let $\rho'\colon G' \to H'$ be another crossed module. A morphism of crossed modules $\rho \to \rho'$ is given by morphisms $G \to G'$ and $H \to H'$ that are compatible with α and α'.

Given a crossed module $\rho\colon G \to H$, a Γ action on ρ consists of actions of Γ on G and H satisfying

$$\rho({}^\sigma g) = {}^\sigma\rho(g)$$

and

$${}^\sigma({}^h g) = {}^{\sigma h}({}^\sigma g)$$

for $g \in G$, $h \in H$ and $\sigma \in \Gamma$.

A Γ action on ρ is continuous, if the actions on G and H are continuous with respect to the discrete topology. A Γ-crossed module is a crossed module equipped with a continuous action. A morphism of Γ-crossed modules is a morphism of crossed modules that is compatible with the Γ actions.

Examples 5. We will mostly use the following two examples.

- Let G be any algebraic group, then $1 \to G$ is a crossed module. The usual Γ action on G defines a Γ action on the crossed module.

- For any algebraic group G, $G \xrightarrow{\mathrm{Int}} \mathrm{Aut}(G)$ is a crossed module. The usual action of Γ on G and $\mathrm{Aut}(G)$ defines a Γ action on the crossed module.

We are now ready to define the hypercohomology groups.

Definition 11. Let $\rho\colon G \to H$ be a Γ-crossed module over F. We define

$$\mathrm{H}^{-1}(F, G \to H) = \ker(\rho)(F).$$

Example 2. For the crossed Γ-module $G \to \mathrm{Aut}(G)$, we have $\mathrm{H}^{-1}(G \to \mathrm{Aut}(G), F) = Z(G)(F)$.

A 0-cocycle of ρ is a tuple (ϕ, h), where $\phi\colon \Gamma \to G(F^{\mathrm{sep}})$ is a continuous map and $h \in H(F^{\mathrm{sep}})$ such that

1. $\phi(\sigma\tau) = \phi(\sigma)\,{}^{\sigma}\phi(\tau)$ for $\sigma, \tau \in \Gamma$,
2. ${}^{\sigma}h = \rho(\phi(\sigma)^{-1}) \cdot h$ for all $\sigma \in \Gamma$.

We denote by Z^0 the set of 0-cocycles of ρ. Note that $G(F^{\mathrm{sep}})$ acts on Z^0: For $(\phi, h) \in Z^0$ and $g \in G(F^{\mathrm{sep}})$ define $(\phi, h).g = (\phi', h')$ via

$$\phi'(\sigma) = g^{-1}\phi(\sigma)\,{}^{\sigma}g$$
$$h' = \rho(g)^{-1}h.$$

Definition 12. Let $\rho\colon G \to H$ be a Γ-crossed module over F. We then define

$$\mathrm{H}^0(F, G \to H) = \mathrm{Z}^0 / G(F^{\mathrm{sep}}). \tag{4.1}$$

We can equip $\mathrm{H}^0(F, G \to H)$ with a group structure. Note that $H(F^{\mathrm{sep}})$ acts on continuous morphisms $\phi\colon \Gamma \to G(F^{\mathrm{sep}})$ via

$$\left({}^{h}\phi\right)(\sigma) := {}^{h}\phi(\sigma)$$

for $h \in H(F^{\mathrm{sep}})$ and $\sigma \in \Gamma$. This allows us to define a group structure on Z^0 via

$$(\phi, h) \cdot (\phi', h') = ({}^{h}\phi'\phi, hh').$$

It is easy to see that this defines a group structure which descends to $\mathrm{H}^0(F, G \to H)$.

Let $\mathrm{Hom}(\Gamma \times \Gamma, G), \mathrm{Hom}(\Gamma, H)$ denote the sets of continuous homomorphisms. Let $\mathrm{Z}^1 \subset \mathrm{Hom}(\Gamma \times \Gamma, G) \times \mathrm{Hom}(\Gamma, H)$ denote subset of tuples (ϕ, ψ) such that for $\sigma, \tau, \gamma \in \Gamma$, the following two conditions are satisfied:

$$\rho\left(\phi(\sigma, \tau)\right)^{-1} \cdot \psi(\sigma\tau) = \psi(\sigma) \cdot {}^{\sigma}\psi(\tau)$$
$$\phi(\sigma, \tau\gamma) \cdot {}^{\psi(\sigma)\sigma}\phi(\tau, \gamma) = \phi(\sigma\tau, \gamma) \cdot \phi(\sigma, \tau)$$

We now define an equivalence relation R on Z^1. We say that (ϕ, ψ) and (ϕ', ψ') are equivalent if there is a continuous map $f \colon \Gamma \to G(F^{\mathrm{sep}})$ and $h \in H(F^{\mathrm{sep}})$ such that

$$\psi'(\sigma) = h^{-1}\rho(f(\sigma))\psi(\sigma)\,{}^{\sigma}h$$
$$\phi'(\sigma, \tau) = {}^{h^{-1}}\left(f(\sigma\tau)\phi(\sigma, \tau)\,{}^{\psi(\sigma)\sigma}f(\tau)^{-1}f(\sigma)^{-1}\right)$$

holds.

Definition 13. Let $\rho \colon G \to H$ be a Γ-crossed module over F. We then define

$$\mathrm{H}^1(F, G \to H) = \mathrm{Z}^1/R.$$

4.2 Properties of hypercohomology

We will now collect results regarding the functoriality of hypercohomology groups.

Proposition 4.2.1 ([Bor98, Section 3.4.2])**.** *Let*

$$1 \to (G_1 \to H_1) \xrightarrow{i} (G_2 \to H_2) \xrightarrow{j} (G_3 \to H_3) \to 1$$

be an exact sequence of complexes of Γ*-groups where* i *is an embedding of crossed modules with* Γ *action. Then, there is an exact sequence of pointed sets*

$$\begin{aligned} 1 &\longrightarrow \mathrm{H}^{-1}(G_1 \to H_1) \longrightarrow \mathrm{H}^{-1}(G_2 \to H_2) \longrightarrow \mathrm{H}^{-1}(G_3 \to H_3) \\ &\longrightarrow \mathrm{H}^{0}(G_1 \to H_1) \longrightarrow \mathrm{H}^{0}(G_2 \to H_2) \longrightarrow \mathrm{H}^{0}(G_3 \to H_3) \\ &\longrightarrow \mathrm{H}^{1}(G_1 \to H_1) \longrightarrow \mathrm{H}^{1}(G_2 \to H_2) \end{aligned}$$

Example 3. We will mostly use Proposition 4.2.1 for the following short exact sequence:

$$1 \to (1 \to \mathrm{Aut}(G)) \xrightarrow{i} (G \to \mathrm{Aut}(G)) \xrightarrow{j} (G \to 1) \to 1$$

where all maps occurring are either the identity or trivial. Then, the corresponding long exact sequence simplifies to the following sequence (cf. [Bre90, Section 4.2.3]).

$$\begin{aligned} 1 &\longrightarrow \mathrm{H}^{0}(Z(G)) \longrightarrow \mathrm{H}^{0}(G) \longrightarrow \mathrm{H}^{0}(\mathrm{Aut}(G)) \\ &\longrightarrow \mathrm{H}^{0}(G \to \mathrm{Aut}(G)) \longrightarrow \mathrm{H}^{1}(G) \longrightarrow \mathrm{H}^{1}(\mathrm{Aut}(G)) \\ &\longrightarrow \mathrm{H}^{1}(G \to \mathrm{Aut}(G))) \end{aligned}$$

We will now describe characterizations of $\mathrm{H}^i(G \to \mathrm{Aut}(G))$ for $i = -1, 0, 1$. Recall that $\mathrm{H}^{-1}(H \xrightarrow{\alpha} G) = \ker(\alpha)^\Gamma$ and hence

$$\mathrm{H}^{-1}(F, G \to \mathrm{Aut}(G)) = Z(G)(F)$$

where $Z(G)$ is the center of G.

As further discussed in Section 6.1.3, $\mathrm{H}^0(G \to \mathrm{Aut}(G))$ classifies G-bitorsors (which we will also introdcue in Section 6.1.3).

Proposition 4.2.2 ([Bre90, Theorem 4.5])**.** *There is a natural isomorphism*

$$\mathrm{H}^0(F, G \to \mathrm{Aut}(G)) \simeq \{\textit{Isomorphism classes of } G\textit{-bitorsors over } F\}\,.$$

Unlike torsors, a G-bitorsor may not be trivial (i.e. isomorphic to G as a bitorsor) even when it admits a point. This phenomenon motives the next definition.

Definition 14. Let $\alpha \in \mathrm{H}^0(G \to \mathrm{Aut}(G))$. We say that α is neutral if a bitorsor representing α admits a point over F.

We now turn our attention to $\mathrm{H}^1(G \to \mathrm{Aut}(G))$, which classifies G-gerbes. (See Chapter 7 for the definition of gerbes).

Proposition 4.2.3 ([Bre90, Theorem 4.5])**.** *There is a natural isomorphism*

$$\mathrm{H}^1(F, G \to \mathrm{Aut}(G)) \simeq \{\textit{Equivalence classes of } G\textit{-gerbes over } F\}\,.$$

Definition 15. Let $\alpha \in \mathrm{H}^1(G \to \mathrm{Aut}(G))$. We say that α is neutral if a corresponding gerbe (and thus every corresponding bitorsor) admits a point over F.

Chapter 5

Patching and local global principles

In this chapter, we will discuss a patching setup first developed by Harbater and Hartmann ([HH10]), and then refined and expanded by Harbater, Hartmann and Krashen ([HHK15a],[HHK15b],[HHK14],[HHK11] and [HHK09]).

5.1 Introduction to field patching

In this section, we want to fix notation and discuss vector space patching problems. Let I denote an indexing set with relations. Throughout, let $\mathcal{F} = \{F_i\}_{i \in I}$ denote a finite inverse systems of fields with inclusions as morphisms. We say that F_k is an overfield of F_i if there is an inclusion $F_i \subset F_k$ in $\mathcal{F}$. Let F denote $\varinjlim \mathcal{F}$. We want to study when the datum of an algebraic object (e.g. a vector space) over F is equivalent to the datum of a collection of algebraic objects over F_i together with isomorphisms between them over common overfields. In other words, we want to be able to think of $\mathcal{F}$ as a *cover* of F. Before we analyze when this is possible, let us make some assumptions on $\mathcal{F}$.

Definition 16. A factorization inverse system over a field F is a finite inverse system of fields such that

1. F is the inverse limit.

2. The index set I can be partitioned as $I = I_v \sqcup I_e$ such that:

   (a) For any $k \in I_e$ there are exactly two elements $i, j \in I_v$ such that $i, j > k$.

   (b) These are the only relations.

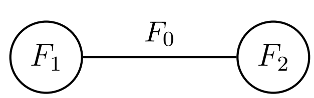
For each index $k \in I_e$, fix a labeling l_k, r_k for the two elements in I_e with $l_k, r_k > k$. Then, let S_I denote the set of triples (l_k, r_k, k).

Given a factorization inverse system, one can associate to it a (multi-)graph Γ, which we will call the graph of $\mathcal{F}$. Its vertices are the elements of I_v, where the edges come from I_e (explaining the subscripts). Given $k \in I_e$, the corresponding edge connects the vertices $i, j \in I_v$ iff $(i, j, k) \in S_I$. Note that Γ is connected, as the inverse limit F would otherwise admit zero-divisors. We will sometimes specialize to the case where Γ is a tree.

Example 4. A basic example of a factorization inverse system is given by fields $F \subset F_1, F_2 \subset F_0$ such that $F = F_1 \cap F_2 \subset F_0$. Pictorially, we get

$$\begin{tikzcd} & F_0 & \\ F_1 \arrow[ur, hook] & & F_2 \arrow[ul, hook'] \\ & F. \arrow[ul, hook] \arrow[ur, hook'] & \end{tikzcd}$$

Note that in this case $I_e = \{0\}, I_v = \{1, 2\}$ and $S_I = \{(1, 2, 0)\}$. The corresponding graph Γ



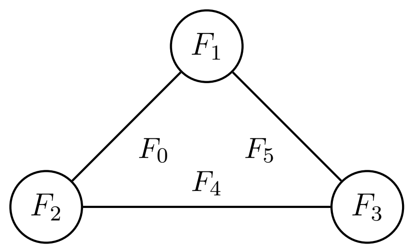
is a tree.

Note, however, that Γ may not always be a tree.

Example 5. Consider $\mathcal{F} = \{F_0, F_1, F_2, F_3, F_4, F_5\}$ with inclusions given by

$$\begin{tikzcd}
 & F_0 & & F_4 & & F_5 & \\
F_1 \arrow[ur, hook] & & F_2 \arrow[ul, hook'] \arrow[ur, hook] & & F_3 \arrow[ul, hook'] \arrow[ur, hook] & & F_1 \arrow[ul, hook'] \\
 & & & F \arrow[ulll, hook'] \arrow[ul, hook'] \arrow[ur, hook] \arrow[urrr, hook] & & &
\end{tikzcd}$$

In this case, $I_e = \{0, 4, 5\}$, $I_v = \{1, 2, 3\}$ and $S_I = \{(1, 2, 0), (2, 3, 4), (3, 1, 5)\}$. The corresponding graph Γ



is *not* a tree.

We are now ready to state the concept of a patching problem. A patching problem roughly consists of data over $\mathcal{F}$ that should give rise to an object over F if we want to think of $\mathcal{F}$ as a cover of F. In other words, a patching problem is an analogue of *descent data* (compare Section 2.3). In fact, it is known (but not published) that one can define a Grothendieck topology based on patches which makes this analogy precise. We will start with the perhaps easiest algebraic object, the vector space. Let $\mathrm{VECT}(F)$ denote the category of finite dimensional vector spaces over F.

Definition 17. A vector space patching problem $\mathcal{V} = \left(\{V_i\}_{i \in I_v}, \{\nu_k\}_{k \in I_e}\right)$ for a factorization inverse system $\mathcal{F}$ is given by a collection of finite dimensional F_i vector spaces V_i together with F_k vector space isomorphisms $\nu_k \colon V_i \otimes_{F_i} F_k \to V_j \otimes_{F_j} F_k$ whenever $(i, j, k) \in S_I$.

A morphism of patching problems $\mathcal{V} \to \mathcal{V}'$ is a collection of F_i-linear transformations $V_i \to V_i'$ for all $i \in I_v$ that are compatible with the isomorphisms ν_k, ν_k'.

The category of vector space patching problems is denoted by $\mathrm{PP}(\mathcal{F})$. By construction, we have

$$\mathrm{PP}(\mathcal{F}) \simeq \prod_{(i,j,k)\in S_I} \mathrm{VECT}(F_i) \times_{\mathrm{VECT}(F_k)} \mathrm{VECT}(F_j).$$

If A/F is a finite product of finite separable field extension, let $\mathrm{PP}(\mathcal{F}_A)$ denote the category of free module patching problems: objects are collections $(\{M_i\}_{i\in I_v}, \{\nu_k\}_{k\in I_e})$ where M_i is a free module over $A_i = F_i \otimes_F A$ of finite rank and $\nu_k \colon M_i \otimes_{A_i} A_k \to M_j \otimes_{A_j} A_k$ are isomorphisms of A_k modules.

Note that we have a canonical functor

$$\beta \colon \mathrm{VECT}(F) \to \mathrm{PP}(\mathcal{F})$$
$$V \mapsto (\{V \otimes_F F_i\}_{i\in I_v}, \{\sigma_k^{\mathrm{can}}\}_{k\in I_e})$$

where σ_k^{can} is the canonical isomorphism $V \otimes_F F_i \otimes_{F_i} F_k \simeq V \otimes_F F_j \otimes_{F_j} F_k$.

Definition 18. A solution to a vector space patching problem $\mathcal{V}$ is an F vector space V such that $\beta(V)$ is isomorphic to $\mathcal{V}$.

If β is an equivalence of categories, then every patching problem has a solution that is unique up to isomorphism.

Untying definitions, we see that V is a solution to $\mathcal{V}$ if and only if there are isomorphisms of vector space $\alpha_i \colon V \otimes_F F_i \to V_i$ for $i \in I_v$ that are compatible with $\{\mu_k\}$, i.e. for each $(i,j,k) \in S_I$, the diagram

$$\begin{tikzcd} V \otimes_F F_k \arrow[r, "\alpha_i\otimes F_k"] \arrow[d, equal] & V_i \otimes_{F_i} F_k \arrow[d, "\nu_k"] \\ V \otimes_F F_k \arrow[r, "\alpha_j\otimes F_k"] & V_j \otimes_{F_j} F_k \end{tikzcd}$$

commutes. This is where the notion of *patching* comes from: We were able to *glue* the vector spaces V_i along the isomorphisms ν_k to get a vector space V over F.

To make this even more concrete, let us pick bases for all vector spaces of a patching problem $\mathcal{V}$. In fact, without loss of generality, let us assume $V_i = F_i^n$ for $i \in I_v$. Then, we can identify ν_k with a matrix $A_k \in \mathrm{GL}_n(F_k)$. (note that n needs to be constant; otherwise, there cannot be an isomorphism ν_k!) The only candidate (up to isomorphism) for a solution is thus given by $V = F^n$. By definition V is a solution if and only if there is a collection of isomorphisms $\{\alpha_i\}_{i \in I_v}$ satisfying the condition outlined above. Identifying the α_i with a matrix $A_i \in \mathrm{GL}_n(F_i)$, we see that the commutativity condition is equivalent to the condition $A_k = A_i A_j^{-1}$. This observation motivates the following definition.

Definition 19. A linear algebraic group G satisfies simultaneous factorization over $\mathcal{F}$ if for any collection of elements $a_k \in G(F_k)$ with $k \in I_e$, there are elements $a_i \in G(F_i)$ for all $i \in I_v$ such that $a_k = a_r^{-1} a_l \in G(F_k)$ for all $(l, r, k) \in S_I$.

In the case where $G = \mathrm{GL}_n$, we simply say that simultaneous factorization holds over $\mathcal{F}$.

The following result provides the basis of many patching results obtained by Harbater, Hartman and Krashen. It is a direct consequence of our discussion above the previous definition.

Proposition 5.1.1 ([HH10, Proposition 2.1])**.** *The functor $\beta\colon$ VECT$(F) \to$ PP$(\mathcal{F})$ is an equivalence of categories if and only if simultaneous factorization holds over $\mathcal{F}$.*

Simultaneous factorization is amenable to many concrete examples. To date, all proofs that β is an equivalence were derived using this characterization. We discuss two concrete examples in Section 5.2.

If $A = \prod_{i=1}^n L_i$ is a finite product of finite separable field extensions of F, then we denote by $\mathcal{F}_A$ the inverse system $\{A_i := A \otimes_F F_i\}_{i \in I}$. It is not necessarily an inverse factorization system but we have $A = \varprojlim \mathcal{F}_A$. Let MOD$(A)$ denote the category of free modules of finite rank over A. Let PP$(\mathcal{F}_A)$ denote the category of patching problems of the form $(\{M_i\}_{i\in I_v}, \{\nu_k\}_{k\in I_e})$ where M_i is a free A_i-module of finite rank and $\nu_k\colon M_i|_{A_k} \to M_j|_{A_k}$ is an isomorphism of A_k-modules for $(i,j,k) \in S_I$. Note that we have a natural functor $\widehat{\beta}\colon \mathrm{MOD}(A) \to \mathrm{PP}(\mathcal{F}_A)$.

The following proposition is a slight variant of [HHK15b, Lemma 2.2.7].

Proposition 5.1.2. *Assume that patching holds over $\mathcal{F}$, i.e. that the functor*

$$\beta\colon \mathrm{VECT}(F) \to \mathrm{PP}(\mathcal{F})$$

is an equivalence. Let $A = \prod_{i=1}^{n} L_i$ be a product of finitely many finite separable field extensions. Then, patching holds over $\mathcal{F}_A$, i.e. the natural functor $\widehat{\beta}\colon \mathrm{MOD}(A) \to \mathrm{PP}(\mathcal{F}_A)$ is an equivalence.

Proof. Given a patching problem $(\{M_i\}_{i\in I_v}, \{\nu_k\}_{k\in I_e})$ in $\mathrm{PP}(\mathcal{F}_A)$, note that M_i is a finite dimensional vector space over F_i for all $i \in I_v$. Also, ν_k is an isomorphism of F_k vector spaces. Hence, by assumption, there is an F-vector space M together with F-vector space isomorphisms $\phi_i\colon M|_{F_i} \to M_i$ for all $i \in I_v$ that are compatible with ν_k. The A_i-module structure of M_i is equivalent to the datum of a morphism $\alpha_i\colon A_i \to \mathrm{End}_{F_i}(M_i)$ of F_i-vector spaces. As the ν_k are A_k module morphisms, they are compatible with α_i. Hence, as the functor β is full, there is a morphism $\alpha\colon A \to \mathrm{End}_F(M)$ of F vector spaces. The resulting A-module M solves the patching problem. This shows that $\widehat{\beta}$ is essentially surjective. As every morphism of A_i-modules is in particular a morphism of F_i-vector spaces, it follows that $\widehat{\beta}$ is faithful. Given two A-modules M, N and a morphism $\widehat{\beta}(M) \to \widehat{\beta}(N)$, note that we can lift it to an F-vector space morphism $\gamma\colon M \to N$. As the image of this morphism in $\mathrm{PP}(\mathcal{F}_A)$ commutes with the A_i-action and as $\widehat{\beta}$ is faithful, it follows that γ is an A-module morphism. $\square$

5.2 Example: Patching over arithmetic curves

In this section, we will discuss two concrete patching setups. We will start by introducing a patching framework for function fields of the projective line over a complete discretely valued field. This patching result is obtained by covering the special fiber of the projective line with (possibly non-open) subsets. This result is then used to prove a patching results for arithmetic curves, stemming from a cover consisting of a finite number of points on the special fiber and their complements. The material discussed here originates in [HH12].

5.2.1 Patching over the projective line

Let T be a complete discretely valued ring with uniformizer t and residue field k. Let K denote its field of fractions. Let $\widehat{X} = \mathbb{P}^1_T$ and $X = \mathbb{P}^1_k$. Let F denote the field $K(x)$, the function field of $\widehat{X}$. We will now construct a system of fields $\mathcal{F}$ over F over which patching will hold.

Let $U \subset X$ be a non-empty subset. We will use R_U to denote the subring of F of functions that are regular on U. The t-adic completion of R_U will be denoted by $\widehat{R}_U$. The fraction field of $\widehat{R}_U$ is denoted by F_U. Analogously, let $R_\emptyset$ denote the ring of rational functions that are regular at the generic point of X. Let again $\widehat{R}_\emptyset$ denote its t-adic completion.

Note that, for $V \subset U \subset X$, we have natural inclusions

$$\widehat{R}_U \subset \widehat{R}_V \subset \widehat{R}_\emptyset \subset F.$$

Given subsets $U_1, U_2 \subset X$, we can form the system $F_{U_1}, F_{U_2} \subset F_{U_1 \cap U_2}$. We will show that this system has inverse limit $F_{U_1 \cup U_2}$. Also, patching holds over this system. To show this, we will prove that GL_n satisfies simultaneous factorization.

Lemma 5.2.1. *Let $U_1, U_2 \subset X$ and $U_0 = U_1 \cap U_2$. Then, for all $a \in \widehat{R}_{U_0}$, there are $b_i \in \widehat{R}_{U_i}$ such that $a \cong b_1 + b_2 \mod t$ in $\widehat{R}_{U_0}$.*

Proof. We can write $a = \sum_{i=0}^{\infty} a_i t^i$ with coefficients $a_i \in A$, where A is a subring of $k(X)$. In particular, a_0 is a rational function on $\mathbb{P}^1_k$ with poles disjoint from U_0. By partial fraction decomposition, we can write $a_0 = b + c$, with $b, c \in k(x)$ such that b has no poles in U_1 and c has no poles in U_2. Hence, $b \in R_{U_1} \subset \widehat{R}_{U_1}$ and, similarly, $c \in R_{U_2} \subset \widehat{R}_{U_2}$. Clearly, $a \cong a_0 = b + c \mod t$, completing the proof. □

Using the additive decomposition from above together with the fact that our rings $\widehat{R}_i$ are t-adically complete, we can prove our first factorization result.

