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Strong u-invariant and Period-Index Bounds

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Integrated M.Sc., University of Hyderabad, India, 2020

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An abstract of
A dissertation submitted to the Faculty of the
James T. Laney School of Graduate Studies of Emory University
in partial fulfilment of the requirements for the degree of
Doctor of Philosophy
in Department of Mathematics
2025

Abstract

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Let K be a field. The u-invariant of K is the maximal dimension of anisotropic quadratic forms over K. For example, the u-invariant of $\mathbb{C}$ is 1, for F a non-real global or local field the u-invariant of F is $1, 2, 4$, or 8, etc. Considerable progress has also been made, particularly in the computation of the u-invariant of function fields of p-adic curves due to Parimala and Suresh in [PS10], [PS14], and by Harbater, Hartmann, and Krashen regarding the u-invariants in the case of function fields of curves over complete discretely valued fields in [HHK09]. Over a finitely generated field extension in m variables over a p-adic field, any quadratic form in more than 2^{m+2} variables has a nontrivial zero was shown by Leep in [Lee13].

For a central simple algebra A over a field K, there are two major invariants, *viz.*, period and index. For a field K, the Brauer-l-dimension of K for a prime number l, is the smallest natural number d such that for every finite field extension L/K and every central simple L-algebra A (of period a power of l), we have that index(A) divides period$(A)^d$.

If K is a number field or a local field, then classical results from class field theory tell us that the Brauer-l-dimension of K is 1. This invariant is expected to grow under a field extension, bounded by the transcendence degree. For F a field finitely generated and of transcendence degree 2 over an algebraically closed field, de Jong in [dJ04] showed that $\mathrm{Brdim}(F) = 1$. Michael Artin conjectured in [Art06] that $\mathrm{Brdim}(k) = 1$ for every C_2 field. Some recent works in this area include that of Saltman [Sal97], Lieblich [Lie11], Harbater-Hartmann-Krashen [HHK09, HHK14] for K a complete discretely valued field, in the good characteristic case. In the bad characteristic case, for such fields K, Parimala-Suresh have given some bounds in [PS14].

In this manuscript, I will present similar bounds for the strong u-invariant and the Brauer-l-dimension of a complete non-Archimedean valued field K with residue field κ.

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Acknowledgments

I owe my deepest gratitude to my supervisor, Professor Suresh Venapally, whose support from the beginning till the end helped me develop an understanding of the subject. The encouragement at each step and guidance towards the bigger picture of the content enabled me to advance in the project. This thesis would not have been successful without his clear instructions. His way of teaching and giving crystal clear insight into concepts – all the while staying rigorous, teaching me the significance of finding elegant solutions, always checking statements with basic examples, etc. helped me a lot and will definitely prove useful in my future too. He uplifted me when necessary and motivated me to work harder, and helped me understand the values of a mentor-student relationship.

When I was contemplating my choices, his support throughout the academic years and beyond, and his patience helped me both academically and personally. This thesis will now form the foundations of my academic and professional life ahead.

I would also like to thank Professor Parimala Raman. I have learned so much from her over the past five years, and I cannot express how much I appreciate the time and energy she spent trying to explain things to me. Her passion and dedication to mathematics research are ever-inspiring, and I hope to continue learning from her. I thank Professors Jim Nagy and David Borthwick for their support over the years, for lending a listening ear to my concerns, and for their assistance with applying for jobs and preparing for interviews. I also thank Terry Ingram for her patience with answering all my administrative questions, and the Department of Mathematics at Emory for providing the atmosphere for my professional growth.

I would be remiss if I didn't thank Dr. Biswajyoti Saha, my master's thesis advisor. He helped me take the first steps into mathematics research, teaching me the value of literature review, reading and understanding results, working effectively under pressure and meeting deadlines, organising and communicating ideas effectively in written and oral presentations to groups, etc – so many things about research that

I could've never learned by myself.

I would also like to thank my collaborators – Brandon Alberts, Helen Grundman, Amanda Tucker, Taylor Dupuy, Alexander Slamen, and Sreejani Choudhary – for broadening my mathematical knowledge, for hitting and getting past research roadblocks together, and for many interesting mathematical conversations.

I have been a graduate student at Emory for five years, and I never would have made it this far without every single person in my life. Thank you all. My move to a completely new continent for doctoral studies, especially during the COVID-19 pandemic, was made easier by my lovely peers in the graduate program at Emory. Alexis Newton, Rohan Nair, Alex Dunbar, Jasmine Camero, Maxwell Auerbach, Roberto Hernandez, Benjamin Yellin, Ariana Brown, and everyone else who has been part of my PhD journey – I am so happy and grateful for the friendships we have cultivated.

I'd also like to thank my friends Hridika Roy, Poulami Seal, Aditya Sarma Phukon, Anwesha Das, Roshni Chattopadhyay, Shiv Dutt Sharma, Manjari Sahay, Ishanika Sharma, Ananya Dash, Sarthak Satpathy, and Abhijit Suprem for all the gatherings, food, movies, celebrations, and love that we've shared over the years. There are many more of you than I can mention here, so if you are not listed, please know that I value your contributions. And lastly, I thank my family for believing in me and helping me become this version of myself.

Contents

Chapter 1

Introduction

My research is in algebra and algebraic arithmetic geometry, particularly focusing on topics like Galois cohomology, Brauer group, and Berkovich theory. My methods have primarily involved the use of tools and techniques from quadratic form theory, algebraic geometry, and ideas from rigid analytic geometry in order to obtain results in algebra and field arithmetic. In my graduate work, I have been looking closely at ultrametric fields through invariants like the universal invariant or the u-invariant and the period-index bound.

Non-Archimedean or rigid-analytic geometry is an analogue of complex analytic geometry over non-Archimedean fields. Tate introduced and formalised it in the 1960s to understand elliptic curves over a p-adic field by means of a uniformisation similar to the familiar description of an elliptic curve over $\mathbb{C}$ as a quotient of the complex plane by a lattice. These spaces developed using rings of convergent power series as opposed to polynomial rings used in algebraic geometry, were much better behaved, admitting, for example, a suitable GAGA principle. It has since gained the status of a foundational tool in algebraic and arithmetic geometry; several other approaches have been found by Raynaud, Berkovich, and Huber. In recent years, it has become even more prominent with the work of Scholze and Kedlaya in p-adic Hodge theory, as well as the non-Archimedean approach to mirror symmetry proposed by Kontsevich.

An important variation, and the one that is my principal focus of study for research, is in the form of Berkovich's k-analytic spaces, developed in the late 1980s. An advantage offered by Berkovich spaces was that it became possible to work directly with the topology of the space itself, as opposed to the 'Grothendieck topology' used in Tate's rigid analytic geometry, a feat made possible by constructively adding additional points to rigid spaces. Presently, Berkovich spaces find profound uses in non-Archimedean analogues for potential theory [BR10], mirror symmetry [KS06], [Nic18], and in the case of my PhD work, to give period-index bounds for complete ultrametric fields [Man24].

My research objective is to use the strength of algebraic and geometric structures/methodologies like local-to-global principles, patching over fields, etc., to better understand complete non-Archimedean valued (ultrametric) fields. Detailed summaries of my past and current projects, as well as potential directions for future research, are included in the upcoming chapters. Chapters 5 and 6 details the work in strong u-invariant and period-index bound for complete ultrametric fields, Appendix A details work on number field counting and a "twisted" variant of Malle's conjecture, and Appendix B describes work on non-Archimedean differential algebra and the efforts to classify all the valuations of a particular differential ring.

1.1 The Initiation

Let k be a field with $\text{char}(k) \neq 2$. The *u-invariant of k*, denoted by $u(k)$, is the maximal dimension of anisotropic quadratic forms over k. We say that $u(k) = \infty$ if there exist anisotropic quadratic forms over k of arbitrarily large dimension. For example, u-invariant of an algebraic closed field K, $u(K) = 1$; $u(\mathbb{R}) = \infty$; for k a finite field, $u(k) = 2$, etc. The u-invariant is a positive integer if it is finite. A key area of research in the theory of quadratic forms is to find all the possible values this invariant can take for a fixed or varying field. For example, it has been established in the literature (see [Lam05, Chapter XI, Proposition 6.8]) that the u-invariant cannot take values 3, 5, and 7. See [Lam05, Chapter XIII, Section 6], for more open problems about this invariant. Considerable progress has also been made, particularly in the computation of the u-invariant of function fields of p-adic curves due to Parimala and Suresh in [PS10], [PS14], and by Harbater, Hartmann, and Krashen regarding the u-invariants in the case of function fields of curves over complete discretely valued fields in [HHK09].

Harbater, Hartmann, and Krashen defined the *strong u-invariant of K*, denoted by $u_s(K)$, as the smallest real number m such that, $u(E) \leq m$ for all finite field extensions E/K, and $u(E) \leq 2m$ for all finitely generated field extensions E/K of transcendence degree 1. We say that $u_s(K) = \infty$ if there exist such field extensions E of arbitrarily large u-invariant. In [HHK09, Theorem 4.10], the same authors prove that for K a complete discretely valued field, whose residue field $\tilde{K}$ has characteristic unequal to 2, $u_s(K) = 2u_s(\tilde{K})$.

Let k be a complete non-Archimedean valued field with residue field $\tilde{k}$ such that $\text{char}(\tilde{k}) \neq 2$. Let $\sqrt{|k^*|}$ denote the divisible closure of the value group $|k^*|$. In [Meh19b], Mehmeti shows that if $\dim_{\mathbb{Q}}\sqrt{|k^*|} = n$, then $u_s(k) \leq 2^{n+1}u_s(\tilde{k})$, and if $|k^*|$ is a free $\mathbb{Z}$-module with $\text{rank}_{\mathbb{Z}}|k^*| = n$, then $u_s(k) \leq 2^n u_s(\tilde{k})$. Mehmeti used field patching in the setting of Berkovich analytic geometry to prove a local-global principle, and provides applications to quadratic forms and the u-invariant. The results she obtained generalise those of [HHK09].

Our main result concerning the u-invariant of a complete non-Archimedean (ultrametric) field is the following theorem.

Theorem 1.1.1 (Theorem 5.4.1)**.** *Let k be a complete ultrametric field, with char$(\tilde{k}) \neq$ 2. Suppose that dim$_{\mathbb{Q}}(\sqrt{| k^* |}) = n$ is finite. Then $u(k) \leq 2^n u(\tilde{k})$.*

Corollary 1.1.2 (Theorem 5.4.3)**.** *Let k be a complete ultrametric field, with char$(\tilde{k}) \neq$ 2. Suppose the dim$_{\mathbb{Q}}(\sqrt{|k^*|}) = n$ is finite. Let C be a curve over k and $F = k(C)$ the function field of C. Then $u(F) \leq 2^{n+1}u_s(\tilde{k})$.*

Given a field k, recall the definitions of period and index of a central simple algebra. The *period* (or *exponent*) of a central simple k-algebra A is the order of class of A in the Brauer group of k. The *index* of A is the degree of the division algebra D_A that lies in the class of A (i.e., A is a matrix ring over D_A). The period and index always have the same prime factors, and the period always divides the index [GS17, Proposition 4.5.13].

The *period-index problem* asks whether all central simple algebras A over a given field k satisfy $\text{ind}(A) \mid \text{per}(A)^d$ for some fixed exponent d depending only on k. In the spirit of [PS14], we make the following definition. Let k be any field. For a prime l, define the *Brauer l-dimension* of k, denoted by $\text{Br}_l\text{dim}(k)$, to be the smallest integer $d \geq 0$ such that for every finite extension L of k and for every central simple algebra A over L of period a power of l, $\text{ind}(A)$ divides $\text{per}(A)^d$. The *Brauer dimension of* k, denoted by $\text{Brdim}(k)$, is defined as the maximum of the Brauer l-dimension of k as l varies over all primes. It is expected that this invariant should increase by one upon a finitely generated field extension of transcendence degree one.

Saltman proved some results in this direction, including the fact that the index divides the period squared for function fields of p-adic curves [Sal97]. Then Harbater, Hartmann, and Krashen in [HHK09, Theorem 5.5] show that for k a complete discretely valued field, its residue field $\tilde{k}$, F the function field of a curve over k, and $l \neq \text{char}(\tilde{k})$, they prove that if $\text{Br}_l\text{dim}(\tilde{k}) \leq d$ and $\text{Br}_l\text{dim}(\tilde{k}(T)) \leq d+1$, then $\text{Br}_l\text{dim}(F) \leq d+2$.

Using [Meh19b] we get the following as a direct consequence.

Theorem 1.1.3 (Theorem 6.4.1)**.** *Let k be a complete ultrametric field with $\dim_{\mathbb{Q}}(\sqrt{|k^*|}) = n$ is finite. Let C be a curve over k and $F = k(C)$ the function field of the curve. Let A be a central simple algebra over F and let $V(F)$ be the set of all non-trivial rank 1 valuations of F. Then $\text{ind}(A)$ is the maximum of the set $\{\text{ind}(A \otimes F_v)\}$ for $v \in V(F)$.*

Our main results concerning the Brauer dimension are the following theorems.

Theorem 1.1.4 (Theorem 6.5.1)**.** *Let k be a complete ultrametric field. Let l be a prime such that $l \neq char(\tilde{k})$. Suppose $dim_{\mathbb{Q}}(\sqrt{|k^*|}) = n$ is finite. If the $Br_l dim(\tilde{k}) = d$, then $Br_l dim(k) \leq d + n$.*

Theorem 1.1.5 (Theorem 6.5.3)**.** *Let k be a complete ultrametric field. Let l be a prime such that $l \neq char(\tilde{k})$. Suppose $dim_{\mathbb{Q}}(\sqrt{|k^*|}) = n$ is finite. Let C be a curve over k and $F = k(C)$ the function field of C. Suppose there exist an integer d such that $Br_l dim(\tilde{k}) \leq d$ and $Br_l dim(\tilde{k}(T)) \leq d + 1$. Then the $Br_l dim(F) \leq d + 1 + n$.*

Remark: Note that if k is a complete discretely valued field, then $n = 1$. Theorem 1.1.4 now gives that $\mathrm{Br}_l\mathrm{dim}(k) \leq d + 1$, which is a classical result [GS17, Corollary 7.1.10]. In the same case, Theorem 1.1.5 now implies that if $l \neq \mathrm{char}(\tilde{k})$, then $\mathrm{Br}_l\mathrm{dim}(F) \leq d + 2$, which is found in [HHK09, Corollary 5.10].

1.2 The Plan

We start with some preliminaries and necessary background in Chapters 2 and 3, where we talk about the known results on local-global principles from [HHK09] and [Meh19b].

In Chapter 4, we prove the main decomposition lemma, which we then use in Chapter 5 to prove that for a complete ultrametric field k with residue field characteristic $\mathrm{char}(\tilde{k}) \neq 2$ and $\mathrm{dim}_{\mathbb{Q}}(\sqrt{|k^*|}) = n$, $u(k) \leq 2^n u(\tilde{k})$. We further prove the result about $\mathrm{Br}_l\mathrm{dim}(k) \leq \mathrm{Br}_l\mathrm{dim}(\tilde{k}) + n$ in Chapter 6.

Using the results from Chapters 5 and 6, we prove our results on the strong u-invariant and Brauer l-dimension of function fields of curves over complete ultrametric fields.

Conventions. Unless stated otherwise, throughout this manuscript, we use the Berkovich approach to non-Archimedean analytic geometry, which is one of the several possible approaches to non-Archimedean analytic geometry. A Berkovich analytic curve will be a separated analytic space of pure dimension 1.

A valued field is a field endowed with a non-archimedean absolute value. For any valued field k, we denote the residue field by $\widetilde{k}$.

The empty set is considered to be connected.

A Berkovich analytic space which is reduced and irreducible is called *integral*. Thus, an *integral affinoid space* is an affinoid space whose corresponding affinoid algebra is an integral domain.

Throughout this document, unless otherwise mentioned, we work over a complete valued base field k.

Chapter 2

Preliminaries

This thesis is written in a comfortable style by preference, in the hope that it is an easy read for anyone possessing some knowledge of commutative and homological algebra, a handful of algebraic geometry jargon (mainly to understand Berkovich theory), some fundamental theorems about central simple algebras, definitions and properties of algebraic groups, and the resolve to chase diagrams. Additionally, we give ample references, which we hope will fill in the missing details in the manuscript that are sought by the keen reader.

Here is a very, very rough framework of the theory of quadratic forms and central simple algebras, which is a topic that might or might not be covered in a first-year PhD course. We hope the reader, if uninitiated in these topics, will still be able to proceed with the rest of this manuscript and progress meaningfully.

2.1 Quadratic forms

An excellent reference for an introduction to this subject matter would be the book [Sch12].

