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Nonuniqueness Properties of Zeckendorf Related Decompositions

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An abstract of
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Abstract

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Zeckendorf's Theorem states that every natural number can be uniquely written as the sum of distinct and nonconsecutive terms of the Fibonacci number sequence. Similarly, every natural number can be written as the sum of distinct and nonconsecutive terms of the Lucas number sequence. Although such decompositions of natural numbers in the Lucas number sequence need not be unique, there has been much progress on categorizing those natural numbers that do not carry this uniqueness property. We investigate the proportion of natural numbers that cannot be uniquely written as the sum of distinct and nonconsecutive terms of the Lucas number sequence. In doing so, we show the limiting value of this proportion and speculate on future research that generalizes the ideas presented in this paper.

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1. Introduction

The Fibonacci and Lucas number sequences have interested mathematicians for centuries through their applications in nature and mathematical theory. In 1972, Belgian mathematician [Kim98] Edouard Zeckendorf published the following theorems in relation to the Fibonacci and Lucas number sequences, Theorem 1.1 being the well–known Zeckendorf's Theorem [Zec72].

Definition 1.1. *Define the Fibonacci number sequence* $\{F_k\}_{k=0}^{\infty}$ *by the second–order linear recurrence* $F_0 = 0$*,* $F_1 = 1$*, and* $F_k = F_{k-2} + F_{k-1}$ *for* $k \geq 2$*. Let* $\{L_k\}_{k=0}^{\infty}$ *denote the Lucas number sequence given by the second–order linear recurrence* $L_0 = 2$*,* $L_1 = 1$*, and* $L_k = L_{k-2} + L_{k-1}$ *for* $k \geq 2$*.*

From the way we defined the Fibonacci number sequence, there is an additional condition on the term's indices in Theorem 1.1 [Zec72].

Theorem 1.1. (Zeckendorf's Theorem)*. Every natural number* n *can be uniquely written as the sum of distinct and nonconsecutive terms whose indices are greater than one of the Fibonacci number sequence (called a Zeckendorf representation of* n*).*

Theorem 1.2. (Zeckendorf)*. Every natural number can be written as the sum of distinct and nonconsecutive terms of the Lucas number sequence.*

Note that the distinction between Theorems 1.1 and 1.2 lies in the *uniqueness* property of the decompositions of natural numbers in the Fibonacci and Lucas number sequences. While five can be uniquely decomposed as F_5 in the Fibonacci number sequence, its decomposition need not be unique in the Lucas number sequence as

$$5 = L_0 + L_2 = 2 + 3 = L_1 + L_3 = 1 + 4.$$

One attempt by Zeckendorf to categorize those natural numbers which cannot be uniquely decomposed as the sum of distinct and nonconsecutive terms of the Lucas number sequence is stated in the following theorem, the proof of which can be found in [Zec72].

Theorem 1.3. *Natural numbers of the form $L_{2v+1} + 1$ cannot be uniquely written as the sum of distinct and nonconsecutive terms of the Lucas number sequence.*

Using five from our previous example, we see that $5 = L_{2\cdot 1+1} + 1 = L_3 + 1$. Although Theorem 1.3 captures a class of natural numbers that do not have a unique decomposition of distinct and nonconsecutive terms of the Lucas number sequence, the result is incomplete as

$$23 = L_0 + L_2 + L_6 = L_1 + L_3 + L_6$$

but 23 cannot be decomposed in the form $L_{2v+1} + 1$ for any integer v.

A quantity we can consider from this incomplete result by Zeckendorf is the proportion α of natural numbers that cannot be uniquely decomposed as the sum of distinct and nonconsecutive terms of the Lucas number sequence. To find this value, we start by defining the following functions, terms, and notations. Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{I}$ denotes the set of all infinite sequence of integers.

Definition 1.2. *Given a natural number n and an infinite sequence of integers $\{a_k\}_{k=0}^{\infty}$, we say n has a decomposition in $\{a_k\}_{k=0}^{\infty}$ if n can be written as the sum of distinct and nonconsecutive terms of $\{a_k\}_{k=0}^{\infty}$. Furthermore, we call a decomposition of n in $\{a_k\}_{k=0}^{\infty}$ unique if it is the only possible decomposition.*

Definition 1.3. *Define the decomposition counting function $D\colon \mathbb{I} \times \mathbb{N} \to \mathbb{Z}^* \cup \{\infty\}$ associated to an infinite sequence of integers $\{a_k\}_{k=0}^{\infty}$ of a natural number n by*

$$D\Big(\{a_k\}_{k=0}^{\infty}, n\Big) := \textit{number of distinct decompositions of } n \textit{ in } \{a_k\}_{k=0}^{\infty}.$$