Proposition 5.2.2. *Let $U_1, U_2 \subset X$ and $U_0 = U_1 \cap U_2$. Then, for any $A_0 \in \mathrm{GL}_n(\widehat{R}_{U_0})$ such that $A_0 \cong \mathrm{I} \mod t$, there are $A_i \in \mathrm{GL}_n(\widehat{R}_{U_i})$ such that $A_0 = A_1^{-1} A_2$.*

Sketch: We will construct a sequence of matrices $\{B_j\}$ with $B_j \in \mathrm{GL}_n(\widehat{R}_{U_1})$ and $\{C_j\}$ with $C_j \in \mathrm{GL}_n(\widehat{R}_{U_2})$ such that $A_1 = \lim B_j$ and $A_2 = \lim C_j$. (The limits are taken in the t-adic topology). We start by setting $B_0 = C_0 = \mathrm{I}_n$. We want to construct B_j, C_j such that

- $B_j \cong B_{j-1} \mod t^j$,
- $C_j \cong C_{j-1} \mod t^j$,
- $A_0 = B_j^{-1} C_j \mod t^{j+1}$,

holds. It is clear then that $\lim B_j$ and $\lim C_j$ give the desired factorization. Fix some $r \geq 0$ and assume that we constructed B_j, C_j for $j \leq r$. In particular, there is a matrix $\tilde{A}_{r+1} \in \mathrm{Mat}_n(\widehat{R}_{U_0})$ such that

$$A_0 - B_r^{-1} C_r = t^{r+1} \tilde{A}_{r+1}$$

holds. Applying Lemma 5.2.1 on every entry of $\tilde{A}_{r+1}$. we obtain matrices B'_{r+1} and C'_{r+1} such that

$$\tilde{A}_{r+1} \cong B'_{r+1} + C'_{r+1} \mod t$$

holds. We then define $B_{r+1} = B_r - t^r B'_{r+1}$ and $C_{r+1} = C_r - t^r C_{r+1}$. We omit the verification that these matrices satisfy the conditions imposed above. □

In order to apply this result to matrices with entries in the F_{U_0} we need an analogue of the Weierstrass Preparation Theorem.

Theorem 5.2.3 (Weierstrass Preparation). *Let $U \subset X$ and $f \in F_U$. Then, we can write $f = a \cdot u$, with $a \in F$, and $u \in \widehat{R}_U^*$.*

Proof. If U contains all closed points of X, then the result is immediate. If U contains no closed points, then we can take $a = t^m$, where m is the t-adic valuation of f in $F_\emptyset$. (Note that, by definition, $F_\eta = F_\emptyset$.)

So, we may assume that U contains some but not all closed points of X. We can write $f = \frac{g}{h}$ with $g, h \in \widehat{R}_U$, and the statement holds for f if it holds for g and h. Hence, we assume that $f \in \widehat{R}_U \setminus \{0\}$. As mentioned before, $\widehat{R}_U = A[\![t]\!]$ for some commutative ring A. Hence, we can write $f = t^r \sum_{i=0}^n a_i t^i$ with $a_i \in A$ and constant term $a_0 \neq 0$. It is enough to prove the proposition for $\sum_{i=0}^n a_i t^i$, so we assume that $r = 0$. Consider the element $f' = f/a_0 \in \widehat{R}_\emptyset$ with constant term 1. We will us the factorization result from Proposition 5.2.2 with $n = 1$, $U_1 = X \setminus U$ and $U_2 = U$. The proposition guarantees the existence of $f_i \in \widehat{R}_{U_i}^*$ such that $f' = f_1 f_2$. In particular, $f_0 f_1 = f f_2^{-1} \in \widehat{R}_{U_1}[f_0] \cap \widehat{R}_{U_2}$. Note that this intersection is contained in F, hence $u = f_2$ and $a = f_0 f_1$ works. □

We are now ready to prove the main results needed for patching: the intersection property and the factorization property.

Theorem 5.2.4. *Let $U_1, U_2 \subset X$ and $U_0 = U_1 \cap U_2$ and $U = U_1 \cup U_2$. Then, $F_U = F_{U_1} \cap F_{U_2}$ inside F_{U_0}.*

Proof. Note first that by definition, for any $V \subset X$, we have $F_V = F_{V \cup \{\eta\}}$. Here η denotes the generic point of X. Hence, we may assume without loss of generality, that U_i contain the generic point.

The statement is clear if either U_i equals X. Hence, we assume that both U_i miss at least one closed point of X.

We have already proven that

$$F_U \subset F_{U_1}, F_{U_2} \subset F_{U_0}$$

holds. Hence, only $F_{U_1} \cap F_{U_2} \subset F$ remains to be shown. Pick an element $f \in F_{U_1} \cap F_{U_2}$. By Weierstrass Preparation (cf. Theorem 5.2.3), we can write $f = f_1 u_1 = f_2 u_2$ for $f_i \in F \subset F_U$ and $u_i \in \widehat{R}^*_{U_i}$.

Let us first assume that $U \subsetneq X$. Then, we can write $f_i = \frac{a_i}{b_i}$ for $a_i, b_i \in \widehat{R}_U$. Thus, we obtain

$$f = \frac{a_1 u_1}{b_1} = \frac{a_2 u_2}{b_2}$$

which implies that

$$a_1 b_2 u_1 = a_2 b_1 u_2 \in \widehat{R}_{U_1} \cap \widehat{R}_{U_2} = \widehat{R}_{R_U}.$$

Therefore, $f = \frac{a_1 b_2 u_1}{b_2 b_1} \in F_U$.

In the case $U = X$, write $\widehat{R}_{U_i} = A_i[\![t]\!]$ for some subrings $A_i \subset k(X)$. As $U = X$, we know that $U_1 \not\subset U_2$ and hence $\widehat{R}_{U_2} \not\subset \widehat{R}_{U_1}$ and therefore $A_2 \not\subset A_1$.

Therefore, we can pick an element $f_0 \in A_2 \subset A_1 \cap A_2$. Let $R' = \widehat{R}_1[f_0] \cap \widehat{R}_2$. We refer the reader to [HH10, Theorem 4.9] for a proof of the fact that $\operatorname{Frac} R' = F$.

Using this, we can write $f_i = \frac{a_i}{b_i}$ with $a_i, b_i \in R'$. The proof now continues as in the case $U \subsetneq X$. □

Theorem 5.2.5. *Let $U_1, U_2 \subset X$ and $U_0 = U_1 \cap U_2$. Then, for any $A_0 \in \mathrm{GL}_n(F_{U_0})$, there are $A_i \in \mathrm{GL}_n(F_{U_i})$ such that $A_0 = A_1^{-1}A_2$.*

Proof. Let us first assume $U_0 = \emptyset$. Then, there is some positive integer r such that $t^r A_0 \in \mathrm{Mat}_n(R_\emptyset)$. Therefore, $A_0^{-1} \in t^{-r}\,\mathrm{Mat}_n(\widehat{R}_\emptyset) \subset \mathrm{Mat}_n(F_{U_0})$. As $R_\emptyset$ is t-adically dense in $\widehat{R}_\emptyset$, there is a matrix $C_{U_0} \in \mathrm{Mat}_n(R_\emptyset)$ such that $C_0 \cong t^r A_0^{-1} \mod t^{r+1}$. Let $C = t^{-r}C_0 \in t^{-r}\,\mathrm{Mat}_n(R_\emptyset) \subset \mathrm{Mat}_n(F_{U_1})$. Then, by construction, $C - A_0^{-1} \in t\,\mathrm{Mat}_n(\widehat{R}_\emptyset)$, and thus $CA_0 - I_n \in t\,\mathrm{Mat}_n(\widehat{R}_{U_0})$. But, this means that $CA_0 \in \mathrm{GL}_n(\widehat{R}_\emptyset)$, so in particular, $C \in \mathrm{GL}_n(F_{U_1})$. As $CA_0 \cong I_n \mod t$, we can conclude by Proposition 5.2.2 that there are $B_i \in \mathrm{GL}_n(F_{U_i})$ such that $CA_0 = B_1^{-1}B_0$. Hence, the matrices $A_1 = B_1C$ and $A_2 = B_2$ satisfy the statement of the theorem.

In the general case, let $U_2' = U_2 \setminus U_0$. We note that

1. $F_{U_2'} \cap F_{U_0} = F_{U_2}$, by Theorem 5.2.4

2. $U_1 \cap U_2' = \emptyset$

hold. Hence, by the first step, there are matrices $A_1 \in \mathrm{GL}_n(F_{U_1})$ and $A_2 \in \mathrm{GL}_n(F_{U_2'})$ such that $A_0 = A_1^{-1}A_2$ holds. We finish the proof by showing that in fact $A_2 \in \mathrm{GL}_n(F_{U_2})$. Note that $A_2 = A_1A_0 \in \mathrm{GL}_n(F_{U_0})$ and thus

$$A_2 \in \mathrm{GL}_n(F_{U_0}) \cap \mathrm{GL}_n(F_{U_2'}) = \mathrm{GL}_n(F_{U_2}).$$

□

As an immediate corollary, we obtain the main result of this subsection.

Theorem 5.2.6. *With the notation from above, let $\mathcal{F} = \{F_{U_1}, F_{U_2}, F_{U_0}\}$ with the natural inclusions. Then, $\varprojlim \mathcal{F} = F$ and patching holds over $\mathcal{F}$.*

We will end the section by stating a generalization of the main result.

Theorem 5.2.7 ([HH10, Theorem 4.14]). *Let T be a complete discrete valuation ring and let $\hat{X}$ be a smooth connected projective T-curve with closed fibre X. Let $U_1, \ldots, U_r$ denote subsets of X such that the pairwise intersection $U_i \cap U_j$ (for $i \neq j$) are all equal to a common subset $U_0 \subset X$. Let $U = \bigcup_i U_i$.*

Consider the finite inverse factorization system $\mathcal{F} = \{F_{U_i}\}_{i=0,\ldots,n}$. Then, the inverse limit of $\mathcal{F}$ is F_U and patching holds over $\mathcal{F}$.

5.2.2 The local case

We will now recall a patching setup for function fields of arithmetic curves obtained by Harbater, Hartmann and Krashen in [HHK14]. Following the notation of [HHK14], let T be a complete discretely valued ring with field of fraction K, uniformizer t and residue field k. Let $\hat{X}$ be a projective, integral and normal T-curve with function field F and let X denote its closed fiber.

For any closed point $p \in X$, let $\widehat{\mathcal{O}}_{\hat{X},p}$ denote the completion of the local ring $\mathcal{O}_{\hat{X},p}$ at its maximal ideal and let F_p denote the fraction field of $\widehat{\mathcal{O}}_{\hat{X},p}$. For a subset $U \subset X$, that is contained in an irreducible component of X and does not meet other components, let R_U denote the subring of F of rational functions regular on U. Let $\widehat{R}_U$ denote its t-adic completion and F_U denote the fraction field of $\widehat{R}_U$. For each branch of X at a closed point p, i.e., for each height one prime b of $\widehat{\mathcal{O}}_{\hat{X},p}$ that contains t, let $\widehat{R}_b$ be the completion of $\widehat{\mathcal{O}}_{\hat{X},p}$ at b and let F_b denote its fraction field.

Let $\mathcal{P} \subset X$ be a non-empty set of closed points of X including all points where distinct irreducible components of X meet and all closed points where X is not unibranched. This implies that $X \setminus \mathcal{P}$ is a disjoint union of finitely many irreducible affine k curves. The set of these curves will be denoted by $\mathcal{U}$.

Let $p \in \mathcal{P}$ and $U \in \mathcal{U}$ be chosen such that p is contained in the closure of U. Then, the ideal defining U induces an ideal in $\mathcal{O}_{\hat{X},p}$. The branches of U at p are the height one prime ideals in R_p containing said induced ideal. Let $\mathcal{B}$ denote the set of all branches. Note that we have inclusions $F \subset F_u, F_p \subset F_b$, whenever b is a branch corresponding to U and p.

Figure 5.1 visualizes this setup in the case where three points were chosen.

These fields form a finite inverse factorization system $\mathcal{F}$. With the notation from the beginning of this chapter, we have $I_v = \mathcal{P} \sqcup \mathcal{U}$ and $I_e = \mathcal{B}$.

Our goal in this section is to discuss the proof of the following theorem.

Theorem 5.2.8 ([HH10, Theorem 6.4], [HHK15b, Proposition 3.2.1])**.** *In the above setup,*

1. *F is the inverse limit of $\mathcal{F}$,*

2. *patching holds for finite dimensional vector spaces over $\mathcal{F}$.*

To simplify exposition, we will focus on the case where $\hat{X}$ is smooth.

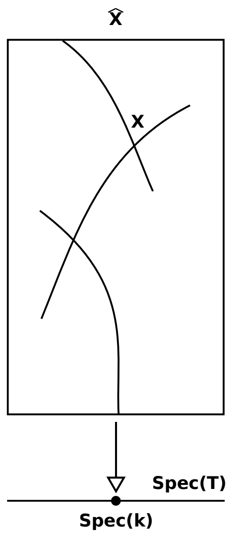
In order to prove the theorem above, we will prove that GL_n satisfies factorization over $\mathcal{F}$. For notational simplicity, let us assume that our patching setup is given by

$$\begin{tikzcd} & F_b & \\ F_p \arrow[ur, hook] & & F_U \arrow[ul, hook'] \\ & F \arrow[ul, hook] \arrow[ur, hook'] & \end{tikzcd}$$

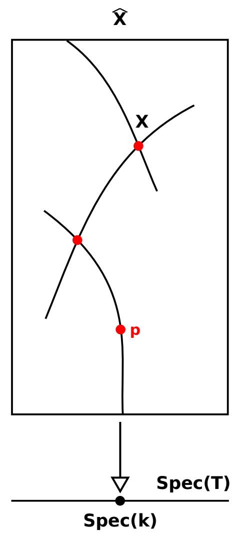
i.e. $\mathcal{U} = \{U\}$, $\mathcal{P} = \{p\}, \mathcal{B} = \{b\}$ with $X = U \cup \{p\}$ and a single branch at p and U. An example of this setup is given by $\widehat{X} = \mathbb{P}^1_T$ and $U = \mathbb{A}^1_k$ and $p = \{\infty\}$.

We want to prove that the natural functor

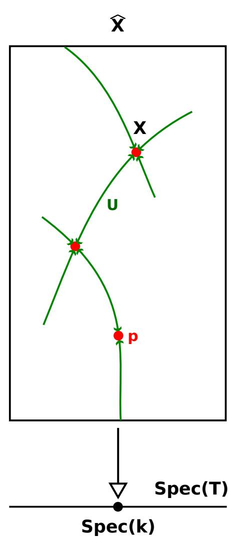
$$\mathrm{VECT}(F) \to \mathrm{VECT}(F_U) \times_{\mathrm{VECT}(F_b)} \mathrm{VECT}(F_p) \tag{5.1}$$



(a) Setup



(b) Choosing closed points p_i



(c) Collecting U_i

Figure 5.1: Local patching setup

is an equivalence. In the last section, we discussed (in the case $\hat{X} = \mathbb{P}^1_T$) that the functor

$$\mathrm{VECT}(F) \to \mathrm{VECT}(F_U) \times_{\mathrm{VECT}(F_\emptyset)} \mathrm{VECT}(F_{\{p\}})$$

is an equivalence. Hence, in order to show (5.1), it is enough to show that

$$\mathrm{VECT}(F_{\{p\}}) \to \mathrm{VECT}(F_p) \times_{\mathrm{VECT}(F_b)} \mathrm{VECT}(F_\emptyset)$$

is an equivalence.

Thus, the second part of Theorem 5.2.8 follows from the next Proposition.

Proposition 5.2.9. *For any $n \geq 1$ and $A \in \mathrm{GL}_n(F_\emptyset)$, there are $A_1 \in \mathrm{GL}_n(F_p)$ and $A_1 \in \mathrm{GL}_n(F_\emptyset)$ such that $A = A_1^{-1} A_2$.*

Let us remark (without proof) some algebraic properties of the rings $\widehat{R}_p, \widehat{R}_{\{p\}}, \widehat{R}_b$ and $\widehat{R}_\emptyset$:

- As $\widehat{X}$ is regular and projective, we can conclude that $\widehat{R}_p$ is a 2-dimensional regular local ring; we will denote its maximal ideal by m_p.
- If f is a lift of a local parameter $\overline{f} \in \mathcal{O}_{X,p}$ under $\widehat{R}_{\{p\}} \twoheadrightarrow \mathcal{O}_{X,p}$, then $\{f, t\}$ is a system of local parameters for $\widehat{X}$ at p.
- We have

  $$R_\emptyset \subset \widehat{R}_{\{p\}} \subset \widehat{R}_\emptyset$$

  and hence

  $$\widehat{R}_\emptyset = \widehat{\widehat{R}_{\{p\}}[f^{-1}]}_{(t)} = \widehat{\widehat{R}_{\{p\}}[f^{-1}]}$$

where all completions are with respect to t.

- As $\widehat{R}_b$ is the t-adic completion of $\left(\widehat{R}_p\right)_{(t)}$ and

$$\left(\widehat{R}_p\right)_{(t)} \subset \widehat{R}_p[f^{-1}]_{(t)} \subset \widehat{R}_b,$$

we can conclude that $\widehat{R}_b$ is the t-adic completion of $\widehat{R}_p[f^{-1}]_{(t)}$ and (equivalently) $\widehat{R}_p[f^{-1}]$.

Let us rename our rings for notational simplicity:

- $\widehat{R} := \widehat{R}_{\{p\}}$
- $\widehat{R}_1 := \widehat{R}_{\emptyset}$
- $\widehat{R}_2 := \widehat{R}_p$
- $\widehat{R}_0 := \widehat{R}_b$

Let $\overline{R} := \widehat{R}/(t)$ and $\overline{R}_i := \widehat{R}/(t)$. The next lemma collects various algebraic properties of the ring $\widehat{R}, \widehat{R}_i$ and their residue rings.

Lemma 5.2.10.

1. *$\widehat{R}_1$ is the f-adic completion of $\widehat{R}$ and $\widehat{R} \subset \widehat{R}_1, \widehat{R}_2 \subset \widehat{R}_0$.*
2. *$t\widehat{R}_i \cap \widehat{R} = t\widehat{R}$ for $i = 0, 1, 2$ and $t\widehat{R}_0 \cap \widehat{R}_i = t\widehat{R}_i$ for $i = 1, 2$.*
3. *The rings $\widehat{R}_2$ and $\widehat{R}_0$ are complete discretely valued rings with parameter t. The rings $\overline{R}_1$ and $\overline{R}$ are discretely valued rings with parameter $\overline{f}$.*
4. *The ring $\overline{R}_1$ is the $\overline{f}$-adic completion of the ring $\overline{R}$. Furthermore,*

$$\overline{R}_2 \simeq \overline{R}[\overline{f}^{-1}] = \mathrm{Frac}(\overline{R})$$
$$\overline{R} \simeq \overline{R}_1[\overline{f}^{-1}] = \mathrm{Frac}(\overline{R}_1)$$

5. *There are natural inclusions* $\overline{R} \subset \overline{R}_2, \overline{R}_1 \subset \overline{R}_0$ *and* $\overline{R}_1 \cap \overline{R}_2 = \overline{R}$ *inside* $\overline{R}_2$.

Proof. 1. As the maximal ideal $m_{\widehat{R}}$ of $\widehat{R}$ is generated by f and t, we have $(f^{2n}, t^{2n}) \subset m^{2n} \subset (f^n, t^n)$. Thus, using that $\widehat{R}$ is t-adically complete, we see that

$$\begin{aligned}\varprojlim_j \widehat{R}/f^j\widehat{R} &= \varprojlim_j \varprojlim_i \left(\widehat{R}/t^i\widehat{R}\right)/f^j\left(\widehat{R}/t^i\widehat{R}\right) \\ &= \varprojlim_j \varprojlim_i \widehat{R}/(t^i, f^j) = \varprojlim_j \widehat{R}/m^j = \widehat{R}_1\end{aligned}$$

holds.

Recall that for any ring A and ideal $I \subset A$ such that $1 + I$ has no zero divisors, we can conclude that $\bigcap_{n\in\mathbb{N}} I^n = 0$.

Also, note that the maps

$$\begin{aligned}\widehat{R} &\to \widehat{R}_1 \\ \widehat{R}[f^{-1}] &\to \widehat{R}_2 \\ \widehat{R}_1[f^{-1}] &\to \widehat{R}_0\end{aligned}$$

are injective as completion maps. Hence, the morphisms

$$\begin{aligned}\widehat{R} &\to \widehat{R}_2 \\ \widehat{R}_1 &\to \widehat{R}_0\end{aligned}$$

are injective.

Let us now show that $\widehat{R}_2 \to \widehat{R}_0$ is injective. For this, let us first note that, as $\widehat{R}_1$ is the f-adic completion of $\widehat{R}$, we can conclude that $\widehat{R} \cap f^n\widehat{R}_1 = f^n\widehat{R}$ for any n. Also, $t^j\widehat{R}$ is f-adically dense in $t^j\widehat{R}_1$. Thus, for $g \in t^j\widehat{R}_1 \cap \widehat{R}$, we can write $g = t^j r + f^m s$ with $r \in \widehat{R}, s \in \widehat{R}_1$. It follows that

$$f^n s \in \widehat{R} \cap f^n \widehat{R}_1 = f^n \widehat{R}$$

and therefore $s \in \widehat{R}$. This in turn implies that $g \in (t^j, f^n) \subset \widehat{R}$ for all n. As $\overline{f} \in \widehat{R}/t^j\widehat{R}$ is in the maximal ideal, the ideal $1 + (\overline{f})$ contains no zero divisors. Hence, $\bigcap_{n\in\mathbb{N}}(\overline{f}^n) = (0)$ in $\widehat{R}/t^j\widehat{R}$ and $\bigcap_{n\in\mathbb{N}}(t^j, f^n) = (t^j) \subset \widehat{R}$. In particular, $g \in t^j\widehat{R}$ and thus $t_j\widehat{R}_1 \cap \widehat{R} = t^j\widehat{R}$. As this implies that

$$t^j\widehat{R}_1[f^{-1}] \cap \widehat{R}[f^{-1}] = t^j\widehat{R}[f^{-1}]$$

for all j, we can conclude that $\widehat{R}_2 \to \widehat{R}_0$ is injective. (Recall that $\widehat{R}_0$ is the the t-adic completion of $\widehat{R}_2[f^{-1}]$.)

2. The equality $t\widehat{R}_1 \cap \widehat{R} = t\widehat{R}$ was proven in the last part.

   In order to prove $t\widehat{R}_0 \cap \widehat{R} = t\widehat{R}$, pick $g \in t\widehat{R}_1[f^{-1}] \cap \widehat{R}$. Then, there is a positive integer n such that $f^n g \in t\widehat{R}_1 \cap \widehat{R} = t\widehat{R}$ holds. As f and t are local parameters, it follows that $g \in t\widehat{R}$. Hence,

$$t\widehat{R}_1 \cap \widehat{R} \subset t\widehat{R}$$

   which, after passing to the t-adic completion, implies

$$t\widehat{R}_0 \cap \widehat{R} = t\widehat{R}.$$

   Note that this immediately implies $t\widehat{R}_2 \cap \widehat{R} = t\widehat{R}$, as we proved $t\widehat{R}_2 \subset t\widehat{R}_0$ in the last step.