Definition 1. Let k be a field of characteristic different from 2 and V a finite-dimensional vector space over k. We say $q : V \to k$ is a *quadratic form* if

(i) q is a quadratic map, *i.e.*, $q(\lambda v) = \lambda^2 q(v)$ for each $\lambda \in k$ and $v \in V$.

(ii) The associated polar form $b_q : V \times V \to k$, such that (u, v) mapping to $\dfrac{q(u+v) - q(u) - q(v)}{2}$, is bilinear. We notice here that b_q is symmetric by definition.

Such a pair (V, q) is called a *quadratic space.*

Any quadratic form over a field k of characteristic $\neq 2$ can be *diagonalised* and written as $q := \langle a_1, a_2, \ldots, a_n \rangle$, for $a_i \in k$. That is, there exists a basis of vectors $e_i \in V$ such that $q(e_i) = a_i$ and $b_q(e_i, e_j) = 0$ for $i \neq j$.

Definition 2. The *orthogonal sum* of two quadratic forms (V, q) and (V', q') over k, denoted by $(V, q) \perp (V', q')$, is the quadratic form $q \perp q' : V \times V' \to k$ such that $(v, v') \mapsto q(v) + q'(v')$.

An *isomtery* between quadratic spaces (V, q) and (V', q') is a k-linear isomorphism $\varphi : V \to V'$ such that $q'(\varphi(v)) = q(v)$ for all $v \in V$.

Definition 3. A quadratic form q is said to be *non-degenerate* (or *regular*) if $\widetilde{b_q} : V \to V^*$, induced by b_q, is injective (and hence an isomorphism since V is finite dimensional). It is called **isotropic** if there is a non-zero $x \in V$ such that $q(x) = 0$, and *anisotropic* otherwise.

A subspace W of V is called *totally isotropic* if $b_q(x, y) = 0$ for all $x, y \in W$.

Corollary 2.1.1. *All maximal totally isotropic subspaces of* (V, q) *have the same dimension.*

Example 2.1.2. Let $\mathbb{H}$ denote the two-dimensional quadratic form $\langle -1, 1 \rangle$, called the *hyperbolic plane.* For $n \in \mathbb{N}$, let $\mathbb{H}^n$ denote the orthogonal sum of n hyperbolic planes, also known as a *hyperbolic space.*

The common dimension of maximal totally isotropic subspaces of a regular quadratic form (V, q) is called the *Witt index* of (V, q).

Theorem 2.1.3 (Witt Decomposition). *Let (V, q) be a quadratic space, with Witt index m. Then there exists an orthogonal direct sum decomposition,*

$$V = \mathbb{H}^m \perp V_1,$$

where V_1 is anisotropic. The form V_1 is uniquely determined up to isometry.

Two non-degenerate quadratic forms q_1 and q_2 are said to be *Witt equivalent* (or *similar*), if there exist integers n, m such that $q_1 \perp \mathbb{H}^n \simeq q_2 \perp \mathbb{H}^m$. The above-mentioned results tell us that every non-degenerate quadratic form q is similar to an anisotropic form q_{an}, which is uniquely determined up to isometry.

Define $W(k)$ as the set of isomorphism classes of non-degenerate quadratic forms modulo Witt equivalence. This set is an abelian group, with the group operation corresponding to the orthogonal direct sum of forms. The Witt group of K, $W(K)$ can also be given a commutative ring structure by using the tensor product of quadratic forms. This is sometimes called the Witt ring of K.

2.2 Central simple algebras

A great reference for an introduction to this topic would be the book [GS17].

Definition 4. A *central simple algebra* over a field k is a finite-dimensional associative k-algebra A, with center k and no non-trivial two-sided ideals.

The notion of central simple algebra over a field generalises to that of an *Azumaya algebra* over a commutative ring. The following theorem is due to Azumaya (over a local ring), Auslander and Goldman (over an arbitrary commutative ring), and Grothendieck (over a scheme).

Theorem 2.2.1 (Wedderburn Structure Theorem). *Let A be a central simple algebra over a field k. Then there exists an integer $n \geq 1$ and a central division* *algebra D over k such that $A \simeq M_n(D)$. Moreover, D is uniquely determined up to isomorphism.*

Proof. See for instance [GS17, p. 22]. □

Notation 1. *For any algebra A over a field k and any field extension K/k, we write A_K for the K-algebra obtained from A by extending scalars to K:*

$$A_K = A \otimes_k K.$$

Theorem 2.2.2 (Wedderburn)**.** *Let A be an algebra over a field k. Then A is central simple if and only if there is a field K, containing k, such that $A_K \simeq M_n(K)$ for some n.*

Proof. See, for instance, [Sch12, Chapter 8]. □

Corollary 2.2.3. *If A is a central simple k-algebra, its dimension over k is a square.*

Definition 5. Let K over k be a field extension such that $A_K \simeq M_n(K)$ for some n, are called *splitting fields* of A. If K is a splitting field of A, we also say that A *splits over* K or K *splits* A.

Definition 6. Let A be a central simple algebra over a field k. We define the *degree* of A, denoted $\deg_k(A)$ or simply $\deg(A)$, to be the integer $\sqrt{\dim_k(A)}$.

2.2.1 The Brauer group

To gain an understanding of the finite-dimensional central division algebras over a field k, it is best to consider the more general central simple algebras over k. This is because central simple algebras are closed under the tensor product, while central division algebras in general are not. For example, if $\mathbb{H}$ is the Hamiltonian quaternion algebra over $\mathbb{R}$, then $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \simeq M_4(\mathbb{R})$.

If A is a central simple algebra over k, then by Wedderburn's Structure Theorem, we have a k-algebra isomorphism $A \simeq M_n(D)$, for some integer $n \geq 1$ and some finite dimensional division algebra D over k, which is uniquely determined up to k-algebra isomorphism. This motivates the following equivalence relation.

Definition 7. Let $A \simeq M_n(D_A)$ and $B \simeq M_m(D_B)$ be two central simple algebras over a field k, where D_A, D_B are the associated central division algebras, respectively. We say A *is similar to* B, denoted by $A \sim B$, if there is a k-algebra isomorphism $D_A \simeq D_B$.

This $\sim$ is an equivalence relation on the set of central simple algebras over k, and let $[A]$ denote the equivalence class of A under this relation.

Definition 8. For any k-algebras A, the *opposite algebra*, A^{op}, is defined by

$$A^{\text{op}} = \{a^{\text{op}} \mid a \in A\},$$

with the operations defined as follows:

- $a^{\text{op}} + b^{\text{op}} = (a + b)^{\text{op}}$,
- $a^{\text{op}} b^{\text{op}} = (ba)^{\text{op}}$,
- $\alpha \cdot a^{\text{op}} = (\alpha \cdot a)^{\text{op}}$,

for all $a, b \in A$ and $\alpha \in k$.

The tensor product gives a binary operation on the set of equivalence classes of central simple algebras, defined as $[A] \cdot [B] := [A \otimes_k B]$, which makes it into an abelian group.

Definition 9. The *Brauer group* of a field k is the set of equivalence classes of central simple k-algebras under the equivalence relation defined above. Denote this group by $\text{Br}(k)$.

The identity element of this abelian group $(\text{Br}(k), \cdot)$ is the class of k, $[k]$, and the inverse element is $[A]^{-1} = [A^{\text{op}}]$, for all classes $[A] \in \text{Br}(k)$.

Remark 2.2.4. There is a one-to-one correspondence between the set of finite-dimensional central division algebras over k and the set of elements of $\mathrm{Br}(k)$.

Theorem 2.2.5. *The Brauer group is a torsion abelian group.*

Notation 2. *We denote the n-torsion subgroup of $\mathrm{Br}(k)$ by ${}_n\mathrm{Br}(k)$.*

Definition 10. Let A be a central simple algebra over a field k. Let D_A be the central division algebra for which $A \simeq M_n(D_A)$. Define the *index* of A, denoted by $\mathrm{ind}(A)$, to be $\deg(D_A)$.

Definition 11. The *period* (or *exponent*) of a central simple k-algebra A, denoted by $\mathrm{per}(A)$, is the order of its class $[A]$ in $\mathrm{Br}(k)$.

Theorem 2.2.6 (Brauer)**.** *Let A be a central simple k-algebra. Then the period of A divides the index of A. Moreover, the period and the index have the same prime factors.*

Proof. See [GS17, Proposition 4.5.13]. □

Definition 12 (Azumaya algebra)**.** Let R be a commutative ring. An R-algebra A is called an *Azumaya algebra* if A is finitely generated and projective (equivalently, locally free) as an R-module, such that the map $A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$ which sends $a \otimes b \in A \otimes_R A^{\mathrm{op}}$ to the A-endomorphism $x \mapsto axb$ is an isomorphism.

Theorem 2.2.7. *Let A be an algebra over a commutative ring R and let $A_S := S \otimes_R A$ for a homomorphism of rings $R \to S$. The following are equivalent conditions.*

(i) The map $A \otimes_R A^{op} \to End_R(A)$ is an isomorphism.

(ii) For every algebraically closed field k and a homomorphism $R \to k$, $A_k \simeq Mat_n(k)$.

(iii) For every maximal ideal $\mathfrak{m} \subset R$, let $k = R/\mathfrak{m}$; the ring A_k is a central simple algebra over k.

Two Azumaya R-algebras are Morita equivalent if there exists finitely generated projective R modules M and N such that $A \otimes \mathrm{End}_R(M) \simeq B \otimes \mathrm{End}_R(N)$ as R-algebras.

Morita equivalence classes of Azumaya algebras over R form a group under tensor product, which is called the *Brauer group of* R, denoted by $\mathrm{Br}(R)$.

2.3 Severi-Brauer varieties

A reference for an introduction to this topic would be Chapter 5 of [GS17] and Chapter I, §1 of [KMRT98].

Definition 13. Let n be a positive integer. A *Severi-Brauer variety* of dimension $n-1$ over a field k is a k-variety X such that there exists a field extension $k \subset K$ and an isomorphism of K-varieties $X \times_k K \cong \mathbb{P}_K^{n-1}$. Such varieties are also called *twisted forms* of $\mathbb{P}_k^{n-1}$.

There is a natural bijection between the isomorphism classes of Severi-Brauer varieties over a field k and the isomorphism classes of central simple k-algebras.

A twisted form of $\mathbb{P}_k^1$ is a smooth, projective, geometrically integral curve C of genus 0. Any smooth plane conic is a twisted form of $\mathbb{P}_k^1$. The automorphism group of projective space $\mathbb{P}_k^{n-1}$ is the algebraic group $\mathrm{PGL}_{n,k}$. The group $\mathrm{PGL}_n(k)$ is the automorphism group of the matrix algebra $M_n(k)$. Galois descent then gives a bijection between the isomorphism classes of twisted forms of $\mathbb{P}_k^{n-1}$ and the isomorphism classes of twisted forms of $M_n(k)$, which are precisely the central simple algebras of degree n over k.

Thus, we get a canonical bijection of pointed sets

$$H^1(k, \mathrm{PGL}_{n,k}) \cong \{\text{central simple algebras over } k \text{ of degree } n\}/\text{iso}$$

$$\cong \{\text{Severi-Brauer varieties over } k \text{ of dimension } n-1\}/\text{iso}$$

and a map of pointed sets $H^1(k, \mathrm{PGL}_{n,k}) \to \mathrm{Br}(k)$, which sends a central simple algebra A of degree n to its class $[A] \in \mathrm{Br}(k)$. For a central simple algebra of degree n, there exists a Severi-Brauer variety X_A of dimension $n-1$, which is uniquely determined up to isomorphism of k-schemes [Jah03, Theorem 5.1].

For a Severi-Brauer variety X of dimension $n-1$, we denote by $[X] \in \mathrm{Br}(k)$ the image of the isomorphism class of X under this map.

Theorem 2.3.1. *Let X be a variety over a field k. The following properties are equivalent:*

(i) X is a Severi-Brauer variety of dimension $n-1$.

(ii) There is an isomorphism $\bar{X} \cong \mathbb{P}^{n-1}_{\bar{k}}$.

(iii) There is an isomorphism $X^s \cong \mathbb{P}^{n-1}_{k^s}$.

(iv) There is a central simple k-algebra A of degree n such that $X \cong X_A$.

The central simple algebra A in (iv) is well-defined up to isomorphism. If $X = X_A$, then $[X] = [A] \in Br(k)$.

Let F be a field and A be a central simple F-algebra of degree n. Let $I \subseteq A$ be a right ideal of A. Then the F-dimension of I, denoted by $\dim_F(I)$ is divisible by the degree of A, $\deg(A)$. The quotient $\dim_F(I)/\deg(A)$ is called the *reduced dimension* of the ideal.

Definition 14. For any integer i, we write $\mathrm{SB}_i(A)$ for the *i-th generalised Severi-Brauer variety* of the right ideals in A of reduced dimension i.

In particular, $\mathrm{SB}_0(A) = \mathrm{Spec}(F) = \mathrm{SB}_{\deg(A)}(A)$ and $\mathrm{SB}_i(A) = \phi$ for i outside of the interval $[0, \deg(A)]$. The variety $\mathrm{SB}_1(A)$ is the usual Severi-Brauer variety of A.

In the same setup let E/F be a field extension and if A_E denotes the algebra $A \otimes_F E$, then the E-points of $\mathrm{SB}_i(A)$ are in bijection with the right ideals in A_E which are of reduced dimension i and are direct summands of A_E.

Also, $\mathrm{SB}_i(A)(E) \neq \phi$ if and only if $\mathrm{ind}(A_E)$ divides i. Since A_E is a central simple algebra over E, we have $A_E \cong M_m(D)$ for some E-division algebra D and some $m \geq 1$. Now, the right ideals of reduced dimension i in A_E are in natural bijection with the subspaces of D^m of D-dimension $i/\mathrm{ind}(A_E)$. Thus, writing D_A for the F-division algebra in the class of A, the F-linear algebraic group $\mathrm{GL}_1(A) = \mathrm{GL}_m(D_A)$ acts transitively on the points of the F-scheme $\mathrm{SB}_i(A)$.

Chapter 3

Patching and Local-Global Principles

In this chapter, we will discuss a patching setup first developed by Harbater and Hartmann in [HH10], and then refined and expanded by Harbater, Hartmann, and Krashen in [HHK09], [HHK11], and [HHK14]. We will also discuss patching in the setup of the Berkovich theory of rigid analytic geometry, as seen in the PhD thesis of Vlerë Mehmeti [MEH19a].

3.1 Introduction to field patching

This section aims to fix notation and discuss vector space patching problems. Let I denote an indexing set with relations. Throughout, let $\mathcal{F} = \{F_i\}_{i \in I}$ denote a finite inverse system of fields with inclusions as morphisms. We call F_j an *overfield* of F_i if there is an inclusion $F_i \subset F_j$ in $\mathcal{F}$. Let $F = \varinjlim \mathcal{F}$.

We want to study when the data of an algebraic object (eg, a vector space) over F is equivalent to the data of a collection of algebraic objects over F_i, together with an isomorphism between them over common overfields. Specifically, we want to be able to think of $\mathcal{F}$ as a *cover* of F. We need to make some assumptions on $\mathcal{F}$ before proceeding.

3.1.1 Necessary definitions

Definition 15. A *factorisation inverse system* over a field F is a finite inverse system of fields such that:

1. F is the inverse limit.

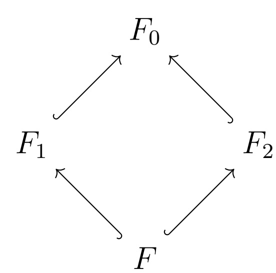
2. The index set I can be partitioned as $I = I_v \sqcup I_e$ such that:

   (i) For any $k \in I_e$, there are exactly two elements $i, j \in I_v$ such that $i, j > k$.

   (ii) These are the only relations in I.

Let's consider a very basic example of a factorisation inverse system.

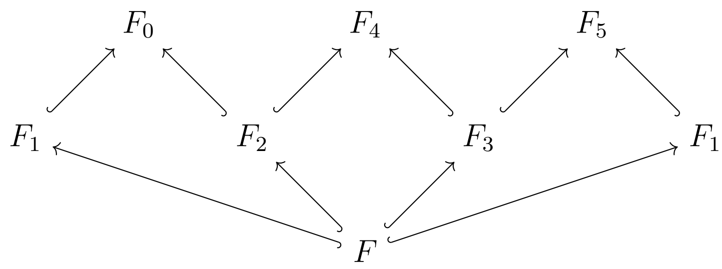
Example 3.1.1. Let F_1, F_2, and $F = F_1 \cap F_2$ be sub-fields of a given field F_0. Then $\mathcal{F} = \{F_i\}_{i \in I}$, with $I = \{0, 1, 2\}$ is a factorisation inverse system with $F = \varprojlim F_i$. Pictorially, we get



In this case, $I_e = \{0\}, I_v = \{1, 2\}$.