Definition 1.4. *Given an infinite sequence of integers $\{a_k\}_{k=0}^{\infty}$, define the nonuniqueness counting function $U\colon \mathbb{I} \times \mathbb{N} \to \mathbb{N}$ associated to $\{a_k\}_{k=0}^{\infty}$ by*

$$U\Big(\{a_k\}_{k=0}^{\infty}, N\Big) := \#\Big\{1 \leq x \leq N\colon D\Big(\{a_k\}_{k=0}^{\infty}, x\Big) \neq 1\Big\}.$$

In Section 2, we present data that motivates our research and exploration of the following limit

$$\lim_{N\to\infty} \frac{U\Big(\{L_k\}_{k=0}^{\infty}, N\Big)}{N}$$

which gives us a better understanding of the proportion α we wish to calculate. We then transition to Section 3 where we give the necessary definitions and notations we use throughout this paper. The major results we obtain are presented in the following theorems below. The proofs and greater explanation of Theorems 1.4 and 1.5 will be provided in Section 4. Finally, we end with a discussion of future research in Section 5 which consists of generalizing ideas presented in this paper.

Theorem 1.4. *The maximum number of decompositions a natural number can have in the Lucas number sequence is two.*

Theorem 1.5. *Let U be the nonuniqueness counting function associated to the Lucas number sequence. Then*

$$\lim_{N\to\infty} \frac{U\Big(\{L_k\}_{k=0}^{\infty}, N\Big)}{N} = \frac{1}{2\Phi^3 - \Phi^2}$$

where $\Phi = \frac{1+\sqrt{5}}{2}$ denotes the golden ratio.

2. Motivation

In this section, we provide data which motivates our desire to study the proportion α discussed in Section 1. To generate data, we implement a computer algorithm written in Java which inputs two integers N_0 and N_1 that determine the second–order linear recurrence integer sequence and a third integer N_2 which sets an upper bound. The algorithm then returns decompositions of all natural numbers between zero and N_2 in the second–order linear recurrence sequence determined by N_0 and N_1. Furthermore, the algorithm also returns the number of natural numbers within the range that do not have unique decomposition. To study the Lucas number sequence, we set $N_0 = 2$, $N_1 = 1$, and let N_2 vary. Let U be the nonuniqueness counting function associated to the Lucas number sequence. The data we collect for various N_2 values is shown in the table below.

N_2	$U\left(\left\{L_k\right\}_{k=0}^{\infty}, N_2\right)$	$\frac{U\left(\left\{L_k\right\}_{k=0}^{\infty}, N_2\right)}{N_2}$
10,000	1,708	17.08%
50,000	8,541	17.082%
100,000	17,082	17.082%
200,000	34,164	17.082%
500,000	85,410	17.082%

Table 1

The algorithm follows the logic of subtracting Lucas sequence terms (which have pairwise nonconsecutive indices and are distinct) from a natural number n until n reaches zero. From Table 1, we see that the proportion α of natural numbers that do not have unique decomposition in the Lucas number sequence approaches 17.082% which roughly equates to $\frac{1}{2\Phi^3-\Phi^2}$ where $\Phi = \frac{1+\sqrt{5}}{2}$ denotes the golden ratio. For our next observation, we list the first fifty natural numbers and their decompositions in the Lucas number sequence.

$1 = L_1$	$2 = L_0$
$3 = L_2$	$4 = L_3$
$5 = L_0 + L_2 = L_1 + L_3$	$6 = L_0 + L_3$
$7 = L_4$	$8 = L_1 + L_4$
$9 = L_0 + L_4$	$10 = L_2 + L_4$
$11 = L_5$	$12 = L_0 + L_2 + L_4 = L_1 + L_5$
$13 = L_0 + L_5$	$14 = L_2 + L_5$
$15 = L_3 + L_5$	$16 = L_0 + L_2 + L_5 = L_1 + L_3 + L_5$
$17 = L_0 + L_3 + L_5$	$18 = L_6$
$19 = L_1 + L_6$	$20 = L_0 + L_6$
$21 = L_2 + L_6$	$22 = L_3 + L_6$
$23 = L_0 + L_2 + L_6 = L_1 + L_3 + L_6$	$24 = L_0 + L_3 + L_6$