   The statement $t\widehat{R}_0 \cap \widehat{R}_1 = t\widehat{R}_1$ follows from $t\widehat{R}_1[f^{-1}] \cap \widehat{R}_1 = t\widehat{R}_1$. To prove this equality, not that the inclusion $\supset$ is trivial. For the other direction, fix some

$g \in f\widehat{R}_1[f^{-1}] \cap \widehat{R}_1$. Then, there is some $n \geq 1$ such that $f^n g \in t\widehat{R}_1 \cap f^n \widehat{R}_1 = tf^n \widehat{R}_1$. Therefore, we have $g \in t\widehat{R}_1$, concluding this part.

Similarly to the last case, the equality $t\widehat{R}_0 \cap \widehat{R}_2 = t\widehat{R}_2$ follows from $t\widehat{R}_1[f^{-1}] \cap \widehat{R}[f^{-1}] = t\widehat{R}[f^{-1}]$ which was proven in the first part of the lemma.

3. As f, t form a s system of local parameters in $\widehat{R}$ and $\widehat{R}_1$, it follows that $\widehat{R}[f^{-1}]$ and $\widehat{R}_1[f^{-1}]$ are regular domains of dimension 1. In particular, their t-adic completions $\widehat{R}_2$ and $\widehat{R}_0$, are complete discrete valued fields with parameter t.

   Similarly, it follows that $\overline{f}$ is a local parameter for the 1 dimensional local domains $\overline{R}$ and $\overline{R}_1$. It follows that $\overline{R}$ and $\overline{R}_1$ are discretely valued rings with parameter $\overline{f}$.

4. The first statement follows immediately from the first part. The equalities $\overline{R}_2 = \overline{R}[\overline{f}^{-1}]$ and $\overline{R}_0 = \overline{R}_1[\overline{f}^{-1}]$ are direct consequences of the last part.

5. The inclusions are clear by the last part. For the intersection, the description of the last part implies

$$\overline{R}_1 \cap \overline{R}_2 = \overline{R}_1 \cap \overline{R}[\overline{f}^{-1}] = \overline{R}$$

   as $\overline{f}$ is a parameter, and thus not a unit, in $\overline{R}_1$.

□

Analogously to the global case, we need an additive decomposition of scalars.

Lemma 5.2.11. *For any $a \in \widehat{R}_0$, there are $b \in \widehat{R}_1$ and $c \in \widehat{R}_2$ such that $a = b + c \mod \widehat{R}_0 t$.*

Proof. The statement is clear when $a = 0$, so let us assume $a \neq 0$. Let $\overline{a} \in \overline{R}_0$ denote the image of a under the natural quotient map $\widehat{R}_0 \to \overline{R}_0$. By Lemma 5.2.10, $\overline{R}_1$ is a discretely valued ring with parameter $\overline{f}$ and fraction field $\overline{R}_0$. Let $n \in \mathbb{Z}$ denote the

$\overline{f}$-adic valuation of $\overline{a}$. If $n \geq 0$, then $a \in \widehat{R}_1$ and the decomposition $b = a$ and $c = 0$ works.

If n $n < 0$, then $\overline{f^n a} \in \widehat{R}_1$. As $\widehat{R} \subset \widehat{R}_1$ is f-adically dense by part 4 of Lemma 5.2.10, there is $\overline{d} \in \widehat{R}$ such that

$$\overline{d} \cong \overline{f^n a} \mod \overline{f}^n \widehat{R}_1$$

holds. Let $\overline{c} = \overline{f}^{-n} \overline{d} \in \overline{R}_2$. Then,

$$\overline{f}^n (\overline{a} + \overline{c}) = \overline{f}^n \overline{d} - \overline{c} \in \overline{f}^n \overline{R}_1$$

and thus, $\overline{b} = \overline{a} - \overline{c} \in \overline{R}_1$. Choosing lifts of $\overline{c}$ and $\overline{b}$ yields the desired decomposition. □

Let us denote $F = \mathrm{Frac}(\widehat{R})$ and $F_i = \mathrm{Frac}(\widehat{R}_i)$.

Theorem 5.2.12. *For all $A \in \mathrm{GL}_n(F_0)$ there exist $A_i \in \mathrm{GL}_n(F_i)$ for $i = 1, 2$ such that $A = A_1 A_2$.*

Proof. The proof is analogous to the proof of Theorem 5.2.5, by first proving a local version of Theorem 5.2.2. Compare [HH10, Theorem 5.4] for details. □

We will now work toward proving the intersection property. Similarly to the last case, we will first prove the intersection property for the underlying rings and use this to prove an analogue of the Weierstrass Preparation Theorem.

Lemma 5.2.13. *We have*

$$\widehat{R}_1 \cap \widehat{R}_2 = \widehat{R},$$

where the intersection is taken inside $\widehat{R}_0$.

Proof. Let $S = \widehat{R}_1 \cap \widehat{R}_2$, note that there is a natural inclusion $R \hookrightarrow S$. By Lemma 5.2.10, we know $\overline{R} = \overline{R}_1 \cap \overline{R}_2$. Hence, the map $\overline{R} \hookrightarrow \overline{S}$ is surjective. Assume now that $\overline{x} \in \overline{R}$ maps to 0 in $\overline{S}$. Then, there exists $y \in S$ such that $ty = x$. Therefore, $ty \in R \cap R_1 t$. Using Lemma 5.2.10 we can conclude $x = ty \in tR$. Hence $\overline{x} = 0$. Thus, the natural map $R \hookrightarrow S$ becomes an isomorphism modulo t.

It follows that $\widehat{R} \cap tS = t\widehat{R}$ and $S = \widehat{R} + tS$. By induction, it follows that $\widehat{R} \cap t^j S = t^j \widehat{R}$ and $S = \widehat{R} + t^j S$. So, for any $x \in S$, there is a $y_n \in \widehat{R}$ such that $x - y_n \in t^j \widehat{R}$. Hence, $y_n - y_m \in t^N \widehat{R}$ for $n, m > N$. Thus, $\{y_n\}$ form a Cauchy sequence with limit x. As $\widehat{R}$ is t-adically complete, this implies $x \in \widehat{R}$. □

We will state the following technical lemma without proof.

Lemma 5.2.14 ([HH10, Lemma 5.5]). *Let $a \in \widehat{R}_0^*$. Then, there are $a_i \in \widehat{R}_i^*$ such that $a = a_1 a_2$.*

Using this Lemma, we can prove a local analogue of the Weierstrass preparation theorem.

Proposition 5.2.15 (Local Weierstrass Preparation). *For $a \in \widehat{R}_1$ there exist $b \in \widehat{R}_1^*$ and $c \in F$ such that $a = bc$.*

Proof. We can assume without loss of generality that a is not 0. Hence, there is a non-negative integer n such that $a = t^n a'$ with $a' \in \widehat{R}_1^*$. If we prove the statement for a', then it clearly also holds for a. Hence, we may assume that a is a unit. By Lemma 5.2.14, there are $a_i \in \widehat{R}_i^*$ such that $a = a_1 a_2$. However, $a_2 = a^{-1} a_1 \in \widehat{R}_1^*$. As $\widehat{R}_1 \cap \widehat{R}_2 = \widehat{R}$, we can conclude that $a_2 \in \widehat{R}$. hence, the choice $a_1 = b$ and $a_2 = c$ works. □

Theorem 5.2.16. *We have*

$$F_1 \cap F_2 = F,$$

where the intersection is taken inside F_0.

Proof. Let $h \in F_1 \cap F_2$. As $h \in F_1$, we can write $h = \frac{a}{b}$ with $a, b \in \widehat{R}_1$. By Weierstrass Preparation (cf. Theorem 5.2.15), we can write $b = uf$ for $u \in \widehat{R}_1^*$ and $f \in F^*$. Thus, $h = \frac{au^{-1}}{f}$. As it is enough to prove that $fh \in F$, we will assume from now on that $h \in \widehat{R}_1$.

Recall that $\widehat{R}_2$ is a complete discretely valued field with parameter t. As $h \in F_2$, there is an integer n such that $t^n h \in \widehat{R}_2$. As $t \in F$, it is enough to prove the statement for $t^n h$, i.e. we may assume that $h \in \widehat{R}_2$. Hence, $h \in \widehat{R}_1 \cap \widehat{R}_2$, so, by use of Lemma 5.2.13, we can conclude that $h \in \widehat{R} \subset F$. $\square$

Having proven the intersection property and the factorization property, we can conclude that patching holds.

Corollary 5.2.17. *The natural functor*

$$\mathrm{VECT}(F) \to \mathrm{VECT}(F_1) \times_{\mathrm{VECT}(F_0)} \mathrm{VECT}(F_2)$$

is an equivalence.

This concludes our discussion of the proof of Theorem 5.2.8. We end the section with an example.

Example 6. Let $T = k[\![t]\!]$ and let $\widehat{X} = \mathbb{P}^1_T$ be the projective T-line. Pick the patching cover given by $U = X \setminus \{0\}$ and $p = 0 \in X$. The corresponding system of fields then corresponds to the following diagram:

$$\begin{tikzcd}
 & \mathrm{Frac}\,(k(\!(x)\!)[\![t]\!]) & \\
\mathrm{Frac}\,(k[x^{-1}][\![t]\!]) \arrow[ur, hook] & & \mathrm{Frac}\,(k[\![x,t]\!]) \arrow[ul, hook'] \\
 & k(\!(t)\!)(x). \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

5.3 Patching for torsors and a local-global principle

In this section, we will describe how patching for vector spaces implies patching for torsors under linear algebraic groups. This will then lead us to a characterization of a local-global principle for such torsors. The material in this section is based on [HHK15a].

We will start by fixing notation and describing a torsor patching problem. Fix a field F and an inverse factorization system $\mathcal{F}$ over F. Let G be a sheaf of groups in the big étale site over F.

Let $\mathrm{TOR}(G)(F)$ denote the category of G-torsors over F. Let G_i denote that base change of G to F_i. A G-torsor patching problem over $\mathcal{F}$ is given by a collection of G_i-torsors T_i over F_i for all $i \in I_v$ and a collection of isomorphisms $\nu_k\colon T_i|_{F_k} \to T_j|_{F_k}$ of G_k-torsors for all $(i,j,k) \in S_I$.

Let $\mathrm{TPP}(G)(\mathcal{F})$ denote the category of torsor patching problems, i.e. the category with objects $\left(\{T_i\}_{i\in I_v}, \{\nu_k\}_{k\in I_e}\right)$. A morphism of torsor patching problems

$$\left(\{T_i\}_{i\in I_v}, \{\nu_k\}_{k\in I_e}\right) \to \left(\{T_i'\}_{i\in I_v}, \{\nu_k'\}_{k\in I_e}\right)$$

is a collection of G_i-torsor morphisms $T_i \to T_i'$ compatible with ν_k and ν_k'.

As in the case of vector spaces, there is a natural functor

$$\beta_G'\colon \mathrm{TOR}(G)(F) \to \mathrm{TPP}(G)(\mathcal{F}).$$

sending a torsors defined over F to its associated trivial patching datum.

Theorem 5.3.1 ([HHK15a, Theorem 2.3]). *Let G be a linear algebraic group over F. If the natural functor $\beta\colon \mathrm{VECT}(F) \to \mathrm{PP}(\mathcal{F})$ is an equivalence of categories, then so is the functor $\beta_G'\colon \mathrm{TOR}(G)(F) \to \mathrm{TPP}(G)(\mathcal{F})$.*

Sketch: We will only prove that β'_G is essentially surjective. Fix a torsor patching problem $\left(\{T_i\}_{i\in I_v}, \{\nu_k\}_{k\in I_e}\right)$ As G is linear algebraic, there is an embedding $G \hookrightarrow \mathrm{GL}_n$. Hence, we have the exact sequence

$$1 \to G(F_i) \to \mathrm{GL}_n(F_i) \to \mathrm{GL}_n/G(F_i) \to \mathrm{H}^1(F_i, G) \to \mathrm{H}^1(F_i, \mathrm{GL}_n)$$

for $i \in I$. Note that $\mathrm{H}^1(F_i, \mathrm{GL}_n)$ is trivial by Hilbert 90. Hence, the morphism $\mathrm{GL}_n/G(F_i) \to \mathrm{H}^1(F_i, G)$ is surjective. Let $[A_i] \in \mathrm{GL}_n/G(F_i)$ with $A_i \in \mathrm{GL}_n(F_i^{\mathrm{sep}})$ be lifts of the classes $[T_i] \in \mathrm{H}^1(F_i, G)$. Consider the matrices $B_k = \nu_k(A_i)A_i^{-1} \in \mathrm{GL}_n(F_k^{\mathrm{sep}})$ for $(i, j, k) \in S_I$. This element defines a morphism

$$A_i G_{F_k^{\mathrm{sep}}} \to \nu_k(A_i) G_{F_k^{\mathrm{sep}}} = A_j G_{F_k^{\mathrm{sep}}}$$

via left multiplication. This morphism identifies with the isomorphism $\nu_k|_{F_k^{\mathrm{sep}}} \colon T_i(F_k^{\mathrm{sep}}) \to T_j(F_K^{\mathrm{sep}})$ and is thus defined over F_k. In particular, it sends A_i to $\nu_k(A_i)$. and we can conclude $B_k \in \mathrm{GL}_n(F_k)$. As patching holds for vector spaces, we can deduce that GL_n satisfies factorization over $\mathcal{F}$ (compare Theorem 5.1.1). Thus, there are matrices $B_i \in \mathrm{GL}_n(F_i)$ and $B_j \in \mathrm{GL}_n(F_j)$ such that $B_k = B_i^{-1}B_j$. Consider now the system $C_i = B_i A_i \in \mathrm{GL}_n(F_i^{\mathrm{sep}})$. Note that the collection $(\{C_i G_{F_k^{\mathrm{sep}}}\}_{i\in I_v}, \{\mathrm{id}_{C_i G_{F_k^{\mathrm{sep}}}}\}_{k\in I_e})$ also forms a system of torsors over $\mathcal{F}$, as $C_i G_{F_k^{\mathrm{sep}}}$ and $C_j G_{F_k^{\mathrm{sep}}}$ agree by construction. Note that the morphisms $C_i G_{F_k^{\mathrm{sep}}} \to A_i G_{F_k^{\mathrm{sep}}}$ defined by left multiplication with B_i^{-1} are compatible with ν_k. Hence, the two patching problems are equivalent. Let $\pi \colon \mathrm{GL}_n \to \mathrm{GL}_n/G$ be the quotient map (note that GL_n/G is a quasi-projective variety). Let $p_i = \pi(C_i) \in \mathrm{GL}_n/G(F_i)$. By construction, the p_i form a family of F_i rational points such that $p_i = p_j \in \mathrm{GL}_n/G(F_k)$ whenever $(i, j, k) \in S_I$. As F is the inverse limit of $\mathcal{F}$, we thus obtain a point $p \in \mathrm{GL}_n/G(F)$. The fiber of p has the shape $CG_{F^{\mathrm{sep}}}$ for $C \in \mathrm{GL}_n(F^{\mathrm{sep}})$ such that $CG_{F_i^{\mathrm{sep}}} = C_i G_{F_i^{\mathrm{sep}}}$ for all $i \in I_v$. Thus, the torsor associated with $CG_{F^{\mathrm{sep}}}$ is a solution to the patching problem.

We will omit the proof of fully faithfulness and refer to [HHK15a, Theorem 2.3]. $\square$

In particular, patching holds for G-torsors in the two examples defined in the last section.

Corollary 5.3.2. *Let F and $\mathcal{F}$ be as in Section 5.2. Then, patching holds for G-torsor over $\mathcal{F}$.*

The following proposition is a slight variant of [HHK15b, Theorem 2.2.4(c)(iii)]. It will later be used to establish an analogous result for bitorsors which in turn will be crucial to prove patching results for gerbes. For an étale F-algebra A, write $\mathrm{TOR}(G)(A)$ for the category of $G \times_F \mathrm{Spec}(A)$-torsors over $\mathrm{Spec}(A)$. Let $\mathrm{TPP}(\mathcal{F}_A)$ denote the category of $G \times_F \mathrm{Spec}(A)$-torsor patching problems over $\mathcal{F}_A$, defined analogously to $\mathrm{PP}(\mathcal{F}_A)$.

Proposition 5.3.3. *Let $A = \prod_{r=1}^{n} L_r$ be a product of finitely many finite separable field extensions L_i/F. Let G be a linear algebraic group over A (i.e. $G|_{L_r}$ is a linear algebraic group over L_i). If the functor $\beta\colon \mathrm{VECT}(F) \to \mathrm{PP}(\mathcal{F})$ is an equivalence of categories, then so is the functor $\widehat{\beta}'_G\colon \mathrm{TOR}(G)(A) \to \mathrm{TPP}(G)(\mathcal{F}_A)$.*

Proof. By Proposition 5.1.2, patching holds for free modules of finite rank over $\mathcal{F}_A$. Hence, GL_n satisfies factorization over $\mathcal{F}_A$, cf. Theorem 5.1.1. The proof is now verbatim to the proof of [HHK15a, Theorem 2.3]. $\square$

Once patching for torsors holds, one can describe local-global principle for torsors in terms of simultaneous factorization. We say that G-torsors satisfy the local-global principle with respect to $\mathcal{F}$ if for any G-torsor P, we have that $P \times_{\mathrm{Spec}(F)} \mathrm{Spec}(F_i) \simeq G_i$ for all $i \in I_v$ if and only if $P \simeq G$ (here, G denotes the trivial G torsor).

We can express torsor patching in the form a Mayer-Vietoris type sequence, that will reveal a characterization of when the local-global principle for torsors hold.

Theorem 5.3.4. *Let G be a linear algebraic group. Assume that patching for torsors holds over $\mathcal{F}$. Then, there is an exact sequence*

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{H}^0(F,G) \arrow[r] & \prod_{i\in I_v} \mathrm{H}^0(F_i,G) \arrow[r] & \prod_{k\in I_e} \mathrm{H}^0(F_k,G) \arrow[dll, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\ & \mathrm{H}^1(F,G) \arrow[r] & \prod_{i\in I_v} \mathrm{H}^1(F_i,G) \arrow[r, shift left] \arrow[r, shift right] & \prod_{k\in I_e} \mathrm{H}^1(F_k,G) \end{tikzcd}$$

of pointed sets.

Sketch. The exactness of the first row follows from the intersection property of $\mathcal{F}$.

The exactness at $\prod_{i\in I_v} \mathrm{H}^1(F_i,G)$ follows immediately from the assumption that patching holds.

Let us now describe the construction of the map δ. Given an element $(a_k) \in \prod_{k\in I_e} G(F_k)$, we want to construct an H-torsor P_a over F. Identifying $a_k \in G(F_k)$ with the torsor isomorphism $G_k \to G_k$ induced by left translation of a_k, we can define the G-torsor patching problem $(\{(G|_{F_i}\}_{i\in I_v}, \{a_k\}_{k\in I_e})$. By assumption, patching holds for G-torsors, let P_a denote a solution to this patching problem. We define δ by mapping (a_k) to the class of P_a in $\mathrm{H}^1(F,G)$. It is easy to check that this is well-defined.

If δ maps (a_k) to the class of zero, then this means that P_a is isomorphic to G. This in turn is equivalent to the torsor patching problems

$$(\{G|_{F_i}\}_{i\in I_v}, \{a_k\}_{k\in I_e})$$

and

$$(\{(G|_{F_i}\}_{i\in I_v}, \left\{\mathrm{id}_{G|_{F_k}}\right\}_{k\in I_e}).$$

being isomorphic in $\mathrm{TPP}(\mathcal{F},G)$. By the definition of morphism, this is equivalent to the existence of $(b_i)_{i\in I_v}$ such that $a_k = b_i^{-1}b_j$ for all $(i,j,k) \in S_I$. Hence, the sequence is exact at $\prod_{k\in I_e} \mathrm{H}^0(F_k, G)$.

Similarly, if the class of a G-torsor P maps to 0 in $\prod_{i\in I_v} \mathrm{H}^1(F_i, G)$, then P corresponds to a patching problem

$$(\{(G|_{F_i}\}_{i\in I_v}, \{\phi_k\}_{k\in I_e}).$$

As the group of automorphisms of $G|_{F_k}$ as a $G|_{F_k}$ torsor is $G(F_k)$, we can identify each ϕ_k with an element $a_k \in G(F_k)$. Hence, the class of P is the image of (a_k) under δ. This proves exactness at $\mathrm{H}^1(F, G)$. □

As an immediate corollary, we see that local-global principle for G-torsors is equivalent to factorization for G.

Theorem 5.3.5 ([HHK15a, Theorem 3.5])**.** *Let G be a linear algebraic group. Then, local-global principle for G-torsors holds if and only if G satisfies simultaneous factorization over $\mathcal{F}$.*

5.3.1 Factorization and local-global in the local case

We have just seen that local-global principles for G-torsors are related to simultaneous factorization of G. Of course, this characterization is only helpful, if we are able to verify factorization in interesting cases. In this subsection, we will recall a factorization theorem for rational linear algebraic groups in the patching setup described in Section 5.2.8. Let F and $\mathcal{F}$ be as described in said section and let G be a linear algebraic group.

Harbater, Hartmann and Krashen proved the following important theorem on factorization for rational linear algebraig groups.

Theorem 5.3.6 ([HHK15a, Corollary 6.5])**.** *Let G be a rational linear algebraic group. Then, G satisfies simultaneous factorization over $\mathcal{F}$ if and only if G is connected or Γ is a tree.*

5.4 Separable Factorization

Fix an abstract inverse factorization system $\mathcal{F}$ with inverse limit F. Let $L = F^{\text{sep}}$ denote the separable closure of F and let G denote the absolute Galois group of F. We will write L_i to denote the separable F_i-algebra $F_i \otimes_F F^{\text{sep}}$.

There is an exact sequence of F-vector spaces

$$0 \to F \to \prod_{i \in I_v} F_i \xrightarrow{\Delta} \prod_{k \in I_e} F_k$$

Here, Δ is given by the product of the maps $\Delta_k \colon F_i \times F_j \to F_k$ for $(i, j, k) \in S_I$ where Δ_k is defined via $(f_i, f_j) \mapsto f_i - f_j$. Recall that we fixed the orientation, i.e. the order of i and j in (i, j, k).

This in turn implies that the sequence

$$0 \to L \to \prod_{i \in I_v} L_i \xrightarrow{\Delta} \prod_{k \in I_e} L_k$$

is also exact. Let G be a linear algebraic group over F. We can conclude that

$$0 \to G(L) \to \prod_{i \in I_v} G(L_i) \xrightarrow{\Delta} \prod_{k \in I_e} G(L_k)$$

is exact as well.

We say that G satisfies separable factorization if Δ is surjective, i.e. if the sequence

$$0 \to G(L) \to \prod_{i \in I_v} G(L_i) \xrightarrow{\Delta} \prod_{k \in I_e} G(L_k) \to 0$$

is exact.

More concretely, the definition says that given group elements $g_k \in G(L_k)$ for every $k \in I_e$, then there are $g_i \in G(L_i)$ for $i \in I_v$ such that $g_k = g_i g_j^{-1}$ for any $(i, j, k) \in S_I$, explaining the term *separable factorization.*

Harbater, Hartmann and Krashen proved that GL_n satisfies factorization if and only if it satisfies separable factorization. Maybe even more surprising, GL_n satisfies (separable) factorization if and only if **any** linear algebraic group satisfies separable factorization.