If $\mathcal{F} = \{F_i\}_{i \in I}$ is a factorisation inverse system, then for each index $k \in I_e$ there is an ordered triple (l_k, r_k, k), where $l_k, r_k \in I_v$ and $l_k, r_k > k$. In fact, the factorisation system is a finite multi-graph with an orientation for each (l_k, r_k, k), see [HHK14, Section 2.1]. Collect these triples (l_k, r_k, k) in a set S_I.

Example 3.1.2. Consider $\mathcal{F} = \{F_0, F_1, F_2, F_3, F_4, F_5\}$ with the following field inclusions



In this example, $I_e = \{0, 4, 5\}$ and $I_v = \{1, 2, 3\}$.

3.1.2 Patching problem

Patching is a principle that allows the study of algebraic objects over a field F by studying corresponding objects over extension fields of F. Think of $\mathcal{F}$ to be a cover of F, then a *patching problem* consists of data over $\mathcal{F}$ that would give rise to an object over F.

Let's start with vector spaces, one of the easiest algebraic objects to study. Let VECT(F) denote the category of finite-dimensional vector spaces over F.

Definition 16. Let $\mathcal{F} = \{F_i\}_{i\in I}$ be a factorisation inverse system over a field F. A *vector space patching problem* for $\mathcal{F}$ is given by a collection of finite dimensional F_i-vector spaces V_i, together with F_k-vector space isomorphisms $\mu_k : V_i \otimes_{F_i} F_k \to V_j \otimes_{F_j} F_k$, whenever $(i, j, k) \in S_I$. Denote this by $\mathcal{V} = (\{V_i\}_{i\in I_v}, \{\mu_k\}_{k\in I_e})$.

Now consider two different vector space patching problems $\mathcal{V} = (\{V_i\}_{i\in I_v}, \{\mu_k\}_{k\in I_e}), \mathcal{V}' = (\{V_i'\}_{i\in I_v}, \{\mu_k'\}_{k\in I_e})$. We can now talk of a *morphism of patching problems* from $\mathcal{V} \to \mathcal{V}'$ – it is a collection of F_i-linear transformations $V_i \to V_i'$ for all $i \in I_v$ that are compatible with the isomorphisms μ_k, μ_k'.

This allows us to define the *category of vector space patching problems*, denoted by PP($\mathcal{F}$). Because of the way we have done the construction, it's an easy exercise to see that,

$$\mathrm{PP}(\mathcal{F}) \simeq \prod_{(i,j,k)\in S_I} \mathrm{VECT}(F_i) \times_{\mathrm{VECT}(F_k)} \mathrm{VECT}(F_j).$$

We also have a canonical functor,

$$\beta : \mathrm{VECT}(F) \to \mathrm{PP}(\mathcal{F})$$

$$V \mapsto (\{V \otimes_F F_i\}_{i \in I_v}, \{\alpha_k\}_{k \in I_e}),$$

where α_k is the canonical isomorphism between $(V \otimes_F F_i) \otimes_{F_i} F_k \simeq (V \otimes_F F_j) \otimes_{F_j} F_k$.

Definition 17. A *solution* to a vector space patching problem $\mathcal{V}$ is an F-vector space V, such that $\beta(V)$ is isomorphic to $\mathcal{V}$.

3.2 Patching over Berkovich curves

This section is dedicated to discussing and proving that patching can be applied to an analytic curve. To do this, we follow closely the deliberations in [MEH19a] and [HHK09]. We will discuss the main patching results.

The purpose of this section is to discuss a matrix decomposition result under conditions which generalise those of the HHK setup in [HHK09, §3, Theorem 3.2]. As a consequence, a generalisation of vector space patching on analytic curves is obtained. We will state lemmas and propositions as required, without proof. We refer the reader to [MEH19a] for an in-depth treatment of the topic.

3.2.1 Necessary definitions

Definition 18. An *absolute value* on a field k is a function $|\cdot|: k \to \mathbb{R}_{\geq 0}$ such that:

a. $\mid 1 \mid = 1$,

b. for $x \in k$, $\mid x \mid = 0$ if and only if $x = 0$,

c. for all $x, y \in k$, $\mid xy \mid = \mid x \mid\mid y \mid$,

d. for all $x, y \in k$, $\mid x - y \mid \leq \mid x \mid + \mid y \mid$.

We will say that $(k, |\cdot|)$ is a field with an absolute value.

If the absolute value satisfies the following stronger condition

$$\text{for all } x, y \in k, \mid x - y \mid \leq \max\{\mid x \mid, \mid y \mid\},$$

then we call $|\cdot|$ a *non-Archimedean* or *ultrametric* absolute value on k. In this case, we say that $(k, |\cdot|)$ is a non-Archimedean valued (or ultrametric) field.

An absolute value which is not ultrametric is called Archimedean.

Let $|\cdot|$ be an absolute value on k. The field k is said to be complete with respect to $|\cdot|$, if it is complete with respect to the metric induced on k. One can now define the completion $(\hat{k}, |\cdot|)$ of $(k, |\cdot|)$ by using the Cauchy sequence construction. Then $(\hat{k}, |\cdot|)$ is complete. A field k equipped with a non-Archimedean valuation $|\cdot| : k \to \mathbb{R}_{\geq 0}$ is called a *non-Archimedean field* or an *ultrametric field.*

Recall that given an abelian group $G \subset \mathbb{R}^\times$, its *divisible closure* is the group $\{a \in \mathbb{R}^\times : \exists n \in \mathbb{Z}, a^n \in G\}$, which we will denote by $\sqrt{G}$.

Notation 3. *For a field k with an absolute value $|\cdot|$, denote by $\mid k \mid := \{r \in \mathbb{R}_{\geq 0} : \exists a \in k, \mid a \mid = r\}$. Set $\mid k^\times \mid = \{r \in \mid k \mid : r \neq 0\}$. This is a multiplicative subgroup of $\mathbb{R}_{>0}$. Denote by $\sqrt{\mid k^\times \mid}$ its divisible closure.*

Definition 19. Let A be a ring. A *semi-norm* on A is a function $|\cdot| : A \to \mathbb{R}_{\geq 0}$ such that:

a. $\mid 0 \mid = 0, \mid 1 \mid = 1$,

b. for all $x, y \in k$, $\mid xy \mid \leq \mid x \mid \mid y \mid$,

c. for all $x, y \in k$, $\mid x - y \mid \leq \mid x \mid + \mid y \mid$.

If the condition b is strengthened to for all $x, y \in k$, $\mid xy \mid = \mid x \mid \mid y \mid$, we will say that $|\cdot|$ is a *multiplicative semi-norm* on A.

If $\ker |\cdot| = \{0\}$, then $|\cdot|$ is a *norm* on A. A multiplicative semi-norm, which is also a norm, is called a *multiplicative norm.*

Let $(A, |\cdot|)$ be a normed ring. It is complete with respect to $|\cdot|$ if any Cauchy sequence in A has a limit in A, in which case, A is said to be a *Banach ring*. In Berkovich's theory of rigid analytic geometry, Banach rings play a role analogous to commutative rings in algebraic geometry.

The rings considered in this manuscript are commutative, with unity. We will now define the Berkovich analogue of the affine spectrum.

Definition 20. Let $(A, ||\cdot||)$ be a semi-normed ring. A semi-norm $|\cdot|$ on A is said to be *bounded* if there exists a positive real number c, such that $|\cdot| \leq c||\cdot||$.

Consider two semi-normed rings $(A, ||\cdot||_A)$ and $(B, ||\cdot||_B)$. A morphism $\varphi : A \to B$ is called *bounded* if there exists a real number $c > 0$, such that for any $x \in A$, $||\varphi(x)||_B \leq c \cdot ||x||_A$.

The morphism φ is said to be *admissible* if the quotient semi-norm on $A/\ker(\varphi)$ induced by $||.||_A$ is the same as the semi-norm induced on $\mathrm{Im}(\varphi)$ by $||.||_B$.

Definition 21 (The Berkovich Spectrum)**.** Let $(A, ||\cdot||)$ be a Banach ring. The *Berkovich spectrum* of A, denoted by $\mathcal{M}(A)$, is the set of all bounded multiplicative semi-norms on A.

Endow $\mathcal{M}(A)$ with the coarsest topology such that the function $v : \mathcal{M}(A) \to \mathbb{R}_{\geq 0}, |\cdot| \mapsto |f|$, is continuous for all $f \in A$.

Convention: For a point x of the space $\mathcal{M}(A)$, we will also use the notation $|\cdot|_x$ when treating it as a semi-norm on A.

For a Banach ring A, the points of $\mathcal{M}(A)$ are the equivalence classes of bounded morphisms $A \to K$, where K is a complete ultrametric field.

Lemma 3.2.1 (Theorem 1.2.1, [Ber90])**.** *Let A be a Banach Ring. Then, $\mathcal{M}(A)$ is a non-empty compact space.*

The fact that $\mathcal{M}(A)$ is compact is one of the key differences with the algebraic setting. The spectra of certain Banach rings form the building blocks of Berkovich spaces.

An important example of a K-algebra is the following:

Example 3.2.2. Let $n \in \mathbb{N}$ and $r_1, r_2, \ldots, r_n \in \mathbb{R}^n_{>0}$. Let

$$K\{\underline{r}^{-1}\underline{T}\} = K\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} := \{\sum_{u \in \mathbb{N}^n} a_u \underline{T}^u : a_u \in K, \lim_{|u| \to +\infty} |a_u| \underline{r}^u = 0\},$$

where for any $u \in \mathbb{N}^n$, $|u| := u_1 + u_2 + \cdots + u_n$.

When $r_1 = r_2 = \cdots = r_n = 1$, the K-algebra is now called the *Tate affinoid algebra* $K\{\underline{T}\}$.

Definition 22. A Banach K-algebra A is called a *K-affinoid algebra* if there exist $n \in \mathbb{N}$ and $\underline{r} = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n_{>0}$, and a surjective admissible morphism $K\{\underline{r}^{-1}\underline{T}\} \twoheadrightarrow A$.

The Banach algebra A is called a *strict K-affinoid algebra* if there exists $n \in \mathbb{N}$, and a surjective admissible morphism $K\{\underline{T}\} \twoheadrightarrow A$.

Affinoid algebras are to Berkovich theory what finite type algebras are to algebraic geometry. Throughout this subsection, let $(k, |\cdot|_k)$ be a complete ultrametric field.

Convention: A *k-affinoid space* is the Berkovich spectrum of a k-affinoid algebra. Let $\psi : A \to B$ be a bounded morphism of Banach rings, and let $\psi' : \mathcal{M}(B) \to \mathcal{M}(A)$ denote the induced continuous morphism.

A *morphism* $X \to Y$ of k-affinoid spaces is induced by a bounded k-linear morphism $A_Y \to A_X$ of corresponding k-affinoid algebras.

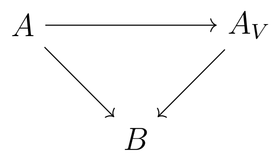
Definition 23. Let A be a k-affinoid algebra, and X the corresponding k-affinoid space. An affinoid domain in X is a pair (V, A_V), such that:

1. V is a closed subset of X, and A_V is a k-affinoid algebra;

2. there exists a bounded morphism $\phi : A \to A_V$, such that $\phi'(\mathcal{M}(A_V)) \subset V \subset X$;

3. the following universal property is satisfied: for any bounded k-linear morphism $\varphi : A \to B$ such that $\varphi'(\mathcal{M}(B)) \subset V$, where B is a K-affinoid algebra for

some complete ultrametric field extension K/k, there exists a unique bounded morphism $A_V \to B$ such that the following diagram commutes.



Proposition 3.2.3 (Proposition 2.2.3, [Ber90])**.** *Suppose k is non-trivially valued. Let X be a strict k-affinoid space. Then the strict k-affinoid domains in X form a basis of neighbourhoods of the topology on X.*

A morphism $\phi : Y \to X$ of k-analytic spaces is called *separated* if the canonical induced morphism $Y \to Y \times_X Y$ is a closed immersion. A *good k-analytic space* is a locally ringed space $(Y, \mathcal{O}_Y)$, where each point has a neighbourhood isomorphic to a k-affinoid space. A good k-analytic space Y is called *separated* if the canonical morphism $Y \to \mathcal{M}(k)$ is separated.

Definition 24. A k-analytic space is called a k-analytic curve if it is separated and of pure dimension 1.

For a Banach ring A and $x \in \mathcal{M}(A)$, the kernel of $|.|_x$ is closed and a prime ideal in A. Thus x induces a semi-norm $||.||_x$ on the domain $A/\ker|.|_x$, and for any $u, v \in A$ such that $|u - v|_x = 0$, then $|u|_x = |v|_x$. Let $\tilde{u}$ be the image of u in $A/\ker|.|_x$, then $|u|_x = ||\tilde{u}||_x$. This implies that the quotient semi-norm in $A/\ker|.|_x$ is a multiplicative norm, which means it can be extended to $\mathrm{Frac}(A/\ker|.|_x)$, the field of fractions of $A/\ker|.|_x$.

Definition 25. Let A be a Banach ring. Let $x \in \mathcal{M}(A)$ and complete the field $\mathrm{Frac}(A/\ker|.|_x)$ with respect to the quotient norm. This completion, denoted by $\mathcal{H}(x)$, is called the *complete residue field* of x.

Let k be an ultrametric field and $|\cdot|$ be the non-Archimedean valuation on this field. Suppose that $\dim_{\mathbb{Q}}\sqrt{|k^*|}$ (i.e., the rational rank of $|k^*|$) is finite, say

$n = \dim_{\mathbb{Q}}(\sqrt{|k^*|})$, and residue field characteristic not 2. Then for any k-analytic space and any of its points x, the field $\mathcal{H}(x)$ also satisfies them.

Remark 3.2.4. If k is a complete non-Archimedean valued field and A is a Banach algebra over k, then for any $x \in \mathcal{M}(A)$, we have a canonical isometric embedding of $k \hookrightarrow \mathcal{H}(x)$. This implies that $\mathcal{H}(x)$ is also a complete non-Archimedean valued field.

There is a full classification of points on a curve $\mathcal{C}$. Let $\mathcal{C}$ be a k-analytic curve. For any $x \in \mathcal{C}$, define $s_x := \operatorname{trdeg}_{\tilde{k}} \widetilde{H(x)}$ and $t_x := \dim_{\mathbb{Q}} \mid \mathcal{H}(x)^{\times} \mid /(\mid k^{\times} \mid \otimes_{\mathbb{Z}} \mathbb{Q})$. Here $\mathcal{H}(x)$ is the completed residue field of x; $\widetilde{k}, \widetilde{\mathcal{H}(x)}$ are the residue fields of $k, \mathcal{H}(x)$, respectively. We know from [Ber93, Lemma 2.5.2] that for points of a good k-analytic space X, $\dim(X) = \sup_{x \in X}(s_x + t_x)$. This tells us that for any $x \in \mathcal{C}$, $s_x + t_x \leq 1$. Fix an algebraic closure $\bar{k}$ of k. The absolute value of k extends uniquely to $\bar{k}$. Now denote by $\widehat{\bar{k}}$ the completion of $\bar{k}$ with respect to this absolute value.

Definition 26. The point $x \in \mathcal{C}$ is called

(a) *type 1*, if $\mathcal{H}(x) \subseteq \widehat{\bar{k}}$; remark that $s_x = t_x = 0$;

(b) *type 2*, if $s_x = 1$;

(c) *type 3*, if $t_x = 1$;

(d) *type 4*, if $s_x = t_x = 0$ and x is not of type 1.

Abhyankar points of a good k-analytic space X are the points $x \in X$ for which $s_x + t_x = \dim(X)$. So the thing to note here is that type 2 and 3 points are the Abhyankar points of $\mathcal{C}$. Also, rigid points are type 1 points (but not vice-versa unless k is algebraically closed or trivially valued).

Definition 27. A finite cover $\mathcal{U}$ of a k-analytic curve is called *nice* if:

1. the elements of $\mathcal{U}$ are connected affinoid domains with only type 3 points in their topological boundaries;

2. for any different $U, V \in \mathcal{U}$, $U \cap V = \partial U \cap \partial V$, or equivalently, $U \cap V$ is a finite set of type 3 points;

3. for any two different elements of $\mathcal{U}$, neither is contained in the other.

We will now define the sheaf of meromorphic functions, which heavily resembles the complex setting.

