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Field Patching and Galois Cohomology

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A dissertation submitted to the Faculty of the Graduate School
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Abstract

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Let T be a complete discrete valuation ring with uniforrmizer t, and $\hat{X}$ a smooth projective curve over $S = \mathrm{Spec}T$. Let $F = K(\hat{X})$ be the function field and let $\hat{F} = \hat{K}(\hat{X})$ be the completion of F with respect to the discrete valuation defined by the closed fibre X.

In this paper, we construct indecomposable and noncrossed product division algebras over F. This is done by defining an index-preserving homomorphism and using this map s to lift indecomposable and noncrossed product division algebras over $\hat{F}$ to indecomposable and noncrossed product division algebras over F, respectively.

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CONTENTS

Part I

INTRODUCTION

1

INTRODUCTION

The primary result of this paper is a construction of indecomposable and noncrossed product division algebras whose centers are function fields of curves over Henselian discrete valuation rings. We briefly recall definitions of these objects here. (See Chapter 2 for the more detailed definitions of division algebras, indecomposable division algebras and noncrossed product division algebras.)

Recall if K is a field, a K-*division algebra* D is a (noncommutative) algebra over K in which every nonzero element has an inverse. The *period* of D is the order of the class $[D]$ in $\mathrm{Br}(K)$, and the *index* $\mathrm{ind}(D)$ is the square root of D's K-dimension. A *noncrossed product is a* K-division algebra which does not contain a Galois maximal subfield, or equivalently, whose structure is not represented by a 2-cocycle. Noncrossed products were first constructed by Amitsur in Amitsur [2], settling a longstanding open problem. Since then there have been several other constructions, including Saltman [29], Jacob and Wadsworth [23], Brussel [5], Brussel [4], Reichstein and Youssin [26], and Hanke [15].

A K-division algebra is *indecomposable* if it cannot be expressed as tensor product of two nontrivial K-division subalgebras. It is not hard to see that all division algebras of equal period and index are indecomposable, and that all division algebras of composite period and index have "Sylow" decomposition, so the problem of producing indecomposable division algebras reduces easily to producing algebras whose period and index are unequal prime powers. Albert constructed decomposable division algebras of unequal (2-power) period and index in the 1930's, but indecomposable division algebras of unequal period and index

The method used here is quite different from that of Brussel et al. [6] and other works on this topic. Our construction relies on the ideas from *patching*, a method that has been used in the past to prove many results about Galois theory (c.f. Harbater [16]). Harbater and Hartmann [17] extended patching to structures over fields rather than over rings, to make the method more amenable to other applications. This approach shows that giving an algebraic structure over certain function fields is equivalent to giving the structure over a suitable collection of overfields.

The application of *patching over fields* to division algebras, or equivalently, to Brauer groups, was introduced in Harbater et al. [18] and Harbater et al. [19]. Considerable amount of results has been produced since then.

In the next a few chapters, we lay out the backgrounds and summarize the main results we will prove later on in this paper. In particular, in Chapter 2, we introduce the main objects of study for this paper, division algebras and central simple algebras; in Chapter 3, we introduce the main technique that we employ for the construction, namely, *patching over fields*.

Part II

BACKGROUND

2

DIVISION ALGEBRAS AND BRAUER GROUPS

In this chapter, we will set up the basic definition and facts about division algebras and Brauer groups, in particular, the fundamental connection between these two notions. Then we will introduce the main objects we will study in this paper, indecomposable and noncrossed product division algebras.

2.1 DEFINITIONS AND BASIC FACTS

We begin by recalling the following

Definition 2.1. Let k be a field. A k-algebra D is called a k-division algebra if every nonzero element of D has a two-sided multiplicative inverse.

Division algebras are closely tied to *central simple algebras*, which we define next.

Definition 2.2. Let k be a field. A k-algebra C is called *simple*, if C has no (two-sided) proper ideal. C is called *central* if its centre equals k. C is a *central simple* k*-algebra* if C is both simple and central.

A division k-algebra is clearly a central simple k-algebra. A deeper relation between k-division algebras and central simple k-division algebras is the following celebrated theorem due to Wedderburn:

Theorem 2.3. *Let A be a finite dimensional simple algebra over a field k. Then there exists an integer $n \geqslant 1$ and a division algebra $D \supset k$ so that A is isomorphic to the matrix ring $M_n(D)$. Moreover, the division algebra D is uniquely determined up to isomorphism.*

