Distribution Agreement

Lingxin Cheng April 13, 2021

Local-to-Global Property of Transitive Subgroups of Sp

by

Lingxin Cheng

David Zureick-Brown
Adviser

Department of Mathematics

David Zureick-Brown

Adviser

Raman Parimala

Committee Member

Effrosyni Seitaridou

Committee Member

2021

Local-to-Global Property of Transitive Subgroups of Sp

By

Lingxin Cheng

David Zureick-Brown

Adviser

An abstract of
a thesis submitted to the Faculty of Emory College of Arts and Sciences
of Emory University in partial fulfillment
of the requirements of the degree of
Bachelor of Science with Honors

Department of Mathematics

2021

Abstract

Local-to-Global Property of Transitive Subgroups of Sp
By Lingxin Cheng

This paper analyzed the local-to-global property of symmetric groups.

Local-to-Global Principle of Transitive Subgroups of Sp

By

Lingxin Cheng

David Zureick-Brown

Adviser

A thesis submitted to the Faculty of Emory College of Arts and Sciences
of Emory University in partial fulfillment
of the requirements of the degree of
Bachelor of Science with Honors

Department of Mathematics

2021

Acknowledgements

I would like to thank my advisor, Dr. David Zureick-Brown, who guided throughout my honor research. He gave my lots of useful suggestions and helped me with the code that I am not familiar with. I would also like to thank Dr. Parimala and Dr. Seitaridou for coming to my thesis defense .

Table of Contents

1. Introduction

The local-to-global question arises from the Galois representation of elliptic curves. Consider the Galois group from to joint with n-torsion points on elliptic curve, then

$$Gal(Q(E[n])/Q) \subseteq GL_2(n)$$

Lets Denote $H(n) := Gal(Q(E[n])/Q)$, B to be the matrix group $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$.

If for any $h \in H(n)$, $\exists g \in GL2(n), b \in B$ s.t. $ghg^{-1} = b$, then there exists some $g \in GL2(n)$ s.t. $gHg^{-1} \subseteq B$.

The local-to-global problem wants to generate the proposition to general group: Let H,B be two subgroups of G. Suppose every element of H is g-conjugate to an element of B, then is H itself g-conjugate to a subgroup of B?

In this paper, I will mainly focus on dihedral groups and symmetric groups with prime degree, i.e. the cases where $G = D_{2n}$ or S_p.

2. Dihedral Groups

Let G be a dihedral group, and let H, B be two subgroups of G.

We say the conjugation of H to B is locally true if for any element $h \in H$, we can find an element $g \in G$, such that $ghg^{-1} \in B$.

We say the conjugation of H to B is globally true if there exists an element g in G such that $gHg^{-1} \leq B$.

If we have locally true implies globally true, then we say the local-to-global property is true.

In this section We got that if G is dihedral group, then Local to Global is true.

Lemma 2.1. *If D_{2n}'s subgroup H has odd order, then H's conjugacy class is itself.*

Proof. If H has odd order a, by Lemma, H is of the form $< r^{\frac{2n}{a}} >$.

If the g we choose from D_{2n} is of the form r^j, with $0 \leq j \leq n-1$, then for any element $h = r^{\frac{2kn}{a}}$ in H, $ghg^{-1} = h$, therefore $gHg^{-1} = H$.

If the g we choose from D_{2n} is of the form sr^j, with $0 \leq j \leq n-1$, then for any element $h = r^{\frac{2kn}{a}}$ in H, $ghg^{-1} = h^{-1}$, therefore we also have $gHg^{-1} = H$. □

Lemma 2.2. *If D_{2n}'s non-cyclic subgroup H has even order, denoted b, and it's index $\frac{2n}{b}$ is odd, then it has $\frac{2n}{b}$ conjugacy groups. The conjugacy groups are all of the form $< r^{\frac{2n}{b}}, sr^j >$, with $0 \leq j \leq \frac{2n}{b} - 1$.*

Proof. If D_{2n}'s non-cyclic subgroup H has even order, denoted b, and it's index $\frac{2n}{b}$ is even, then given $H = < r^{\frac{2n}{b}}, sr^j >$, it conjugates with all the groups of the form $< r^{\frac{2n}{b}}, sr^{j\prime} >$, with $0 \leq j\prime \leq \frac{2n}{b} - 1$ and $j \equiv j\prime \pmod 2$.