Definition 28. Let X be a good k-analytic space. Let $\mathcal{S}_X$ be the presheaf of functions on X, which associates to any analytic domain U the set $\{$analytic functions on U whose restriction to any affinoid domain in it is not a zero-divisor$\}$. Let $\mathscr{M}_{-}$be the presheaf on X that associates to any analytic domain U the ring $\mathcal{S}_X(U)^{-1}\mathcal{O}_X(U)$. the sheafification $\mathscr{M}_X$ of the presheaf $\mathscr{M}_{-}$ is the *sheaf of meromorphic functions on* X.

When there is no risk of ambiguity, we will simply denote $\mathcal{O}$, respectively $\mathscr{M}$, for the sheaf of analytic, respectively meromorphic functions on X.

If X is a good k-analytic space, then for any $x \in X$, $\mathscr{M}_{X,x}$ is the total ring of fractions of $\mathcal{O}_{X,x}$. In particular, if $\mathcal{O}_{X,x}$ is a domain, then $\mathscr{M}_{X,x} = \mathrm{Frac}(\mathcal{O}_{X,x})$.

Before going further into the Berkovich theory analogue of the analytification functor from classical algebraic geometry, let's take a moment to discuss a fundamental example of Berkovich spaces (as a topological space). This next example was originally defined and explored in [Ber90, Section 1.5].

Example 3.2.5. (The analytic affine space) Let A be a Banach ring and let $\mathbb{A}_k^{n,an}$ define thew following set

$$\mathbb{A}_k^{n,an} := \{\text{multiplicative semi-norms on } A[T_1, T_2, \ldots, T_n] \text{ that are bounded on } A\}.$$

Now give the set $\mathbb{A}_k^{n,an}$ the coarsest topology such that the map $\mathbb{A}_k^{n,an} \to \mathbb{R}_{\geq 0}$, which sends $x \mapsto |p|_x$, is continuous for all $p \in A[T_1, T_2, \ldots, T_n]$.

This topological space $\mathbb{A}_k^{n,an}$ is called the *n-dimensional analytic affine space* over A. This space has neat topological properties, for more details see [Poi10, Theorem 1.1.13].

Definition 29. Let k be a complete ultrametric field. Let X be a scheme of locally finite type over k. The *Berkovich Analytification* of X, denoted by X^{an}, is a good k-analytic space together with a morphism of k-locally ringed spaces $X^{an} \to X$. This morphism represents the functor which sends any good k-analytic space Y to $\mathrm{Hom}(Y, X)$, where $\mathrm{Hom}(\cdot, \cdot)$ represents the morphisms in the category of k-locally ringed spaces.

Let k be a complete ultrametric field (aka. complete non-Archimedean valued field). A field extension l over k is called a *complete ultrametric field extension* if l is complete with respect to an absolute value that extends the absolute value on k.

Theorem 3.2.6 ([Ber90])**.** *Let X be a scheme of locally finite type over k. The Berkovich analytification X^{an} of X exists.*

1. *For any complete ultrametric field extension l/k, $X^{an}(l) \simeq X(l)$. Furthermore, the canonical morphism $\varphi : X^{an} \to X$ is surjective and induces a bijection between the rigid points of X^{an} and the closed points of X.*

2. *For any $x \in X^{an}$, the canonical morphism $\varphi_x : \mathcal{O}_{X^{an},\varphi(x)} \to \mathcal{O}_{X,x}$ is faithfully flat. Moreover, if x in X^{an} is a rigid point, then φ_x indices an isomorphism of completions $\widehat{\mathcal{O}_{X^{an},\varphi(x)}} \to \widehat{\mathcal{O}_{X,x}}$.*

Let's quickly discuss how the space X^{an} is constructed. If $X = \mathbb{A}_k^n$ for some $n \in \mathbb{N}$, then the Berkovich analytification is $\mathbb{A}_k^{n,an}$, and the canonical map is a kernel map $\varphi : \mathbb{A}_k^{n,an} \to \mathbb{A}_k^n$ such that $x \mapsto \ker(|\cdot|_x)$. This permits the construction of analytifications of closed subschemes of $\mathbb{A}_k^n$, *viz.*, for any finitely generated k-algebra A, the analytification of $X = \mathrm{Spec}(A)$ is

$$X^{an} = (\mathrm{Spec}(A))^{an} := \{\text{multiplicative semi-norms on } A \text{ that extend the norm on } k\}.$$

The canonical map $\varphi : X^{an} \to X$ is still the kernel map. If I is the ideal sheaf corresponding to X as a Zariski closed subset of $\mathbb{A}_k^{n,an}$, then the analytic structure on X is given by $\mathcal{O}_X := \mathcal{O}_{\mathbb{A}_k^{n,an}}/\varphi^* I$. In the general case when X is any locally finite type scheme over k, we get X^{an} and the canonical map φ by glueing together the analytifications and the canonical maps of any open affine cover of X.

3.2.2 A local-global principle over Berkovich curves

Let k be a complete non-Archimedean valued field. Let C/k be a normal irreducible projective algebraic curve with function field F. We shall discuss local-global principles of certain homogeneous spaces defined over F from an analytic point of view. Unless mentioned otherwise, we assume that $\sqrt{|\, k^\times \,|} \neq \mathbb{R}_{>0}$.

If a linear algebraic group G acts on a variety H over a field F, we will say that *G acts transitively on the points of H* if for every field extension E of F the induced action of the group $G(E)$ on the set $H(E)$ is transitive.

Following [HHK09], we have the definition -

Definition 30. Let k be a field. Let X be a k-variety and G a linear algebraic group over k. We say that G acts *strongly transitively* on X if G acts on X and, for any field extension L/k, either $X(L) = \phi$ or $G(L)$ acts transitively on $X(L)$.

We can start by discussing some patching results over nice covers. Before that, we need to recall the following definition from [MEH19a].

Definition 31. [MEH19a, Definition 3.1.6] Let C be a k-analytic curve. Let $\mathcal{D}$ be a nice cover of C. The function $T_{\mathcal{D}} : \mathcal{D} \to \{0, 1\}$ is called a *parity function* for $\mathcal{D}$ if it satisfies the following:

$$\text{for any different } U, V \in \mathcal{D} \text{ such that } U \cap V \neq \phi, \text{ then } T_{\mathcal{D}}(U) \neq T_{\mathcal{D}}(V).$$

Proposition 3.2.7. *[MEH19a, Proposition 3.2.2] Let D be $\mathbb{P}_k^{1,an}$ or a connected affinoid domain of $\mathbb{P}_k^{1,an}$. Let $\mathcal{D}$ be a nice cover of D and $T_{\mathcal{D}}$ a parity function for*

$\mathcal{D}$. Let $G/\mathscr{M}(D)$ be a connected rational linear algebraic group. Then, for any $(g_s)_{s\in S_{\mathcal{D}}} \in \prod_{s\in S_{\mathcal{D}}} G(\mathscr{M}(\{s\}))$, there exists $(g_U)_{U\in\mathcal{D}} \prod_{U\in\mathcal{D}} G(\mathscr{M}(U))$ satisfying: for any $s \in S_{\mathcal{D}}$, if $U_0, U_1 \in \mathcal{D}$ contain s and $T_{\mathcal{D}}(U_0) = 0$, then $g_s = g_{U_0} \cdot g_{U_1}^{-1}$ in $G(\mathscr{M}(\{s\}))$.

Proposition 3.2.8. *[MEH19a, Proposition 3.2.3] Let Y be an integral strict k-affinoid curve. Set $K = \mathscr{M}(Y)$. Let G/K be a connected rational linear algebraic group. For any open cover $\mathcal{V}$ of Y, there exists a nice refinement $\mathcal{U}$ of $\mathcal{V}$ with a parity function $T_{\mathcal{U}}$ such that for any given $(g_y)_{y\in S_{\mathcal{U}}} \in \prod_{y\in S_{\mathcal{U}}} G(\mathscr{M}(\{y\}))$, there exists $(g_U)_{U\in\mathcal{U}} \prod_{U\in\mathcal{U}} G(\mathscr{M}(U))$ satisfying: for any $y \in S_{\mathcal{U}}$, if $U', U'' \in \mathcal{U}$ contain y and $T_{\mathcal{U}}(U') = 0$, then $g_y = g_{U'} \cdot g_{U''}^{-1}$ in $G(\mathscr{M}(\{y\}))$.*

Lemma 3.2.9. *[MEH19a, Lemma 3.2.4] For any point s of type 3 in $\mathbb{D}$, $\mathscr{M}(\{s\}) \otimes_{\mathscr{M}(\mathbb{D})} \mathscr{M}(Y) = \prod_{x\in f^{-1}(s)} \mathscr{M}(\{x\})$.*

Proposition 3.2.10. *[MEH19a, Proposition 3.2.5] Let Y be a normal irreducible strict k-affinoid curve. Set $K = \mathscr{M}(Y)$. Let X/K be a variety and G/K a connected rational linear algebraic group acting strongly transitively on X. The following local-global principles hold:*

- *$X(K) \neq \phi$ if and only if $X(\mathscr{M}_x) \neq \phi$ for all $x \in Y$;*
- *for any open cover $\mathcal{P}$ of Y, $X(K) \neq \phi$ if and only if $X(\mathscr{M}(U)) \neq \phi$ for all $U \in \mathcal{P}$.*

Theorem 3.2.11. *[MEH19a, Theorem 3.2.9] Suppose that k is non-trivially valued. Let Y be a normal irreducible k-affinoid curve. Set $K = \mathscr{M}(Y)$. Let X/K be a variety and G/K a connected rational linear algebraic group acting strongly transitively on X. The following local-global principles hold:*

- *$X(K) \neq \phi$ if and only if $X(\mathscr{M}_x) \neq \phi$ for all $x \in Y$;*
- *for any open cover $\mathcal{P}$ of Y, $X(K) \neq \phi$ if and only if $X(\mathscr{M}(U)) \neq \phi$ for all $U \in \mathcal{P}$.*

Theorem 3.2.12. *[Meh19b, Theorem 3.11] Let k be a complete ultrametric field. Let C be an irreducible normal projective k-analytic curve. Set $F = \mathscr{M}(C)$. Let X/F be a variety and G/F a connected rational linear algebraic group acting strongly transitively on X. The following local-global principles hold:*

- *$X(F) \neq \phi$ if and only if $X(\mathscr{M}_x) \neq \phi$ for all $x \in C$;*
- *for any open cover $\mathcal{P}$ of C, $X(F) \neq \phi$ if and only if $X(\mathscr{M}(U)) \neq \phi$ for all $U \in \mathcal{P}$.*

Now, Theorem 3.2.12 can be applied to the projective variety X defined by a quadratic form q over F. In [HHK09, Theorem 4.2], HHK show that for a regular quadratic form q over F, if $\text{char}(F) \neq 2$, $SO(q)$, the special orthogonal group of q, acts strongly transitively on X when $\dim(q) \neq 2$. So in that case, we can take $G = SO(q)$.

Because of the relation of Berkovich points to valuations of the function field of a curve, as a result of Theorem 3.2.12, we will obtain a local-global principle with respect to completions. Recall that an irreducible compact analytic curve is either projective or affinoid.

Corollary 3.2.13. *[Meh19b, Corollary 3.18] Let k be a complete ultrametric valued field. Let C be a compact irreducible normal k-analytic curve. Set $F = \mathscr{M}(C)$. Let X/F be a variety and G/F a connected rational linear algebraic group acting strongly transitively on X. The following local-global principles hold:*

(1) if C is affinoid and $\sqrt{|\, k^\times \,|} \neq \mathbb{R}_{>0}$, then $X(F) \neq \phi$ if and only if $X(F_v) \neq \phi$ for all $v \in V_{\mathcal{O}(C)}(F)$;

(2) if C is projective, $X(F) \neq \phi$ if and only if $X(F_v) \neq \phi$ for all $v \in V(F)$.

Corollary 3.2.14. *[Meh19b, Corollary 3.19] Let k be a complete non-Archimedean valued field. Let C be a compact irreducible normal k-analytic curve. Set $F = \mathscr{M}(C)$. Suppose that $\text{char}(F) \neq 2$. Let q be a quadratic form over F of dimension different from 2. The following local-global principles hold:*

(1) if C is affinoid and $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$, then q is isotropic over F if and only if it is isotropic over all completions F_v, for all $v \in V_{\mathcal{O}(C)}(F)$, of F;

(2) if C is projective, q is isotropic over F if and only if it is isotropic over all completions F_v, for all $v \in V(F)$, of F.

Remark 3.2.15. For any finitely generated field extension F/k of transcendence degree 1, there exists a unique normal projective k-algebraic curve C^{alg} with function field F. Let C be the Berkovich analytification of C^{alg}. Then $\mathscr{M}(C) = F$, so the local-global principles apply to any such field F.

One can draw a comparison between the local-global principles proved by Mehmeti in [MEH19a, Theorem 3.2.12] and the one proven by HHK in [HHK09, Theorem 3.7]. More precisely, the overfields appearing in [HHK09] can be interpreted in the Berkovich setting, to show that [HHK09, Theorem 3.7] can be obtained as a consequence of Theorem 3.2.12. For a 'fine' enough model, one can also prove the converse. We will refer the reader to Chapter 3, Section 3 of [MEH19a] for the comparison of overfields.

In this manuscript, we use field patching in the setting of Berkovich analytic geometry. By patching over analytic curves, Mehmeti proved a local-global principle and provided applications to quadratic forms and the u-invariant.

We will use in the upcoming chapters the following main result due to Vlerë Mehmeti on the local-global principle in Berkovich theory.

Theorem 3.2.16 ([Meh19b])**.** *Let k be a complete ultrametric field. Let C be a normal irreducible projective k-algebraic curve. Denote by F the function field of C. Let X be an F-variety and G a connected rational linear algebraic group over F acting strongly transitively on X.*

Let $V(F)$ be the set of all non-trivial rank 1 valuations of F which either extend the valuation of k or are trivial when restricted to k.

If F is a perfect field or X is a smooth variety, then

$$X(F) \neq \phi \textit{ if and only if } X(F_v) \neq \phi \textit{ for all } v \in V(F),$$

where F_v denotes the completion of F with respect to v.

Chapter 4

Complete Ultrametric Fields

Let $M \subset \mathbb{R}^*$ be a subgroup and let $\sqrt{M} := \{a \in \mathbb{R}^* | \ a^z \in M \text{ for some } z \in \mathbb{Z}_{\neq 0}\}$ be the divisible closure of M. It is a $\mathbb{Q}$-vector space. Suppose $dim_{\mathbb{Q}}(\sqrt{M})$ is finite. For a prime l, set $M^l = \{m^l | m \in M\}$.

Let $n = \dim_{\mathbb{Q}}(\sqrt{M})$. There exists a $\mathbb{Q}$-basis $z_1, z_2, \ldots, z_n$ of $\sqrt{M}$. By definition of $\sqrt{M}$, there exist non-zero integers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $z_1^{\alpha_1}, z_2^{\alpha_2}, \ldots, z_n^{\alpha_n}$ are in M. Let $t_i := z_i^{\alpha_i} \in M$. Then $t_1, \ldots, t_n$ is also a basis. Thus, for any $t \in M$, there exist unique $p_1, p_2, \ldots, p_n \in \mathbb{Q}$ such that

$$t = \prod_{i=1}^{n} {t_i}^{p_i}.$$

Definition 4.0.1. Let $t \in M$. Let $t = \prod_{i=1}^{n} {t_i}^{s_i/r_i}$ for $\frac{s_i}{r_i} \in \mathbb{Q}$ with $gcd(s_i, r_i) = 1$ for $i = 1, 2, \ldots, n$. Let r be the lcm of $r_i, i = 1, 2, \ldots, n$. We will say that r is the *order of* t.

Let k be an ultrametric field and $| \cdot |$ be the non-Archimedean valuation on k. Suppose that $\dim_{\mathbb{Q}} \sqrt{|k^*|}$ (i.e., the rational rank of $|k^*|$) is finite, say $n = \dim_{\mathbb{Q}}(\sqrt{|k^*|})$, and residue field characteristic not 2.

In particular, from the above discussion we get that when $M = |k^*|$, there exist $\pi_1, \pi_2, \ldots, \pi_n \in k^*$ with $|\pi_i| = t_i$ such that for any $t \in \sqrt{|k^*|}$, there exist unique

$p_1, p_2, \ldots, p_n \in \mathbb{Q}$ such that

$$t = \prod_{i=1}^{n} |\pi_i|^{p_i} = \prod_{i=1}^{n} t_i^{p_i}.$$

Let us fix such elements $\pi_1, \pi_2, \ldots, \pi_n$ of k^*.

We use the following notation throughout this document.