Theorem 5.4.1 ([HHK14, Theorem 2.2.4]). *The following are equivalent:*

1. *Patching holds over $\mathcal{F}$,*

2. GL_n *satisfies factorization over $\mathcal{F}$,*

3. *Any linear algebraic group over F satisfies separable factorization over $\mathcal{F}$.*

Sketch: We will only sketch why 2) implies 1). If GL_n satisfies separable factorization, then, by definition, the sequence

$$0 \to \mathrm{GL}_n(L) \to \prod_{i \in I_v} \mathrm{GL}_n(L_i) \xrightarrow{\Delta} \prod_{k \in I_e} \mathrm{GL}_n(L_k) \to 0$$

is exact. The resulting long exact sequence starts as

$$0 \to \mathrm{GL}_n(F) \to \prod_{i \in I_v} \mathrm{GL}_n(F_i) \xrightarrow{\Delta} \prod_{k \in I_e} \mathrm{GL}_n(F_k) \to \mathrm{H}^1(F, \mathrm{GL}_n).$$

But, by Hilbert 90, $\mathrm{H}^1(F, \mathrm{GL}_n) = 0$, proving that GL_n satisfies factorization. $\square$

One immediate application of this result is that any linear algebraic group satisfies separable factorization in the patching setup over arithmetic curves.

Theorem 5.4.2 ([HHK14, Theorem 3.1.1]). *Let F and $\mathcal{F}$ be as in Section 5.2. Every linear algebraic group G over F satisfies separable factorization over $\mathcal{F}$.*

5.5 Galois Cohomology

Our final goal in this chapter is to sketch patching results in Galois cohomology. The material in this section is based on [HHK14]. Let us start by defining what we mean by patching for Galois cohomology.

Let A be an abelian linear algebraic group over F and let $\mathrm{H}^n(F, A)$ denote the n-th Galois cohomology group. For every $(i, j, k) \in S_I$, we have a map $\mathrm{H}^n(F_i, A) \times \mathrm{H}^n(F_j, A) \to \mathrm{H}^n(F_k, A)$ given by $(\alpha_i, \alpha_j) \mapsto \alpha_i|_{F_k} \alpha_j|_{F_k}^{-1}$. The collection of these maps gives a map $\prod_{i\in I_v} \mathrm{H}^n(F_i, A) \to \prod_{k\in I_e} \mathrm{H}^n(F_k, A)$.

Definition 20. We say that patching holds for Galois cohomology with values in A over $\mathcal{F}$ if the sequence

$$\mathrm{H}^n(F, A) \to \prod_{i\in I_v} \mathrm{H}^n(F_i, A) \to \prod_{k\in I_e} \mathrm{H}^n(F_k, A)$$

is exact for any $n \geq 0$.

Assume now that vector space patching holds for $\mathcal{F}$ over F. Recall from the last section, that this implies that any linear algebraic group satisfies separable factorization. Thus, we know that the natural sequence

$$0 \to \mathrm{GL}_n(F^{\mathrm{sep}}) \to \prod_{i\in I_v} \mathrm{GL}_n(F_i \otimes_F F^{\mathrm{sep}}) \xrightarrow{\Delta} \prod_{k\in I_e} \mathrm{GL}_n(F_k \otimes_F F^{\mathrm{sep}}) \to 0$$

is exact. It is tempting to conclude that patching holds for Galois cohomology with values in A by looking at the long exact sequence associated to this short exact sequence. However, the problem lies in the fact that, generally, F_i^{sep} and $F_i \otimes_F F^{\mathrm{sep}}$ are quite different. Fix separable closures F_i^{sep} for $i \in I$ and let F_i^{gd} denote the compositum of F^{sep} and F_i. One can show that

$$\mathrm{H}^n(F, A(F_i \otimes_F F^{\mathrm{sep}})) \simeq \mathrm{H}^n(F_i^{\mathrm{gd}}/F_i, A(F_i^{\mathrm{gd}}))$$

holds, compare [HHK14, Lemma 2.3.2]. (In fact, this holds for an arbitrary field extension)

This motivates the following definition.

Definition 21. Let E/F be a field extension. Fix a separable closure E^{sep} of E and let F^{sep} denote the separable closure of F inside E^{sep}. Let E^{gd} denote the compositum of F^{sep} and E inside E^{sep}.

We say that the cohomology of A over E is globally dominated with respect to F if $\mathrm{H}^n(E^{\text{gd}}, A) = 0$ for all $n > 0$.

Thus, if cohomology is globally dominated, then the *difference* between $\mathrm{H}^n(E^{\text{gd}}/E, A(E^{\text{gd}}))$ and $\mathrm{H}^n(E, A(E^{\text{sep}})$ is zero. This can be made precise by use of the Hochschild-Serre spectral sequence, and thus proving the next proposition.

Proposition 5.5.1 ([HHK14, Proposition 2.3.4])**.** *Let E/F be a field extension. Assume that Galois cohomology of A over E is globally dominated with respect to F. Then, we can identify*

$$\mathrm{H}^n(F, A(E \otimes_F F^{\text{sep}}) = \mathrm{H}^n(E^{\text{gd}}/E, A(E^{\text{gd}})) = \mathrm{H}^n(E, A(E^{\text{sep}})$$

for all $n \geq 0$.

If we assume that cohomology is globally dominated, then the proof outlined above actually goes through to show the next theorem.

Theorem 5.5.2 ([HHK14, Theorem 2.5.1])**.** *Suppose that, for all $i \in I$, the cohomology of A is globally dominated over F_i. Then, we have a long exact sequence in Galois cohomology:*

$$\begin{array}{l} 1 \longrightarrow \mathrm{H}^0(F, A) \longrightarrow \prod_{i \in I_v} \mathrm{H}^0(F_i, A) \longrightarrow \prod_{k \in I_e} \mathrm{H}^0(F_k, A) \longrightarrow \\ \qquad \longrightarrow \mathrm{H}^1(F, A) \longrightarrow \prod_{i \in I_v} \mathrm{H}^1(F_i, A) \longrightarrow \prod_{k \in I_e} \mathrm{H}^1(F_k, A) \longrightarrow \dots \end{array}$$

In particular, patching holds for Galois cohomology with values in A over $\mathcal{F}$.

Harbater, Hartmann and Krashen have proven that in the local patching setup, global domination is often satisfied. We mention their main result and refer the reader to the original paper [HHK14] for details.

Theorem 5.5.3 ([HHK14, Theorem 3.1.3]). *Let F and $\mathcal{F}$ be given as in the local case of Section 5.2. Let G be an abelian linear algebraic group. If $\operatorname{char}(k) = p > 0$, assume furthermore that $p \nmid |A| < \infty$. Then, for any $n \geq 0$, patching holds for Galois cohomology with values in A over $\mathcal{F}$.*

Chapter 6

Bitorsors

Our main goal in this chapter is to prove that patching torsors implies patching for bitorsors. We will start by stating the definition of bitorsors. Our main tool to prove our principal result in this chapter is a semi-cocyclic description of bitorsors developed by Breen ([Bre90, Section 2]). Let S be a scheme and fix a Grothendieck topology for Sch/S. When we speak of a sheaf, torsor or bitorsor over S we will always refer to the fixed topology.

Let G and H be group sheaves over S. Let Y be a scheme over S. For ease of noation, we will write $G|_Y$ for $G \times_S Y$ throughout the next chapters. We will use similary notation for bitorsors and gerbes.

Definition 22. A (G, H)-bitorsor P over S is a left G-torsor and a right H-torsors such that the actions commute, i.e. we have $(g.x).h = g.(x.h)$ for all $x \in P(Y), g \in G(Y)$ and $h \in H(Y)$ for all $Y \in \mathrm{Sch/S}$.

If P is a (G, H) bitorsor we define P^{op} to be the (H, G)-bitorsor obtained by switching the actions of G and H.

A morphism of (G, H)-bitorsors $P \to P'$ is a morphism of sheaves that is simultaneously a morphism of left G-torsors and right H-torsors. We denote the category of (G, H)-bitorsors over S by $\mathrm{BIT}(G, H)(S)$. We will often just write $\mathrm{BIT}(G, H)$ when the base is clear from context. A (G,H)-bitorsor P is neutral if $P(S) \neq \emptyset$.

A G-bitorsor is a (G,G)-bitorsor. The G-bitorsor G with the actions defined via left- and right-translation is the trivial G-bitorsor. We denote the category of G-bitorsors by $\mathrm{BIT}(G)$.

Remark 4.

- Note that, unlike in the case of torsors, a neutral G-bitorsor may not be trivial. Take any group sheaf G which admits a non-trivial isomorphism of group sheaves $\lambda\colon G \to G$. Consider the G-bitorsor G whose left action is given by translation, but with the twisted right action defined via: $x \diamond g := x.\lambda(Y)(g)$ (here . corresponds to translation) for $x, g \in G(Y)$ and $Y \in \mathrm{Sch/S}$. This bitorsor is clearly neutral and not isomorphic to the trivial bitorsor.

- Note that the existence of a (G,H)-bitorsor implies that G and H are locally isomorphic: If P is a (G,H)-bitorsor, then any section $p \in P(Y)$ induces a group isomorphism

$$G|_Y \xrightarrow{p} P|_Y \xrightarrow{p} H|_Y$$

so that G and H are forms of each other.

- If P is a (G,H)-bitorsor with G abelian, then one can see that $H = G$. Furthermore, the datum of a G-bitorsor for G abelian is equivalent to the datum of a G-torsor. For the first claim, let P be a (G,H)-bitorsor with G abelian. Fix some $Y \in \mathrm{Sch/S}$ and $p \in P(Y)$. Then, this point induces an isomorphism

$$G|_Y \to P|_Y \to H|_Y$$

as remarked in the last paragraph. As G is abelian, H is also abelian and this isomorphism is independent of the point $p \in P(Y)$. This can be seen as follows: for $p, p' \in P(Y)$ with $h.p = p'$, there is a $g \in G(Y)$ such that $g.p = p.g'$.

Therefore, $g.p' = g.h.p = h.g.p = h.p.g' = p'.g$ and these equations are precisely the equations defining the isomorphisms $G|_Y \to H|_Y$ (see below for a more detailed explanation of this part). Hence, the local isomorphisms glue together and it follows that $G \simeq H$.

The second claim, that a G-bitorsor is, up to isomorphism, defined by its underlying (left) G-torsor, follows easily from the hypercohomological interpretation given in Section 4. Alternatively, let G be the trivial left G-torsor. We want to equip it with a right action so that it becomes a G-bitorsor. If we can show that our only option is given by the usual right translation, then this means that every G-bitorsor is a form of the trivial G-bitorsor, thus proving our claim. Let $\star$ be a right action on G making it a G-bitorsor. Fix a scheme $Y \in \mathrm{Sch/S}$ and a section $g \in G(Y)$. For any $h \in G(Y)$ there is $h' \in G(Y)$ such that $h.g = g \star h'$. This defines a morphism $G \to \mathrm{Aut}(G)$. We will see below in greater generality that this map is G-equivariant with respect to the conjugation map $G \to \mathrm{Aut}(G)$. As G is abelian, this map is trivial. In other words, each g yields the same automorphism. Clearly, the identity e of $G(S)$ maps to $\mathrm{id} \in \mathrm{Aut}(G)(S)$. Hence, any g maps to the identity and thus $g \star h = g.h$.

Let G' be another group sheaf and consider a (G, H)-bitorsor P and a (H, G')-bitorsor P'. We define the wedged product $P \wedge^H P'$ as the sheafification of

$$U \mapsto P(U) \times P'(U)/\sim$$

where $(p, p'), (q, q') \in P(U) \times P'(U)$ are equivalent if there is $h \in H(U)$ such that $(p.h, h.p') = (q, q')$. Note that $P \wedge^H P'$ inherits the left G action of P and the right G'-action of P' making it a (G, G')-bitorsor.

In particular, if P and P' are G-bitorsors, then $P \wedge^G P'$ is again a G-bitorsor. This binary operation descends to isomorphism classes, equipping the set of isomorphism classes of G-bitorsors with a group structure. The neutral element is given by the class of G and the inverse of the class represented by a G-bitorsor P is represented by P^{op}.

This group structure is compatible with the group structure of $\mathrm{H}^0(F, G \to \mathrm{Aut}(G))$ described in Chapter 4.

6.1 A semi-cocyclic description

The material in this section is based on [Bre90, Section 2], see also [Bre09, Section 1]. Our goal is to describe a (G, H)-bitorsor as a left G-torsor together with a G-equivariant morphism $P \to \mathrm{Isom}(H, G)$.

Let P be a (G, H)-bitorsor and fix be a section p of P over some $U \in$ Sch/S. Define an isomorphism of group sheaves

$$u_p \colon H|_U \to G|_U$$

via

$$p|_V h = u_p(V)(h)p|_V$$

for $V \in$ Sch/U and $h \in H|_U(V)$.

Let p' be another section over the same U. As P is a left G-torsor, there is some $\gamma \in G(U)$ such that $p' = \gamma p$. Note that then

$$u'_p = i_\gamma u_p$$

where i_γ is conjugation by g ($g \mapsto \gamma g \gamma^{-1}$).

This follows immediately from the definition: Given $V \to U$ and $h \in G(V)$, we obtain:

$$\gamma|_V u_p(V)(h)\gamma|_V^{-1} p'|_V = \gamma|_V u_p(V)(h) p|_V = \gamma|_V p|_V h = p'|_V h|_V = u_{p'}(V)(h) p'|_V.$$

Thus, we obtain a morphism of group sheaves

$$u\colon P \longrightarrow \mathrm{Isom}(H, G)$$
$$p \mapsto u_p$$

that is equivariant with respect to the conjugation map

$$i\colon G \longrightarrow \mathrm{Aut}(G).$$

This process is reversible.

Lemma 6.1.1. *For a left G-torsor P, the following are equivalent:*

1. *the data of a right H action making it a bitorsor,*
2. *a morphism of sheaves $u\colon P \longrightarrow \mathrm{Isom}(H, G)$ equivariant with respect to $i\colon G \to \mathrm{Aut}(G)$.*

Proof. We have already seen that a bitorsor yields the equivariant morphism.

For the other direction, fix some $V \to S$ and define the right H-action over V via

$$p.h = u_p(V)(h).p$$

for $h \in G(V)$ and $p \in P(V)$.

Let us check that right and left actions commute: For $p \in P(V)$ and $g, h \in G(V)$, we obtain

$$(gp)h = gu_p(V)(h)g^{-1}gp = gu_p(V)(h)p$$
$$g(ph) = gu_p(V)(h)p.$$

Let us now check that this right action is associative:

$$p(hh') = u_p(V)(hh')p = u_p(h)u_p(V)(h')p$$
$$(ph)h' = (u_p(V)(h)p)h' = u_p(V)(h)(ph') = u_p(V)(h)u_p(V)(h')p.$$

It remains to show that the right H-action is simply transitive. However, as u_p is an isomorphism, this follows from the definition and the fact the left G-action is simply transitive. $\square$

The following Lemma continues this equivalence with respect to morphisms.

Lemma 6.1.2. *Let $f\colon P \to P'$ be a morphism of left G-torsors. Assume that P and P' are (G, H)-bitorsors and let $u\colon P \to \mathrm{Isom}(H, G)$, $u'\colon P' \to \mathrm{Isom}(H, G)$ denote the equivariant morphisms of sheaves from Lemma 6.1.1. Then, the following are equivalent:*

1. *f is a morphism of bitorsors*
2. *$u = u'f$*

Proof. For the first implication, assume that f is a morphism of bitorsors. Then, for any $Y \in \mathrm{Sch/S}$ and $p \in P(Y)$, we have

$$u_p(h)f(Y)(p) = f(Y)(u_p(h)p) = f(Y)(ph) = f(Y)(p)h = u'_{f(Y)(p)}(h)f(Y)(p)$$

so we get $u_p = u'_{f(p)}$, i.e. $u = u'f$.

For the reverse direction, observe

$$f(Y)(p)h = u'_{f(Y)(p)}(h)f(Y)(p) = f(u'_{f(Y)(p)}(h)p) = f(Y)(u_p(h)p) = f(Y)(ph)$$

so f is compatible with the right action, i.e. a morphism of bitorsors. $\square$

If we fix a *cover* of $Y \to S$ and a section p over that cover, we can reinterpret the two lemmas above:

Lemma 6.1.3. *Let (P, p) be a left G-torsor with a section p over a cover $Y \to S$. Then, the following data are equivalent*

1. *a right H-action on P making it a (G, H)-bitorsor*
2. *a sheaf isomorphism $u_p \colon H|_Y \to G|_Y$*

Let now (P', p') be another such tuple where p' is also a section over Y. Assume that both P and P' are (G, H)-bitorsors. Let $f \colon P \to P'$ be a morphism of left G-torsors. Let $g \in G(Y)$ be such that $f(p) = gp'$ holds. Then, the following are equivalent

1. *f is a morphism of bitorsors*
2. *$u_p = i_g u'_p$*

Proof. The second assertion follows readily from the first one.

For the first claim, note that the implication 1) $\Rightarrow$ 2) is clear. For the other direction, observe that u_p induces a unique equivariant isomorphism of sheaves $P \to \mathrm{Isom}(H, G)$: As we want this to be a morphism of sheaves, it is enough to define it over the cover Y. Over said cover, the equivariance only allows us to define $u_{p'}$ as $i_g u_p$ for $p' = gp$ which does indeed make it equivariant. The claim follows. $\square$

6.2 Patching for bitorsors

In this section, fix a base field F and let $\mathcal{C} = \mathrm{Sch/F}$ be equipped with the big étale topology. Also, let $\mathcal{F}$ denote a finite inverse factorization system with limit F and indexing set I. Our goal in this section is to prove that bitorsor patching holds over $\mathcal{F}$ whenever torsor patching does. Our main tool will be the semi-cocyclic description of Section 6.1.

Lemma 6.2.1. *Let K be a finite separable extension of F and let $\mathcal{K}$ denote the corresponding inverse system obtained from base change. Let G, H be linear algebraic groups over K. Given isomorphism of group schemes $u_i \colon G|_{K_i} \to H|_{K_i}$ for all $i \in I_v$ such that*

$$u_i|_{K_k} = u_j|_{K_k}$$

whenever $(i, j, k) \in S_I$, there is an isomorphism of group schemes $u \colon G|_K \to H_K$ satisfying $u|_{K_i} = u_i$ for all $i \in I$.

Proof. Let A, B be K-algebras such that $G|_K = \mathrm{Spec}(A)$ and $H|_K = \mathrm{Spec}(B)$. Then, the u_i induce ring isomorphisms $f_i \colon B_i \to A_i$. Fix some $b \in B$. Consider the elements $f_i(b)$ for $i \in I$. We have $f_i(b) = f_j(b) \in A_k$ for $(i, j, k) \in S_I$ by assumption. Hence, the elements $f_i(b)$ determine a unique element $f(b)$ in A which define a morphism $f \colon B \to A$. It is clear that this is a ring isomorphism. Hence, we get an isomorphism of schemes $u \colon G \to H$. For u to be compatible with the group structure, we need f to be compatible with Hopf algebra structure. But, as each f_i is compatible with the Hopf algebra structure, it follows that f is as well, by the faithfulness of the patching functor. □

Let $\mathrm{BPP}(G, H)(\mathcal{F})$ denote the category of (G, H)-bitorsor patching problems over $\mathcal{F}$, defined analogously to $\mathrm{TPP}(G)(F)$:

1. Objects: An object is a tuple $(\{P_i\}_{i\in I_v}, \{\phi_k\}_{i\in I_e})$ where P_i is a (G_i, H_i)-bitorsor over F_i and $\phi_k \colon P_i|_{F_k} \to P_j|_{F_k}$ is an isomorphism of (G_i, H_i)-bitorsors for $(i, j, k) \in S_I$.

2. Morphisms: A morphism

$$(\{P_i\}_{i\in I_v}, \{\phi_k\}_{i\in I_e}) \to (\{P_i'\}_{i\in I_v}, \{\phi_k'\}_{i\in I_e})$$

is given by a collection of bitorsor morphisms $P_i \to P_i'$ for $i \in I_v$ that are compatible with ϕ_k and ϕ_k'.

Just like there is a natural functor

$$\beta_G' \colon \mathrm{TOR}(G)(F) \to \mathrm{TPP}(G)(\mathcal{F}),$$

there also is a natural functor

$$\beta_{(G,H)}'' \colon \mathrm{BIT}(G, H)(F) \to \mathrm{BPP}(G, H)(\mathcal{F}).$$

given by the trivial gluing datum induced by the universal property of pullback.

Definition 23. We say that patching holds for (G, H)-bitorsors over $\mathcal{F}$ if $\beta_{(G,H)}''$ is an equivalence.

We will now see that we can patch bitorsors whenever we can patch torsors.

Theorem 6.2.2. *Assume that patching holds for G-torsors over $\mathcal{F}$. Then, patching holds for (G, H)-bitorsors, i.e. if β_G' is an equivalence of categories, then so is $\beta_{(G,H)}''$.*

Proof. We need to prove essential surjectivity and fully faithfulness. Let us start with essential surjectivity.

Fix some $\mathcal{P} \in \mathrm{BPP}(G, H)(\mathcal{F})$. We have a commuting diagram of functors

$$\begin{array}{ccc}
\mathrm{BIT}(G,H)(F) & \xrightarrow{\beta''_{(G,H)}} & \mathrm{BPP}(G,H)(\mathcal{F}) \\
\downarrow & & \downarrow \\
\mathrm{TOR}(G)(F) & \xrightarrow{\beta'_G} & \mathrm{TPP}(G)(\mathcal{F}),
\end{array}$$

where the vertical functors are the forgetful ones.

By assumption β'_G is essentially surjective, so there is some left G-torsor P defined over F together with isomorphisms $\phi_i\colon P|_{F_i} \to P_i$ for all $i \in I_v$ that are compatible with the morphisms ν_k (as morphisms of torsors). Let K/F be a finite separable field extension, such that $P(K) \neq \emptyset$. Fix some $p_0 \in P(K)$. Let $K_i := F_i \otimes_F K$ for all $i \in I$. Set $p_i = \phi_i(K_i)(p_0|_{K_i})$, which defines a trivializing family $\{p_i\}_{i\in I}$. Note that, by construction, $\nu_k(p_i|_{K_k}) = p_j|_{K_k}$. By Lemma 6.1.3, we get sheaf isomorphisms $u_{p_i}\colon H|_{K_i} \to G|_{K_i}$ for all $i \in I$. Recall that this morphism is defined over $V \to K_k$ via $p_i|_V.h = u_{p_i}(V)(h).p_i|_V$ for $h \in H(V)$. We claim that $u_{p_i}|_{K_k} = u_{p_j}|_{K_k}$ for all $(i,j,k) \in S_I$. This follows from

$$\begin{aligned}
u_{p_i}(V)(h).p_j|_V &= u_{p_i}(h).\nu_k(V)(p_i|_V) \\
&= \nu_k(V)(u_{p_i}(V)(h).p_i|_V) \\
&= \nu_k(p_i|_V.h) \\
&= \nu_k(p_i|_V).h \\
&= p_k.h \\
&= u_{p_j}(V)(h).p_j|_V.
\end{aligned}$$

By Lemma 6.2.1, we get a global isomorphism $u_{p_0}\colon G \to H$. By Lemma 6.1.3, this in turn equips P with a (G,H)-bitorsor structure.

In order to show that $\beta''_{(G,H)}(P)$ is isomorphic to the given bitorsor patching problem $\mathcal{P}$, it is enough to show that the morphisms ϕ_i are in fact morphisms of bitorsors.

By Lemma 6.1.3 and the choice of p and p_i, this is equivalent to checking that

$$u_{p_0}|_{K_i} = u_{p_i}$$

for all $i \in I_v$ (note that $\phi_i(p_0) = p_i$). But, this is clear by construction of u_{p_0}. Hence, $\beta''_{(G,H)}$ is essentially surjective.

Let us now show that $\beta''_{(G,H)}$ is fully faithful. It is clearly faithful as β'_G is faithful. So, we only need to prove that it is full. Fix two (G, H)-bitorsors P, P' over F and let $\beta''_{(G,H)}(P) = \mathcal{P}$ and $\beta''_{(G,H)}(P') = \mathcal{P}'$. Let $\alpha\colon \mathcal{P} \to \mathcal{P}'$ be a morphism in $\mathrm{BPP}(G, H)(\mathcal{F})$. Note that α is also a morphism in $\mathrm{TPP}(G)(\mathcal{F})$. Hence, there is a morphism $\alpha\colon P \to P'$ of left G-torsors inducing the morphisms $\alpha_i\colon P_i \to P'_i$. A straightforward check shows that α is a morphism of bitorsors. If the morphism was not compatible with the right action, then there would be a scheme X/F such that $P(X) \neq \emptyset \neq P'(X)$ with $p \in P(X)$ such that $a(p.h) \neq a(p).h$ for some $h \in H(X)$.