Two central simple k-algebras A and A' are called *Brauer equivalent* or *similar* if $A \otimes_k M_m(k) \cong A' \otimes_k M_{m'}(k)$ for some $m, m' > 0$. This defines an equivalence relation on the set of all central simple k-algebras. By Theorem 2.3, each equivalence class is uniquely represented by a division algebra. The set of equivalence classes of central simple k-algebras is denoted by $\mathrm{Br}(k)$. Since a tensor product of central simple k-algebras is still a central simple k-algebra, we could equip the set $\mathrm{Br}(k)$ with the tensor product operation. And in fact, the tensor product makes the set $\mathrm{Br}(k)$ into an abelian group, which we state in the following

Proposition 2.4. *The set $\mathrm{Br}(k)$ equipped with the tensor product operation is an abelian group, called the* Brauer group *of k.*

Thus the study of division algebras over the field k amounts the same to studying the Brauer group $\mathrm{Br}(k)$. This is the crucial perspective that we will take to study division algebras throughout the sequel because $Br(k)$ can be studied using cohomological method.

Theorem 2.5. *Let k be a field and k_s a fixed separable closure of k. Then there exists a natural isomorphism of abelian groups*

$$Br(k) \cong H^2(k, k_s^\times).$$

We also have the following cohomological interpretation of the m-torsion part ${}_mBr(k)$ of the Brauer group.

Corollary 2.6. *For each positive integer m prime to the characteristic of k we have a canonical isomorphism*

$${}_mBr(k) \cong H^2(k, \mu_m).$$

where μ_m denotes the group of m-th roots of unity in k_s equipped with its canonical Galois action.

2.2 INDEX AND PERIOD

In this section we use the cohomological theory of the Brauer group to derive basic results of Brauer concerning two important invariants for central simple algebras. They play important role in the sequel.

The first invariant is the following.

Definition 2.7. Let A be a central simple algebra over a field k. The *index* $\text{ind}_k(A)$ *is defined to be the degree of the* D *over* k*, where* D *is the division algebra for which* $A = M_n(D)$ *according to Theorem 2.3.*

Remark 2.8. We have the following facts regarding the index of a central simple algebra:

1. For a division algebra index and degree are the same thing.

2. The index of a central simple k-algebra A depends only on the class of A in the Brauer group $\text{Br}(k)$. Indeed, this class depends only on the division algebra D associated with A by Theorem 2.3. Therefore, index can be viewed as a cohomological invariant associated to the elements of Brauer groups.

Now we come to the second invariant.

Definition 2.9. Let A be a central simple k-algebra. Then the *period* or *exponent* of A is the order of the corresponding Brauer element in $\text{Br}(k)$.

2.3 NONCROSSED PRODUCT DIVISION ALGEBRAS

Let K be a Galois extension of a field k with $[K : k] = n \leqslant \infty$, and let $G = \text{Gal}(K : k)$. Recall that from any 2-cocycle $f \in$

$Z^2(G, K^\times)$ one can build a *crossed product algebra* $(K/k, G, f)$ as $\bigoplus_{\sigma \in G} Kx_\sigma$ with multiplication given by

$$(cx_\sigma)(dx_\tau) = c\sigma(d)f(\sigma, \tau)x_{\sigma\tau}.$$

This $(K/k, G, f)$ is a central simple k-algebra of dimension n^2 over F, and it contains a copy of K as a maximal subfield. Conversely, one deduces from the Skolem-Noether theorem that if A is a central simple k-algebra of dimension n^2 and A contains a Galois extension field K' of k with $[K' : k] = n$, then $A \cong (K'/k, \mathrm{Gal}(K'/k), f)$ for some 2-cocycle f, whose cohomology class in $H^2(\mathrm{Gal}(K'/f), K'^\times)$ is uniquely determined. The crossed product construction provides an explicit description of the isomorphism between the Brauer group $\mathrm{Br}(k)$ and the continuous cohomology group $H^2(\mathrm{Gal}(k_s/k), k_s^\times)$ as given in Theorem 2.5. Besides this, knowing that a specific central simple algebra A is a crossed product gives a concrete description of the multiplication in A that can help us understand A.

For several decades, the biggest open question in the theory of finite-dimensional division algebras was whether every such algebra D is a crossed product. Restated, the question was: Does every such D contain a maximal subfield which is Galois over the center of D? This possibility seemed plausible in light of Reiner [27, Theorem 7.15(ii)] which says that D has a maximal subfield which is separable over the center. Moreover, there was the great theorem of the 1930's, (cf. Hasse et al. [20] and Albert and Hasse [1]), which says that every central simple algebra over an algebraic number field is a cyclic algebra. But that has not ruled out the possibility of crossed products with noncyclic Galois groups. Also it had been proved by Wedderburn and Albert that every division algebra of degree 2, 3, 4, 6 or 12 is a crossed product (cf. Rowen [28, PP. 180-183]).

In 1972 Amitsur [2] finally settled the crossed product question which had been lingering since the 1930's, by producing counterexamples. Since then there has been several other constructions, including Saltman [29], Jacob and Wadsworth [23], Brussel [5].