Notation 4. *For any valued field E, let E° denote the ring of integers, $E^{\circ\circ}$ the corresponding maximal ideal, and $\widetilde{E}$ the residue field.*

Lemma 4.0.2. *Let k be a field and m, n be positive integers. Suppose m is coprime to n. Let $a \in k^*$. Then there exists an integer m' such that $a = a^{mm'} \in k^*/(k^*)^n$.*

Proof. Since m and n are coprime, there exists $m', n' \in \mathbb{Z}$ such that $mm' + nn' = 1$. Hence $a = a^{mm'+nn'} = a^{mm'} \in k^*/(k^*)^n$. □

Theorem 4.0.3 (Main Decomposition Lemma)**.** *Let k be an ultrametric valued field, $dim_{\mathbb{Q}}(\sqrt{|k^*|}) = n$ is finite, and l a prime such that $char(\widetilde{k}) \neq l$. Let $a_1, a_2, \ldots, a_s \in k^*$, there exist elements $c_1, c_2, \ldots, c_n$ in k^* such that any $a_i = u_i \prod_{j=1}^{n} c_j^{\mu_{ij}} (b_i)^l$, $\mu_{ij} \in \mathbb{Z}$, $0 \leq \mu_{ij} \leq l-1$, $b_i \in k$, and u_i are units in k°.*

Proof. Let us fix elements $\pi_1, \pi_2, \ldots, \pi_n$ in k^* such that $\{t_i = |\pi_i|\}$ is a basis of $\sqrt{|k^*|}$.

Since $a_i \in k^*$,$|a_i| \in \sqrt{|k^*|}$. So there exist unique $p_1, p_2, \ldots, p_n \in \mathbb{Q}$ such that, $|a_i| = \prod_{j=1}^{n} |\pi_j|^{p_{ij}} = \prod_{j=1}^{n} t_j^{p_{ij}} = t_1^{r_{i1}} \prod_{j=2}^{n} t_j^{r_{ij}}$, where $r_{ij} \in \mathbb{Q}$.

Let y_{i1} be the highest power of l that divides the denominator of r_{i1}. Then we can write $|a_i| = t_1^{\frac{x_{i1}}{l^{y_{i1}} z_{i1}}} \prod_{j=2}^{n} t_j^{r_{ij}}$, where $x_{ij}, y_{i1}, z_{i1} \in \mathbb{Z}$, and $r_{ij} \in \mathbb{Q}$. Thus, $|a_i^{z_{i1}}| = t_1^{x_{i1}/l^{y_{i1}}} \prod_{j=2}^{n} t_j^{s_{ij}}$, where $s_{ij} \in \mathbb{Q}$.

From Lemma 4.0.2, there exists z'_{i1} such that $a_i \equiv a_i^{z_{i1} z'_{i1}} (\mod (k^*)^l)$. Since l does not divide z_{i1}, it also does not divide z'_{i1}. It now follows that $|a_i^{z_{i1} z'_{i1}}| = t_1^{\frac{x_{i1} z'_{i1}}{l^{y_{i1}}}} \prod_{j=2}^{n} t_j^{s_{ij}} = t_1^{\widehat{x_{i1}}/l^{y_{i1}}} \prod_{j=2}^{n} t_j^{s_{ij}}$, where $\widehat{x_{i1}} \in \mathbb{Z}$ and $l \nmid \widehat{x_{i1}}$.

Without loss of generality, we can say that any $a_i \in k^*$ has $|a_i^{z_{i1}z'_{i1}}| = t_1^{x_{i1}/l^{y_{i1}}} \prod_{j=2}^{n} t_j^{r_{ij}}$, where $r_{ij} \in \mathbb{Q}$. Note that $\gcd(x_{i1}, l^{y_{i1}}) = 1$.

Case(i): Suppose $y_{i1} = 0$ for all i. So, $|a_i| = t_1^{x_{i1}} \prod_{j=2}^{n} t_j^{r_{ij}}$. Considering the element $\pi_1^{-x_{i1}} a_i \in k^*$, $|\pi_1^{-x_{i1}} a_i| = \prod_{j=2}^{n} t_j^{r_{ij}}$, where $r_{ij} \in \mathbb{Q}$. Choose $c_1 := \pi_1$, so for all i, then $|\pi_1^{-x_{i1}} a_i| = \prod_{j=2}^{n} t_j^{r_{ij}}$. From here, the choices for c_j are clear.

Case(ii): Suppose $y_{i1} \geq 1$ for some i. Without loss of generality, say $y_{11} = \max_i\{y_{i1}\} \geq 1$. So, $|a_1| = t_1^{x_{11}/l^{y_{11}}} \prod_{j=2}^{n} t_j^{r_{1j}}$. We also know that $l \nmid x_{11}$, so we can choose $x'_{11}, x''_{11} \in \mathbb{Z}$ such that $l \nmid x'_{11}$ and $x_{11}x'_{11} + l^{y_{11}+1}x''_{11} = 1$.

So, $a_1 \equiv a_1^{x'_{11}} (\mod (k^*)^l)$, implying $|a_1^{x'_{11}}| = t_1^{x_{11}x'_{11}/l^{y_{11}}} \prod_{j=2}^{n} t_j^{r_{1j}x'_{11}}$. Now consider the exponent of t_1:

$$\frac{x_{11}x'_{11}}{l^{y_{11}}} = \frac{1 - l^{y_{11}+1}x''_{11}}{l^{y_{11}}} = \frac{1}{l^{y_{11}}} - lx''_{11}.$$

So $|a_1| \equiv |a_1^{x'_{11}}| \equiv t^{1/l^{y_{11}}} \prod_{j=2}^{n} t_j^{\lambda_{ij}} (\mod (k^*)^l)$. Without loss of generality, say $|a_1| = t_1^{1/l^{y_{11}}} \prod_{j=2}^{n} t_j^{r_{1j}}$ and $|a_2| = t_1^{x_{21}/l^{y_{21}}} \prod_{j=2}^{n} t_j^{r_{2j}}$. Now consider the element $|a_1^{-x_{21}l^{y_{11}-y_{21}}} a_2| = t_1^{\frac{-l^{y_{11}-y_{21}}}{l^{y_{11}}} + \frac{x_{21}}{l^{y_{21}}}} \prod_{j=2}^{n} t_j^{\lambda_j}$, where $\lambda_j \in \mathbb{Q}$.

Consider again the exponent of t_1:

$$\frac{-x_{21}l^{y_{11}-y_{21}}}{l^{y_{11}}} + \frac{x_{21}}{l^{y_{21}}} = 0.$$

So for $x_{21}l^{y_{11}-y_{21}} \in \mathbb{Z}$, we have $|a_1^{-x_{21}l^{y_{11}-y_{21}}} a_2| = \prod_{j=2}^{n} t_j^{\lambda_j}$, where $\lambda_j \in \mathbb{Q}$. Choose $c_1 := a_1$. So for all i, $|a_1^{-x_{i1}l^{y_{11}-y_{i1}}} a_i| = \prod_{j=2}^{n} t_j^{\lambda_{ij}}$, where $\lambda_{ij} \in \mathbb{Q}$.

Continuing this process, we get $c_2, \ldots, c_n$, $b_i \in k^*$ and $\mu_{ij} \in \mathbb{Z}$ such that

$$|\, a_i b_i^{\ell} \,| = |\prod_{j=1}^{n} c_j^{\mu_{ij}} \,| \,.$$

Since for any $u \in k^*$, u is a unit in the valuation ring of k if and only if $|u| = 1$, we have

$$a_i = u_i b_i^\ell \prod_j c_j^{\mu_{ij}}$$

for some units $u_i \in k^\circ$, the valuation ring of k, $b_i \in k^*$ and $\mu_{ij} \in \mathbb{Z}$.

□

Chapter 5

Strong u-invariant

5.1 Introduction

Definition 5.1.1. Let k be a field with $\mathrm{char}(k) \neq 2$. The *u-invariant of* k, denoted by $u(k)$, is the maximal dimension of anisotropic quadratic forms over k. We say that $u(k) = \infty$ if there exist anisotropic quadratic forms over k of arbitrarily large dimension.

Example 5.1.2. u-invariant of an algebraic closed field K, $u(K) = 1$.

Example 5.1.3. $u(\mathbb{R}) = \infty$.

Example 5.1.4. For k a finite field, $u(k) = 2$.

The u-invariant is a positive integer if it is finite. A key area of research in the theory of quadratic forms is to find all the possible values this invariant can take for a given field. For example, it has been established in the literature (see [Lam05, Chapter XI, Proposition 6.8]) that the u-invariant cannot take values 3, 5, and 7. See [Lam05, Chapter XIII, Section 6], for more open problems about this invariant. Considerable progress has also been made, particularly in the computation of the u-invariant of function fields of p-adic curves due to Parimala and Suresh in [PS10], [PS14], and by Harbater, Hartmann, and Krashen regarding the u-invariants in the case of function fields of curves over complete discretely valued fields in [HHK09].

Harbater, Hartmann, and Krashen make the following definition:

Definition 5.1.5. The *strong u-invariant of* K, denoted by $u_s(K)$, as the smallest real number m such that, $u(E) \leq m$ for all finite field extensions E/K, and $u(E) \leq 2m$ for all finitely generated field extensions E/K of transcendence degree 1.

We say that $u_s(K) = \infty$ if there exists such field extension E of arbitrarily large u-invariant. In [HHK09, Theorem 4.10], the same authors prove that for K a complete discretely valued field, whose residue field $\tilde{K}$ has characteristic away from 2, $u_s(K) = 2u_s(\tilde{K})$.

The following discussion can be found in Commutative Algebra by Bourbaki [Bou98, Chap. 6, §10, n○2]. For an abelian group Γ, define the *rational rank* of Γ, denoted by *rat.rank*(Γ), to be the $\dim_{\mathbb{Q}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{Q})$. The rational rank is an element of $\mathbb{N} \cup \{+\infty\}$. If Γ' is a subgroup of Γ, then since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, we have that *rat.rank*$(\Gamma) =$ *rat.rank*(Γ') + *rat.rank*(Γ/Γ'). The rational rank of a group Γ is zero if and only if Γ is a torsion group. If Γ is the value group of a valuation, then its rational rank is zero if and only if the valuation is the trivial valuation. Also, if the field L is an algebraic extension of the valued field k, then the quotient group $|L^*|/|k^*|$ is a torsion group, and the residue field $\tilde{L}$ is an algebraic extension of $\tilde{k}$ [Vaq06, Proposition 1.16].

5.2 Review of literature: Complete discretely valued fields

Kaplansky asked if *the* u-*invariant of a field is always a power of* 2. It is well known in the literature that the u-invariant does not take the values $3, 5, 7$. In the 1990s, Merkurjev constructed examples of fields k with $\mathrm{u}(k) = 2m$ for any $m \geq 1$ in [Mer92]. Since $m = 3$ was the first open case, this answered Kaplansky's question in the negative. It has been shown that the u-invariant could be odd. Izhboldin proved in [Izh01] that there exist fields k with $\mathrm{u}(k) = 9$, and then in [Vis07], Vishik has shown that there are fields k with $\mathrm{u}(k) = 2^r + 1$ for all $r \geq 3$. Merkurjev's, Izhboldin's and Vishik's constructions yield fields k, which are not of arithmetic type, i.e., they are not finitely generated over a number field or a p-adic field. It is still an interesting question whether $\mathrm{u}(k)$ is a power of 2 if k is of arithmetic type.

Theorem 5.2.1 (Theorem 4.10,[HHK09])**.** *For k a complete discretely valued field, whose residue field $\tilde{k}$ has characteristic unequal to 2,* $\mathrm{u}_s(k) = 2\mathrm{u}_s(\tilde{k})$.

5.3 Review of literature: Complete non-Archimedean valued fields

Let k be a complete non-Archimedean valued field with residue field $\tilde{k}$ and $\mathrm{char}(\tilde{k}) \neq 2$. Let $\sqrt{|k^*|} := \{a \in \mathbb{R}^* : a^z \in |k^*| \text{ for some } z \in \mathbb{Z}_{\neq 0}\}$ denote the *divisible closure of the value group* $|k^*|$.

Theorem 5.3.1 ([Meh19b])**.** *If $\dim_{\mathbb{Q}} \sqrt{|k^*|} = n$ is finite, then $u_s(k) \leq 2^{n+1} u_s(\tilde{k})$. And if $|k^*|$ is a free $\mathbb{Z}$-module with $\mathrm{rank}_{\mathbb{Z}} |k^*| = n$, then $u_s(k) \leq 2^n u_s(\tilde{k})$.*

Mehmeti used field patching in the setting of Berkovich analytic geometry to prove a local-global principle, and provides applications to quadratic forms and the u-invariant. The results she obtained generalise those of [HHK09].

5.4 Results regarding strong u-invariant

Theorem 5.4.1. *Let k be a complete ultrametric field with $\mathrm{char}(\tilde{k}) \neq 2$. Suppose that $\dim_{\mathbb{Q}}(\sqrt{| k^* |}) = n$ is finite. Then $u(k) \leq 2^n u(\tilde{k})$.*

Proof. Given a quadratic form $q = \langle a_1, \ldots, a_s \rangle$ over k of dimension $s > 2^n u(\tilde{k})$, such that $| a_i | \in \sqrt{| k^* |}$ for all i. From the main decomposition theorem, we have elements $c_1, \ldots, c_n$ in k^* such that each $a_i = u_i \prod_{j=1}^{n} {c_j}^{\lambda_{ij}} b_i^2$, for u_i unit in $(k^\circ)^*$, $b_i \in k^*$, and $\lambda_{ij} \in \{0, 1\}$.

So there are 2^n possibilities for $\prod_{j=1}^{n} c_j^{\lambda_{ij}}$. Let's call them $\theta_1, \theta_2, \ldots, \theta_{2^n}$. Thus each $a_i \equiv u_i\theta_j (\mod (L^*)^2)$, for some j with $1 \leq j \leq 2^n$.

So after re-indexing, we have, $\langle a_1, \ldots, a_s\rangle \cong \langle u_1, \ldots, u_{s_1}\rangle\theta_1 \perp \langle u_{s_1+1}, \ldots, u_{s_2}\rangle\theta_2 \perp \cdots \perp \langle u_{s_{2^n-1}+1}, \ldots, u_{s_{2^n}}\rangle\theta_{2^n}$, where all the s_i's add up to s. Consequently, for any θ_i as above, there exists a diagonal quadratic form σ_{θ_i} with coefficients in $(k^\circ)^*$ such that q is k-isometric to $\perp_{\sigma \in Q} \theta_i \cdot \sigma_{\theta_i}$.

Since $s > 2^n u(\tilde{k})$, there exists i such that $\dim(\sigma_{\theta_i}) = \dim(\langle u_{s_i+1}, \ldots, u_{s_{i+1}}\rangle) = s_{i+1} - s_i > u(\tilde{k})$. Thus σ_{θ_i} is isotropic over $\tilde{k}$. So σ_{θ_i} is isotropic over k. Thus q is isotropic over k, making $u(k) \leq 2^n u(\tilde{k})$. □

Remark 5.4.2. If the value group $\mid k^* \mid$ is a finitely generated free module of rank n, then the above result is a consequence of Pumplün, [Pum09, Theorem 4].

This gives us a refinement of the results on u-invariant from Mehmeti's paper [Meh19b].

Recall the discussion regarding the rational rank of abelian groups from Section 5.1. In our situation, consider a complete ultrametric valued field k. Let L be a valued field extension of k. Let $|L^*|$, $|k^*|$ be the value groups of L and k, respectively. We thus have $\dim_{\mathbb{Q}}(\sqrt{|L^*|}) = \dim_{\mathbb{Q}}(\sqrt{|k^*|}) + \dim_{\mathbb{Q}}(\frac{|L^*|}{|k^*|} \otimes_{\mathbb{Z}} \mathbb{Q})$. If L is an algebraic extension of k, then the quotient group $|L^*|/|k^*|$ is a torsion group. Thus $\dim_{\mathbb{Q}}(|L^*|/|k^*| \otimes_{\mathbb{Z}} \mathbb{Q}) = 0$, which implies that the two divisorial closures $\sqrt{|L^*|}$ and $\sqrt{|k^*|}$ have the same dimension over $\mathbb{Q}$.

Theorem 5.4.3. *Let k be a complete ultrametric field with char$(\tilde{k}) \neq 2$. Suppose dim$_{\mathbb{Q}}(\sqrt{\mid k^* \mid}) = n$ is finite. Let C be a curve over k and $F = k(C)$ the function field of the curve. Let q be a quadratic form over F with dimension d. If $d > 2^{n+1}u_s(\tilde{k})$, then q is isotropic. In particular, $u(F) \leq 2^{n+1}u_s(\tilde{k})$.*

Proof. Let $V(F)$ be the set of all non-trivial rank 1 valuations of F which either extend the valuation of k or are trivial when restricted to k. Let F_v denote the completion of F with respect to v. We will first show that for a quadratic form q over F with $\dim(q) > 2^{n+1}u_s(\tilde{k})$, q is isotropic over F_v for all $v \in V(F)$. Then we will apply Corollary 3.19 from [Meh19b] to say that q is isotropic over F.