This inequality must hold on some affine subscheme of X, so we may assume without loss of generality that X is affine. Note that, as A is free as an F vector space, A injects into $\prod_{i\in I_v} A_i$. Hence, there is some $i \in I_v$ such that $a_i(p_i.h_i) \neq a_i(p_i).h_i$ where the subscript denotes restriction. This contradicts the fact that a_i is a morphism of bitorsors. Hence, a must be compatible with the right action. $\square$

In the context of gerbe patching, we will have to patch bitorsors over covers of F. These covers are formed by finite products of finite separable field extensions. We thus need to extend our patching results to this setup.

Corollary 6.2.3. *Let A be a finite product of finite separable field extensions of F and let G, H be linear algebraic groups over A. If vector space patching holds over $\mathcal{F}$ then (G, H)-bitorsor patching holds over $\mathcal{F}_A$.*

Proof. Follows from Theorem 6.2.2 and Proposition 5.3.3. (Note that the proof of Theorem 6.2.2 goes through verbatim if we replace $\mathcal{F}$ by $\mathcal{F}_A$.) $\square$

6.2.1 A Mayer-Vietoris sequence and a local-global principle for bitorsors

Using patching for bitorsors, we can construct the start of our Mayer-Vietoris sequence, which will be extended in section 7.4. Fix a group sheaf G and let Z denote its center.

We say that G-bitorsors satisfy the local-global principle over $\mathcal{F}$ if for any G-bitorsor T, we have that $T \simeq G$ if and only if $T_i \simeq G_i$ for all $i \in I_v$.

Recall that automorphisms of the trivial G-bitorsor G can be identified with $Z(F)$. Furthermore, note that $\mathrm{H}^{-1}(F, G \to \mathrm{Aut}(G)) = Z(F)$ and that $\mathrm{H}^0(F, G \to \mathrm{Aut}(G))$ classifies bitorsors up to isomorphism (cf. Chapter 4).

There area natural maps

$$\mathrm{H}^0(F_i, G \to \mathrm{Aut}(G)) \times \mathrm{H}^0(F_j, G \to \mathrm{Aut}(G)) \to \prod_{i \in I_k} \mathrm{H}^0(F_k, G \to \mathrm{Aut}(G))$$

for $(i, j, k) \in S_I$ defined via $(P_i, P_j) \mapsto P_i \wedge^G P_j^{\mathrm{op}}$. This induces a map

$$\prod_{i \in I_v} \mathrm{H}^0(F_i, G \to \mathrm{Aut}(G)) \to \prod_{i \in I_k} \mathrm{H}^0(F_k, G \to \mathrm{Aut}(G)).$$

Lemma 6.2.4. *There is a map of pointed sets*

$$\prod_{i \in I_k} \mathrm{H}^{-1}(F_k, G \to \mathrm{Aut}(G)) \to \mathrm{H}^0(F, G \to \mathrm{Aut}(G))$$

Proof. We can define the map as follows: Given elements $e_k \in Z(F_k)$, consider the G-bitorsor patching problem $(\{G_i\}_i, \{e_k\}_k)$. By Theorem 6.2.2, there is a G-bitorsor P over F in the essential preimage of the patching problem. Map $(e_k)_k$ to the equivalence class of P. Note that this is independent of the choice of P. $\square$

For notational simplicity, let us abbreviate

$$G^{\mathrm{Aut}} = G \to \mathrm{Aut}(G).$$

Theorem 6.2.5 (Mayer-Vietoris for non-abelian hypercohomology (1))**.** *Assume that G-bitorsor patching holds over $\mathcal{F}$. Then, there is an exact sequence*

$$\begin{aligned} 1 \longrightarrow \mathrm{H}^{-1}(F, G^{\mathrm{Aut}}) \longrightarrow \prod_{i \in I_v} \mathrm{H}^{-1}(F_i, G^{\mathrm{Aut}}) \longrightarrow \prod_{k \in I_e} \mathrm{H}^{-1}(F_k, G^{\mathrm{Aut}}) \\ \hookrightarrow \mathrm{H}^{0}(F, G^{\mathrm{Aut}}) \longrightarrow \prod_{i \in I_v} \mathrm{H}^{0}(F_i, G^{\mathrm{Aut}}) \longrightarrow \prod_{k \in I_e} \mathrm{H}^{0}(F_k, G^{\mathrm{Aut}}) \end{aligned}$$

Proof. Exactness in the first row follows from the description $\mathrm{H}^{-1}(F, G^{\mathrm{Aut}}) = Z(F)$ and the fact that $\mathcal{F}$ is a factorization inverse system with limit F. Exactness at $\prod_{i \in I_v} \mathrm{H}^{0}(F_i, G^{\mathrm{Aut}})$ follows from Theorem 6.2.2. Finally, exactness at $\mathrm{H}^{0}(F, G^{\mathrm{Aut}})$ also follows from Theorem 6.2.2 and the fact that the automorphism group of G as a bitorsor is Z. □

The sequence above allows us to characterize when the local-global principle for bitorsors with respect to patches holds.

Corollary 6.2.6. *Assume that G-bitorsor patching holds over $\mathcal{F}$. Then, G-bitorsors satisfy local-global principle over $\mathcal{F}$ iff $Z(G)$ satisfies factorization over $\mathcal{F}$.*

6.3 Bitorsor Factorization

Let $\mathcal{F}$ be an inverse factorization system over F and let G be a linear algebraic group over F. Recall that we say that G satisfies factorization over $\mathcal{F}$ if for every tuple $(g_k)_{k \in I_e}$ with $g_k \in G(F_k)$ there exists a tuple $(g_i)_{i \in I_v}$ with $g_i \in G(F_i)$ and $g_k = g_i^{-1} g_j$ for every $(i, j, k) \in S_I$. We saw in Theorem 5.3.5 that this property is equivalent to local-global principle for G-torsors.

In this section, we will introduce the similar notion of factorization for G-bitorsors. We will also prove a first result on when bitorsor factorization holds. In Chapter 7, we will link this property to local-global principle for gerbes. We will also prove more results on bitorsor factorization in the arithmetic curve setting in Chapter 8.

Definition 24. Let G be a linear algebraic group over F and let $\mathcal{F}$ be a finite inverse factorization system over F. We say that factorization holds for G-bitorsors if for any tuple $(P_k)_{k\in I_e}$ of $G|_{F_k}$-bitorsors P_k over F_k, there is a tuple $(P_i)_{i\in I_v}$ of $G|_{F_i}$-bitorsors P_i over F_i such that $P_k \simeq P_i|_{F_i} \wedge^{G|_{F_k}} P_j|_{F_k}$ for any $(i,j,k) \in S_I$.

Our main goal for this section is to prove that, under some under assumptions on G, *neutral* bitorsors always satisfy factorization.

Note that a G-bitorsor P over F corresponds to the data of (A, g, λ) where

- A is an étale F-algebra
- $g \in G(A \otimes_F A)$ satisfies the cocycle condition $g_{12}g_{23} = g_{13}$ in $G(A \otimes_F A \otimes_F A)$.
- $\lambda \in \mathrm{Aut}(G)(A)$ satisfies $\lambda_1 = \mathrm{Int}(g)\lambda_2 \in \mathrm{Aut}(G)(A \otimes_F A)$.

Here, the subscripts correspond to the natural projections.

Given two such cocycles $(A, g, \lambda), (A, g', \lambda)$ corresponding to G-bitorsors P and P', the wedged product $P \wedge^G P'$ corresponds to the cocycle $(A, g\lambda(A \otimes_F A)(g'), \lambda\lambda')$.

Theorem 6.3.1. *Let G be a linear algebraic group which does not admit automorphisms. Assume furthermore that* $\mathrm{Aut}(G)$ *is linear algebraic and that* Γ *is a tree. Then, factorization holds for G-bitorsors if vector space patching holds over $\mathcal{F}$.*

Proof. To simplify notation, we assume that $\mathcal{F}$ has the shape

$$\begin{tikzcd} & F_b & \\ F_U \arrow[ur] & & F_p. \arrow[ul] \end{tikzcd}$$

Fix a neutral $G|_{F_k}$-bitorsor P over F_k. Note that a neutral G-bitorsor over F_b has the form (F_b, e, λ) for some $\lambda \in \mathrm{Aut}(G)(F_k$.

So, we are looking for a finite separable K-algebra A and $m_U \in G(A_U \times A_U)$, $m_p \in G(A_p \times A_p)$ as well as $\lambda_U \in \mathrm{Aut}(G)(A_U)$ and $\lambda_p \in \mathrm{Aut}(G)(A_p)$ such that

- $\lambda_U^1 = \mathrm{Int}(g_U)\lambda_U^2$
- $m_U^{12} m_U^{23} = m_U^{13}$
- $m_p^{12} m_p^{23} = m_p^{13}$
- $\lambda_p^1 = \mathrm{Int}(m_p)\lambda_p^2$
- $m_U \lambda_U^2(m_p) = e$
- $\lambda_U \lambda_p = \lambda$

holds. These relations ensure that $(A|_U, m_u, \lambda_U)$ and $(A|_p, m_p, \lambda_p)$ define G-bitorsors over F_U and F_p respectively and that their wedged product over F_k is isomorphic to P.

Recall that any linear algebraic group over F satisfies separable factorization over $\mathcal{F}$ by assumption. Thus, there is a a finite separable extension L/F and $\lambda_U \in \mathrm{Aut}(G)(L_U)$ and $\lambda_p \in \mathrm{Aut}(G)(L_p)$ such that $\lambda = \lambda_U \lambda_p \in \mathrm{Aut}(G)(L_b)$. As G admits no outer automorphisms, there are $h_U \in G(L_U)$ and $h_p \in G(L_p)$ such that

- $\lambda_U = \mathrm{Int}(h_U)$
- $\lambda_p = \mathrm{Int}(h_p)$

holds.

Therefore,

$$\lambda_U^1(\lambda_U^2)^{-1} = \mathrm{Int}\left(h_U^1(h_U^2)^{-1}\right)$$
$$\lambda_p^1(\lambda_p^2)^{-1} = \mathrm{Int}\left(h_p^1(h_p^2)^{-1}\right)$$

so that we set $g_U = h^1_U(h^2_U)^{-1}$ and $g_p = h^1_p(h^2_p)^{-1}$. As they are coboundaries (in the Čech complex of G with respect to the cover L/F), g_U and g_p satisfy the cocycle condition. They are also compatible with λ_U and λ_p by construction. Hence, (L_U, g_U, λ_U) defines a G-bitorsor over F_U and (L_p, g_p, λ_p) defines a G-bitorsor over F_p.

Hence, we only have to worry about $g_U\lambda^2_U(g_p) = e$. Consider the following

$$\mathrm{id} = \lambda^1(\lambda^2)^{-1} = \lambda^1_U\lambda^1_p(\lambda^2_p)^{-1}(\lambda^2_U)^{-1} = \mathrm{Int}(h^1_U h^1_p(h^2_p)^{-1}(h^2_U)^{-1})$$

so that

$$h^1_U h^1_p(h^2_p)^{-1}(h^2_U)^{-1} = z$$

for some $z \in Z(G)(L_b \times L_b)$. This can be rewritten as

$$g_p = z(h^1_U)^{-1}h^2_U.$$

Considering that $\lambda^2_U = \mathrm{Int}(h^2_U)$, we can conclude that

$$\lambda^2_U(g_p) = zh^2_U(h^1_U)^{-1}h^2_U(h^2_U)^{-1} = zg_U^{-1}$$

holds. Therefore

$$g_U\lambda^2_U(g_p) = z$$

Let us now prove that z is a cocycle. For this, note that the identity

$$\lambda^1_U = \mathrm{Int}(g_U)\lambda^2_U$$

over $L_b \times L_b$ pulls back to

$$\lambda_U^2 = \mathrm{Int}(g_U^{23})\lambda_U^3$$

over $L_b \times L_b \times L_b$ via pr_{23}. With this in mind, we calculate

$$\begin{aligned}
\left(g_U \lambda_U^2(g_p)\right)_{12} \left(g_U \lambda_U^2(g_p)\right)_{23} &= g_U^{12}\lambda_U^2(g_p^{12})g_U^{23}\lambda_U^3(g_p^{23}) \\
&= g_U^{12}g_U^{23}\lambda_U^3(g_p^{12})(g_U^{23})^{-1}g_U^{23}\lambda_U^3(g_p^{23}) \\
&= g_U^{13}\lambda_U^3(g_p^{12}g_p^{23}) \\
&= g_U^{13}\lambda_U^3(g_p^{13}) \\
&= \left(g_U \lambda_U^2(g_p)\right)_{13}
\end{aligned}$$

so that $z = g_U\lambda_U^2(g_p)$ is in fact a cocycle.

Hence, there is a finite separable extension $E/L/F$ such that z splits, i.e. $z = c_1(c_2)^{-1}$ with $c \in Z(G)(E_b)$. After another such base change to $A/E/L/F$, we may assume the existence of $c_U \in Z(G)(A_U)$ and $c_p \in Z(G)(A_p)$ such that $c^{-1} = c_U c_p$.

We define $m_U = c_U^1(c_U^2)^{-1}g_U$ and $m_p = c_p^1(c_p^2)^{-1}g_p$. Then, as c_U, c_p are central and $c_U^1(c_U^2)^{-1}$ are coboundaries, m_U, m_p are still coboundaries and still satisfy

$$\lambda_U^1 = \mathrm{Int}(m_U)\lambda_U^2$$
$$\lambda_p^1 = \mathrm{Int}(m_p)\lambda_p^2.$$

So, (m_U, λ_U) and (m_p, λ_p) define bitorsors. Note that λ^2 is an inner automorphism and as such trivial on central elements. Hence,

$$
\begin{aligned}
m_U\lambda^2(m_p) &= c_U^1(c_U^2)^{-1}g_U\lambda_U^2\left(c_p^1(c_p^2)^{-1}g_p\right) \\
&= c_U^1c_p^1(c_U^2c_p^2)^{-1}g_U\lambda_U^2\left(g_p\right) \\
&= (c_1)^{-1}c_2z = (c_1)^{-1}c_2c_1(c_2)^{-1} = e
\end{aligned}
$$

as desired. □

Chapter 7

Gerbes

The main goal of this chapter is to prove that, under an additional assumption, patching for vector spaces implies patching for gerbes. We will see in the next chapter that this additional assumption is satisfied in the local patching setup over arithmetic curves in equicharacteristic 0. In the case that the characteristic of the residue field is positive, we also show that gerbe patching often holds.

We will start the chapter by recalling the definition of gerbes and bands. We then proceed and describe a semi-cocyclic description of gerbes in terms of bitorsors developed by Breen ([Bre90]). This description is crucial, as it allows us to reduce gerbe patching to bitorsor patching. We prove this in the following section. The last section interprets the patching result in terms of a Mayer-Vietoris sequence in non-abelian hypercohomology. This also allows us to characterize a local-global principle for gerbes in terms factorization for bitorsors.

7.1 Generalities

Let us fix the big étale site Sch/F of the scheme Spec(F) for some fixed field F. As before, we will often just write F to denote Spec(F). Note that covers of F can all be taken to be of the form $Y \to F$ for a single arrow where Y is a finite product of finite separable field extensions of F. We will use this throughout this section as it will simplify notation.

We will first recall the definition of a gerbe as a special kind of stack. Our main reference for gerbes is [Gir71].

Definition 25. Let $\mathcal{G} \to \mathrm{Sch/F}$ be a stack. We say that $\mathcal{G}$ is a gerbe if the following two conditions are satisfied:

1. For all $X \in \mathrm{Sch/F}$, there is a cover $Y \to X$ such that $\mathcal{G}(Y) \neq \emptyset$
2. For any $x, x' \in \mathcal{G}(X)$, there is a cover $Y \to X$ such that the pullbacks of x and x' are isomorphic in $\mathcal{G}(Y)$.

Note the similarity between the definition of gerbes and torsors, which is why we often think of gerbes as a 1-categorical analogue of torsors.

Example 7. Let G be a sheaf of groups over Sch/F. Then, the classifying stack BG is a gerbe. Its fiber over $Z \to F$ is given by $BG(Z) = \mathrm{TOR}(G)(Z)$.

Just like torsors, a torsor is trivial if it admits an F-rational point.

Definition 26. We say that a gerbe $\mathcal{G} \to \mathrm{Sch/F}$ is trivial (or neutral) if $\mathcal{G}(F) \neq \emptyset$.

Example 8. The gerbe BG is trivial as the trivial left G-torsor G is in $BG(F)$.

Remark 5. Let $x, y \in \mathcal{G}(Z)$ for some $Z \in \mathrm{Sch/F}$. The sheaf $\mathrm{Isom}_Z(x, y)$ on $\mathrm{Sch/Z}_{\acute{e}t}$ is a $(\mathrm{Aut}_Z(y), \mathrm{Aut}_Z(x))$-bitorsor.

Recall that a 2-category Ç consists of a class of objects $\mathrm{obj}(\mathcal{C})$ together with categories $\mathrm{Hom}(A, B)$ for $A, B \in \mathrm{obj}(\mathcal{C})$ and composition functors $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ satisfying the usual identities. Additionally, we require $\mathrm{Hom}(A, A)$ to contain an identity element with respect to composition. Similarly, for $f \in \mathrm{Hom}(A, B)$, we require there to be an identity element in $\mathrm{Hom}(f, f)$. We will call an object in $\mathrm{Hom}(A, B)$ a 1-morphism and a morphism in $\mathrm{Hom}(A, B)$ a 2-morphism.

A typical example of a 2-category is the category of categories, where for two categories $\mathcal{A}, \mathcal{B}$, the category $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ has functors as objects and natural transformations as morphisms.

Recall from Section 2.4 that, as stacks are in particular categories, we consider the 2-category of stacks over some site as the category with objects being stacks, 1-morphisms being functors of fibered categories and 2-morphisms being natural transformations of functors.

Similarly, we will consider the 2-category of gerbes in this section and will later define the 2-category of gerbe patching problems.

Definition 27. A 1-morphism of gerbes $\mathcal{G} \to \mathcal{G}'$ over Sch/F is a morphism of stacks $\mathcal{G} \to \mathcal{G}'$ over Sch/F.

A 2-morphism of gerbes is a natural transformation of functors.

By Gerbe(F), we denote the 2-category of gerbes with 1-morphisms given by equivalences and 2-morphisms given by natural isomorphisms of functors.

There is a beautiful analogue of Morita equivalence for morphisms between gerbes of the form BG. Let G and H be group sheaves on Sch/F and let P be a (H, G)-bitorsor. Recall that given a a left G-torsor T, the wedged product $T \wedge^G P$ is a left H-torsor. As this construction is functorial, it induces a morphism

$$\begin{aligned} \Psi_P \colon BG &\to BH \\ (S, T) &\mapsto (S, P_S \wedge^{G_S} T) \end{aligned}$$

(where S denotes the scheme over which T is a torsor). This morphism is an equivalence, as a quasi-inverse is given by the morphism $\Psi_{P^{\mathrm{op}}}$. Note also that if P is a (H, G)-bitorsor and P' is a (H', H)-bitorsor, then there is a natural isomorphism $\Psi_{P' \wedge^H P} \simeq \Psi'_P \circ \Psi_P$.

Conversely, given a morphism $\Psi \colon BG \to BH$, consider the image of the trivial

G-torsor over F, $P = \psi((F, G))$. Note that, as $\mathrm{Aut}_{BG}((F, G)) \simeq G$ and as Ψ is fully faithful, we also have that $G \simeq \mathrm{Aut}_{BH}(P)$, so that G acts on P on the right. This action makes P a right G-torsor. As it is compatible with left H-action, we see that P has a natural structure of a (H, G)-bitorsor. It is easy to check that $\Psi \simeq \Psi_P$, and that this isomorphism is natural.

As a consequence of these observations, we can see that the category $\mathrm{Hom}(BG, BH)$ in $\mathrm{Gerbe}(F)$ is equivalent to the category of (G, H)-bitorsors.

Theorem 7.1.1 ([Gir71, Giraud]). *Let P be a (H, G)-bitorsor over F. Then, the functor*

$$BG \to BH$$

$$(S, T) \mapsto (S, T \wedge^{G|_S} P_S)$$

is an equivalence of categories. Furthermore, any equivalence between BG and BH is of this form.

7.1.1 Bands and patching of non-abelian H^2

In this section, we will continue to work over the big étale site of schemes over F. Before we can patch gerbes, we will first investigate when we can patch equivalence classes of gerbes, i.e. elements in the non-abelian second cohomology set of a band.

Given a gerbe $\mathcal{G}$, one can associate to it a band L. Let us quickly review the definition of a band. For more details, we refer again to [Gir71],but compare also the appendix of [DM82].

Let H, G be group sheaves and denote $G^{\mathrm{ad}} = G/Z$, where Z is the center of G. Then, G^{ad} acts on $\mathrm{Isom}(H, G)$ via conjugation. Let $\mathrm{Isex}(H, G)$ denote the quotient of $\mathrm{Isom}(H, G)$ by the G^{ad}-action. Then, a band L consists of a triple (Y, G, φ) where Y is a cover of F, G is a group sheaf defined over Y and $\varphi \in \mathrm{Isex}(G_1, G_2)$ where

$G_i = \mathrm{pr}_i\, G$ for $\mathrm{pr}_i \colon Y \times_F Y \to Y$. We require φ to satisfy the cocycle condition $\mathrm{pr}_{31}^* \varphi = \mathrm{pr}_{32}^* \varphi \circ \mathrm{pr}_{21}^* \varphi$ over $Y \times_F \times_F Y$.

Thus, a band is a descent datum for a group sheaf over a cover $Y \to F$ modulo inner automorphisms. Every group sheaf over F defines a band with trivial gluing datum. Furthermore, any abelian band (i.e. G is abelian) is induced by a group sheaf over F as the datum of a band in this case just gives descent datum since inner automorphisms of G are all trivial.

The center $Z(L)$ of a band $L = (Y, G, \varphi)$ is defined as (Y, Z, φ_Z) and we identify it with the group sheaf over F determined by the descent datum of the band. For a cover $f \colon Y' \to Y$, we identify the bands (Y, G, φ) and $(Y', f^*G, f^*\varphi)$. An isomorphism between two bands $(Y, H, \varphi) \to (Y, G, \tau)$ is given by an element $g \in \mathrm{Isex}(H, G)$ compatible with φ and τ.

We say that a band $L = (Y, G, \varphi)$ is linear algebraic, if G is a linear algebraic group, i.e. if $Y = \mathrm{Spec}(R)$ for $R = \prod_i L_i$ where L_i is a finite separable field extension, G_{L_i} is a linear algebraic group for all i.

The reason for the definition of a band is its natural appearance in the context of gerbes. It turns out that the gerbe axioms put strong conditions on the automorphisms of the elements in the gerbe. In particular, as two objects are locally isomorphic, all automorphism groups are forms of each other. In fact, they even induce a unique band.

Given a gerbe $\mathcal{G}$, we can define an associated band. Pick a cover $Y \to F$ and an object $x \in \mathcal{G}(Y)$. Let $G = \mathrm{Aut}(x)$ be the sheaf of automorphisms of x. Let $x_i = \mathrm{pr}_i^* x$ denote the two pullbacks along the projections $\mathrm{pr}_i \colon Y \times_F Y \to Y$. By the definition of gerbes, there is a cover $U \to Y \times_F Y$ such that $x_1|_U$ and $x_2|_U$ are isomorphic in $\mathcal{G}(U)$. An isomorphism $f \colon x_1|_U \to x_2|_U$ defines an isomorphism $\lambda_f \colon G_1|_U \to G_2|_U$. If $g \colon x_1|_U \to x_2|_U$ is another isomorphism, then λ_f, λ_g differ by an inner automorphism of G_2. Thus, there is a well defined element $\lambda \in \mathrm{Isex}(G_1, G_2)(U)$.