2.4 INDECOMPOSABLE DIVISION ALGEBRAS

A division algebra D/k is said to be *decomposable* if $D \cong D_1 \otimes_k D_2$ where each $\deg(D_i) > 1$. It is obviously desirable to know whether a given D is decomposable, since if so D can be studied in terms of the smaller division algebras D_i. Of course, one always has the primary decomposition of D (also called the "Sylow decomposition" of D): if $\deg(D) = p_1^{r_1} \cdots p_l^{r_l}$, where the p_i are distinct primes, then $D \cong D_1 \otimes_k \cdots \otimes_k D_l$ where $\deg(D_i) = p_i^{r_i}$, and each D_i is uniquely determined up to isomorphism (although typically there are many different copies of each D_i in D). Thus, the study of decomposability is immediately reduced to the case where $\deg(D)$ is a prime power.

The first serious investigation of decomposability in the prime power case seems to have been given by Saltman [30]. Suppose $\deg(D) = p^n$, where p is a prime. It is immediate that if $\exp(D) = \deg(D)$, then D is indecomposable. So, the interesting division algebras for decomposability questions are those of degree p^n and exponent p^m where $m < n$. Saltman [30] gave the first example of such a division algebra which is indecomposable. Since then there have been several other constructions such as Tignol [34], Jacob and Wadsworth [22], Jacob [21], Schofield and Van den Bergh [32], Brussel [3] and McKinnie [25].

3

PATCHING OVER FIELDS

3.1 NOTATION

In this chapter, we will briefly recall the main technique used in this paper; namely, *patching over fields* which was introduced in Harbater and Hartmann [17] and then applied to the study of division algebras, Brauer groups and quadratic forms in Harbater et al. [18] and Harbater et al. [19].

Throughout this chapter, T will be a complete discrete valuation ring with uniformizer t, fraction field K and residue field k. Let $\hat{X}$ be a smooth projective T-curve with function field F.

We follow Harbater and Hartmann [17, Section 6] to introduce the notation. Given an irreducible component X_0 of X with generic point η, consider the local ring of $\hat{X}$ at η. For a (possibly empty) proper subset U of X_0, we let R_U denote the subring of this local ring consisting of rational functions that are regular at each point of U. In particular, $R_\emptyset$ is the local ring of $\hat{X}$ at the generic point of the component X_0. The t-adic completion of R_U is denoted by

$\hat{R}_U$. If P is a closed point of X, we write R_P for the local ring of $\hat{X}$ at P, and $\hat{R}_P$ for its completion at its maximal ideal. A height 1 prime ideal $\mathfrak{p}$ of $\hat{R}_P$that contains t determines a *branch* of X at P, i.e., an irreducible component of the pullback of X to $\mathrm{Spec}(\hat{R}_P)$. Similarly the contraction of $\mathfrak{p}$ to the local ring of $\hat{X}$ at P determines an irreducible component X_0 of X, and we say that $\mathfrak{p}$ *lies on* X_0. Note that a branch $\mathfrak{p}$ uniquely determines a closed point P and an irreducible component X_0. In general, there can be several branches $\mathfrak{p}$ on X_0 at a point P; but if X_0 is smooth at P then there is a unique branch $\mathfrak{p}$ on X_0 at P. We write $\hat{R}_{\mathfrak{p}}$ for the completion of the localization of $\hat{R}_P$ at $\mathfrak{p}$; thus $\hat{R}_P$ is contained in $\hat{R}_{\mathfrak{p}}$, which is a complete discrete valuation ring.

Since $\hat{X}$ is normal, the local ring R_P is integrally closed and hence unibranched; and since T is a complete discrete valuation ring, R_P is excellent and hence $\hat{R}_P$ is a domain (cf. Grothendieck and Dieudonné [13, Scholie 7.8.3(ii,iii,vii)]). For nonempty U as above and $Q \in U$, $\hat{R}_U/t^n\hat{R}_U \to \hat{R}_Q/t^n\hat{R}_Q$ is injective for all n and hence $\hat{R}_U \to \hat{R}_Q$ is also injective (cf. Harbater et al. [19, P. 241]). Thus $\hat{R}_U$ is also a domain. Note that the same is true if U is empty. The fraction fields of the domains $\hat{R}_U, \hat{R}_P$ and $\hat{R}_{\mathfrak{p}}$ will be denoted by $F_U, F_P$and $F_{\mathfrak{p}}$.