$25 = L_4 + L_6$ $\quad$ $26 = L_1 + L_4 + L_6$

$27 = L_0 + L_4 + L_6$ $\quad$ $28 = L_2 + L_4 + L_6$

$29 = L_7$ $\quad$ $30 = L_0 + L_2 + L_4 + L_6 = L_1 + L_7$

$31 = L_0 + L_7$ $\quad$ $32 = L_2 + L_7$

$33 = L_3 + L_7$ $\quad$ $34 = L_0 + L_2 + L_7 = L_1 + L_3 + L_7$

$35 = L_0 + L_3 + L_7$ $\quad$ $36 = L_4 + L_7$

$37 = L_1 + L_4 + L_7$ $\quad$ $38 = L_0 + L_4 + L_7$

$39 = L_2 + L_4 + L_7$ $\quad$ $40 = L_5 + L_7$

$41 = L_0 + L_2 + L_4 + L_7 = L_1 + L_5 + L_7$ $\quad$ $42 = L_0 + L_5 + L_7$

$43 = L_2 + L_5 + L_7$ $\quad$ $44 = L_3 + L_5 + L_7$

$45 = L_0 + L_2 + L_5 + L_7 = L_1 + L_3 + L_5 + L_7$ $\quad$ $46 = L_0 + L_3 + L_5 + L_7$

$47 = L_8$ $\quad$ $48 = L_1 + L_8$

$49 = L_0 + L_8$ $\quad$ $50 = L_2 + L_8$

In the list above, we notice that the natural numbers which do not have unique decomposition in the Lucas number sequence carry the Lucas sequence terms L_0 and L_2 in their representations. This observation leads to the result by Brown which gives conditions for when decompositions of natural numbers in the Lucas number sequence are unique [BJ69].

Theorem 2.1. (Brown). *Let n be a nonnegative integer satisfying $0 < n < L$, for some $k > 1$. Then*

$$n = \sum_{0}^{k-1} \alpha_i L_i \tag{1}$$

where L_i corresponds to the i^{th} term of the Lucas number sequence and α_i binary digits satisfying

1. $\alpha_i \alpha_{i+1} = 0$ *for* $i \geq 0$

2. $\alpha_0 \alpha_2 = 0$.

Further, the representation of n in this form is unique. If $k - 1 < 2$ in Equation (2), we define $\alpha_2 = 0$ so that the condition $\alpha_0\alpha_2 = 0$ is automatically satisfied.

Something interesting to note about Theorem 2.1 is the condition $\alpha_0\alpha_2 = 0$ which as Brown states, reflects the particularity of the Lucas sequence [BJ69]. We see this particularization further in our list of decompositions. The characterization by Brown and our table of data affirm our hypothesis that there is a way to determine the proportion α of natural numbers that do not have unique decomposition in the Lucas number sequence.

3. Preliminaries and Background

In this section, we present definitions, notations, and the proof of Zeckendorf's Theorem which will give us intuition of how we prove our major results. Recall that we define the Fibonacci and Lucas number sequences in the following manner.

Definition 1.1. *Define the Fibonacci number sequence* $\{F_k\}_{k=0}^{\infty}$ *by the second–order linear recurrence* $F_0 = 0$, $F_1 = 1$, *and* $F_k = F_{k-2} + F_{k-1}$ *for* $k \geq 2$. *Let* $\{L_k\}_{k=0}^{\infty}$ *denote the Lucas number sequence given by the second–order linear recurrence* $L_0 = 2$, $L_1 = 1$, *and* $L_k = L_{k-2} + L_{k-1}$ *for* $k \geq 2$.

Definition 1.2. *Given a natural number* n *and an infinite sequence of integers* $\{a_k\}_{k=0}^{\infty}$, *we say* n *has a decomposition in* $\{a_k\}_{k=0}^{\infty}$ *if* n *can be written as the sum of distinct and nonconsecutive terms of* $\{a_k\}_{k=0}^{\infty}$. *Furthermore, we call a decomposition of* n *in* $\{a_k\}_{k=0}^{\infty}$ *unique if it is the only possible decomposition.*

Definition 1.3. *Define the decomposition counting function* $D\colon \mathbb{I} \times \mathbb{N} \to \mathbb{Z}^* \cup \{\infty\}$ *associated to an infinite sequence of integers* $\{a_k\}_{k=0}^{\infty}$ *of a natural number* n *by*

$$D\Big(\{a_k\}_{k=0}^{\infty}, n\Big) := \textit{number of distinct decompositions of } n \textit{ in } \{a_k\}_{k=0}^{\infty}.$$

Definition 1.4. *Given an infinite sequence of integers* $\{a_k\}_{k=0}^{\infty}$, *define the nonuniqueness counting function* $U\colon \mathbb{I} \times \mathbb{N} \to \mathbb{N}$ *associated to* $\{a_k\}_{k=0}^{\infty}$ *by*

$$U\Big(\{a_k\}_{k=0}^{\infty}, N\Big) := \#\Big\{1 \leq x \leq N \colon D\Big(\{a_k\}_{k=0}^{\infty}, x\Big) \neq 1\Big\}.$$