An easy calculation shows that the pullbacks of λ_f on $U \times_{Y \times_F Y} U$ differ by an inner automorphism. Hence, the pullbacks of λ agree on $U \times_{Y \times_F Y} U$ and we thus obtain an element $\lambda' \in \mathrm{Isex}(G_1, G_2)(Y \times_k Y)$ whose restriction to U equals λ.

It is not hard to see that λ' satisfies the cocycle condition and thus (Y, G, λ') defines a band. Given a gerbe, the associated band is unique up to unique isomorphism and we denote it by $\mathrm{Band}(\mathcal{G})$. Note that a morphism $\mathcal{G} \to \mathcal{G}'$ induces a map $\mathrm{Band}(\mathcal{G}) \to \mathrm{Band}(\mathcal{G}')$. Given a band L, a gerbe banded by L is a tuple $(\mathcal{G}, \theta)$ where $\mathcal{G}$ is a gerbe and $\theta\colon \mathrm{Band}(\mathcal{G}) \to L$ is an isomorphism of bands. We often suppress θ from the notation.

Given two gerbes $\mathcal{G}, \mathcal{G}'$ banded by L, an L-morphism of gerbes is a morphism $\alpha\colon \mathcal{G} \to \mathcal{G}'$ that is compatible with the morphisms θ and θ', i.e. the diagram

$$\begin{tikzcd} \mathrm{Band}(\mathcal{G}) \arrow[rr, "\alpha"] \arrow[dr, "\theta"'] & & \mathrm{Band}(\mathcal{G}') \arrow[dl, "\theta'"] \\ & L & \end{tikzcd}$$

commutes. Any morphism of L-gerbes is an equivalence, just as any G-equivariant morphism of G-torsors is an isomorphism.

Let $\mathrm{H}^2(F, L)$ denote the set of L-equivalence classes of gerbes banded by L (this means that we only consider equivalences of L-gerbes that are compatible with the band L). If L is abelian coming from the group sheaf A over F, then this definition coincides with the usual definition of $\mathrm{H}^2(F, A)$ as a Galois cohomology group.

Furthermore, there is a remarkable relation between $\mathrm{H}^2(F, L)$ and $\mathrm{H}^2(F, Z(L))$.

Theorem 7.1.2 ([Gir71, Chapter IV, Theorem 3.3.3])**.** *If* $\mathrm{H}^2(F, L)$ *is not empty, then it is a principal homogeneous space under* $\mathrm{H}^2(F, Z(L))$.

Sketch. Given a class α in $\mathrm{H}^2(F, L)$ represented by the L-gerbe $\mathcal{G}$ and given a class in $\beta \in \mathrm{H}^2(F, Z(L))$ represented by the $Z(L)$-gerbe $\mathcal{H}$, we define $\beta.\alpha$ to be the class in $\mathrm{H}^2(F, L)$ represented by $\mathcal{G} \wedge^{Z(L)} \mathcal{H}$. We refer the reader to [Gir71, Chapter IV] for details and definitions. $\square$

Harbater, Hartmann and Krashen have proved patching for Galois cohomology groups of abelian algebraic groups over arithmetic curves under some mild compatibility assumptions between the characteristic of F and the order of the group (compare Section 5.5). Using the above theorem, we can deduce patching for non-abelian Galois cohomology over arithmetic curves whenever the center of the band admits patching. We will pursue this in Section 8.3. For the purpose of gerbe patching, let us note this immediate consequence.

We define patching for $\mathrm{H}^2(\circ, L)$ over a finite inverse factorization system $\mathcal{F}$ analogously to the case of patching for Galois cohomology, see Section 5.5. Note that for $(i,j,k) \in S_I$, we have two maps $\mathrm{H}^2(F_i, L) \times \mathrm{H}^2(F_j, L) \rightrightarrows \mathrm{H}^2(F_k, L)$ given by restriction of the first and the second factor respectively. Hence, we have two maps $\prod_{i \in I_v} \mathrm{H}^2(F_i, L) \rightrightarrows \prod_{k \in I_e} \mathrm{H}^2(F_k, L)$.

Definition 28. Let $\mathcal{F}$ be a finite inverse factorization system with inverse limit F. For a band L over F, we say that patching holds for $\mathrm{H}^2(\circ, L)$ over $\mathcal{F}$ if the the following sequence is an equalizer diagram

$$\mathrm{H}^2(F, L) \to \prod_{i \in I_v} \mathrm{H}^2(F_i, L) \rightrightarrows \prod_{k \in I_e} \mathrm{H}^2(F_k, L).$$

Proposition 7.1.3. *Let $\mathcal{F}$ be a finite inverse system for fields with inverse limit F and let L be a band over F. If patching holds for $\mathrm{H}^2(\circ, Z(L))$ over $\mathcal{F}$ and $\mathrm{H}^2(F, L) \neq \emptyset$, then patching holds for $\mathrm{H}^2(\circ, L)$ over $\mathcal{F}$.*

Proof. By assumption, there is a class $\alpha \in \mathrm{H}^2(F, L)$. Given a patching problem $\{\beta_i\}_{i \in I_v}$ with $\beta_i \in \mathrm{H}^2(F_i, L)$, there are elements $\gamma_i \in \mathrm{H}^2(F_i, Z(L))$ such that $\beta_i = \gamma_i.\alpha|_{F_i}$ by Theorem 7.1.2. As the action of $\mathrm{H}^2(F_i, Z(L))$ on $\mathrm{H}^2(F_i, L)$ is simply transitive, it follows that $\{\gamma_i\}_{i \in I_v}$ defines a patching problem for $\mathrm{H}^2(\circ, Z(L))$. By assumption, there is $\gamma \in \mathrm{H}^2(F, Z(L))$ such that $\gamma|_{F_i} = \gamma_i$ for all $i \in I_v$. The element $\beta := \gamma.\alpha$ solves the patching problem $\{\beta_i\}_{i \in I_v}$. $\square$

7.2 A semi-cocyclic description

We will now follow sections 2.3-2.6 of [Bre94] to introduce a cocyclic description of a gerbe $\mathcal{G} \to \mathrm{Sch/F}$.

Let $Y \to F$ be a cover such that there is an element $y \in \mathcal{G}(Y)$. Let $G = \mathrm{Aut}_Y(y)$ denote the sheaf of automorphisms of y over Sch/Y. The choice of an object y defines an equivalence

$$\Phi \colon \mathcal{G}|_Y \longrightarrow BG$$

defined on fibers over $f \colon Z \to Y$ via

$$\begin{aligned} \Phi(Z) \colon \mathcal{G}(Z) &\longrightarrow \mathrm{TOR}(G)(Z) \\ z &\mapsto \mathrm{Isom}_Z(f^*y, z) \end{aligned}$$

where $\mathrm{Isom}_Z(z, f^*y)$ is a $G_y|_Z$-torsor by the natural action on the left.

We fix the following notation: Let $\mathrm{pr}_i \colon Y^2 = Y \times_F Y \to Y$ denote the natural projections on the i-th factor for $i = 1, 2$. We will denote by $G_i = \mathrm{pr}_i^* G$ the pullbacks of G to Y^2. Let $\mathrm{pr}_{ij} \colon Y^3 \to Y^2$ denote the natural projections on the i-th and j-th component for $1 \le i < j \le 3$. We will denote by G^i the pullbacks of G to Y^3 along the natural projections $Y^3 \to Y$. Finally, let $\mathrm{pr}_{ijk} \colon Y^4 \to Y^3$ denote the natural projection on the i-th, j-th and k-th component for $1 \le i < j < k \le 4$. Let $G_{(i)}$ denote the pullbacks of G to Y^4 along the natural projections $Y^4 \to Y$. Note that Φ induces an equivalence

$$\varphi = \mathrm{pr}_1^* \Phi \circ (\mathrm{pr}_2^* \Phi)^{-1} : BG_2 \longrightarrow BG_1.$$

where $G_i = \mathrm{pr}_i^* G$.

We obtain an isomorphism of functors

$$\psi\colon\ \mathrm{pr}_{12}^*\,\varphi \circ \mathrm{pr}_{23}^*\,\varphi \Rightarrow \mathrm{pr}_{13}^*\,\varphi$$

rather than an equality. The various pullbacks to Y^4 of this isomorphism make the following diagram of natural transformations commute:

$$\begin{tikzcd}
BG|_{Y^4} \arrow[rrr] \arrow[dr] \arrow[drr] & & & BG|_{Y^4} \\
& BG|_{Y^4} \arrow[r] \arrow[urr] & BG|_{Y^4} \arrow[ur] &
\end{tikzcd}$$

where $Y^4 = Y \times_F Y \times_F Y \times_F Y$.

The gerbe $\mathcal{G}$ is completely determined by the tuple (G, φ, ψ).

We will now reinterpret the above description in terms of bitorsors. Note that, by Mortia equivalence, the equivalence φ can be identified with the (G_2, G_1)-bitorsor $E = \mathrm{Isom}(\mathrm{pr}_2^*\,y, \mathrm{pr}_1^*\,y)$. Then, the isomorphism ψ corresponds to an isomorphism of (G^1, G^3)-bitorsors

$$\mathrm{pr}_{12}^*\,E \wedge^{G^2} \mathrm{pr}_{23}^*\,E \longrightarrow \mathrm{pr}_{13}^*\,E$$

where G^i denotes the pullback of G along $Y \times_F Y \times_F Y \to Y$ on the i-th component.

Finally, the compatibility condition translates to the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{pr}_{12}^*\,E \wedge^{G_{(2)}} \mathrm{pr}_{23}^*\,E \wedge^{G_{(3)}} \mathrm{pr}_{34}^*\,E \arrow[r] \arrow[d] & \mathrm{pr}_{13}^*\,E \wedge^{G_{(3)}} \mathrm{pr}_{34}^*\,E \arrow[d] \\
\mathrm{pr}_{12}^*\,E \wedge^{G_{(2)}} \mathrm{pr}_{24}^*\,E \arrow[r] & \mathrm{pr}_{14}^*\,E
\end{tikzcd}$$

where the arrows are induced by the various pullbacks of ψ and $G_{(i)}$ denotes the pullback of G along $Y^4 \to Y$ onto the i-th component. So, a gerbe can be described by the triple (G, E, ψ) and we will henceforth go back and forth between the categorical and the cocyclic description. We call the triple (G, E, ψ) the semi-cocyclic description of $\mathcal{G}$. (We could further describe E as a cocyle yield a cocyclic description of $\mathcal{G}$)

Let us now turn to morphisms of gerbes.

Let $\rho\colon \mathcal{G} \to \mathcal{G}'$ be a morphism of gerbes over a field F. Let K/F be a finite separable extension such that there are $x \in \mathcal{G}(K)$ and $y \in \mathcal{G}'(K)$. After possibly replacing K by a further finite separable extension, we may assume that $\rho(x)$ and y are isomorphic in $\mathcal{G}'(K)$. Using the cocyclic description, we get $\mathcal{G} = (G, E, \psi)$ and $\mathcal{G}' = (G', E', \psi')$ where we choose the descriptions induced by x and y. In particular, $\mathrm{Aut}(x) = G$ and $\mathrm{Aut}(y) = G'$.

Observe that ρ induces a map

$$\rho'\colon G \longrightarrow G'$$

and a $\rho'|_{K\times_F K}$-equivariant map

$$\alpha\colon E \to E'.$$

Note that α is compatible with ψ and ψ'. Conversely, given a map α compatible with ψ and ψ', one gets an induced morphism of gerbes $\mathcal{G} \to \mathcal{G}'$.

Lemma 7.2.1 ([Bre94, Section 2.6])**.** *Let $\rho\colon \mathcal{G} \to \mathcal{G}'$ be a morphism of gerbes. Assume that there is a finite separable cover K/F such that $\mathcal{G}(K) \neq \mathcal{G}'(K)$. Fix some $x \in \mathcal{G}(K)$ and $y \in \mathcal{G}(K)$. Assume that there is a morphism $\rho(K)(x) \to y$ in $\mathcal{G}'(K)$. Rewrite $\mathcal{G} = (\mathrm{Aut}(x), E, \psi)$ and $\mathcal{G} = (\mathrm{Aut}(y), E', \psi')$ using the construction described above coming from x and y. Then, ρ induces an equivariant isomorphism of bitorsors $E \to E'$ that is compatible with ψ and ψ'. Conversely, given an isomorphism $E \to E'$ compatible with the gluing data, one can construct a morphism of gerbes.*

7.3 Patching for gerbes

Let $\mathcal{F}$ be a factorization inverse system of fields with inverse limit F. Let L be a band over F. Let $\mathrm{Gerbe}(F, L)$ denote the 2-category of L-banded gerbes $\mathcal{G}$ over F. Here, morphisms are given by equivalences of gerbes and 2-morphisms are given by natural isomorphisms. Let $\mathrm{GPP}(\mathcal{F}, L)$ denote the 2-category of L-gerbe patching problems, i.e. an object consists of a collection of gerbes L-banded gerbes $\mathcal{G}_i$ over F_i together with L-equivalences $\sigma_k \colon \mathcal{G}_i|_{F_k} \to \mathcal{G}_j|_{F_k}$ for $(i, j, k) \in S_I$.

A 1-Morphisms $(\{\mathcal{G}_i\}, \{\sigma_k\}) \xrightarrow{(\alpha,f)} (\{\mathcal{G}'_i\}, \{\sigma'_k\})$ is given by a collection of equivalences of gerbes $\alpha_i \colon \mathcal{G}_i \to \mathcal{G}'_i$ and natural isomorphisms $f_k \colon \sigma'_k \circ \alpha_i|_{F_k} \Rightarrow \alpha_j|_{F_k} \circ \sigma_k$, pictorially:

$$\begin{tikzcd}
\mathcal{G}_i|_{F_k} \arrow[r, "\alpha_i|_{F_k}"] \arrow[d, "\sigma_k"'] \arrow[dr, phantom, "\Downarrow f_k"] & \mathcal{G}'_i|_{F_k} \arrow[d, "\sigma'_k"] \\
\mathcal{G}_j|_{F_k} \arrow[r, "\alpha_j|_{F_k}"'] & \mathcal{G}'_j|_{F_k}
\end{tikzcd}$$

Composition of morphisms is given by composing equivalences of gerbes and by horizontally composing the natural isomorphisms.

Given two morphisms $(\alpha, f), (\beta, g) \colon \{\mathcal{G}_i\}, \{\sigma_k\}) \to (\{\mathcal{G}'_i\}, \{\sigma'_k\})$, a 2-morphism $\psi = (\{\psi_i\})$ is given by a collection of natural isomorphisms

$$\begin{tikzcd}
\mathcal{G}_i \arrow[r, bend left, "\alpha_i"] \arrow[r, bend right, "\beta_i"'] \arrow[r, phantom, "\Downarrow \psi_i"] & \mathcal{G}'_i
\end{tikzcd}$$

such that the diagram of natural transformations commutes:

$$\begin{tikzcd}
\mathcal{G}_i|_{F_k} \arrow[r, "\alpha_i", ""{name=A}] \arrow[d, "\sigma_k"'] \arrow[dr, phantom, "\Downarrow f_k"] & \mathcal{G}'_i|_{F_k} \arrow[d, "\sigma'_k"] & \mathcal{G}_i|_{F_k} \arrow[r, "\beta_i", ""{name=B}] \arrow[d, "\sigma_k"'] \arrow[dr, phantom, "\Downarrow g_k"] & \mathcal{G}'_i|_{F_k} \arrow[d, "\sigma'_k"] \\
\mathcal{G}_j|_{F_k} \arrow[r, "\alpha_j"', ""{name=C}] & \mathcal{G}'_j|_{F_k} & \mathcal{G}_j|_{F_k} \arrow[r, "\beta_j"', ""{name=D}] & \mathcal{G}'_j|_{F_k}
\arrow[Rightarrow, from=A, to=B, bend left, "\psi_i|_{F_k}"]
\arrow[Rightarrow, from=C, to=D, bend right, "\psi_j|_{F_k}"']
\end{tikzcd}$$

i.e.

$$\begin{tikzcd}
\sigma'_k \circ \alpha_i|_{F_K} \arrow[r, Rightarrow, "f_k"] \arrow[d, Rightarrow, "\psi'_i"] & \alpha_j|_{F_k} \circ \sigma_k \arrow[d, Rightarrow, "\psi'_j"] \\
\sigma'_k \circ \beta_i|_{F_k} \arrow[r, Rightarrow, "g_k"] & \beta_j|_{F_k} \circ \sigma_k
\end{tikzcd}$$

commutes. Here ψ'_i is the natural isomorphism induced by ψ_i and σ'_k. Composition of 2-morphisms is given by vertical composition of the various natural transformations.

Lemma 7.3.1. *Every 1-morphism in* $\mathrm{GPP}(\mathcal{F}, L)$ *admits a quasi-inverse.*

Proof. Let $(\alpha, f)\colon (\{\mathcal{G}_i\}, \{\sigma_k\}) \to (\{\mathcal{G}'_i\}, \{\sigma'_k\})$ be a 1-morphism. Let β_i denote a quasi-inverse of α_i. Then, we get natural isomorphisms $g_k\colon \beta_j|_{F_k} \circ \sigma'_k \Rightarrow \sigma_k \circ \beta_i|_{F_k}$ defined via

$$\beta_j|_{F_k} \circ \sigma'_k \Rightarrow \beta_j|_{F_k} \circ \sigma'_k \circ \alpha_i|_{F_k} \circ \beta_i|_{F_k} \Rightarrow \beta_j|_{F_k} \circ \alpha_j|_{F_k} \circ \sigma_k \circ \beta_i|_{F_k} \Rightarrow \sigma_k \circ \beta_i|_{F_k}.$$

Here, the first and last natural transformation are induced by fixed natural isomorphisms $\phi_i\colon \alpha_i \circ \beta_i \Rightarrow \mathrm{id}$, while the middle arrow is induced by f_k. Hence, we can define a 1-morphism $(\beta, g)\colon (\{\mathcal{G}'_i\}, \{\sigma'_k\}) \to (\{\mathcal{G}_i\}, \{\sigma_k\})$. It remains to check that $(\alpha, f) \circ (\beta, g)$ and $(\beta, g) \circ (\alpha, f)$ are 2-isomorphic to the identity morphism. We will prove that $(\alpha, f) \circ (\beta, g)$ is isomorphic to the identity, the other case is analogous. For this, we need to give a collection of 2-isomorphisms $\psi_i\colon \alpha_i \circ \beta_i \Rightarrow \mathrm{id}$ that are compatible with g_k and f_k. It is tedious but straightforward to check that the choice $\psi_i = \phi_i$ works. $\square$

Note that there is a natural functor of 2-categories

$$\beta'''_L\colon \mathrm{Gerbe}(F, L) \longrightarrow \mathrm{GPP}(\mathcal{F}, L)$$

induced by base change. We say that patching holds for L-gerbes over $\mathcal{F}$ if β'''_L is an equivalence.

In order to prove our main result, we need the following elementary lemma from the general theory of stacks. Let $\mathcal{X}$ and $\mathcal{Y}$ be stacks in groupoids over Sch/F and fix a cover $\{U \to X\}$ in $\mathcal{C}$.

Let $\mathrm{pr}_i \colon U \times_X U \to U$ and $\mathrm{pr}_{ij} \colon U \times_X U \times_X U \to U \times_X U$ denote the usual projections.

Lemma 7.3.2. *The following data are equivalent:*

1. *a morphism of stacks* $\mathcal{X} \to \mathcal{Y}$

2. *a morphism of stacks* $\alpha \colon \mathcal{X}|_U \to \mathcal{Y}|_U$ *together with a natural transformation* $\psi \colon \mathrm{pr}_1^* \alpha \to \mathrm{pr}_2^* \alpha$ *such that the diagram*

$$\begin{tikzcd} \mathrm{pr}_{12}^* \mathrm{pr}_1^* \alpha \arrow[r, "\mathrm{pr}_{12}^* \psi"] \arrow[d, equal] & \mathrm{pr}_{12}^* \mathrm{pr}_2^* \alpha \arrow[r, equal] & \mathrm{pr}_{23}^* \mathrm{pr}_1^* \alpha \arrow[d, "\mathrm{pr}_{23}^* \psi"] \\ \mathrm{pr}_{13}^* \mathrm{pr}_1^* \alpha \arrow[r, "\mathrm{pr}_{13}^* \psi"] & \mathrm{pr}_{13}^* \mathrm{pr}_2^* \alpha \arrow[r, equal] & \mathrm{pr}_{23}^* \mathrm{pr}_2^* \alpha \end{tikzcd}$$

commutes.

Proof. This immediately follows from the fact that the category of morphisms $\mathcal{X} \to \mathcal{Y}$ is a stack itself. □

Definition 29. Let $(\{\mathcal{G}_i\}, \{\nu_k\})$ be a gerbe patching problem. We say that $(\{\mathcal{G}_i\}, \{\nu_k\})$ has property D if there is a cover $Z \to F$ such that $\mathcal{G}_i(Z_i) \neq \emptyset$ for all $i \in I_v$ and that there are elements $x_i \in \mathcal{G}_i(Z_i)$ such that $\nu_k(x_i|_{Z_k})$ is isomorphic to $x_j|_{Z_k}$ for all $(i, j, k) \in S_I$.

Proposition 7.3.3. *Let* $L = (Y, G, \psi)$ *for some cover* $Y \to F$*. Let* $(\{\mathcal{G}_i\}, \{\nu_k\})$ *be an* L*-gerbe patching problem with property D. Then, if patching holds for* G*-torsors, there is* $\mathcal{G} \in \mathrm{Gerbe}(F, L)$ *such that* $\beta_L'''(\mathcal{G}) \simeq (\{\mathcal{G}_i\}, \{\nu_k\})$.

Proof. Since the gerbe patching problem has property D, there is a cover $Z \to F$ and elements $x_i \in \mathcal{G}_i(Z_i)$ such that $\nu_k(x_i|_{Z_k}) \simeq x_j|_{Z_k}$ for $(i, j, k) \in S_I$. We may assume without loss of generality that $Z \to F$ factors through $Y \to F$ as we could replace Z by $Z \times_F Y$. Hence, we may assume $L = (Z, G, \psi)$ and that we can patch G-torsors over Z. Let $\mathrm{pr}_j \colon Z \times_F Z \to Z$ denote the natural projection for $j = 1, 2$. Let $G_j = \mathrm{pr}_j^* G$.

Let $A = Z \times_F Z$ and note that patching holds for (G_1, G_2)-bitorsors over $\mathcal{F}_A$ by assumption and Corollary 6.2.3.

By use of the cocyclic description of gerbes and their morphisms, we will show that the given gerbe patching problem induces a bitorsor patching problem in

$$\mathrm{BPP}(G_2, G_1)(\mathcal{F}_A).$$

Describing the gerbes $\mathcal{G}_i$ with respect to Z_i as a cocycle, we get tuples $(G|_{Z_i}, P_i, \psi_i)$ where P_i are (G_1, G_2)-bitorsors over A_i. The equivalences of the various $\mathcal{G}_i|_k$ with the $\mathcal{G}_j|_k$ for $(i, j, k) \in S_I$ translate to isomorphisms $P_i|_{A_k} \to P_j|_{A_k}$ by Lemma 7.2.1. Hence, we get an element in $\mathrm{BPP}(G_2, G_1)(\mathcal{F}_A)$.