If $\mathfrak{p}$ is a branch at P lying on the closure of $U \subset X_0$, then there are natural inclusions of $\hat{R}_P$ and $\hat{R}_U$ into $\hat{R}_{\mathfrak{p}}$, and hence of F_P and F_U into $F_{\mathfrak{p}}$. The inclusion of $\hat{R}_P$ was observed above; for $\hat{R}_U$, note that the localization of R_U and of $R_{\mathfrak{p}}$ at the generic point of X_0 are the same; and this localization is naturally contained in the t-adically complete ring $\hat{R}_{\mathfrak{p}}$. Thus so is R_U and hence its t-adic completion $\hat{R}_U$.

3.2 MAIN RESULTS FROM PATCHING OVER FIELDS

We will use the following notation $\mathfrak{p} = (U, Q)$ when $\mathfrak{p}$ is a branch at Q lying on the closure of U.The inclusions of $\hat{R}_U$ and of $\hat{R}_Q$ into $\hat{R}_{\mathfrak{p}}$, for $\mathfrak{p} = (U, Q)$, induce inclusions of the corresponding fraction fields F_U and F_Q into the fraction field $F_{\mathfrak{p}}$ of $\hat{R}_{\mathfrak{p}}$. Let I be the index set consisting of all $U, Q, \mathfrak{p}$ described above. Via the above inclusions, the collection of all F_ξ, for $\xi \in I$, then forms an inverse system with respect to the ordering given by setting $U \succ \mathfrak{p}$ and $Q \succ \mathfrak{p}$ if $\mathfrak{p} = (U, Q)$.

Under the above hypotheses, suppose that for every field extension L of F, we are given a category $\mathfrak{A}(L)$ of algebraic structures over L(i.e. finite dimensional L-vector spaces with additional structure, e.g. associative L-algebras), along with base-change functors $\mathfrak{A}(L) \to \mathfrak{A}(L')$ when $L \subseteq L'$. An $\mathfrak{A}$-*patching problem* for $(\hat{X}$,S$)$ consists of an object V_ξ in $\mathfrak{A}(F_\xi)$ for each $\xi \in I$, together with isomorphisms $\phi_{U,\mathfrak{p}} : V_U \otimes_{F_U} F_{\mathfrak{p}} \to V_{\mathfrak{p}}$ and $\phi_{Q,\mathfrak{p}} : V_Q \otimes_{F_Q} F_{\mathfrak{p}} \to V_{\mathfrak{p}}$ in $\mathfrak{A}(F_{\mathfrak{p}})$. These patching problems form a category, denoted by $PP_{\mathfrak{A}}(\hat{X}, S)$, and there is a base change functor $\mathfrak{A}(F) \to PP_{\mathfrak{A}}(\hat{X}, S)$.

If an object $V \in \mathfrak{A}(F)$ induces a given patching problem up to isomorphism, we will say that V is a *solution* to that patching problem, or that it is *obtained by patching* the objects V_ξ. We similarly speak of obtaining a morphism over F by patching morphisms in $PP_{\mathfrak{A}}(\hat{X}, S)$. The next result is given by Harbater and Hartmann [17, Theorem 7.2].

Theorem 3.1. *Let T be a complete discrete valuation ring. Let $\hat{X}$ be a smooth connected projective T-curve with closed fibre X. Let $U_1, U_2 \subseteq X$, let $U_0 = U_1 \cap U_2$, and let $F_i := F_{U_i} (i = 0, 1, 2)$. Let $U = U_1 \cup U_2$ and form the fibre product of groups $\mathrm{Br}(F_1) \times_{\mathrm{Br}(F_0)} \mathrm{Br}(F_2)$ with respect to the maps $\mathrm{Br}(F_i) \to \mathrm{Br}(F_0)$ induced by $F_i \hookrightarrow F_0$. Then the base change map $\beta : \mathrm{Br}(F_U) \to \mathrm{Br}(F_1) \times_{\mathrm{Br}(F_0)} \mathrm{Br}(F_2)$ is a group isomorphism.*

The above Theorem says that giving a Brauer class over a function field F is equivalent to giving compatible Brauer classes over the patches. The nice thing about patching Brauer classes over a function field F is that we have good control of the index, which is stated in Harbater et al. [19, Theorem 5.1].

Theorem 3.2. *Under the above notation, let A be a central simple F-algebra. Then $\mathrm{ind}(A) = \mathrm{lcm}_{\xi \in \mathfrak{P} \cup \mathfrak{U}}(\mathrm{ind}(A_{F_\xi}))$.*

To conclude this section, we record a variant of Hensel's Lemma from Harbater et al. [19, Lemma 4.5] that will be used over and over again in the index computation.