Definition 3.1. *Let* $A = \{a_0, a_1, \ldots, a_m\}$ *be the set consisting of the first* $m+1$ *terms of the sequence* $\{a_k\}_{k=0}^{\infty}$. *We say a proper subset* H *of* A *is a nonconsecutive subset of* A *if the indices of the elements of* H *are pairwise nonconsecutive.*

Definition 3.2. *Let* $A = \{a_0, a_1, \ldots, a_m\}$ *be the set consisting of the first* $m+1$ *terms of the sequence* $\{a_k\}_{k=0}^{\infty}$. *We call a sum* S *of* A *nonconsecutive if* S *is the sum of distinct elements of* A *whose indices are pairwise nonconsecutive.*

The proof of Theorem 1.1 (Zeckendorf's Theorem) is adapted from [Hen16]. To give the full proof, we need the following lemma, the proof of which can be found in [Hen16].

Lemma 3.1. *For any increasing sequence* $\{s_i\}_{i=0}^{k}$ *such that* $s_i \geq 2$ *and* $s_{i+1} > s_i + 1$ *for* $i \geq 0$, *we have*

$$\sum_{i=0}^{k} F_{s_i} < F_{s_k} + 1.$$

Theorem 1.1. (Zeckendorf's Theorem). *Every natural number* n *can be uniquely written as the sum of distinct and nonconsecutive terms whose indices are greater than one of the Fibonacci number sequence (called a Zeckendorf representation of* n*).*

Proof. To prove Theorem 1.1, we first show the existence portion and then the uniqueness portion of Zeckendorf representations for all natural numbers. For existence, we proceed by strong induction. We see that $1 = F_2$, $2 = F_3$, $3 = F_4$, and $4 = F_2 + F_4$. This shows the base case. Assume the existence portion of Theorem 1.1 holds for all natural numbers less than or equal to k. If $k+1$ is a Fibonacci sequence number, then we are done. Otherwise, we have that $k+1$ satisfies $F_j < k+1 < F_{j+1}$ for some natural number j greater than one. Let $a = k+1-F_j = k-(F_j-1)$, this implies $a \leq k$. By our inductive hypothesis, we have that a carries a Zeckendorf representation. Furthermore, $a+F_j = k+1 < F_{j+1} = F_{j-1}+F_j$ by definition, implying $a < F_{j-1}$. Therefore, $k+1$ has a Zeckendorf representation as $k+1 = F_j + a$ and a has a Zeckendorf representation that does not carry F_{j-1}, asserting that we do not contradict the definition of Zeckendorf representations in that we do not use consecutive Fibonacci sequence terms. This proves the existence portion of Zeckendorf's Theorem.

To show uniqueness, let S and T be sets which contain Fibonacci sequence terms that make up two Zeckendorf representations of an arbitrary natural number n. Consider the sets $S' = S - T$ and $T' = T - S$. From these sets, we get the equation

$$\sum_{x \in S} x - \sum_{a \in S \cap T} a = \sum_{y \in T} y - \sum_{b \in S \cap T} b$$

which implies

$$\sum_{x \in S'} x = \sum_{y \in T'} y. \tag{2}$$

Without loss of generality, suppose S' is empty. Then $\sum_{x \in S'} x$ and $\sum_{y \in T'} y$ will be equal to zero. This implies T' is empty as well since T' contains only nonnegative integers. Hence we have that $S = T$ as S' and T' are both empty. For our next case, suppose S' and T' are both nonempty and let F_s and F_t be the maximum elements of S' and T' respectively. Without loss of generality, let $F_s < F_t$. From Lemma 3.1, we have that

$$\sum_{x \in S'} x < F_{s+1} \leq F_t. \tag{3}$$

By Equation 2, $\sum_{x \in S'} x = \sum_{y \in T'} y$ which yields a contradiction as Inequality 3 asserts

$$\sum_{x \in S'} x < \sum_{y \in T'} y.$$

Therefore, S' and T' must be empty, implying $S = T$. This proves the uniqueness portion of Zeckendorf's Theorem. $\square$

The proof of existence for decompositions the Lucas number sequence is approached similarly to the proof of existence for Zeckendorf representations in the Fibonacci number sequence, although we must exercise caution when setting conditions for the base case and the inductive step as the Lucas number sequence is not an increasing sequence of integers due to the initial two terms $L_0 = 2$ and $L_1 = 1$.

4. Main Results

In this section, we present our results. To prove our main results which are stated in Theorems 1.4 and 1.5, we present several lemmas.