Thus, there is a (G_2, G_1)-bitorsor P defined over $A = Z \times_F Z$. The morphisms ψ_i from the cocyclic description of G_i glue together by bitorsor patching to give a global isomorphism $\psi \colon P_{12} \wedge^{G^2} P_{23} \to P_{13}$ of (G^1, G^3)-bitorsors. Also, again by bitorsor patching, the morphism ψ satisfies the coherence condition. Hence, we get a cocycle (G, P, ψ) defining an L-gerbe $\mathcal{G}$ in $\mathrm{Gerbe}(F, L)$. By construction, we obtain a 1-morphism $\beta'''_{K,G}(\mathcal{G}) \to (\{\mathcal{G}_i\}, \{\nu_k\})$. □

Proposition 7.3.4. *Let $L = (Y, G, \psi)$ for some cover $Y \to F$. Assume that patching holds for G-torsors. Then, the functor β'''_L is essentially surjective on 1-morphisms.*

Proof. Let $\mathcal{G}, \mathcal{G}' \in \mathrm{Gerbe}(F, L)$ and let $(\{\mathcal{G}_i\}, \{\sigma_k\})$ and $(\{\mathcal{G}'_i\}, \{\sigma'_k\})$ denote the images in $\mathrm{GPP}(\mathcal{F}, L)$. Given a morphism $(\{\alpha_i\}, \{f_k\}) \colon (\{\mathcal{G}_i\}, \{\sigma_k\}) \to (\{\mathcal{G}'_i\}, \{\sigma'_k\})$, we want to construct a morphism $\mathcal{G} \to \mathcal{G}'$ whose image in $\mathrm{GPP}(\mathcal{F}, L)$ is isomorphic to (α, f).

Let $Z \to F$ be a cover such that $\mathcal{G}(Z) \neq \emptyset \neq \mathcal{G}'(Z)$. Thus, $\mathcal{G}|_Z, \mathcal{G}'|_Z \simeq BG$, and therefore $(\{\mathcal{G}_i\}, \{\sigma_k\})|_Z = (\{BG_i\}, \{\sigma_k|_{Z_k}\})$ and we can identify $\sigma_k|_{Z_k}$ with the trivial bitorsor G_k. The same conclusion holds for $\mathcal{G}'$ and $(\{\mathcal{G}'_i\}, \{\sigma'_k\})$. Therefore, over Z_i, α_i corresponds to a G-bitorsor P_i by Theorem 7.1.1. Over Z_k, the natural transformation f_k corresponds to the diagram

$$\begin{tikzcd} BG_i|_{Z_k} \arrow[r, "P_i"] \arrow[d, "G_k"'] \arrow[dr, phantom, "\Downarrow f_k"] & BG_i|_{Z_k} \arrow[d, "G_k"] \\ BG_j|_{Z_k} \arrow[r, "P_j"'] & BG_j|_{F_k} \end{tikzcd}$$

and thus corresponds to an isomorphism of G_k-bitorsors $P_i|_{Z_k} \to P_j|_{Z_K}$. By Theorem 6.2.2, we get a G-bitorsor P defined over Z together with isomorphisms $\phi_i \colon P|_{Z_i} \to P_i$. This bitorsor in turn defines a morphism $\alpha \colon \mathcal{G}|_Z \to \mathcal{G}'|_Z$. We claim that it actually descends to a morphism $\mathcal{G} \to \mathcal{G}'$. According to Lemma 7.3.2, we need an isomorphism of functors $\psi \colon \operatorname{pr}_1^* \alpha \to \operatorname{pr}_2^* \alpha$ such that

$$\begin{tikzcd} \operatorname{pr}_{12}^* \operatorname{pr}_1^* \alpha \arrow[r, "\operatorname{pr}_{12}^* \psi"] \arrow[d, equal] & \operatorname{pr}_{12}^* \operatorname{pr}_2^* \alpha \arrow[r, equal] & \operatorname{pr}_{23}^* \operatorname{pr}_1^* \alpha \arrow[d, "\operatorname{pr}_{23}^* \psi"] \\ \operatorname{pr}_{13}^* \operatorname{pr}_1^* \alpha \arrow[r, "\operatorname{pr}_{13}^* \psi"] & \operatorname{pr}_{13}^* \operatorname{pr}_2^* \alpha \arrow[r, equal] & \operatorname{pr}_{23}^* \operatorname{pr}_2^* \alpha \end{tikzcd}$$

commutes. In terms of bitorsors, this means that we need an isomorphism of bitorsors $\psi \colon \operatorname{pr}_1^* P \to \operatorname{pr}_2^* P$ making the analogous diagram commute. Such a morphism clearly exists for each P_i as these bitorsors come from morphisms defined over F_i. Furthermore, as these morphisms are compatible with the gluing data, these isomorphisms glue to give a global $\psi \colon \operatorname{pr}_1^* P \to \operatorname{pr}_2^* P$ by Theorem 6.2.2. Hence, we get a morphism of gerbes $\mathcal{G} \to \mathcal{G}'$ and it is easy to see that its image is isomorphic to $(\{\alpha_i\}, \{f_k\})$. □

Proposition 7.3.5. *Let $L = (Y, G, \psi)$ for some cover $Y \to F$. Assume that patching holds for G-torsors. Then, the functor β_L''' is fully faithful on 2-morphisms.*

Proof. Fix two 1-morphisms of gerbes $\alpha, \beta \colon \mathcal{G} \to \mathcal{G}'$ and let

$$(\{\alpha_i\}, \{f_k\}), (\{\beta_i\}, \{g_k\}) \colon (\{\mathcal{G}_i\}, \{\sigma_k\}) \to (\{\mathcal{G}'_i\}, \{\sigma'_k\})$$

denote their images in $\mathrm{GPP}(\mathcal{F}, L)$.

Given two 2-morphisms $\psi, \psi' \colon \alpha \to \beta$ whose image in $\mathrm{GPP}(\mathcal{F}, L)$ are the same, we want to prove that $\psi = \psi'$. It is enough to show this after base change, i.e. to prove $\psi|_Z = \psi'|_Z$. Thus, we may assume $\mathcal{G} = \mathcal{G}' = BG$. Thus, α and β correspond to G-bitorsors P, Q and ψ, ψ' are bitorsor isomorphisms $P \to Q$. We then obtain $\psi = \psi'$ immediately from Theorem 6.2.2.

Finally, we need to check fullness. Given a 2-morphism

$$(\{\psi_i\}) \colon (\{\alpha_i\}, \{f_k\}) \to (\{\beta_i\}, \{g_k\})$$

we first base change to Z. Then, ψ_i correspond to bitorsor isomorphisms and the compatibility condition for 2-morphisms ensures that these isomorphisms glue. Hence, we get $\psi' \colon \alpha|_Z \to \beta|_Z$ by Theorem 6.2.2. It remains to show that this morphism descends to a morphism $\psi \colon \alpha \to \beta$. This follows from arguments analogous to the argument to lift the 1-morphism in the proof of Proposition 7.3.4. $\square$

Theorem 7.3.6. *Let $L = (Y, G, \psi)$ for some cover $Y \to F$. Assume that any L-gerbe patching problem $(\{\mathcal{G}_i\}, \{\nu_k\})$ has property D. If patching holds for G-torsors, then it also holds for L-gerbes, i.e. if the functor β'_G is an equivalence of 1-categories, then β'''_L is an equivalence of 2-categories.*

Proof. By Proposition 7.3.3 and assumption, β'''_L is essentially surjective on objects. By Proposition 7.3.4 β'''_L is essentially surjective on 1-morphisms and by Proposition 7.3.5, β'''_L is fully faithful on 2-morphisms. Hence, β'''_L is an equivalence. $\square$

Remark 6. Let L be a band and assume that L-gerbe patching holds over $\mathcal{F}$. Let $\mathcal{G}, \mathcal{G}'$ be L-gerbes over $\mathcal{F}$ such that $\beta'''_{L,\mathcal{F}}(\mathcal{G})$ and $\beta'''_{L,\mathcal{F}}(\mathcal{G}')$ are isomorphic. Then, there is an equivalence $\mathcal{G} \simeq \mathcal{G}'$, unique up to unique isomorphism, as $\beta'''_{L,\mathcal{F}}$ is an equivalence of 2-categories.

We will now conclude this section with a sufficient condition for the technical assumption of the above theorem to be satisfied.

Proposition 7.3.7. *Let $L = (Y, G, \psi)$ be a band over F. Assume that patching holds for $\mathrm{H}^2(\circ, L)$ and that bitorsor factorization holds for any cover $Z \to Y$. Then, any L-gerbe patching has property D.*

Proof. Let $\alpha_i \in \mathrm{H}^2(F_i, L_i)$ denote the equivalence class of $\mathcal{G}_i$. Then, we have that $\alpha_i = \alpha_j \in \mathrm{H}^2(F_k, L_k)$ by assumption. Hence, there is some $\alpha \in \mathrm{H}^2(F, L)$ inducing α_i for all $i \in I_v$. Let $\mathcal{G}$ be an L-gerbe representing α and let $Z \to Y$ be a cover such that $\mathcal{G}(Z) \neq \emptyset$. Then, $\mathcal{G}_i(Z_i) \neq \emptyset$ by construction. Fix equivalences $\mathcal{G}_i \to BG_i$. Then, for $(i, j, k) \in S_I$, we have a morphism

$$\begin{array}{ccc} BG_i & & BG_j \\ \downarrow & & \uparrow \\ \mathcal{G}_i & \xrightarrow{\sigma_k} & \mathcal{G}_j \end{array}$$

defined over Z_k. Let P be the G_k-bitorsor corresponding to the composition $BG_i \to BG_j$. By assumption, we there is a G_i-bitorsor P_i over Z_i and a G_j-bitorsor G_j over Z_j such that $P \simeq P_i \wedge^{G_k} P_j$ over Z_k. Let $x_i \in \mathcal{G}_i(Z_i)$ correspond to P_i^{op} with respect to the chosen equivalence $\mathcal{G}_i \to BG_i$ and let x_j correspond to P_j. Then, it is straightforward to check that $\sigma_k(x_i|_{Z_k})$ and $x_j|_{Z_k}$ are isomorphic in $\mathcal{G}_j(Z_k)$. □

Corollary 7.3.8. *Under the assumptions of Proposition 7.3.7, L-gerbe patching holds over $\mathcal{F}$.*

7.4 A Mayer-Vietoris sequence and a local-global principle for gerbes

In this section, we will fix a finite inverse factorization system $\mathcal{F}$ with inverse limit F. We will also fix a group sheaf G defined over F for which G-torsor patching holds. Let L be the band induced by G.

Definition 30. We say that G-gerbes satisfy the local-global principle with respect to $\mathcal{F}$ if for any G-gerbe $\mathcal{G}$ over F,

$$\mathcal{G}_i \simeq BG|_{F_i}$$

for all $i \in I_v$ implies

$$\mathcal{G} \simeq BG.$$

Recall that G-bitorsors over F are classified by $\mathrm{H}^0(F, G \to \mathrm{Aut}(G))$ and G gerbes are classified by $\mathrm{H}^1(F, G \to \mathrm{Aut}(G))$ (cf. Chapter 4).

Note that we obtain two maps

$$\prod_{i \in I_v} \mathrm{H}^1(F_i, G \to \mathrm{Aut}(G)) \rightrightarrows \prod_{k \in I_e} \mathrm{H}^1(F_k, G \to \mathrm{Aut}(G)).$$

via base change.

Lemma 7.4.1. *There is a map of pointed sets*

$$\prod_{k \in I_k} \mathrm{H}^0(F_k, G \to \mathrm{Aut}(G)) \to \mathrm{H}^1(F, G \to \mathrm{Aut}(G))$$

Proof. We can define the map as follows: Given G-bitorsors P_k over F_k, consider the G-gerbe patching problem $(\{BG|_{F_i}\}_i, \{P_k\}_k)$. By Theorem 7.3.6, there is a G-gerbe $\mathcal{G}$ over F in the essential preimage of the patching problem. We map the equivalence class of $(P_k)_k$ to the equivalence class $\mathcal{G}$. Note that this is independent of the choice of $\mathcal{G}$ (cf. Remark 6). □

We can put these maps in a Mayer-Vietoris type sequence. Recall that we use the notation $G^{\mathrm{Aut}} := G \to \mathrm{Aut}(G)$.

Theorem 7.4.2 (Mayer-Vietoris for non-abelian hypercohomology (2))**.** *Assume patching holds for G-torsors and G-gerbes over $\mathcal{F}$. Then, there is an exact sequence*

$$\begin{aligned}
1 \longrightarrow \mathrm{H}^{-1}(F, G^{\mathrm{Aut}}) &\longrightarrow \textstyle\prod_{i \in I_v} \mathrm{H}^{-1}(F_i, G^{\mathrm{Aut}}) \longrightarrow \textstyle\prod_{k \in I_e} \mathrm{H}^{-1}(F_k, G^{\mathrm{Aut}}) \\
\hookrightarrow \mathrm{H}^{0}(F, G^{\mathrm{Aut}}) &\longrightarrow \textstyle\prod_{i \in I_v} \mathrm{H}^{0}(F_i, G^{\mathrm{Aut}}) \longrightarrow \textstyle\prod_{k \in I_e} \mathrm{H}^{0}(F_k, G^{\mathrm{Aut}}) \\
\hookrightarrow \mathrm{H}^{1}(F, G^{\mathrm{Aut}}) &\longrightarrow \textstyle\prod_{i \in I_v} \mathrm{H}^{1}(F_i, G^{\mathrm{Aut}}) \rightrightarrows \textstyle\prod_{k \in I_e} \mathrm{H}^{1}(F_k, G^{\mathrm{Aut}})
\end{aligned}$$

Proof. The exactness in the first two rows is the content of Theorem 6.2.5. The exactness at $\prod_{i \in I_v} \mathrm{H}^1(F_i, G^{\mathrm{Aut}})$ follows from gerbe patching (Theorem 7.3.6). The exactness at $\mathrm{H}^1(F, G^{\mathrm{Aut}})$ follows immediately from Theorem 7.1.1. □

From this exact sequence, we can deduce a necessary and sufficient criterion for the local-global principle for gerbes to hold in terms of bitorsor factorization.

Theorem 7.4.3. *Assume that gerbe patching holds for L-gerbes. Then, L-gerbes satisfy a local-global principle with respect to patches if and only G satisfies bitorsor factorization.*

Remark 7. These results are analogous to the results in [HHK15a] concerning local-global principles for G-torsors. They prove that local-global principle for $\mathrm{H}^1(F, G)$ is equivalent to factorization of $\prod_k \mathrm{H}^0(F_k, G)$. In other words, local-global principle for G-torsors is equivalent to G satisfying factorization.

Note that when G is abelian, our result recovers the beginning of the Mayer-Vietoris sequence described in Section 5.5.

Chapter 8

Arithmetic Curves

The goal of this chapter is to apply the results on bitorsor patching and gerbe patching to the patching setup described in Section 5.2.2. We will also discuss bitorsor factorization over arithmetic curves and use these results to prove local-global principles for certain gerbes.

Throughout this section, let F be the function field of an arithmetic curve and let $\mathcal{F}$ be a finite inverse factorization system as described in Section 5.2.2.

8.1 Bitorsors Patching

Let G, H be linear algebraic groups over F.

Recall the natural functor

$$\beta''_{(G,H)} \colon \mathrm{BIT}(G,H)(F) \to \mathrm{BPP}(G,H)(\mathcal{F}).$$

introduced in Chapter 6.1.3.

Theorem 8.1.1. *Let G, H be linear algebraic groups over F. Then, patching holds for (G,H)-bitorsors over $\mathcal{F}$, i.e. the functor $\beta''_{(G,H)}$ is an equivalence*

Proof. Follows from Theorem 5.3.2 and Theorem 6.2.2. □

As a corollary, we immediately obtain a criterion for when G-bitorsors satisfy a local-global principle.

Corollary 8.1.2. *The local-global principle for G-bitorsors holds over $\mathcal{F}$ iff $Z(G)$ satisfies factorization over $\mathcal{F}$.*

Proof. Follows from Corollary 6.2.6 and Theorem 8.1.1. □

As $Z(G)$ is a linear algebraic group itself, we can apply the results from Section 5.3.1 to prove the local-global principles for bitorsors whose center is rational.

Theorem 8.1.3. *Let G be a linear algebraic group whose center Z is rational. Then, G-bitorsors satisfy local-global principle over $\mathcal{F}$ iff Z is connected or Γ is a tree.*

Proof. In Theorem 5.3.6, we have seen that H satisfies factorization over $\mathcal{F}$ iff H is connected or Γ is a tree. Hence, the result follows from Corollary 8.1.2. □

8.2 Bitorsor Factorization

We will now investigate which group schemes G over F admit G-bitorsor factorization over $\mathcal{F}$. Let $\mathcal{F}$ be indexed by $I = I_v \sqcup I_e$ with associated graph Γ (cf. Chapter 5 for details). The short exact sequence $1 \to (1 \to \mathrm{Aut}(G)) \to (G \to \mathrm{Aut}(G)) \to (G \to 1) \to 1$ of crossed modules induces the long exact sequence

$$\begin{array}{c} 1 \longrightarrow \mathrm{H}^0(Z(G)) \longrightarrow \mathrm{H}^0(G) \longrightarrow \mathrm{H}^0(\mathrm{Aut}(G)) \longrightarrow \\ \longrightarrow \mathrm{H}^0(G \to \mathrm{Aut}(G)) \longrightarrow \mathrm{H}^1(G) \longrightarrow \mathrm{H}^1(\mathrm{Aut}(G)) \longrightarrow \mathrm{H}^1(G \to \mathrm{Aut}(G)) \end{array} \tag{8.1}$$

(compare Example 3 in Chapter 4).

We will first consider the case of finite constant group schemes with trivial center.

Theorem 8.2.1. *Assume that G is a finite constant group scheme over F with trivial center and that Γ is a tree. Then, G satisfies bitorsor factorization over $\mathcal{F}$.*

Proof. Given a collection $\{P_b\}_{b\in\mathcal{B}}$ with P_b a G-bitorsor over F_b, we need to show that there are $\{P_U\}_{U\in\mathcal{U}}$ and $\{P_p\}_{p\in\mathcal{P}}$ such that P_U is a G-bitorsor over F_U, P_p is a G-bitorsors over F_P and $P_U|_{F_b} \wedge^G P_p|_{F_b} \simeq F_b$ whenever b is a branch at U and p.

By Theorem 4.2.2, this is equivalent to showing that

$$\prod_{U\in\mathcal{U}} \mathrm{H}^0(F_U, G \to \mathrm{Aut}(G)) \times \prod_{p\in\mathcal{P}} \mathrm{H}^0(F_p, G \to \mathrm{Aut}(G)) \to \prod_{b\in\mathcal{B}} \mathrm{H}^0(F_b, G \to \mathrm{Aut}(G))$$

is surjective. Since $Z(G) = \{e\}$, the sequence

$$1 \to G \to \mathrm{Aut}(G) \to \mathrm{Aut}(G)/G \to 1$$

is exact. Since G is constant, so is $\mathrm{Aut}(G)$ and the long exact sequence associated to the short exact sequence above reads

$$1 \to \mathrm{H}^1(F_b, G) \to \mathrm{H}^1(F_b, \mathrm{Aut}(G)) \to \dots$$

for any $b \in \mathcal{B}$. Since $\mathrm{H}^1(F_b, G) \to \mathrm{H}^1(F_b, \mathrm{Aut}(G)$ is injective, it follows from sequence (8.1) that $\mathrm{H}^0(F_b, \mathrm{Aut}(G)) \to \mathrm{H}^0(F_b, G \to \mathrm{Aut}(G))$ is surjective. It is thus enough to show that

$$\prod_{U\in\mathcal{U}} \mathrm{H}^0(F_U, \mathrm{Aut}(G)) \times \prod_{p\in\mathcal{P}} \mathrm{H}^0(F_p, \mathrm{Aut}(G)) \to \prod_{b\in\mathcal{B}} \mathrm{H}^0(F_b, \mathrm{Aut}(G))$$

is surjective, i.e. that $\mathrm{Aut}(G)$ satisfies factorization. Since $\mathrm{Aut}(G)$ is also a finite, constant group scheme, it satisfies factorization by assumption and Theorem 5.3.6. $\square$

Examples 6. Examples of group schemes satisfying the assumptions of theorem 8.2.1

include S_n for any $n > 2$, A_n for $n > 3$ and any finite, nonabelian simple group.

As bitorsors are classified by hypercohomology with values in $G \to \mathrm{Aut}(G)$, the properties of this morphism are important for bitorsor factorization. This is of course also closely related to the center of G and the (non-)existence of outer automorphisms. This is why it is useful to study bitorsor factorization for semisimple groups dependent on their type, as discussed in Section 3.2.

Theorem 8.2.2. *Let G be an algebraic group over F such that the natural map $G \to \mathrm{Aut}(G)$ is an isomorphism. If G satisfies factorization over $\mathcal{F}$, then G satisfies bitorsor factorization over $\mathcal{F}$.*

Proof. Using the assumption, we can see that the map $\mathrm{H}^0(F, G) = \mathrm{H}^0(F, \mathrm{Aut}(G)) \to \mathrm{H}^0(F, G \to \mathrm{Aut}(G))$ is surjective. Hence, the claim follows immediately. $\square$

Examples 7. Let G be semisimple adjoint, Then, $G \to \mathrm{Aut}(G)$ is an isomorphism if G is of type A_1, B_n, C_n, E_7, E_8, F_4, and G_2 (compare 24.A and [KMTR98, Proposition 25.15]).

Thus, if G satisfies factorization over $\mathcal{F}$, then it also satisfies bitorsor factorization.

Let now G be a semisimple group whose adjoint group admits no outer automorphism. Then, we have a short exact sequence of crossed modules (see Chapter 4 for the definition and see Corollary 25.17 in [KMTR98] for exactness):

$$1 \to (Z \to 1) \to (G \to \mathrm{Aut}(G)) \to (G/Z \to \mathrm{Aut}(G/Z)) \to 1$$

We thus obtain the following exact sequence of hypercohomology groups.

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{H}^0(F, Z) \arrow[r] & \mathrm{H}^0(F, Z) \arrow[r] & 1 \arrow[dll, rounded corners] \\
 & \mathrm{H}^1(F, Z) \arrow[r] & \mathrm{H}^0(F, G \to \mathrm{Aut}(G)) \arrow[r] & \mathrm{H}^0(F, G/Z \to \mathrm{Aut}(G/Z)) \arrow[dll, rounded corners] \\
 & \mathrm{H}^2(F, Z) \arrow[r] & \mathrm{H}^1(F, G \to \mathrm{Aut}(G)) &
\end{tikzcd} \tag{8.2}$$

Lemma 8.2.3. *Let G be a semisimple group whose adjoint group G/Z admits no outer automorphisms. Then, the map*

$$\mathrm{H}^0(F, G/Z \to \mathrm{Aut}(G/Z)) \to \mathrm{H}^2(F, Z)$$

from the exact sequence (8.2) is the zero map.

Proof. Consider the diagram

$$\begin{tikzcd}
 & \mathrm{H}^1(F, G/Z) \arrow[dr, "\delta"] & \\
\mathrm{H}^0(F, G/Z \to \mathrm{Aut}(G/Z)) \arrow[ur] \arrow[rr] & & \mathrm{H}^2(F, Z).
\end{tikzcd}$$

Note that this diagram does not commute: a cocycle $(f, \lambda) \in Z^0(F, G \to \mathrm{Aut}(G))$ maps to $\lambda(\delta(f))$ as opposed to $\delta(f)$. However, as $\delta(f) = 0$ implies $\lambda(\delta(f)) = 0$, it is enough to show that

$$\mathrm{H}^0(F, G/Z \to \mathrm{Aut}(G/Z)) \to \mathrm{H}^1(F, G/Z)$$

is the zero map.