Since we want to say that q is isotropic over F_v for all $v \in V(F)$, we have two cases. If $v \in V(F)$ is such that v restricted to k is the trivial valuation, then F_v is a discrete valued field. Let $\widetilde{F_v}$ be the residue field of F_v. By Springer's theorem on non-dyadic complete discrete valuation fields (see [Lam05], VI.1.10 and XI.6.2(7)), we have $u(F_v) = 2u(\widetilde{F_v})$. Since $\widetilde{F_v}$ is a finite extension of k, $\widetilde{F_v}$ is a complete ultrametric field with residue field $\widetilde{\widetilde{F_v}}$ a finite field extension of $\tilde{k}$. Using the fact that $\widetilde{F_v}$ is an algebraic extension of k, then the quotient group $|\widetilde{F_v}^*|/|k^*|$ is a torsion group. Thus $\dim_{\mathbb{Q}}(|\widetilde{F_v}^*|/|k^*|\otimes_{\mathbb{Z}}\mathbb{Q}) = 0$, which in turn implies that the two divisorial closures $\sqrt{|\widetilde{F_v}^*|}$ and $\sqrt{|k^*|}$ have the same dimension over $\mathbb{Q}$. It then follows from Theorem 5.4.1 that $u(\widetilde{F_v}) \leq 2^n u(\widetilde{\widetilde{F_v}}) \leq 2^n u_s(\tilde{k})$. Thus, $u(F_v) \leq 2^{n+1}u_s(\tilde{k})$.

In the case that v restricted to k is the valuation on k, then $\widetilde{F_v}$ is an extension of $\tilde{k}$ of transcendence degree ≤ 1. Let $s = \dim_{\mathbb{Q}}\Big(\frac{|\ F_v^*\ |}{|\ k^*\ |} \otimes_{\mathbb{Z}} \mathbb{Q}\Big)$ and $t = \operatorname{tr\,deg}_{\tilde{k}}(\widetilde{F_v})$. By Abhyankar's inequality [Abh56], we have $0 \leq s + t \leq 1$.

Suppose $t = 0$. Then $\widetilde{F_v}$ is a finite extension of $\tilde{k}$, and hence $u(\widetilde{F_v}) \leq u_s(\tilde{k})$. Since $t = 0$ and $0 \leq s + t \leq 1$, we have $s \leq 1$. When $s = 0$, the discussion before Theorem 5.4.3 tells us that, $\dim_{\mathbb{Q}}(\sqrt{|F_v^*|}) = \dim_{\mathbb{Q}}(\sqrt{|k^*|}) + \dim_{\mathbb{Q}}(\frac{|F_v^*|}{|k^*|} \otimes_{\mathbb{Z}} \mathbb{Q}) = \dim_{\mathbb{Q}}(\sqrt{|k^*|}) + s$. Thus $\dim_{\mathbb{Q}}(\sqrt{|F_v^*|}) = \dim_{\mathbb{Q}}(\sqrt{|k^*|}) = n$. From Theorem 5.4.1, we thus have $u(F_v) \leq 2^n u(\widetilde{F_v}) \leq 2^n u_s(\tilde{k})$. When $s = 1$, again the same discussion implies that, $\dim_{\mathbb{Q}}(\sqrt{|F_v^*|}) = \dim_{\mathbb{Q}}(\sqrt{|k^*|}) + s$. Thus $\dim_{\mathbb{Q}}(\sqrt{|F_v^*|}) = n+1$. Applying Theorem 5.4.1 again, we have $u(F_v) \leq 2^{n+1}u(\widetilde{F_v}) \leq 2^{n+1}u_s(\tilde{k})$.

For the case when $t = 1$ and $s = 0$, the extension $\widetilde{F_v}$ over $\tilde{k}$ is finitely generated of transcendence degree 1. Thus, $u(\widetilde{F_v}) \leq 2u_s(\tilde{k})$. Once again applying Theorem 5.4.1, we get that $u(F_v) \leq 2^n u(\widetilde{F_v}) \leq 2^{n+1}u_s(\tilde{k})$.

Now applying 3.2.16, we get that q is isotropic over F. This proves the claim in the theorem that $u(F) \leq 2^{n+1}u_s(\tilde{k})$. □

Corollary 5.4.4. *Let k be a complete ultrametric field with char$(\tilde{k}) \neq 2$. Suppose dim$_{\mathbb{Q}}(\sqrt{\mid k^* \mid}) = n$ is finite. Then $u_s(k) \leq 2^n u_s(\tilde{k})$.*

Proof. Let K/k be a finite field extension. Then K is an ultrametric field and $\tilde{K}/\tilde{k}$ is a finite extension. Let q be a K-quadratic form of dimension $d > 2^n u_s(\tilde{k})$. Since char$(\tilde{k}) \neq 2$, we may assume q to be diagonal. Further, $\dim_{\mathbb{Q}}(\sqrt{|K^*|}) = \dim_{\mathbb{Q}}(\sqrt{|k^*|}) = n$ is finite. This equality follows from the discussion before Theorem 4.1, and noting that since K is a finite extension of k, the quotient group $|K^*|/|k^*|$ is torsion. Then applying Theorem 5.4.1, we have $u(K) \leq 2^n u(\tilde{K}) \leq 2^n u_s(\tilde{k})$.

Let C be a curve over k and $F = k(C)$ the function field of C. Then it follows from Theorem 5.4.3 that $u(F) \leq 2^{n+1} u_s(\tilde{k})$. Then by the definition of strict u-invariant, we have that $u_s(k) \leq 2^n u_s(\tilde{k})$. □

One of the next steps is going to be to try to prove equalities in all of the above results. In a recent arXiv upload (independently from the work done in this manuscript), Becher-Daans-Mehmeti in [BDM25] show a more general result on the u-invariant of function fields in one variable over a henselian valued field with arbitrary value group and with residue field of characteristic different from 2.

Chapter 6

Brauer l-dimension

6.1 Introduction

Definition 6.1.1. Let A be a central simple algebra over a field k. The *period* (or *exponent*) of A is the order of the class of A in the Brauer group of k.

Definition 6.1.2. Let A be a central simple algebra over a field k. The *index* of A is the degree of the division algebra D that lies in the class of A (i.e., A is a matrix ring over D).

The period and index always have the same prime factors, and the period always divides the index [GS17, Proposition 4.5.13].

The *period-index problem* asks whether all central simple algebras A over a given field k satisfy $\mathrm{ind}(A) \mid \mathrm{per}(A)^d$ for some fixed exponent d depending only on k. In the spirit of [PS14], we make the following definition. Let k be any field. For a prime l, define the *Brauer l-dimension* of k, denoted by $\mathrm{Br}_l\mathrm{dim}(k)$, to be the smallest integer $d \geq 0$ such that for every finite extension L of k and for every central simple algebra A over L of period a power of l, $\mathrm{ind}(A)$ divides $\mathrm{per}(A)^d$. The *Brauer dimension of* k, denoted by $\mathrm{Brdim}(k)$, is defined as the maximum of the Brauer l-dimension of k as l varies over all primes. It is expected that this invariant should increase by one upon a finitely generated field extension of transcendence degree one.

Saltman proved some results in this direction, including the fact that the index divides the period squared for function fields of p-adic curves [Sal97]. Harbater, Hartmann, and Krashen in [HHK09, Theorem 5.5] consider a complete discretely valued field k, its residue field $\tilde{k}$, F the function field of a curve over k, and $l \neq \text{char}(\tilde{k})$. They prove that if $\text{Br}_l\text{dim}(\tilde{k}) \leq d$ and $\text{Br}_l\text{dim}(\tilde{k}(T)) \leq d+1$, then $\text{Br}_l\text{dim}(F) \leq d+2$. In 2006, Michael Artin conjectured in [Art06] that $\text{Brdim}(k) = 1$ for every C_2 field. In the same paper, he also proved that $\text{Br}_2\text{dim}(k) = \text{Br}_3\text{dim}(k) = 1$ for such fields.

6.2 Review of literature: Complete discretely valued fields

Let k be a p-adic field and let K over k be a finitely generated field extension of transcendence degree 1. For such fields K, Saltman in [Sal97] showed that $\text{Br}_l\text{dim}(K) = 2$ for every prime $l \neq p$. This is known as the *good characteristic* case.

More generally, if k is a complete discretely valued field with residue field $\tilde{k}$ such that $\text{Br}_l\text{dim}(\tilde{k}) \leq d$ for all primes $l \neq \text{char}(\tilde{k})$, then $\text{Br}_l\text{dim}(k) \leq d+1$ for all $l \neq \text{char}(\tilde{k})$, which is a classical result [GS17, Corollary 7.1.10].

Let F be the function field of a curve over a complete discretely valued field k, with residue field $\tilde{k}$. One of the long-standing period-index questions is to compute, or even show the finiteness of, $\text{Br}_l\text{dim}(F)$.

In the good characteristic case, *i.e.*, when $\text{char}(\tilde{k}) \neq l$, suppose there exists $d \in \mathbb{N}$ such that the Brauer l-dimension of the of the residue field, $\text{Br}_l\text{dim}(\tilde{k}) \leq d$ and the Brauer l-dimension of the function field for every curve C over $\tilde{k}$, $\text{Br}_l\text{dim}(\tilde{k}(C)) \leq d+1$. Then $\text{Br}_l\text{dim}(F) \leq d+2$, by Harbater-Hartmann-Krashen [HHK09].

In the bad characteristic case, *i.e.*, when $\text{char}(\tilde{k}) = l$. The Brauer l-dimension, $\text{Br}_l\text{dim}(F)$ was again shown to be at most 2 when k is an l-adic field, by the work of Parimala and Suresh in [PS14]. In fact, in the same paper, they also investigate the Brauer l-dimension of function fields whose residue fields are not necessarily perfect and obtain a more general result. Combining the results of Saltman and Parimala-Suresh, we can now conclude that the Brauer dimension of the function field of a curve over a p-adic field is equal to 2.

6.3 Necessary results from literature

We will need the following theorem (stated here without proof) to develop the rest of the manuscript. As in Milnor's Algebraic K-Theory, let L be an arbitrary field, n a positive integer with a non-zero image in L. The K-theory of L forms a graded anti-commutative ring $\oplus_{i\geq 0}K_i(L)$ with $K_0(L) \cong \mathbb{Z}$, $K_1(L) \cong \mathrm{GL}_1(L)$. Denote by (a) for an element of $K_1(L)$ that corresponds with $a \in L^*$, such that $(a) + (b) = (ab)$. Multiplying $(a), (b) \in K_1(L)$ yields an element of $K_2(L)$ that is written as (a, b). One calls (a, b) a Steinberg symbol, or simply, a *symbol*.

In the literature, there are two descriptions of the main theorem of Merkurjev and Suslin, one with Galois cohomology and one with Brauer groups. Let μ_n be the n-th roots of unity in an algebraic closure $\bar{L}$ of L and denote by L^s the separable closure of L in $\bar{L}$. Now consider μ_n as a module for the absolute Galois group $\mathrm{Gal}(L^s/L)$. By Hilbert Theorem 90, the Galois cohomology group $H^1(L, \mu_n)$ is isomorphic with $k_2(L) := K_1(L)/nK_1(L)$ (see [GS17, Section 8.4]). Using a theorem of Matsumoto, and the product structures in K-theory and some Galois cohomology, it gives the following homomorphism called the *Galois symbol* or the *norm-residue homomorphism*

$$\alpha_{L,n} : k_2(L) \to H^2(L, \mu_n^{\otimes 2}).$$

The Merkurjev-Suslin theorem now simply reads

Theorem 6.3.1 (Merkurjev-Suslin)**.** *For all L, n as above, and the n-th roots of unity contained in L, $\alpha_{L,n}$ is an isomorphism.*

Recall the definition of a cyclic algebra over a field. For a positive integer n, let L be a field in which n is invertible such that L contains a primitive n-th root of unity

ω. For nonzero elements a and b of L, the associated *cyclic algebra* is the central simple algebra of degree n over L defined by

$$(a,b)_\omega = L\langle x,y\rangle/(x^n = a, y^n = b, xy = \omega yx).$$

We know from literature that the n-torsion of the Brauer group, ${}_n\mathrm{Br}(L)$ is isomorphic with $H^2(L,\mu_n)$ [CTS21, Section 1.3.4]. If $\mu_n \subset L$, then $\mu_n \simeq \mu_n^{\otimes 2}$; thus, ${}_n\mathrm{Br}(L)$ is isomorphic with $H^2(L,\mu_n^{\otimes 2})$. So, we find that $\alpha_{L,n} : k_2(L) \to {}_n\mathrm{Br}(L)$ sends the coset of the symbol (a,b) to the class of the the cyclic algebra $(a,b)_\omega$, where ω is the primitive n-th root of unity in L. We thus get the following corollary, which conveys the surjectivity of $\alpha_{L,n}$.

Corollary 6.3.2. *Let $\mu_n \subset L$ and let A be a central simple algebra over L, with $[A] \in {}_nBr(L)$. Then A is similar to a tensor product of cyclic algebras $(a_i,b_i)_\omega$, $a_i, b_i \in L^*$.*

The case $n = 2$ is actually Merkurjev's theorem.

To define a cyclic algebra over a commutative ring R, let n be an invertible positive integer in R. Suppose R contains a primitive n-th root of unity ω. For $a, b \in R^*$, the associated cyclic algebra is an Azumaya algebra of degree n over R defined by

$$(a,b)_\omega = R\langle x,y\rangle/(x^n = a, y^n = b, xy = \omega yx).$$

6.4 Preliminary results

The next theorem follows from the proof of Harbater, Hartmann, and Krashen's Theorem 5.1 in [HHK09] and Mehmeti's Corollary 3.19 in [Meh19b].

Theorem 6.4.1. *Let k be a complete, non-trivially valued ultrametric field and suppose $dim_{\mathbb{Q}}(\sqrt{|k^*|}) = n$ is finite. Let C be a normal irreducible projective k-algebraic curve. Denote by F the function field of C. Let A be a central simple algebra over*

F and let $V(F)$ be the set of all non-trivial rank 1 valuations of F. Then ind(A) is the maximum of the set $\{ind(A \otimes F_v)\}$ for $v \in V(F)$.

Proof. Let n be the degree of A. Then $\mathrm{GL}_1(A)$ is a Zariski open subset of $\mathbb{A}_F^{n^2}$. So, it is a rational connected linear algebraic group.

For $1 \le i \le n$, let $\mathrm{SB}_i(A)$ be the i-th generalised Severi-Brauer variety of A. Then $\mathrm{GL}_1(A)$ acts transitively on the points of $\mathrm{SB}_i(A)$.

Further if E is a field extension of F, then $\mathrm{SB}_i(A)(E) \neq \phi$ if and only if $\mathrm{ind}(A_E)$ divides i [KMRT98, Proposition 1.17]. So, the local-global principle proved by Mehmeti in [Meh19b] implies that $\mathrm{ind}(A) \mid i$ if and only if $\mathrm{ind}(A \otimes F_v) \mid i$ for each $v \in V(F)$. Thus, $\mathrm{ind}(A) = \max_{v \in V(F)} \{\mathrm{ind}(A \otimes F_v)\}$ as claimed. □

We also record here the following well-known result.

Proposition 6.4.2. *Let R be a complete local domain with field of fractions F and residue field k. Let A be an Azumaya algebra over R. Then*

$$ind(A \otimes_R F) \le ind(A \otimes_R k).$$

Proof. Note that $A \otimes_R k$ is a central simple algebra over k. Let $n = \mathrm{ind}(A \otimes_R k)$. Then there exists a separable field extension L/k of degree n such that $(A \otimes_R k) \otimes L \simeq M_n(L)$ [GS17, Proposition 2.25]. Since L/k is a finite separable extension, $L = k(\alpha)$ for some $\alpha \in L$. Let $f(x) \in k[X]$ be the monic minimal polynomial of α over k. Let $g(X) \in R[X]$ be a monic polynomial of degree n which maps to $f(X)$ in $k[X]$. Since $f(X)$ is irreducible in $k[X]$, $g(X)$ is irreducible in $R[X]$. Let $S = R[X]/(g(X))$ and $E = F[X]/(g(X))$. Since R is a complete local ring, S is also a complete local ring with residue field L. By [Mil80, I, Proposition 4.5], every complete local ring is Henselian. Since the map $Br(S) \to Br(L)$ is injective [Mil80, IV, Corollary 2.13] and $(A \otimes_R S) \otimes_S L \simeq (A \otimes_R k) \otimes_k L$ is a matrix algebra, $A \otimes_R S$ represents the trivial element in $Br(S)$. Hence $(A \otimes_R F) \otimes E$ is a matrix algebra. In particular $\mathrm{ind}(A \otimes_R F) \le [E : F] = n$. □

Theorem 6.4.3. *Let k be an ultrametric field such that $dim_{\mathbb{Q}}(\sqrt{|k^*|}) = n$ is finite. Let $\tilde{k}$ be the residue field of k, k° the valuation ring, and l such that $l \neq char(k)$. Let $\mu_l \subset k$. Let $A \in {}_lBr(k)$. Then there exists $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ in k; $A_0 \in {}_lBr(k^\circ)$ such that*

$$A = A_0 \otimes \prod_{i=1}^{n} (a_i, b_i)_l.$$

Proof. Let $A \in {}_l\text{Br}(k)$. Merkurjev-Suslin's theorem implies that $A = \prod_{i=1}^{m} (\alpha_i, \beta_i)_l$ for some $m \in \mathbb{N}$. By Theorem 4.0.3, there exists $c_1, c_2, \ldots, c_n$ in k^* such that $\alpha_i = u_i \prod c_j^{r_{ij}}$ and $\beta_i = v_i \prod c_j^{s_{ij}}$, where r_{ij} and s_{ij} are integers and u_i, v_i are units.