Lemma 4.1. *Let* $A = \{a_0, a_1, \ldots, a_{m-1}\}$ *be the set consisting of the first* m *terms of the infinite sequence* $\{a_k\}_{k=0}^{\infty}$. *There exist* F_{m+2} *distinct and nonconsecutive subsets of* A *where* F_{m+2} *denotes the* $(m+2)^{nd}$ *term of the Fibonacci number sequence.*

Proof. Lemma 4.1 follows directly from the well–known result that there are F_{m+2} subsets of $\{1, 2, \ldots, m\}$ which do not contain a pair of consecutive integers. For a formal proof of Lemma 4.1, we proceed by strong induction. For $m - 0$, A is equal to the empty set and we can form $F_2 = 1$ distinct and nonconsecutive subset which is the empty set itself. When $m = 1$, $A = \{a_0\}$ and we can form $F_3 = 2$ distinct and nonconsecutive subsets which are the empty set and the singleton set consisting of a_0. This shows the base case. Assume Lemma 4.1 holds for all natural numbers less than or equal to $m = k$. For the set $A = \{a_0, a_1, \ldots, a_k\}$, we have that the total number of distinct and nonconsecutive subsets of A we can form is the sum of the total number of distinct and nonconsecutive subsets that contain the term a_k and those that do not. From our inductive hypothesis, we know that there are F_{k+2} distinct and nonconsecutive subsets which do not contain a_k. For those distinct and nonconsecutive subsets which contain a_k, we need only consider the subset $\{a_0, a_1, \ldots, a_{k-2}, a_k\}$ of A. From our inductive hypothesis, we can form F_{k+1} distinct and nonconsecutive subsets which contain a_k. Hence there are $F_{k+3} = F_{k+2} + F_{k+1}$ distinct and nonconsecutive subsets of A. This completes the inductive step. $\square$

Lemma 4.2. *Let $A = \{L_0, L_1, \dots, L_m\}$ be the set consisting of the first $m+1$ terms of the Lucas number sequence and S be a nonconsecutive sum of A. Then*

1. *if $m \geq 1$ is odd, then S assumes all values between zero and $L_{m+1} - 1$ inclusive and*
2. *if $m \geq 0$ is even, then S assumes all values between zero and $L_{m+1} + 1$ inclusive excluding L_{m+1}.*

Proof. To prove Lemma 4.2, we proceed by strong induction. For $m = 0$, we have the singleton set $A = \{L_0\}$ which forms the nonconsecutive sums: 0 and $L_{0+1} + 1$ as the empty set results in a sum of zero. For $m = 1$, we have the set $A = \{L_0, L_1\}$ which forms the nonconsecutive sums: $0, L_1$, and $L_{1+1} - 1$. This shows the base case. Assume Lemma 4.2 holds for all integers less than or equal to $m = k$. Without loss of generality, suppose k is an odd integer. Consider the set $A = \{L_0, L_1, \dots, L_{k+1}\}$. Since the nonconsecutive sums of A that include L_{k+1} cannot contain L_k, we need only consider the subset $A_0 = \{L_0, L_1, \dots, L_{k-1}, L_{k+1}\}$ of A. From our inductive hypothesis, the nonconsecutive sums we can form from the initial terms $L_0, L_1, \dots, L_{k-1}$ are the values between zero and $L_k + 1$ inclusive excluding L_k. By adding L_{k+1} to these values, we have that the following nonconsecutive sums we can form from A_0 range from zero to $L_{k+2} + 1$ inclusive.

To show that L_{k+2} cannot be formed as a possible nonconsecutive sum of A_0, consider B_0 which is a nonconsecutive subset of $\{L_0, L_1, \ldots, L_{2j}\}$ where j is an non-negative integer such that $2j < k$. For sake of contradiction, suppose the sum of the elements of B_0 is equal to L_{2j+1}. In our first case, suppose L_{2j} is not in B_0. This implies B_0 is a nonconsecutive subset of $\{L_0, L_1, \ldots, L_{2j-1}\}$ and that the sum of the terms of B_0 are less than or equal to $L_{2j+1} - 1$ from our inductive hypothesis. Hence we have a contradiction as $L_{2j+1} - 1 < L_{2j+1}$ from our initial assumption. This implies L_{2j} must be in B_0. Consider the set $B_1 = B_0/\{L_{2j}\}$ which is a nonconsecutive subset of $\{L_0, L_1, \ldots, L_{2j-2}\}$. We have that the sum of the elements of B_1 is equal to the difference between the sum of the elements of B_0 and L_{2j}. This implies that the sum of the elements of B_1 equals L_{2j-1} which cannot be formed as a nonconsecutive sum from the set $\{L_0, L_1, \ldots, L_{2j-2}\}$ by our inductive hypothesis. Therefore, we have a contradiction and L_{2j+1} cannot be formed as a nonconsecutive sum of $\{L_0, L_1, \ldots, L_{2j}\}$.