From the exact sequence (8.1), we obtain that

$$\mathrm{H}^0(F, G/Z \to \mathrm{Aut}(G/Z)) \to \mathrm{H}^1(F, G/Z) \to \mathrm{H}^1(F, \mathrm{Aut}(G/Z))$$

is exact. But, as G/Z is isomorphic to $\mathrm{Aut}(G/Z)$ by assumption, the claim follows. $\square$

Theorem 8.2.4. *Let G be a semisimple group such that G/Z admits no outer automorphisms. If G/Z and Z satisfy bitorsor factorization over $\mathcal{F}$, then so does G.*

Proof. Let us start with $\alpha \in \mathrm{H}^0(F_b, G \to \mathrm{Aut}(G))$. Let β denote its image in $\mathrm{H}^1(F_b, G/Z \to \mathrm{Aut}(G/Z))$. By assumption, there are $\beta_U \in \mathrm{H}^0(F_U, G/Z \to \mathrm{Aut}(G/Z))$ and $\beta_p \in \mathrm{H}^0(F_p, G/Z \to \mathrm{Aut}(G/Z))$ such that $\beta_U\beta_p = \beta_b \in \mathrm{H}^0(F_b, G/Z \to \mathrm{Aut}(G/Z))$. By Sequence (8.2) and Lemma 8.2.3, we can lift β_U and β_P to elements $\tilde{\alpha}_U$ and $\tilde{\alpha}_p$ in $\mathrm{H}^0(F_U, G \to \mathrm{Aut}(G))$ and $\mathrm{H}^0(F_p, G \to \mathrm{Aut}(G))$ respectively. Consider the element $\tilde{\alpha} = \tilde{\alpha}_U^{-1}\alpha\tilde{\alpha}_p^{-1} \in \mathrm{H}^0(F_b, G \to \mathrm{Aut}(G))$. By construction, $\tilde{\alpha}$ comes from an element $\gamma \in \mathrm{H}^1(F_b, Z)$. By assumption, there are $\gamma_U \in \mathrm{H}^1(F_U, Z)$ and $\gamma_p \in \mathrm{H}^1(F_p, Z)$ satisfying $\gamma_U\gamma_p = \gamma \in \mathrm{H}^1(F_b, Z)$. Consider the elements $\alpha_U = \gamma_U\tilde{\alpha}_u \in \mathrm{H}^0(F_U, G \to \mathrm{Aut}(G))$ and $\alpha_p = \gamma_p\tilde{\alpha}_p \in \mathrm{H}^0(F_p, G \to \mathrm{Aut}(G))$. Then, $\alpha = \alpha_U\alpha_p$. This follows from

$$
\begin{aligned}
\alpha_U^{-1}\alpha\alpha_p^{-1} &= \gamma_U^{-1}\tilde{\alpha}_U^{-1}\alpha\tilde{\alpha}_p^{-1}\gamma_p^{-1} \\
&= \gamma_U^{-1}\gamma\gamma_p^{-1} \\
&= 1.
\end{aligned}
$$

$\square$

We will now turn our attention to groups of type A_n for $n > 1$. For this, let D be a central simple algebra over F. Our goal is to prove bitorsor factorization for $\mathrm{SL}_1(D)$-bitorsors. We need two auxiliary results.

Lemma 8.2.5. *The group $\mathrm{Aut}(\mathrm{SL}_1(D))$ satisfies factorization over $\mathcal{F}$ if Γ is a tree.*

Proof. Note that we have a short exact sequence

$$1 \to \mathrm{PGL}_1(D)(K) \to \mathrm{Aut}(\mathrm{SL}_1(D))(K) \to \mathrm{Z}/2\,\mathrm{Z} \to 1$$

for any field K/F. The result now follows from noting that $\mathrm{PGL}_1(D)$ and $\mathrm{Z}/2\,\mathrm{Z}$ satisfy factorization over $\mathcal{F}$ as they are both rational and Γ is a tree (see Theorem 5.3.6). $\square$

Proposition 8.2.6. *Let D be a central simple algebra over K. Then, the map of pointed sets*

$$\mathrm{H}^1(K, \mathrm{SL}_1(D) \to \mathrm{H}^1(K, \mathrm{Aut}(\mathrm{SL}_1(D)))$$

has trivial image. In particular, the map $\mathrm{H}^0(\mathrm{SL}_1(D) \to \mathrm{Aut}(\mathrm{SL}_1(D))) \to \mathrm{H}^1(\mathrm{SL}_1(D))$ *is surjective.*

Proof. Note that a piece of sequence (8.1) for $G = \mathrm{SL}_1(D)$ reads

$$\mathrm{H}^0(\mathrm{Aut}(\mathrm{SL}_1(D)) \to \mathrm{H}^0(\mathrm{SL}_1(D) \to \mathrm{Aut}(\mathrm{SL}_1(D))) \to \mathrm{H}^1(\mathrm{SL}_1(D)) \to \mathrm{H}^1(\mathrm{Aut}(\mathrm{SL}_1(D))). \tag{8.3}$$

Thus, injectivity of

$$\mathrm{H}^1(K, \mathrm{SL}_1(D) \to \mathrm{H}^1(K, \mathrm{Aut}(\mathrm{SL}_1(D)))$$

implies surjectivity of

$$\mathrm{H}^0(\mathrm{SL}_1(D) \to \mathrm{Aut}(\mathrm{SL}_1(D))) \to \mathrm{H}^1(\mathrm{SL}_1(D)).$$

Note that the map

$$\mathrm{SL}_1(D) \to \mathrm{Aut}(\mathrm{SL}_1(D))$$

factors through

$$\mathrm{SL}_1(D) \to \mathrm{GL}_1(D).$$

Hence,

$$\mathrm{H}^1(K, \mathrm{SL}_1(D) \to \mathrm{H}^1(K, \mathrm{Aut}(\mathrm{SL}_1(D)))$$

factors through

$$\mathrm{H}^1(K, \mathrm{SL}_1(D) \to \mathrm{H}^1(K, \mathrm{GL}_1(D)).$$

The claim thus follows from $\mathrm{H}^1(K, \mathrm{GL}_1(D)) = 0$, i.e. Hilbert 90. □

Theorem 8.2.7. *Let D be a a central simple algebra over F and let Γ be a tree. Then, $\mathrm{SL}_1(D)$ satisfies bitorsor factorization.*

Proof. Fix a collection of

$$\alpha_b \in \mathrm{H}^0(F_b, \mathrm{SL}_1(D) \to \mathrm{Aut}(\mathrm{SL}_1(D)))$$

for $b \in \mathcal{B}$ and let β_b denote their images in $\mathrm{H}^1(F_b, \mathrm{SL}_1(D))$ (along sequence 8.3) when b corresponds to a branch at p. Recall that

$$\mathrm{H}^1(\mathcal{F}_b, \mathrm{SL}_1(D)) = F_b^* / \operatorname{Nrd}(D_b)$$

holds. Let n be the index of D. Since F_b is a completion of F_p at a discrete valuation, the map $F_p^* \to F_b^*/F_b^{*n}$ is surjective. As $F_n^{*n} \subset \operatorname{Nrd}(D_b^*)$, it follows that $F_p^* \to F_b^*/\operatorname{Nrd}(D_b^*)$ is surjective whenever b is a branch at p. Thus, by weak approximation, there are $\beta_p \in \mathrm{H}^1(F_p, \mathrm{SL}_1(D))$ for all $p \in \mathcal{P}$ such that $\beta_p = \beta_b \in \mathrm{H}^1(\mathcal{F}_b, \mathrm{SL}_1(D))$ whenever b is a branch at p.

By Proposition 8.2.6, there are $\alpha_p \in \mathrm{H}^0(F_p, \mathrm{SL}_1(D) \to \mathrm{Aut}(\mathrm{SL}_1(D)))$ mapping onto β_p for all $p \in \mathcal{P}$. Let now b be a branch at p and U. Then, $\alpha_b\alpha_p^{-1}$ maps onto 0 in $\mathrm{H}^1(F_b, \mathrm{SL}_1(D))$ by construction. Hence, there is $\nu_b \in \mathrm{H}^0(F_b, \mathrm{Aut}(\mathrm{SL}_1(D_b)))$ mapping onto $\alpha_b\alpha_p^{-1}$. By Lemma 8.2.5, there exist $\nu_U \in \mathrm{H}^0(F_U, \mathrm{Aut}(\mathrm{SL}_1(D)))$ and $\nu_p \in \mathrm{H}^0(F_p, \mathrm{Aut}(\mathrm{SL}_1(D)))$ such that their product in $\mathrm{H}^0(F_b, \mathrm{Aut}(\mathrm{SL}_1(D)))$ is ν_b for all $(U, p, b) \in S_I$.

Let τ_U and τ_p denote the images of ν_U and ν_p in $\mathrm{H}^0(F_U, \mathrm{SL}_1(D) \to \mathrm{Aut}(\mathrm{SL}_1(D))))$ and $\mathrm{H}^0(F_p, \mathrm{SL}_1(D) \to \mathrm{Aut}(\mathrm{SL}_1(D)))$ respectively. Then, the collection of $\{\tau_U\}_{U\in\mathcal{U}}$ and $\{\tau_p\alpha_p\}_{p\in\mathcal{P}}$ give a factorization of $\{\alpha_b\}_{b\in\mathcal{B}}$. □

8.3 Gerbe Patching and Mayer-Vietoris

We will now investigate, when gerbe patching holds over arithmetic curves. Let us first collect some results related to the technical assumption in Theorem 7.3.6.

Theorem 8.3.1. *Assume that* $\mathrm{char}(K) = 0 = \mathrm{char}(k)$. *Let* $(L_i)_{i\in I}$ *be a collection of finite separable field extensions* L_i/F_i. *Then, there is a finite separable field extension* E/F *such that* E_i *dominates* L_i.

Proof. This follows from Propositions 2.4-2.5 and Theorem 2.6 in [HHK+17]. □

Corollary 8.3.2. *Assume that* $\mathrm{char}(k) = 0$. *Then, every* L*-gerbe patching problem*

$$(\{\mathcal{G}_i\}_{i\in I_v}, \{\nu_k\}_{k\in I_e})$$

over $\mathcal{F}$ *has property D.*

Proof. Pick covers $Z_i \to F_i$ for $i \in I_v$ and $z_i \in \mathcal{G}_i(Z_i)$. Note that Z_i is a product of finite separable field extensions of F_i. By Theorem 8.3.1, there is a cover $Z' \to F$ such that Z'_i dominates Z.

While $\nu_k(x_i|_{Z'_k})$ and $x_j|_{Z'_k}$ may not be isomorphic in $\mathcal{G}_j(Z'_k)$, they are locally isomorphic, so there are covers $Y_k \to Z'_k$ such $\nu_k(x_i|_{Y_k})$ and $x_j|_{Y_k}$ are isomorphic. Again using Theorem 8.3.1, there is a cover $Y \to F$ dominating $Z \to F$ such that $Y'_k \to F$ dominates $Y_k \to F_k$.

Then, the choice $Y' \to F$ and $x_i|_{Y'_i}$ proves the claim. □

In the case where $\mathrm{char}(k) = p$, we want to use Proposition 7.3.7 to prove that every gerbe patching problem has property D. The next proposition proves that this is true under some mild assumptions.

Proposition 8.3.3. *Let L be a band over F such that $Z(L)$ is a linear algebraic group over F with finite order not divided by $\mathrm{char}(k)$. Then, if $\mathrm{H}^2(F, L) \neq \emptyset$, patching holds for $\mathrm{H}^2(\circ, L)$ over $\mathcal{F}$.*

Proof. By [HHK14, Theorem 3.1.3], patching holds for $\mathrm{H}^2(\circ, Z(L))$. Thus, the conclusion follows by Proposition 7.1.3. $\square$

We are now ready to state our main result on gerbe patching over arithmetic curves.

Theorem 8.3.4. *Let $L = (Y, G, \psi)$ be a band over F with G being a linear algebraic group over Y. Assume either*

- $\mathrm{char}(k) = 0$ *or*
- $\mathrm{char}(k) = p > 0$ *and $Z(L)$ has finite order not divisible by $\mathrm{char}(k)$ and G-bitorsor factorization holds over $\mathcal{F}_Z$ for every cover $Z \to Y$.*

Then, gerbe patching holds over $\mathcal{F}$, i.e. the functor β_L''' is a 2-equivalence.

Proof. If $\mathrm{char}(k) = 0$, this follows from Theorem 5.3.2, Theorem 7.3.6 and Theorem 8.3.2. If $\mathrm{char}(k) = p$, then it follows from Theorem 5.3.2, Theorem 7.3.6 and Proposition 8.3.3. $\square$

Corollary 8.3.5 (Mayer-Vietoris of non-abelian hypercohomology over curves)**.** *Let G be a linear algebraic group defined over F and let L denote the associated band. Under the assumption of Theorem 8.3.4, there is an exact sequence of pointed sets*

$$
\begin{aligned}
1 \longrightarrow \mathrm{H}^{-1}(F, G^{\mathrm{Aut}}) \longrightarrow \textstyle\prod_{i \in I_v} \mathrm{H}^{-1}(F_i, G^{\mathrm{Aut}}) \longrightarrow \textstyle\prod_{k \in I_e} \mathrm{H}^{-1}(F_k, G^{\mathrm{Aut}}) \\
\hookrightarrow \mathrm{H}^{0}(F, G^{\mathrm{Aut}}) \longrightarrow \textstyle\prod_{i \in I_v} \mathrm{H}^{0}(F_i, G^{\mathrm{Aut}}) \longrightarrow \textstyle\prod_{k \in I_e} \mathrm{H}^{0}(F_k, G^{\mathrm{Aut}}) \\
\hookrightarrow \mathrm{H}^{1}(F, G^{\mathrm{Aut}}) \longrightarrow \textstyle\prod_{i \in I_v} \mathrm{H}^{1}(F_i, G^{\mathrm{Aut}}) \rightrightarrows \textstyle\prod_{k \in I_e} \mathrm{H}^{1}(F_k, G^{\mathrm{Aut}})
\end{aligned}
$$

Proof. Follows from Theorem 7.4.2 and Theorem 8.3.4. □

8.4 Local-global principles for gerbes

Building on our results on bitorsor factorization, we now use Theorem 7.4.3 to obtain local-global principles for gerbes.

Theorem 8.4.1. *Let G be a linear algebraic group over F with center Z. Then, the local global principle for G-gerbes with respect to patching holds if*

- $\mathrm{char}(k) = 0$ *and one of the following hold:*
  - Γ *is a tree and G is a finite constant group scheme with trivial center,*
  - G *is connected, rational, semisimple, adjoint of type $A_1, B_n, C_n, E_7, E_8, F_4$ or G_4,*
  - G *is semisimple such that G/Z admits no outer automorphism and $Z, G/Z$ satisfy bitorsor factorization,*
  - Γ *is a tree and $G = \mathrm{SL}_1(D)$ where D is a central simple algebra over F,*
- $\mathrm{char}(k) = p > 0$, *Z has finite order not divided by* $\mathrm{char}(k)$ *and G and Γ are as in the case of* $\mathrm{char}(k) = 0$.

Proof. All results use Theorem 7.3.6 and Theorem 7.4.3. The results follow (in order) from Theorem 8.2.1, Example 7 and Theorem 5.3.6, Theorem 8.2.4, and Theorem 8.2.7. □

Chapter 9

Homogeneous Spaces

In this section, we will discuss local-global principles for the existence of rational points on homogeneous spaces over arithmetic curves. Throughout this chapter, let F denote the function field of an arithmetic curve and let $\mathcal{F}$ denote an inverse factorization system as discussed in Section 5.2.2. The results derived here will rely on the local-global principles derived for gerbes in the last chapter.

Let H be a linear algebraic group and let X be a variety over F admitting an H action. We say that H is a homogeneous space if $H(F^{\text{sep}})$ acts transitively on $X(F^{\text{sep}})$.

Definition 31. We say that the local-global principle holds for X over $\mathcal{F}$ if $X(F) \neq \emptyset$ if and only if $X(F_i) \neq \emptyset$ for all $i \in I_v$.

We will link the existence of rational points on X to rational points on a moduli stack. To do this, let us note that if X admits a rational point $p \in X(F)$, then this point defines an H-equivariant surjection

$$H \to X$$

where we think of H as the trivial H-torsor.

Based on this observation, we say that X admits a torsor if there is an H-equivariant morphism

$$P \to X$$

for an H-torsor P (note that this is necessarily a surjection by transitivity). By our observation, admitting a torsor is necessary for the existence of a rational point. We will thus divide our study of local-global principles for rational points into two steps:

1. Study local-global principles for admitting a torsors.

2. Study when admitting a torsor is sufficient for admitting a rational point.

This plan of action is not novel, it was pioneered in the case where F is a number field (and local-global is understood with respect to valuations) by Springer and Borovoi, compare [Spr65], [Bor92b] and [Bor95]. It was also used by Flicker, Scheiderer and Sujatha to prove a formally real analogue of Grothendieck's Theorem regarding the second cohomology group, see [FSS98].

In order to study the first point, we will turn to the moduli stack of torsors over H. This moduli stack will turn out to be a gerbe and is thus amenable to the tools developed in Section 7.4.

We will assume throughout this chapter that the stabilizer of a geometric point $x \in X(F^{\mathrm{sep}})$ is isomorphic to $G|_{F^{\mathrm{sep}}}$ for some linear algebraic group $G \subset H$ defined over F.

The moduli stack of torsors over X is the well-known quotient stack $[X/H]$. Recall that objects of this stack are given by tuples (Y, P, ϕ) where Y is an F-scheme, P is a torsor under $H|_Y$ over Y and $\phi\colon P \to X|_Y$ is an H-equivariant map. By étale descent for torsors, this category is a stack.

Let Y be an F-scheme. If $X(Y) \neq \emptyset$, pick a point $x \in X(Y)$. This defines an H-equivariant morphism $\phi_x\colon H|_Y \to X|_Y$ which gives an element $(Y, H|_Y, \phi_x) \in$

$[X/H](Y)$. Hence, $X(Y) \neq \emptyset$ implies$[X/H](Y) \neq \emptyset$. In particular, $[X/H]$ locally admits points in the étale topology.

Given two elements (Y, P, ϕ) and (Y, P', ϕ') in $[X/H](Y)$, let $U \to Y$ be an étale cover such that P and P' are both trivial on U. Fixing points $p \in P(U)$ and $p' \in P'(U)$ on U, the morphisms $\phi \colon P \to X$ can be identified with a morphism $H \to H/G$ and likewise for ϕ'. Let $\phi(U)(p) = [h]$ and $\phi(U)(p') = [h']$. Consider the isomorphism of $H|_U$ torsors $H|_U \to H|_U$ given by left multiplication by $h'h^{-1}$. This morphism yields a morphism $P|_U \to P|'_U$ that is compatible with ϕ and ϕ' by construction. Hence, (Y, P, ϕ) and (Y, P', ϕ') are locally isomorphic.

Therefore, $[X/H]$ is a gerbe. In general the band of this gerbe is induced by the stabilizers of geometric point. As we assume that these are induced by a F-group G, it is easy to check that $[X/H]$ is in fact a G-gerbe.

The stack $[X/H]$ is a G-gerbe and it is trivial if and only if there is a H-equivariant map $P \to X$ where P is principal homogeneous space under H over F. In particular, if X admits a rational point, then so does $[X/H]$. The question whether the converse is true is generally delicate and ties to the second point of our outline. Clearly, it would be enough to prove that a torsor lying over X admits a rational point. So, if $\mathrm{H}^1(F, H)$ is trivial, i.e. if every H-torsor admits a point, then the converse holds. This seemingly trivial case was actually successfully used by Borovoi in the number field case as this assumption could be made after several reductions ([Bor92b]).

One class of groups for which this is true are special groups. Recall that that H is special (i.e. $\mathrm{H}^1(K, H) = \{e\}$ for all field extensions K/F). Examples for special groups are given by GL_n and SL_n, as seen in Section 3.5.2. This leads us to our first theorem on homogeneous spaces.

Theorem 9.0.2. *Let H be special and let X be a homogeneous space under H. Let $G \subset H$ be a linear algebraic group over F such that $G|_{F^{\mathrm{sep}}}$ is isomorphic to the stabilizer of a geometric point of X. If* $\mathrm{char}(k) = 0$*, the local-global principle for X*

holds if and only if bitorsor factorization holds for G.

If $\mathrm{char}(k) = p > 0$*, assume that*

- $p \nmid |Z(G)| < \infty$*,*
- *G-bitorsor factorization holds over $\mathcal{F}_Z$ for every cover $Z \to F$.*

Then, the local-global principle holds for X over $\mathcal{F}$.

Proof. Since H is special, we have $X(F) \neq \emptyset$ iff $[X/H](F) \neq \emptyset$ as well as $X(F_i) \neq \emptyset$ iff $[X/H](F_i) \neq \emptyset$ by the discussion above. Hence, the local-global principle for X holds iff it holds for $[X/H]$. The result follows from Theorem 7.4.3 and Theorem 8.3.4. $\square$

We can prove a similar result when G is special.

Theorem 9.0.3. *Let H be linear algebraic group and let X be a homogeneous space under H. Let $G \subset H$ be a special linear algebraic group over F such that $G|_{F^{\mathrm{sep}}}$ is isomorphic to the stabilizer of a geometric point of X. If* $\mathrm{char}(k) = 0$*, the local-global principle for X holds if bitorsor factorization holds for G and factorization holds for H. If* $\mathrm{char}(k) = p > 0$*, assume that*

- $p \nmid |Z(G)| < \infty$*,*
- *G-bitorsor factorization holds over $\mathcal{F}_Z$ for every cover $Z \to F$,*
- *H satisfies factorization.*

Then, the local-global principle holds for X over $\mathcal{F}$.

Proof. If $X(F_i) \neq \emptyset$ for all $i \in I_v$, then $[X/H](F_i) \neq \emptyset$. By the local-global principle for gerbes (Theorem 8.4.1) and the assumptions, we can conclude that $[X/H](F) \neq \emptyset$. Hence, there is an H-equivariant map $P \to X$, where P is a H-torsor over F.

The restriction of $P \to X$ to F_i induces an element $\alpha \in [X/H](F_i)$. As $X(F_i) \neq \emptyset$, there is also $\beta \in [X/H](F_i)$ corresponding to a morphism $H|_{F_i} \to X|_{F_i}$. As $[X/H]$ is a G-gerbe, $\mathrm{Isom}(\alpha, \beta)$ is a $G|_{F_i}$-torsors over F_i. As G is special, the torsor must admit a F_i-rational point. In particular, $P|_i \simeq H|_{F_i}$ and thus P is locally trivial (over $\mathcal{F}$). As H satisfies factorization, local-global principle for H-torsors holds over $\mathcal{F}$ by Theorem 5.3.5. Thus, P and hence X admit an F-rational point. □

Combining this with the results on gerbe factorization, we obtain the following result.

Corollary 9.0.4. *Let H be a linear algebraic group let X be a homogeneous space under H. Let $G \subset H$ be a linear algebraic group over F such that $G|_{F^{\mathrm{sep}}}$ is isomorphic to the stabilizer of a geometric point of X. Assume that either*

- *H is special or*
- *G is special and H satisfies factorization.*

Then,

- *If* $\mathrm{char}(k) = 0$, *assume that one of the following holds:*
  - *Γ is a tree and G is a finite constant group scheme with trivial center,*
  - *G is connected, rational, semisimple, adjoint of type $A_1, B_n, C_n, E_7, E_8, F_4$ or G_4,*
  - *G is semisimple such that G/Z admits no outer automorphism and $Z, G/Z$ satisfy bitorsor factorization,*
  - *Γ is a tree and $G = \mathrm{SL}_1(D)$ where D is a central simple algebra over F,*
- *If* $\mathrm{char}(k) = p > 0$, *assume that $Z(G)$ has finite order not divided by* $\mathrm{char}(k)$ *and G and Γ are as in the case of* $\mathrm{char}(k) = 0$.

Then, the local-global principle holds for X over $\mathcal{F}$.

Proof. Follows from Theorem 8.4.1 and Theorem 9.0.2 and Theorem 9.0.3. $\square$

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