Lemma 3.3. *Let* R *be a ring and* I *an ideal such that* R *is* I*-adically complete. Let* X *be an affine* R*-scheme with structure morphism* $\phi : X \to \operatorname{Spec} R$*. Let* $n \geqslant 0$*. If* $s_n : \operatorname{Spec}(R/I^n) \to X \times_R (R/I^n)$ *is a section of* $\phi_n : \phi \times_R (R/I^n)$ *and its image lies in the smooth locus of* ϕ*, then* s_n *may be extended to a section of* ϕ*.*

Part III

THE MAIN CONSTRUCTION

4

SPLITTING MAP

Let T be a complete discrete valuation ring with uniformizer t and residue field k. By a smooth curve $\hat{X}$ over T, we will mean a scheme $\hat{X}$ which is projective and smooth of relative dimension 1 over Spec(T). In particular, $\hat{X}$ is flat and of finite presentation over Spec(T). Let $F = K(\hat{X})$ be the function field of $\hat{X}$. Note that since $\hat{X}$ is smooth, the closed fibre X is smooth, integral and of codimension 1. In addition, X is connected (c.f. Grothendieck and Dieudonné [14, 18.5.19]) , hence X determines a discrete valuation ring on F. Let $\hat{F} = \hat{K}(\hat{X})$ be the completion of F with respect to this discrete valuation.

Throughout the paper, n will denote an integer which is prime to the characteristic of k. We will be using the following notation for cohomology groups in the sequel: For an integer r, we let

$$\mu_n^r = \begin{cases} \mu_n^{\otimes r} & \text{for } r \geqslant 0, \\ \hom(\mu_n^{\otimes -r}, \mu_n) & \text{for } r < 0. \end{cases}$$

For a fixed integer n, and for any field K, we will let $H^q(K, r) = H^q(K, \mu_n^r)$. In the more special case when $r = q - 1$, we can shorten the notation even further as follows:

$H^q(K) = H^q(K, q-1) = H^q(K, \mu_n^{q-1})$. In particular, $H^2(K) = {}_n\mathrm{Br}(K)$ will be the n-torsion part of the Brauer group of K; and $H^1(K)$ will be the n-torsion part of the character group of K.

Adopting the above notation, in this section we will define a map $s : H^2(\hat{F}) \to H^2(F)$ and show that s has the following properties:

- s is a group homomorphism;
- s splits the restriction;
- s preserves index of Brauer classes.

Once such a map s is defined, we could use it to construct indecomposable division algebras and noncrossed product division algebras over F, as in Chapter 6.

4.1 CONSTRUCTION OVER AN OPEN AFFINE SUBSET

Given an element $\hat{\gamma} \in H^2(\hat{F})$, we will define a lift γ_U to F_U of $\hat{\gamma}$. Note that since $\hat{F}$ is a complete discretely valued field with t a unifomizer, and with $k(X)$ the residue field. We have an exact Witt Sequence as in Garibaldi et al. [10, II.7.10 and II.7.11],

$$0 \to H^2(k(X)) \to H^2(\hat{F}) \to H^1(k(X)) \to 0. \qquad (4.1)$$

The sequence is is split (non-canonically) by the cup product with $(t) \in H^1(k(X), 1)$. Hence each element $\hat{\gamma} \in H^2(\hat{F})$ can be written as a sum $\gamma_0 + (\chi_0, t)$, with $\gamma_0 \in H^2(k(X))$ and $\chi_0 \in H^1(k(X))$ (Note that we are identifying $H^r(k(X))$ as a subgroup of $H^r(\hat{F})$, for $r = 1, 2$, as in Garibaldi et al. [10, II.7.10 and II.7.11]). Here we use the notation (χ_0, t) to denote the cup product $\chi_0 \cup (t)$, and we will use this notation throughout the paper without further explanation. Also we will fix the decomposition $\hat{\gamma} = \gamma_0 + (\chi_0, t)$ throughout the sequel.

Let U be an open affine subset of X so that neither γ_0 nor χ_0 ramifies at any closed point of U. This implies that $\gamma_0 \in H^2(k[U])$ and $\chi_0 \in H^1(k[U])$ by purity(cf, Colliot-Thélène [8]), where $k[U]$ denotes the ring of regular functions of the affine scheme U.

By Cipolla [7], there exists a canonical isomorphism $H^2(\hat{R}_U) \to H^2(k[U])$ since $\hat{R}_U$ is t-adically complete and $k[U] \cong \hat{R}_U/(t)$; therefore there is a unique lift of γ_0 to $H^2(\hat{R}_U)$. According to Harbater et al. [19], Grothendieck and Raynaud [12, Théorèm 8.3] implies that there is a unique lift of χ_0 to $H^1(\hat{R}_U)$ as well. Taking $\tilde{\gamma}_0$ and $\tilde{\chi}_0$ as the lifts of γ_0 and χ_0 to $\hat{R}_U$, we will let

$$\gamma_U = \tilde{\gamma}_0 + (\tilde{\chi}_0, t) \tag{4.2}$$

be the lift of $\hat{\gamma}$ to $H^2(F_U)$.