Applying this result to our induction step, we have that L_k cannot be formed as a nonconsecutive sum from the subset $\{L_0, L_1, \ldots, L_{k-1}\}$ of A. Therefore, there is no possible way to form $L_{k+2} = L_k + L_{k+1}$ as a nonconsecutive sum of A. This completes the inductive step. $\square$

Lemma 4.3. *The Lucas sequence terms have unique decomposition in the Lucas number sequence.*

Proof. It suffices to show that for all Lucas sequence terms L_m, L_m does not have a decomposition in $A = \{L_0, L_1, \ldots, L_{m-1}\}$. To prove Lemma 4.3, we proceed by strong induction. For $m = 0$, 1, and 2, the Lucas sequence terms L_0, L_1, and L_2 do not have a decomposition in their respective A sets. This shows the base case. Assume Lemma 4.3 holds for all integers less than or equal to $m = k$. Without loss of generality, suppose k is an odd integer. Consider the set $A = \{L_0, L_1, \ldots, L_k\}$ and let B_0 be a nonconsecutive subset of A. For sake of contradiction, suppose the sum of the elements of B_0 is equal to L_{k+1}. In our first case, suppose L_k is not in B_0, this implies B_0 is a nonconsecutive subset of $\{L_0, L_1, \ldots, L_{k-1}\}$. Using Lemma 4.2, we have that the sum of the elements of B_0 is less than or equal to $L_k + 1$, implying $L_{k-1} \leq 1$. This implies $L_{k-1} = 1 = L_1$. From our base case, we showed that L_m for $m = 2$ has unique decomposition in the Lucas number sequence, hence yielding a contradiction. This implies L_k is in B_0. Consider the set $B_1 = B_0/\{L_k\}$ which is a nonconsecutive subset of $\{L_0, L_1, \ldots, L_{k-2}\}$. From Lemma 4.2, we have that the sum of the elements of B_1 is less than or equal to $L_{k-1} - 1$. By definition, the sum of the elements of B_1 is equal to the difference between the sum of the elements of B_0 and L_k, this implies that the sum of the elements of B_1 is equal to L_{k-1}. Therefore we have a contradiction which completes the inductive step. $\square$

The following lemma will be used to prove Theorem 1.4. The ideas behind the proofs of Lemma 4.4 and Theorem 1.4 is adapted from [Her20a].

Lemma 4.4. *Natural numbers of the form $L_{2m+1} + 1$ where m is a natural number and L_{2m+1} represents the $(2m+1)^{st}$ term of the Lucas number sequence have exactly* *two decompositions in the Lucas number sequence.*

Proof. It suffices to show that every natural number of the form $L_{2m+1}+1$ has no more than one decomposition in the set $A = \{L_0, L_1, \ldots, L_{2m}\}$. To prove Lemma 4.4, we proceed by strong induction. When $m = 1$, $A = \{L_0, L_1, L_2\}$ and when $m = 2$, $A = \{L_0, L_1, L_2, L_3, L_4\}$. In each case, we see that $L_{2\cdot 1+1} + 1$ and $L_{2\cdot 2+1} + 1$ have no more than one decomposition in each case respectively. This shows the base case. Assume Lemma 4.4 holds for all integers less than or equal to $m = k$. Consider the set $A = \{L_0, L_1, \ldots, L_{2k+2}\}$. From Lemma 4.2, the only possible nonconsecutive sums we can form from A are the values from zero to $L_{2k+3}+1$ inclusive excluding L_{2k+2}. Let B be a nonconsecutive subset of $\{L_0, L_1, \ldots, L_{2k+2}\}$. For sake of contradiction, suppose that the sum of the elements of B is equal to $L_{2k+3} + 1$ and that B does not contain the term L_{2k+2}. This implies B is a subset of $\{L_0, L_1, \ldots, L_{2k+1}\}$. From Lemma 4.2, the nonconsecutive sums we can form from $\{L_0, L_1, \ldots, L_{2k+1}\}$ are the values from zero to $L_{2k+2} - 1$ inclusive. Hence we have a contradiction as the sum of the elements of B is larger than $L_{2k+2} - 1$. This implies B contains L_{2k+2} and from our induction hypothesis, $L_{2k+1} + 1$ has no more than one decomposition in $\{L_0, L_1, \ldots, L_{2k}\}$. Since $L_{2k+3} + 1 = L_{2k+2} + (L_{2k+1} + 1)$ by definition and B cannot contain both L_{2k+2} and L_{2k+1}, this implies $L_{2k+3}+1$ has no more than one decomposition in A. This completes the inductive step. $\square$