If D_{2n}'s cyclic subgroup H is of the form $< sr^j >$, then its conjugacy classes are all the groups of the form $< sr^{\pm j - 2m} >$. □

Lemma 2.3. *Given non-cyclic subgroup H of D_{2n} has odd index b, then H has global property to a subgroup B of D_{2n} if and only if there exists an integer m, such that $m|H| = |B|$.*

Proof. Let $P = gHg^{-1}$ for any $g \in D_{2n}$, we have $|H| = |P|$. Therefore if the global property hold for H to some B, we have $P \leq B$, implies that there exists an integer m, such that $m|H| = |B|$.

Conversely, let B be a subgroup of D_{2n} such that $m|H| = |B|$ for some integer m. Then H is of the form $< r^{mk}, sr^j >$, where $k = \frac{2n}{mb}$ and $0 \leq j \leq mk-1$, and B is of the form $< r^k, sr^{j\prime} >$, where $0 \leq j \leq k-1$.

Let $Q =< r^{mk} >$, no matter how we choose $g \in D_{2n}$, we have $gQg^{-1} = Q \leq B$. For the set $M = H\ Q$, we have elements of $gMg^{-1} = N$ are all of the form sr^{amk-2j} with $0 \leq a \leq b$, or all of the form sr^{amk+2j} with $0 \leq a \leq b$. For any a, we have $amk - 2j \equiv -2j$ (mod mk) and therefore $amk - 2j \equiv -2j$ (mod k). There fore we can find one $B\prime$, such that all the elements of N is in $B\prime$. More specifically, the $B\prime$ is the one whose the $j\prime$ of the $sr^{j\prime}$ is congruent to $-2j$. Therefore we get a $B\prime$ such that $gHg^{-1} = g(Q \cup M)g^{-1} = gQg^{-1} \cup gMg^{-1} = Q \cup N \leq B\prime$. Given that H is non-cyclic and has odd index, we know from the Lemma that H conjugates to all the group of the same form as it, that is, $< r^{mk}, sr^j >$, with $k = \frac{2n}{mb}$ and $0 \leq j \leq mk-1$.

Therefore, given any B_i of the form $< r^k, sr^{j_i\prime} >$, where $0 \leq j_i\prime \leq k-1$. We can find a subgroup H_i of B_i, which is of the form $< r^{mk}, sr^{j_i\prime} >$, which is of the same order of the corresponding H of this type B. Therefore for all this type H, we are able to find a $g \in D_{2n}$, such that H_i conjugate to it. Therefore all this type H has global property to the corresponding B. $\square$

Lemma 2.4. *Given non-cyclic subgroup H of D_{2n} has even index b and given that $< r^{\frac{2n}{b}}, sr^j >$, then if a subgroup B of D_{2n} of even index, B is the conjugate of H if B is* *of the form* $< r^{\frac{2n}{mb}}, sr^{j'} >$ *for some integer* m *with* $j' \equiv j \pmod 2$ *and* $0 \leq j' \leq \frac{2n}{bm} - 1$.

Proof. Similar to previous proof. □

Theorem 2.1. *There are no subgroup H of D_{2n} that only has local property but does not have global property.*

Proof. Combine all the previous lemma, we can easily get this theorem.

□

3. Symmetric Group

Let G be a symmetric group, and let H, B be two subgroups of G.

We say the conjugation of H to B is locally true if for any element $h \in H$, we can find an element $g \in G$, such that $ghg^{-1} \in B$.

We say the conjugation of H to B is globally true if there exists an element g in G such that $gHg^{-1} \leq B$.

Theorem 3.1. *Every transitive permutation group of prime degree p must be one of the following* [J.D96].

(i) S_p or A_p

(ii) subgroup of $AGL_1(p)$.

(iii) a permutation representation of $PSL_2(11)$ of degree 11.

(iv) one of Mathieu groups M_{11} or M_{23} of degree 11 or 23.

(v) a projective group G with $PSL_d(q) \leq G \leq P\Gamma L_d(q)$, where $\frac{q^d-1}{q-1} = p$.