4.2 CONSTRUCTION OVER CLOSED POINTS

Fix an open affine subset U of X and let $\mathfrak{P} = X \backslash U$. In order to apply the patching result we recalled in Chapter 3, we need to define a γ_P for each $P \in \mathfrak{P}$ in such a way that when $\mathfrak{p} = (U, P)$ is the unique branch of U at P, the restriction to $F_\mathfrak{p}$ of γ_P and γ_U agree with each other, i.e., $\mathrm{res}_{F_\mathfrak{p}}(\gamma_P) = \mathrm{res}_{F_\mathfrak{p}}(\gamma_U)$ (Recall there are field embeddings $F_P \hookrightarrow F_\mathfrak{p}$ and $F_U \hookrightarrow F_\mathfrak{p}$ for $\mathfrak{p} = (U, P)$, as in Chapter 3, hence there are restrictions $\mathrm{res} : H^2(F_U) \to H^2(F_\mathfrak{p})$ and $\mathrm{res} : H^2(F_P) \to H^2(F_\mathfrak{p})$. For more details on these restriction maps, see Serre [33]).

Note that since $\hat{X}$ is regular and the closed fibre X is smooth, the maximal ideal of the local ring R_P is generated by two generators, t and π. So is $\hat{R}_P$.

We define γ_P in the following way: There is a field embedding $F_U \to F_\mathfrak{p}$, hence a canonical restriction $\mathrm{res} : H^2(F_U) \to H^2(F_\mathfrak{p})$. Let $\gamma_\mathfrak{p}$ be the image of γ_U under this restriction. Observe that $F_\mathfrak{p}$ is a complete discretely valued field with residue field $\kappa(\mathfrak{p})$; furthermore, $\kappa(\mathfrak{p})$ is also a complete discretely valued field with residue field $\kappa(P)$. Therefore, applying Garibaldi et al. [10, II.7.10 and II.7.11] twice, we get the following decomposition of $H^2(F_\mathfrak{p})$:

$$H^2(F_\mathfrak{p}) \cong H^2(\kappa(P)) \oplus H^1(\kappa(P)) \oplus H^1(\kappa(P)) \oplus H^0(\kappa(P)). \quad (4.3)$$

In other words, each element $\gamma_\mathfrak{p} \in H^2(F_\mathfrak{p})$ can be written as $\gamma_\mathfrak{p} = \gamma_{0,0} + (\chi_{1,}\pi) + (\chi_2 + (\pi) \cup \xi_r, t)$, where $\gamma_{0,0} \in H^2(\kappa(P)), \chi_{1,}\chi_2 \in H^1(\kappa(P)), \xi_r \in H^0(\kappa(P))$. Note that here $(\pi) \cup \xi_r \in H^1(\kappa(P))$ is a character.

In order to define a lift for $\gamma_\mathfrak{p}$ to F_P, we first show that all characters in $H^1(\kappa(\mathfrak{p}))$ can be lifted by proving the following lemma.

Lemma 4.1. *Let $\chi \in H^1(\kappa(\mathfrak{p}))$ be a character. Then there is a unique $\tilde{\chi} \in H^1(F_P)$ that lifts χ.*

Proof. Since $\kappa(\mathfrak{p})$ is a complete discretely valued field with residue field $\kappa(P)$, we have the classical Witt's decomposition for χ,

$$\chi = \chi_0 + (\pi) \cup \xi_r,$$

where $\chi_0 \in H^1(\kappa(P))$, $\xi_r \in H^0(\kappa(P))$ and $(\pi) \in H^1(\kappa(P), 1)$ denotes the image of π under the Kummer map. Note that χ_0 can be lifted without any difficulty by Grothendieck and Raynaud [12, Théorèm 8.3]; the only trouble comes from the totally ramified part, $(\pi) \cup \xi_r$.

Let $L, L_0/\kappa(\mathfrak{p})$ be the field extension determined by χ, χ_0 respectively. Then L_0 is the maximal unramified subextension of $\kappa(\mathfrak{p})$ inside L and L/L_0 is a totally ramified extension determined by the character $(\pi) \cup \xi_r$. Note that Fesenko and Vostokov [9, Theorem II.3.5] implies that $(\pi) \cup \xi_r$ can be lifted to $H^1(F_P)$ in a unique way as well, since $\kappa(\mathfrak{p})$ is a complete discretely valued field. □