Theorem 1.4. *The maximum number of decompositions a natural number can have in the Lucas number sequence is two.*

Proof. It suffices to show that for every m, there is no natural number with more than two decompositions in the set $A = \{L_0, L_1, \dots, L_m\}$. To prove Theorem 1.4, we proceed by strong induction. When $m = 0$, $A = \{L_0\}$ and when $m = 1$, $A = \{L_0, L_1\}$. In both cases, no natural has more than two decompositions in A. This shows the base case. Assume Theorem 1.4 holds for all integers less than or equal to $m = k$. For our first case, suppose k is an odd integer and let $A = \{L_0, L_1, \dots, L_k\}$. From Lemma 4.2, all nonconsecutive sums that can be formed from A are the values from zero to $L_{k+1} - 1$ inclusive. Hence when we add the term L_{k+1} to A, all new nonconsecutive sums that can be formed must be at least L_{k+1}. This implies there is no possible way in which we can form a third decomposition for any natural number in A as there is no intersection between the old and the new nonconsecutive sums in which we can form after the addition of the term L_{k+1}. We consider the next case when k is even. From Lemma 4.2, all nonconsecutive sums that can be formed from A are the values from zero to $L_{k+1} + 1$ inclusive excluding L_{k+1}. When we add the term L_{k+1} to A, all new nonconsecutive sums that can be formed are at least L_{k+1} with $L_{k+1}+1$ being formed again, namely $L_{k+1} + L_1$. By Lemma 4.4, we know that $L_{k+1} + 1$ has exactly two decompositions in the Lucas number sequence. Therefore, there is no possible way we can form a third decomposition for any natural number in A which completes the inductive step. □

The idea behind the proof of Theorem 1.5 is adapted from [Her20b].

Theorem 1.5. *Let U be the nonuniqueness counting function associated to the Lucas* *number sequence. Then*

$$\lim_{N\to\infty} \frac{U\Big(\{L_k\}_{k=0}^{\infty}, N\Big)}{N} = \frac{1}{2\Phi^3 - \Phi^2}$$

where $\Phi = \frac{1+\sqrt{5}}{2}$ *denotes the golden ratio.*

Proof. Let U be the nonuniqueness counting function associated to the Lucas number sequence, $\Phi = \frac{1+\sqrt{5}}{2}$ be the golden ratio, and $\beta = -\frac{1}{\Phi}$. Consider the set $A = \{1, 2, \ldots, L_{m+1}\}$ consisting of the first L_{m+1} natural numbers. To determine which natural numbers between one and L_{m+1} inclusive do not have unique decomposition in the Lucas number sequence, we need only consider elements in the subset $B = \{L_0, L_1, \ldots, L_m\}$ of A as L_{m+1} has unique decomposition in the Lucas number sequence by Lemma 4.3. We first consider the case when m is odd. From Lemma 4.1, we know that we can form F_{m+3} distinct and nonconsecutive subsets of B, implying there are $F_{m+3} - 1$ distinct sums of Lucas sequence terms which are natural numbers as the empty set results in a sum of zero. By Lemma 4.2 and Theorem 1.4, we have that

$$(F_{m+3} - 1) - (L_{m+1} - 1)$$

gives the total number of natural numbers in A that do not have unique decomposition in the Lucas number sequence. From the well-known identity $L_m = F_{m-1} + F_{m+1}$, $(F_{m+3} - 1) - (L_{m+1} - 1)$ becomes F_{m-1} [Aza12].

We next consider the case when m is even. Using a similar argument, we have that

$$(F_{m+3} - 1) - (L_{m+1})$$

gives the total number of natural numbers in A that do not have unique decomposition in the Lucas number sequence. From the identity used in the odd case, $(F_{m+3}-1)-(L_{m+1})$ becomes $F_{m-1}-1$ [Aza12]. When computing the limit for asymptotic density, the difference of one between F_{m-1} and $F_{m-1}-1$ is negligible. Hence on the subsequence of Lucas numbers up to L_{m+1}, we have the following equation

$$\lim_{m\to\infty} \frac{U\Big(\big\{L_k\big\}_{k=0}^{\infty},\, L_{m+1}\Big)}{L_{m+1}} = \lim_{m\to\infty} \frac{F_{m-1}}{L_{m+1}}. \tag{4}$$