So,

$$(\alpha_i, \beta_i) = (u_i \prod c_j^{r_{ij}}, v_i \prod c_j^{s_{ij}}) = (u_i, v_i) \otimes \sum (u_i, c_j^{s_{ij}}) \otimes \sum (c_j^{r_{ij}}, v_i) \otimes \sum (\prod c_j^{r_{ij}}, c_j^{s_{ij}}).$$

Let $\prod c_j^{r_{ij}} = d_i$. So we get that

$$\begin{aligned}(\alpha_i, \beta_i) &= (u_i, v_i) \otimes \sum (u_i, c_j^{s_{ij}}) \otimes \sum (c_j^{r_{ij}}, v_i) \otimes \sum (d_i, c_j^{s_{ij}}) \\ &= (u_i, v_i) \otimes \sum (u_i^{s_{ij}}, c_j) \otimes \sum (v_i^{-r_{ij}}, c_j) \otimes \sum (d_i^{s_{ij}}, c_j),\end{aligned}$$

the last equality coming from properties of symbols, i.e., $(\alpha, \beta^j) = (\alpha^j, \beta)$ and $(\alpha, \beta) = -(\beta, \alpha) = (\beta^{-1}, \alpha)$. Now, combining the terms, we get the following equality

$$(\alpha_i, \beta_i) = (u_i, v_i) \otimes \sum_{j=1}^{n} (u_i^{s_{ij}} v_i^{-r_{ij}} d_i^{s_{ij}}, c_j).$$

Let $x_j = u_i^{s_{ij}} v_i^{-r_{ij}} d_i^{s_{ij}}$ in k^*. Thus, we have $A = \prod_{i=1}^{m} \big((u_i, v_i) \otimes \sum_{j=1}^{n} (x_j, c_j)\big)$.

Let $A_0 = \otimes (u_i, v_i)_{k^\circ}$. Since each cyclic algebra $(u_i, v_i)_{k^\circ}$ is an Azumaya algebra over k°, $A_0 \in \text{Br}(k^\circ)$ and we thus have $A = A_0 \otimes \prod_{i=1}^{n} (a_i, b_i)$.

□

Once again, recall the discussion regarding the rational rank of abelian groups from Section 5.1. In our situation, consider a complete ultrametric valued field k. Let L be a valued field extension of k. Let $|L^*|$, $|k^*|$ be the value groups of L and k, respectively. We thus have $\dim_{\mathbb{Q}}(\sqrt{|L^*|}) = \dim_{\mathbb{Q}}(\sqrt{|k^*|}) + \dim_{\mathbb{Q}}(\frac{|L^*|}{|k^*|} \otimes_{\mathbb{Z}} \mathbb{Q})$. If L is an algebraic extension of k, then the quotient group $|L^*|/|k^*|$ is a torsion group. Thus $\dim_{\mathbb{Q}}(|L^*|/|k^*| \otimes_{\mathbb{Z}} \mathbb{Q}) = 0$, which implies that the two divisorial closures $\sqrt{|L^*|}$ and $\sqrt{|k^*|}$ have the same dimension over $\mathbb{Q}$.

6.5 Results regarding Brauer l-dimension

Theorem 6.5.1. *Let k be a complete ultrametric field. Suppose $dim_{\mathbb{Q}}(\sqrt{|k^*|}) = n$ is finite. Let l be a prime such that $l \neq char(\tilde{k})$ and $\mu_l \subset k$. Then the $Br_l dim(k) \leq Br_l dim(\tilde{k}) + n$.*

Proof. Consider the l-torsion of $\mathrm{Br}(k)$, denoted by ${}_l\mathrm{Br}(k)$. Then for a given $A \in {}_l\mathrm{Br}(k)$, by Theorem 6.4.3, there exists $A_0 \in {}_l\mathrm{Br}(k^\circ)$ and $c_1, c_2, \ldots, c_n \in k^*$ such that $(A - A_0) \otimes k(\sqrt[l]{c_1}, \sqrt[l]{c_2}, \ldots, \sqrt[l]{c_n})$ is split. Let L be the field $k(\sqrt[l]{c_1}, \sqrt[l]{c_2}, \ldots, \sqrt[l]{c_n})$. Then the algebra $(A - A_0) \otimes L$ is split.

This implies that $\mathrm{ind}(A)$ divides $\mathrm{ind}(A_0)l^n$. Since k° is complete, it follows from Proposition 6.4.2 that $\mathrm{ind}(A_0) \leq \mathrm{ind}(A_0 \otimes_{k^\circ} \tilde{k})$, hence divides $\mathrm{Br}_l\mathrm{dim}(\tilde{k})$. So, $\mathrm{Br}_l\mathrm{dim}(k) \leq \mathrm{Br}_l\mathrm{dim}(\tilde{k}) + n$.

□

Corollary 6.5.2. *Let k be a complete ultrametric field. Suppose $dim_{\mathbb{Q}}(\sqrt{|k^*|}) = n$ is finite. Let l be a prime such that $l \neq char(\tilde{k})$. Then the $Br_l dim(k) \leq Br_l dim(\tilde{k}) + n$.*

Proof. Let $A \in {}_l\mathrm{Br}(k)$. Let ρ_l be the l-th primitive root of unity. So, the degree of the extension $k(\rho_l)$ over k is at most $l - 1$, i.e., $l \nmid [k(\rho_l) : k]$. Thus, $\mathrm{ind}(A) = \mathrm{ind}(A \otimes k(\rho_l))$ and $\mathrm{per}(A) = \mathrm{per}(A \otimes k(\rho_l))$, by [PP82], Propositions 13.4(vi) and 14.4b(v). Hence, the corollary follows from Theorem 6.5.1.

□

Theorem 6.5.3. *Let k be a complete ultrametric field. Suppose $dim_{\mathbb{Q}}(\sqrt{|k^*|}) = n$ is finite. Let l be a prime such that $l \neq char(\tilde{k})$. Let C be a curve over k and $F = k(C)$ the function field of the curve. Suppose there exist an integer d such that $Br_l dim(\tilde{k}) \leq d$ and $Br_l dim(\tilde{k}(t)) \leq d+1$. Then the $Br_l dim(F) \leq d+1+n$.*

Proof. Let $V(F)$ be the set of all non-trivial rank 1 valuations of F which either extend the valuation of k or are trivial when restricted to k. Let F_v denote the completion of F with respect to v.

If $v \in V(F)$ is such that v restricted to k is the trivial valuation, then F_v is a discrete valued field. Let $\widetilde{F_v}$ be the residue field of F_v. Since $\widetilde{F_v}$ is a finite extension of k, it is a complete ultrametric field with residue field $\widetilde{\widetilde{F_v}}$ a finite extension of $\tilde{k}$. By the definition of $\mathrm{Br}_l\mathrm{dim}$, we have that $\mathrm{Br}_l\mathrm{dim}(\widetilde{\widetilde{F_v}}) \leq \mathrm{Br}_l\mathrm{dim}(\tilde{k})$. Note that F_v is a complete discretely valued field, thus $\mathrm{Br}_l\mathrm{dim}(F_v) \leq \mathrm{Br}_l\mathrm{dim}(\widetilde{F_v}) + 1$, which follows from [HHK09, Theorem 5.5]. Now utilising the fact that $\widetilde{F_v}$ is an algebraic extension of k, the quotient group $|\widetilde{F_v}^*|/|k^*|$ is a torsion group. Thus, $\dim_{\mathbb{Q}}(|\widetilde{F_v}^*|/|k^*| \otimes_{\mathbb{Z}} \mathbb{Q}) = 0$, which in turn implies that the two divisorial closures $\sqrt{|\widetilde{F_v}^*|}$ and $\sqrt{|k^*|}$ have the same dimension over $\mathbb{Q}$. Applying Theorem 6.5.1 to $\widetilde{F_v}$, we have that $\mathrm{Br}_l\mathrm{dim}(\widetilde{F_v}) \leq \mathrm{Br}_l\mathrm{dim}(\widetilde{\widetilde{F_v}}) + n$. Putting it all together gives $\mathrm{Br}_l\mathrm{dim}(F_v) \leq \mathrm{Br}_l\mathrm{dim}(\widetilde{F_v})+1 \leq \mathrm{Br}_l\mathrm{dim}(\widetilde{\widetilde{F_v}})+n+1 \leq \mathrm{Br}_l\mathrm{dim}(\tilde{k})+n+1 \leq d+n+1$.

In the case that v restricted to k is the valuation on k, then $\widetilde{F_v}$ is an extension of $\tilde{k}$ of transcendence degree ≤ 1. Let $s = \dim_{\mathbb{Q}}\Big(\frac{|\ F_v^*\ |}{|\ k^*\ |} \otimes_{\mathbb{Z}} \mathbb{Q}\Big)$ and $t = \mathrm{tr}\ \mathrm{deg}_{\tilde{k}}(\widetilde{F_v})$. By Abhyankar's inequality [Abh56], we have $0 \leq s+t \leq 1$.

Suppose $t = 0$. Then $\widetilde{F_v}$ is a finite extension if $\tilde{k}$. Thus, $\mathrm{Br}_l\mathrm{dim}(\widetilde{F_v}) \leq \mathrm{Br}_l\mathrm{dim}(\tilde{k})$. Since $t = 0$ and $0 \leq s+t \leq 1$, we have $s \leq 1$. When $s = 0$, the discussion at the beginning of this section implies that $\dim_{\mathbb{Q}}(\sqrt{|F_v^*|}) = \dim_{\mathbb{Q}}(\sqrt{|k^*|}) + \dim_{\mathbb{Q}}(\frac{|F_v^*|}{|k^*|} \otimes_{\mathbb{Z}} \mathbb{Q}) = \dim_{\mathbb{Q}}(\sqrt{|k^*|}) + s$. Thus $\dim_{\mathbb{Q}}(\sqrt{|F_v^*|}) = \dim_{\mathbb{Q}}(\sqrt{|k^*|}) = n$. Applying Theorem 6.5.1 now gives that, $\mathrm{Br}_l\mathrm{dim}(F_v) \leq \mathrm{Br}_l\mathrm{dim}(\widetilde{F_v}) + n \leq \mathrm{Br}_l\mathrm{dim}(\tilde{k}) + n \leq d+n$. When $s = 1$, once again the discussion at the beginning of this section tells us that, $\dim_{\mathbb{Q}}(\sqrt{|F_v^*|}) = \dim_{\mathbb{Q}}(\sqrt{|k^*|}) + s$. Thus $\dim_{\mathbb{Q}}(\sqrt{|F_v^*|}) = n+1$. By Theorem 6.5.1,

$\mathrm{Br}_l\mathrm{dim}(F_v) \leq \mathrm{Br}_l\mathrm{dim}(\widetilde{F_v}) + \dim_{\mathbb{Q}}(\sqrt{|F_v^*|}) \leq \mathrm{Br}_l\mathrm{dim}(\tilde{k}) + n + 1 \leq d + n + 1.$

For the case when $t = 1$ and $s = 0$, the extension $\widetilde{F_v}$ over $\tilde{k}$ is finitely generated of transcendence degree 1, *i.e.,* $\widetilde{F_v}$ is a finite extension of $\tilde{k}(T)$. So, $\mathrm{Br}_l\mathrm{dim}(\widetilde{F_v}) \leq \mathrm{Br}_l\mathrm{dim}(\tilde{k}(T))$. By Theorem 6.5.1, $\mathrm{Br}_l\mathrm{dim}(F_v) \leq \mathrm{Br}_l\mathrm{dim}(\widetilde{F_v}) + n \leq \mathrm{Br}_l\mathrm{dim}(\tilde{k}(T)) + n \leq d + 1 + n$.

Thus for all $v \in V(F)$, we have $\mathrm{Br}_l\mathrm{dim}(F_v) \leq d + n + 1$. From Theorem 6.4.1, we see that $\mathrm{Br}_l\mathrm{dim}(F) \leq \mathrm{Br}_l\mathrm{dim}(F_v)$, proving the claim of the theorem.

□

6.6 Future directions

In the future, we hope to develop specialised methods for the bad characteristic case, i.e., give a bound for $\mathrm{Br}_l\mathrm{dim}(k)$ when $\mathrm{char}(\tilde{k}) = l$. In particular, try to adapt the techniques from [PS14], which are based on bounding $\mathrm{Br}_l\mathrm{dim}(k)$ in terms of the l-rank of the residue field $\tilde{k}$.

Appendix A

Embedding Problem

I am working with Brandon Alberts, Helen Grundman, Amanda Tucker, and Alexander Slamen on a project that aims at counting number fields. The archetypal question in number field counting asks

Question 1. Given a fixed number field K and a numerical invariant $\mathrm{inv}(L/K)$ of finite extensions of K, what is the asymptotic behaviour of the counting function $N(K;X) = \#\{L/K \mid \mathrm{inv}(L/K) < X\}$ as $X \to \infty$?

Classically, one takes the absolute discriminant as the invariant $\mathrm{inv}(L/K) = |\mathrm{N}_{K/\mathbb{Q}}(\mathrm{disc}(L/K))|$ since a theorem of Hermite [Neu99] tells us $N(K;X) < \infty$ for every X. This counting function is often subdivided further into functions $N_d(K;X)$, which count only those extensions of degree d. Folklore conjectures assert that $N_d(K;X) \sim c_{d,K}X$ as $X \to \infty$ for some positive constant $c_{d,K}$. Classical work of Davenport and Heilbronn [DH71] shows that $N_3(\mathbb{Q};X) \sim \frac{1}{3\zeta(3)}X$.

The link between cycle types in the Galois group of the Galois closure $\tilde{L}/K$ and powers of primes in $\mathrm{disc}(L/K)$ [Mal02] motivates a further stratification of our count. Indeed, we often consider $N_d(K,G;X)$, which counts those degree d extensions whose Galois closures have Galois group G. Class field theoretic techniques of Mäki [M8̈5] and Wright [Wri89] give precise asymptotics for $N_d(K,G;X)$ with G abelian. Motivated by these works and the heuristic that led us to count by Galois group, Gunter Malle conjectured a general form for $N_d(K,G;X)$ [Mal04].

Conjecture 1 (Malle's Strong Conjecture). *There are constants* $a(G), b(K,G) \in \mathbb{Z}_{>0}$, $c > 0$ *such that* $N_d(K,G;X) \sim cX^{1/a(G)} \log(X)^{b(K,G)-1}$ *as* $X \to \infty$*. Moreover, the constants* $a(G), b(K,G)$ *have explicit formulas.*

Despite known issues with the power of the logarithm [Kl5], the conjecture has proven massively influential in number field counting. It is known to be true for abelian groups [M8̈5, Wri89] and in many non-abelian cases [Bha05, Bha10, CDyDO02, Wan21].