Now we are ready to define a lift for $\hat{\gamma}$ in $H^2(F_P)$. Again Cipolla [7] implies that $H^2(\kappa(P)) \cong H^2(\hat{R}_P)$ and Lemma 4.1 implies that $\chi_1, \chi_2 + (\pi) \cup \xi_r$ can be lifted to $H^1(\hat{R}_P)$ uniquely. Hence each component of $H^2(F_\mathfrak{p})$ can be lifted to $\hat{R}_P$, and thus we will set

$$\gamma_P = \tilde{\gamma}_{0,0} + (\tilde{\chi}_1, \pi) + (\tilde{\chi}_2 + \tilde{\chi}', t). \tag{4.4}$$

where $\tilde{\gamma}, \tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}'$ are the lifts of $\gamma_{0,0,}\chi_1, \chi_2, (\pi) \cup \xi_r$ to $\hat{R}_P$ (and hence to F_P), respectively. Therefore this γ_P is a unique lift of $\gamma_\mathfrak{p}$ to F_P. The assignment of $s_P(\gamma_\mathfrak{p}) = \gamma_P$ will yield a map $s_P : H^2(F_\mathfrak{p}) \to H^2(F_P)$. It is not hard to see that s_P is a group homomorphism, since it is a group homomorphism on each of the components.

In this section we show that γ_U and γ_P that we constructed in Section 4.1 and Section 4.2 are compatible in the sense of patching, that is $\mathrm{res}_{F_\mathfrak{p}}(\gamma_U) = \mathrm{res}_{F_\mathfrak{p}}(\gamma_P)$ for each $P \in \mathfrak{P} = X\backslash U$ when $\mathfrak{p} = (U, P)$ is the unique branch of U at P.

We claim that the compatibility will be proved if we can show that s_P splits the restriction map $\mathrm{res}_{F_\mathfrak{p}} : H^2(F_P) \to H^2(F_\mathfrak{p})$, or equivalently, $\mathrm{res}_{F_\mathfrak{p}} \circ s_P$ is the identity map. This is true because $\gamma_P = s_P(\gamma_\mathfrak{p}) = s_P \circ \mathrm{res}_{F_\mathfrak{p}}(\gamma_U)$, hence we would have that $\mathrm{res}_{F_\mathfrak{p}}(\gamma_P) = \mathrm{res}_{F_\mathfrak{p}}(\gamma_U)$ if $\mathrm{res}_{F_\mathfrak{p}} \circ s_P$ is the identity map. So it suffices to prove the following

Proposition 4.2. s_P *as defined in 4.2 splits the restriction* $\mathrm{res} : H^2(F_P) \to H^2(F_\mathfrak{p})$, *that is,* $\mathrm{res} \circ s_P$ *is the identity map.*

Proof. Take an arbitrary element $\gamma_\mathfrak{p} \in H^2(F_\mathfrak{p})$. As in section 4.2, we write $\gamma_\mathfrak{p} = \gamma_{0,0} + (\chi_1, \pi) + (\chi_2 + (\pi) \cup \xi_r, t)$. Therefore it is easily checked that

$$\begin{aligned}
\mathrm{res} \circ s_P(\gamma_\mathfrak{p}) &= \mathrm{res} \circ s_P(\gamma_{0,0} + (\chi_1, \pi) + (\chi_2 + (\pi) \cup \xi_r, t)) \\
&= \mathrm{res}(\tilde{\gamma}_{0,0} + (\tilde{\chi}_1, \pi) + (\tilde{\chi}_2 + \tilde{\chi}', t)) \\
&= \gamma_{0,0} + (\chi_1, \pi) + (\chi_2 + (\pi) \cup \xi_r, t) \\
&= \gamma_\mathfrak{p}.
\end{aligned}$$

□

Thus γ_U, γ_P will patch and yield $\gamma \in H^2(F)$, by Harbater and Hartmann [17, Theorem 7.2]. To see that we do have a map $s : H^2(\hat{F}) \to H^2(F)$, it remains to show that γ is independent of the choice of the open affine subset U of X. In order to do this, we prove the following

Lemma 4.3. *Let T be a complete discrete valuation ring with residue field k; let $\hat{X}$ be a smooth projective T-curve with function field F and closed fibre X. Let $\hat{F}$ be the completion of F with respect to the discrete valuation induced by X, and denote by $k(X)$ the corresponding residue field. Take an element $\hat{\gamma} = \gamma_0 + (\chi_0, t) \in H^2(\hat{F})$, where $\gamma_0 \in H^2(k(X))$ and $\chi_0 \in H^1(k(X))$. Assume that U_1, U_2 are two open affine subsets of X so that neither γ_0, χ_0 is ramified on any point of $U_1 \cup U_2$. Let $\mathfrak{P}_1, \mathfrak{P}_2$ be the complements of U_1, U_2 respectively. We construct two Brauer classes $\gamma, \gamma' \in H^2(F)$ by patching as we did above, while using U_1 and U_2 as the open affine subset in the construction, respectively, then γ, γ' denote the same Brauer class in $H^2(F)$.*