Binet's formula for Fibonacci and Lucas sequence terms enables us to rewrite Equation 4 as

$$\begin{aligned}\lim_{m\to\infty} \frac{F_{m-1}}{L_{m+1}} &= \lim_{m\to\infty} \frac{\frac{\Phi^{m-1}-\beta^{m-1}}{\Phi-\beta}}{\Phi^{m+1}+\beta^{m+1}} \\ &= \lim_{m\to\infty} \frac{\frac{\Phi^{m-1}-\beta^{m-1}}{\Phi-\beta}\cdot\frac{1}{\Phi^{m-1}}}{\Phi^{m+1}+\beta^{m+1}\cdot\frac{1}{\Phi^{m-1}}} \\ &= \frac{1}{2\Phi^3-\Phi^2}\end{aligned}$$

as $\lim\limits_{m\to\infty}\left(\dfrac{\Phi}{\beta}\right)^{m-1} = 0$ and $\Phi-\beta = 2\Phi-1$. Therefore,

$$\lim_{N\to\infty} \frac{U\Big(\big\{L_k\big\}_{k=0}^{\infty},\, N\Big)}{N} = \frac{1}{2\Phi^3-\Phi^2}.$$

□

5. Future Work

In this section we discuss potential research which generalizes the ideas presented in this paper. An interesting concept we stumble across in Brown's paper is a *complete* sequence of integers [BJ61].

Definition 5.1. *An arbitrary sequence $\{f_i\}_{i=0}^{\infty}$ of positive integers is complete if and only if every positive integer n can be represented in the form*

$$\sum_{i=1}^{\infty} \alpha_i f_i$$

where each α_i is either zero or unity.

Notable complete sequences we have explored in great depth within this paper include the Fibonacci and Lucas number sequences. Brown also proves in his paper the following theorem [BJ61].

Theorem 5.1. (Brown). *Let $\{f_i\}_{i=0}^{\infty}$ be a nondecreasing sequence of positive integers with $f_1 = 1$. Then $\{f_i\}$ is complete if and only if $f_{p+1} \leq 1 + \sum_{i=1}^{p} f_i$ for $p = 1, 2, \ldots$.*

We can also think of the Lucas number sequence as a *swapped* Fibonacci number sequence. The Lucas number sequence is obtained by swapping the order of $F_2 = 1$ and $F_3 = 2$. We can consider another swapped Fibonacci number sequence by switching the order of $F_3 = 2$ and $F_4 = 3$. From this, we have the sequence $\{S_k\}_{k=0}^{\infty}$ where $S_0 = -1$, $S_1 = 3$, and $S_k = S_{k-2} + S_{k-1}$ for $k \geq 2$. The proof that $\{S_k\}_{k=0}^{\infty}$ is a complete (we alter the definition of complete sequences to be sequences which can carry non–positive integers as well) sequence follows similarly to the proof of Zeckendorf's Theorem described in Section 3. From our computer algorithm discussed in Section 2, we have the following table of data for varying values of N_2.

N_2	$U\left(\left\{S_k\right\}_{k=0}^{\infty}, N_2\right)$	$\frac{U\left(\left\{S_k\right\}_{k=0}^{\infty}, N_2\right)}{N_2}$
10,000	991	9.91%
50,000	4,955	9.91%
100,000	9,910	9.91%
200,000	19,821	9.9105%

Table 2

From Table 2, we observe that the proportion α of natural numbers that do not have unique decomposition in $\{S_k\}_{k=0}^{\infty}$ approaches 9.91%. If we generalize the ideas in this paper to any complete sequence of integers $\left\{a_k\right\}_{k=0}^{\infty}$, then we can determine the proportion of natural numbers that do not have unique decomposition in $\left\{a_k\right\}_{k=0}^{\infty}$.

References

[Aza12] Mohammad K Azarian, *Identities Involving Lucas or Fibonacci and Lucas Numbers as Binomial Sums*, Int. J. Contemp. Math. Sciences **7** (2012), no. 45, 2221–2227. ↑19, 20

[BJ61] J. L. Brown Jr., *Note on Complete Sequences of Integers*, The American Mathematical Monthly **68** (1961), no. 6, 557–560. ↑21

[BJ69] ______, *Unique Representations of Integers as Sums of Distinct Lucas Numbers*, The Fibonacci Quarterly **7** (1969), no. 3, 243–252. ↑7, 8

[Hen16] Nik Henderson, *What is Zeckendorf's Theorem?* (2016). `https://math.osu.edu/sites/math.osu.edu/files/henderson_zeckendorf.pdf`. ↑10

[Her20a] Curtis D. Herink, *Honors Thesis*, 2020. E–mail to David C. Luo. ↑16

[Her20b] ______, *Subsequence Argument*, 2020. E–mail to David C. Luo. ↑18

[Kim98] Clark Kimberling, *Edouard Zeckendorf*, The Fibonacci Quarterly **36** (1998), 416–418. ↑1

[Zec72] E. Zeckendorf, *Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. Roy. Sci. Liège **41** (1972), 179–182. ↑1, 2