Based on this Theorem we can sketch how to solve the problem. We can just discuss the combination of (i) to (v) case by case.

H,B	(i)	(ii)	(iii)	(iv)	(v)
(i)					
(ii)					
(iii)					
(iv)					
(v)					

Lemma 3.1. *If B is S_p, then Local-to-Global holds for all H.*

Proof. Obvious. □

Lemma 3.2. *Let G be a transitive permutation group of prime degree p, then the following is equivalent* [J.D96]

(i) H is solvable

(ii) H has a normal Sylow p-subgroup

(iii) H has permutation isomorphic to a subgroup of affine group $AGL_1(p)$

Lemma 3.3. *if H and B are both solvable transitive permutation subgroup of Sp, then the Local-Global property is hold.*

Proof. $AGL(1,p) = C_p \rtimes C_{p-1}$. Therefore, Let a be a p-cycle, b be a $p-1$ cycle, then any subgroup of $AGL(1,p)$ will be one of the following form: $< a >$, $< b^k >$,$< a, b^k >$, with $k|p-1$.Since we are talking about transitive groups, therefore we only need to consider $< a >$ and $< a, b^k >$. Given $H =< a, b^{k_1} >$ and $B =< a, b^{k_2} >$, if $k_2|k_1$, then H is subgroup of B, L-G holds. If $k_2 \nmid k_1$, then local definitely fail, global also fail because the size of H can't divide B. And the case for $H =< a, b^k >$ and $B =< a >$, both local and global fail, therefore L-G holds. If $B =< a, b^k >$ and $H =< a >$, then H subgroup of B, therefore L-G also holds. □

Lemma 3.4. *If H is subgroup of $AGL_1(p)$, and B is A_p, then Local-to-Global holds.*

Proof. We inherit the same notation in the previous lemma for $AGL(1, p)$, then we have a is even permutation, b is odd permutation. If H is $< a >$, then L-G definitely holds. If $H =< a, b^k >$ with k even. Then both a and b^k are even permutations, therefore L-G holds. If $H =< a, b^k >$ with k odd. Then b^k is odd permutation. Therefore both local and global fail, therefore L-G holds. □

Lemma 3.5. *If H is a Mathieu group M_{11}, then Local-to-Global holds.*

Proof. By Magma Computation. □

Lemma 3.6. *If H is a Mathieu group M_{23}, then Local-to-Global holds.*

Proof. By Magma Computation.

□

Lemma 3.7. *If H is a permutation representation of $PSL_2(11)$ of degree 11, and B is a transitive subgroup of S_p, then Local-to-Global holds.*

Proof. By Magma Computation. □

Theorem 3.2. *If both H and B belongs to the (i) to (iv) category, then L-G holds.*

Proof. Combine those previous lemma. □

Lemma 3.8. *If H is a projective group with $PSL_d(q) \leq H \leq P\Gamma L_d(q)$, and B is $AGL(1,p)$, then Global fail.*

Proof. The size of H is too big to make global true. □

Lemma 3.9. *If B is a projective group with $PSL_d(q) \leq H \leq P\Gamma L_d(q)$, and H is A_p, then Global fail.*

Proof. The size of H is too big to make global true. □

Lemma 3.10. *If H is a projective group with $PSL_d(q) \leq H \leq P\Gamma L_d(q)$, and B is M_{11} or M_{23}, then Local-to-Global holds.*

Proof. Can't find any q s.t. $\frac{q^d-1}{q-1}$ is 11 or 23. □

4. Future Work

Once we solve the cases of transitive subgroups of S_p, We can try to analyze the Local-to-Global properties for the transitive subgroups of S_{pq} and S_{p^2}. There are theorems characterize the transitive subgroups of S_{pq} and S_{p^2} in a similar way as what we are using in our paper. We can discuss the case for S_{pq} and S_{p^2}, and finally proceed to $S_n = S_{p_1^a{}_1 p_2^a{}_2 p_3^a{}_3 \ldots p_k^a{}_k}$.

References

[J.D96] B.Mortimer J.Dixon, *Permutation groups* (1996), 91–96. ↑4, 5