This project considers a further stratification of Malle's conjecture. In the spirit of [CDyDO02, Wan21] one can split up a count of extensions L/K by considering all possible (strict) intermediate extensions $L/M/K$ and counting the possible extensions L/M, M/K. More concretely, following the setup of [Alb21], if we fix a normal subgroup $T \lhd G$ with $G/T = B$ (for "top" and "bottom") any Galois extension L/K with Galois group G can be split as $L/L^T/K$ where L/L^T has Galois group T and L^T/K has Galois group B. Now one can count the number of L/K by considering all choices for L^T and computing the "twisted" counts

$$N(M/K, T \lhd G; X) = \#\{L/K \mid \mathrm{Gal}(L/K) \simeq G,\ L^T = M,\ |\mathrm{N}_{K/\mathbb{Q}}(\mathrm{disc}(L/K))| < X\}.$$

Conditional on the existence of a G-extension L/K with $L^T = M$, Alberts proves a Galois correspondence between such extensions L'/K, $(L')^T = M$ and crossed homomorphisms $Z^1(\mathrm{Gal}(\overline{K}/K), T)$, where T carries a Galois action that factors through $\mathrm{Gal}(M/K)$ [Alb21, Lemma 1.3], hence the use of the word "twisted". He then proposes the following twisted version of Malle's conjecture

Conjecture 2 (Twisted Malle Conjecture). *There exist constants* $a(T), b(K,T,G) \in \mathbb{Z}_{>0}$, $c > 0$ *such that* $N(M/K, T \lhd G; X) \sim cX^{1/a(T)} \log(X)^{b(K,T,G)-1}$ *as* $X \to \infty$*. Moreover, the constants* $a(T), b(K,T,G)$ *have explicit formulas.*

Alberts–O'Dorney prove this conjecture when T is abelian (and thus a Galois module) in [AO21] using class field theory type results. Lacking an analogue of class field theory, no such progress has been made when T is non-abelian. We consider this conjecture in the case $G = D_8$ (the dihedral group of 16 elements), T a normal subgroup isomorphic to D_4, $K = \mathbb{Q}$, and M an arbitrary quadratic field. In our motivating example, the center $N = Z(D_8) \cong C_2$ is a central subgroup of $T \cong D_4$, and we are proving the following

Expected Theorem. *Conjecture 2 is true for $D_4 \trianglelefteq D_8$ over $\mathbb{Q}$, i.e., there exist constants $a(D_4)$, $b(\mathbb{Q}, D_4, D_8)$ in $\mathbb{Z}_{>0}$, $c(\mathbb{Q}, D_4, D_8) > 0$ such that*

$$N(M/\mathbb{Q}, D_4 \lhd D_8; X) \sim c(\mathbb{Q}, D_4, D_8)\ X^{1/a(D_4)} \log(X)^{b(\mathbb{Q},D_4,D_8)-1}$$

as $X \to \infty$. Moreover, the constants $a(D_4), b(\mathbb{Q}, D_4, D_8)$ have explicit formulas.

Our approach is based on local-to-global principles in embedding problems — we do the twisted count by counting the number of solutions to a twisted embedding problem. In an embedding problem, one is given $N \lhd G$ and a G/N-extension $M/\mathbb{Q}$ and asked to find a G-extension $L/M/\mathbb{Q}$. By the Galois correspondence, this amounts to lifting a surjection $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to G/N$ to a surjection $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to G$. Results in homological algebra show that for particular choices of central kernel $N \subseteq Z(G)$, such a lift exists if and only if it exists locally everywhere [Ser92]. In the twisted setting, these problems become about lifting *crossed homomorphisms*, leading to the natural question

Question 2. Does our twisted embedding problem obey a local-to-global principle? More generally, is there a local-to-global principle for twisted central embedding problems?

Using a lemma of Alberts [Alb21, Lemma 1.3], we have found that there is a crossed local-to-global principle for central extensions of groups with a Galois action.

Moreover, we have evidence that the solubility of the corresponding local embedding problems is dictated by values of Legendre symbols with modulus equal to the discriminant of the T/N-extension. Simple transformations translate these solubility conditions into indicator functions detecting solutions to embedding problems, so by summing over possible T/N-discriminants, we count all solutions. Such sums of characters closely match those considered by Fouvry–Klüners in their work on 4-ranks of the class groups [FK07], which suggests that similar sieving techniques can be used to determine the asymptotic growth rate of $N(M/\mathbb{Q}, D_4 \lhd D_8; X)$.

We will then sum over all the choices for the intermediate extension $M/\mathbb{Q}$, which satisfies

$$N(\mathbb{Q}, D_8; X) = \sum_{\substack{M/\mathbb{Q} \\ \mathrm{Gal}(M/\mathbb{Q}) \simeq C_2}} N(M/\mathbb{Q}, D_4 \lhd D_8; X).$$

Being able to add these together would lead to new results for Malle's conjecture, as this would be the first non-abelian twisted count in the literature, with the Galois group acting on T by a nontrivial outer automorphism of D_4.

A.0.1 Future Directions

Recalling that our interest in twisted counts stemmed from a desire to break Malle's conjecture into pieces to later be summed together, a crucial consideration will be – *What error terms appear in the asymptotic* $N(M/\mathbb{Q}, D_4 \lhd D_8; X) \sim cX^{1/a(D_4)} \log(X)^{b(\mathbb{Q}, D_4, D_8)-1}$*? What can be said in general about desired error terms in twisted counts?* Since if the error terms are too large they may overpower the predicted main term when summed.

Another relevant question to consider is – *Can the twisted Malle's conjecture be verified with* $T \subseteq Z_2(G)$*, the second term in the lower central series for* G*?* The upshot here is that $T \cap Z(G)$ is a central subgroup of T and $T/T \cap Z(G) \subseteq Z(G/Z(G))$ is abelian, so T is "one central embedding problem" away from an abelian extension.

Appendix B

Non-Archimedean Differential Algebraic Geometry

This project is related to attacking properness from a non-Archimedean/tropical point of view. This is with Sreejani Chaudhury and Taylor Dupuy. Famously, the degree map on polynomials (or rather its negative) can be thought of as the valuation at infinity of $\mathbb{P}^1$. In connection with the Kolchin-Schmidt Conjecture on differential homogeneous polynomials, there is a notion of differential degree. The example we are interested in is $K\{x\} = K[x, x', x'', \dots]$, where K is a differential field. So, $K\{x\}$ is the set of polynomials in all these indeterminates, with the natural derivations (each polynomial involves only a finite number of indeterminates). The differential degree of a differential polynomial is the smallest d such that $f(\frac{x}{y})y^d$ is an element of $K\{x, y\}$. The negative of differential degree defines a valuation $v : K(\{x\}) \to \mathbb{Z} \cup \{\infty\}$. We have been wondering about the geometry of this valuation and wanted to see what other things we could find.

We have found a number of interesting valuations. First, the negative of the differential degree is definitely a valuation – it satisfies a certain compatibility condition appearing in the model theory literature (see van den Dries–Achenbrenner [AvdDvdH17]), and we can figure out what its residue field and completion are. Second, since maximal rank valuations come from term orderings, it turns out that the special term ordering used by Mourtada and his collaborators (see [Mou23]) has a valuation with special properties. This property is different from the previous property, and it is almost like a morphism of semi-rings, which is a differential morphism. This property was more or less isolated in some foundational papers for “Tropical Differential Algebraic Geometry”– the way they state it is for power-series rings (viewed as differential rings) and the way we have it for higher rank valuations associated with terms orderings seems to say this is something that can work “for generic points” too.

We are piecing together these new examples in [CDM24]. We know a little bit about what sort of subsets of the “full Riemann-Zariski space” (of the coordinate ring of the arc scheme for $\mathbb{A}^1$, say) one of our compatibility conditions needs to satisfy but can’t say anything conclusive about properness/non-properness of such a space yet. We have made some progress on specialising and generalising valuations in these examples. We have also started thinking a little bit about how these valuations fit into the perspective of tropicalisation as polytopes coming from changes in the initial ideal (Kapranov’s Theorem [MS21, Theorem 3.1.3]) as we vary the ordering and what this has to do with the Fundamental Theorem of Tropical Differential Algebraic Geometry [AGT16], $(Trop(V(I)) = V(Trop(I))$.

B.0.1 Future Directions

We are looking forward to exploring the “compatibility” conditions that we need for valuations on $K\{x\}$ to be agreeable with the differential operator. More generally, is it possible to comment about the Riemann-Zariski spaces of all valuations for a differential ring? In particular, what does the space of all valuations look like in the case of $K\{x$?

Bibliography

[Abh56] Shreeram Abhyankar, *On the valuations centered in a local domain*, American Journal of Mathematics **78** (1956), no. 2, 321–348.

[AGT16] Fuensanta Aroca, Cristhian Garay, and Zeinab Toghani, *The fundamental theorem of tropical differential algebraic geometry*, Pacific Journal of Mathematics **283** (2016), no. 2, 257–270.

[Alb21] Brandon Alberts, *Statistics of the first Galois cohomology group: a refinement of Malle's conjecture*, Algebra Number Theory **15** (2021), no. 10, 2513–2569. MR 4377858

[AO21] Brandon Alberts and Evan O'Dorney, *Harmonic analysis and statistics of the first Galois cohomology group*, Res. Math. Sci. **8** (2021), no. 3, Paper No. 50, 16. MR 4298097

[Art06] Michael Artin, *Brauer-severi varieties*, Brauer Groups in Ring Theory and Algebraic Geometry: Proceedings, University of Antwerp UIA, Belgium, August 17–28, 1981, Springer, 2006, pp. 194–210.

[AvdDvdH17] Matthias Aschenbrenner, Lou van den Dries, and Joris van der Hoeven, *Asymptotic differential algebra and model theory of transseries*, Annals of Mathematics Studies, vol. 195, Princeton University Press, Princeton, NJ, 2017. MR 3585498

[BDM25] Karim Johannes Becher, Nicolas Daans, and Vlerë Mehmeti, *The u-invariant of function fields in one variable*, arXiv preprint arXiv:2502.13086 (2025).

[Ber90] Vladimir G Berkovich, *Spectral theory and analytic geometry over non-archimedean fields*, no. 33, American Mathematical Soc., 1990.

[Ber93] ______, *Étale cohomology for non-archimedean analytic spaces*, Publications Mathématiques de l'IHÉS **78** (1993), 5–161.

[Bha05] Manjul Bhargava, *The density of discriminants of quartic rings and fields*, Ann. of Math. (2) **162** (2005), no. 2, 1031–1063. MR 2183288

[Bha10] ______, *The density of discriminants of quintic rings and fields*, Ann. of Math. (2) **172** (2010), no. 3, 1559–1591. MR 2745272

[Bou98] Nicolas Bourbaki, *Commutative algebra: chapters 1-7*, vol. 1, Springer Science & Business Media, 1998.

[BR10] Matthew Baker and Robert S Rumely, *Potential theory and dynamics on the berkovich projective line*, no. 159, American Mathematical Soc., 2010.

[CDM24] Sreejani Chaudhury, Taylor Dupuy, and Shilpi Mandal, *Investigating valuations on differential rings using non-archimedean differential algebraic geometry*, (in preparation).

[CDyDO02] Henri Cohen, Francisco Diaz y Diaz, and Michel Olivier, *Enumerating quartic dihedral extensions of* $\mathbb{Q}$, Compositio Math. **133** (2002), no. 1, 65–93. MR 1918290

[CTS21] Jean-Louis Colliot-Thélène and Alexei N Skorobogatov, *The brauer-grothendieck group*, vol. 71, Springer, 2021.

[DH71] H. Davenport and H. Heilbronn, *On the density of discriminants of cubic fields. II*, Proc. Roy. Soc. London Ser. A **322** (1971), no. 1551, 405–420. MR 491593

[dJ04] Aise Johan de Jong, *The period-index problem for the brauer group of an algebraic surface.*

[FK07] Étienne Fouvry and Jürgen Klüners, *On the 4-rank of class groups of quadratic number fields*, Invent. Math. **167** (2007), no. 3, 455–513. MR 2276261

[GS17] Philippe Gille and Tamás Szamuely, *Central simple algebras and galois cohomology*, vol. 165, Cambridge University Press, 2017.

[HH10] David Harbater and Julia Hartmann, *Patching over fields*, Israel Journal of Mathematics **176** (2010), no. 1, 61–107.

[HHK09] David Harbater, Julia Hartmann, and Daniel Krashen, *Applications of patching to quadratic forms and central simple algebras*, Inventiones mathematicae **178** (2009), no. 2, 231–263.

[HHK11] ______, *Patching subfields of division algebras*, Transactions of the American Mathematical Society **363** (2011), no. 6, 3335–3349.

[HHK14] ______, *Local-global principles for galois cohomology.*, Commentarii Mathematici Helvetici **89** (2014), no. 1.

[Izh01] Oleg T Izhboldin, *Fields of u-invariant 9*, Annals of Mathematics (2001), 529–587.

[Jah03] Jörg Jahnel, *The brauer-severi variety associated with a central simple algebra.*

[Kl5] Jürgen Klüners, *A counterexample to Malle's conjecture on the asymptotics of discriminants*, C. R. Math. Acad. Sci. Paris **340** (2005), no. 6, 411–414. MR 2135320

[KMRT98] MA Knus, A Merkurjev, M Rost, and JP Tignol, *The book of involutions. colloquium publication 44*, American Mathematical Society, Providence (1998).

[KS06] Maxim Kontsevich and Yan Soibelman, *Affine structures and non-archimedean analytic spaces*, Springer, 2006.

[Lam05] Tsit-Yuen Lam, *Introduction to quadratic forms over fields*, vol. 67, American Mathematical Soc., 2005.

[Lee13] David B Leep, *The u-invariant of p-adic function fields*, Journal für die reine und angewandte Mathematik (Crelles Journal) **2013** (2013), no. 679, 65–73.

[Lie11] Max Lieblich, *Period and index in the brauer group of an arithmetic surface*.

[M8̈5] Sirpa Mäki, *On the density of abelian number fields*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes (1985), no. 54, 104. MR 791087

[Mal02] Gunter Malle, *On the distribution of Galois groups*, J. Number Theory **92** (2002), no. 2, 315–329. MR 1884706

[Mal04] ———, *On the distribution of Galois groups. II*, Experiment. Math. **13** (2004), no. 2, 129–135. MR 2068887

[Man24] Shilpi Mandal, *Strong u-invariant and period-index bound for complete ultrametric fields*, arXiv preprint arXiv:2407.12169 (2024).

[MEH19a] Vlerë MEHMETI, *Patching on berkovich spaces and the local-global principle*, Ph.D. thesis, Université de Caen, 2019.

[Meh19b] Vlerë Mehmeti, *Patching over berkovich curves and quadratic forms*, Compositio Mathematica **155** (2019), no. 12, 2399–2438.

[Mer92] AS Merkur'ev, *Simple algebras and quadratic forms*, Mathematics of the USSR-Izvestiya **38** (1992), no. 1, 215.

[Mil80] James S Milne, *Etale cohomology (pms-33)*, no. 33, Princeton university press, 1980.

[Mou23] Hussein Mourtada, *Jet schemes and their applications in singularities, toric resolutions and integer partitions*, Handbook of Geometry and Topology of Singularities IV (2023), 211–249.

[MS21] Diane Maclagan and Bernd Sturmfels, *Introduction to tropical geometry*, vol. 161, American Mathematical Society, 2021.

[Neu99] J. Neukirch, *Algebraic number theory*, Springer, Berlin, 1999.

[Nic18] Johannes Nicaise, *Igusa zeta functions and the non-archimedean syz fibration*, Acta Mathematica Vietnamica **43** (2018), 31–44.

[Poi10] Jérôme Poineau, *La droite de berkovich sur z*, Société mathématique de France, 2010.

[PP82] Richard S Pierce and Richard S Pierce, *The associative algebra*, Springer, 1982.

[PS10] Raman Parimala and Venapally Suresh, *The u-invariant of the function fields of p-adic curves*, Annals of Mathematics (2010), 1391–1405.

[PS14] ———, *Period-index and u-invariant questions for function fields over complete discretely valued fields*, Inventiones mathematicae **197** (2014), no. 1, 215–235.

[Pum09] S Pumplün, *u-invariants for forms of higher degree*, Expositiones Mathematicae **27** (2009), no. 1, 37–53.

[Sal97] David J Saltman, *Division algebras over p-adic curves*, JOURNAL-RAMANUJAN MATHEMATICAL SOCIETY **12** (1997), 25–48.

[Sch12] Winfried Scharlau, *Quadratic and hermitian forms*, vol. 270, Springer Science & Business Media, 2012.

[Ser92] J-P. Serre, *Topics in galois theory*, Jones and Bartlett Publishers, London, 1992.

[Vaq06] Michel Vaquié, *Valuations and local uniformization*, Singularity theory and its applications, vol. 43, Mathematical Society of Japan, 2006, pp. 477–528.

[Vis07] A Vishik, *Fields of u-invariant* $2^r + 1$*, algebra, arithmetic and geometry-manin festschrift*, 2007.

[Wan21] Jiuya Wang, *Malle's conjecture for* $S_n \times A$ *for* $n = 3, 4, 5$, Compos. Math. **157** (2021), no. 1, 83–121. MR 4219215

[Wri89] David J. Wright, *Distribution of discriminants of abelian extensions*, Proc. London Math. Soc. (3) **58** (1989), no. 1, 17–50. MR 969545