Proof. We first deal with the case where U_1 is contained in U_2. In this case we have $R_{U_2} \hookrightarrow R_{U_1}$ and consequently a field embedding $F_{U_2} \hookrightarrow F_{U_1}$. Let γ_1 be the restriction of γ to $H^2(F_{U_1})$ and γ_2 be the restriction of γ' to $H^2(F_{U_1}$, respectively. We must have $\gamma_1 = \gamma_2$ by the construction in Section 4.1. Also by the construction in Section 4.2, it follows that for every $P \in \mathfrak{P}_2$, $\mathrm{res}_{F_P}(\gamma) = \mathrm{res}_{F_P}(\gamma')$. Therefore it follows that $\gamma = \gamma'$, by Harbater and Hartmann [17, Theorem 7.1 & 7.2]. This proves the Lemma in the case where U_1 is contained in U_2.

In the general case, let U_3 be an open affine subset of $U_1 \cap U_2$. Clearly γ_0 and χ_0 are both unramified at every point of U_3. Let $\gamma'' \in H^2(F)$ be the Brauer class constructed by patching as above, using U_3 as the open affine subset in the construction. It follows that $\gamma'' = \gamma$ and $\gamma'' = \gamma'$ since U_3 is contained in both U_1 and U_2, by what we just proved for the case where one open affine subset is contained in the other. Hence $\gamma = \gamma' = \gamma'' \in H^2(F)$, which proves the Lemma in the general case. □

4.4 s splits the restriction map

Recall the notation: let T be a complete discrete valuation ring with residue field k and uniformizer t. Let $\hat{X}$ be a smooth projective T-curve with function field F and closed fibre X. Let $\hat{F}$ be the completion of F with respect to the discrete valuation induced by X. Let $s : H^2(\hat{F}) \to H^2(F)$ be the map defined by patching as in section 4.1 and section 4.2. We will show that s splits the restriction map $\text{res} : H^2(F) \to H^2(\hat{F})$. Hence index of Brauer classes cannot go down under the map s, because restriction can never raise index. In particular, we prove Proposition 4.4. But we need to recall *unramified cohomology* first: For a field E, $H^2_{nr}(E)$ denotes the unramified part of $H^2(E)$, or equivalently, $H^2_{nr}(E) =$

$\cap_\nu H^2(E_\nu)$, where ν runs through all discrete valuations on E, and E_ν denotes the completion of E at ν. See Colliot-Thélène [8] for more details on the unramified cohomology.

Proposition 4.4. *The map s is a section to the restriction map $\mathrm{res}_{\hat{F}} : H^2(F) \to H^2(\hat{F})$.*

Proof. It suffices to show that $\mathrm{res} \circ s$ is the identity map on $H^2(\hat{F})$. Since $H^2(\hat{F}) \cong H^2(k(X)) \oplus H^1(k(X))$, it suffices to show that $\mathrm{res}_{\hat{F}} \circ s$ is the identity map on both components; that is, given $\hat{\gamma} = \gamma_0 + (\chi_0, t)$ where $\gamma_0 \in H^2(k(X))$ and $\chi_0 \in H^1(k(X))$, the Proposition will follow if we can show that $\mathrm{res}_{\hat{F}} \circ s(\gamma_0) = \gamma_0$ and $\mathrm{res}_{\hat{F}} \circ s((\chi_0, t)) = (\chi_0, t)$.

Let's first show that s splits restriction on $H^2(k(X))$. Let's call the restriction of s to $H^2(k(X))$ s' to simplify notation. We have the following commutative diagram:

$$\begin{array}{ccc} H^2(\hat{R}_U) & \overset{f}{\hookrightarrow} & H^2(k(X)) \\ \big\downarrow g & & \big\downarrow s' \\ H^2(F_U) & \underset{\mathrm{res}_{F_U}}{\longleftarrow} & H^2(F) \end{array}$$

In the above diagram, f is the composition of the restriction $H^2(k[U]) \to H^2(k(X))$ and the isomorphism $H^2(\hat{R}_U) \to H^2(k[U])$ as in Cipolla [7]; and g is the inflation. The commutativity of the diagram follows from the construction we outlined in Section 4.1. Note that since $\gamma_0 \in H^2(k(X))$ is unramified at any point of U, we have that $\mathrm{res}_{F_U} \circ s'(\gamma_0)$ has to lie in $H^2(\hat{R}_U)$. This implies that we can find an inverse image under g for $\mathrm{res}_{F_U} \circ s(\gamma_0)$ even though in general, g does not have an inverse. Using g^{-1} as a shorthand notation(keep in mind g^{-1} does not exist at all), we could define a map $f \circ g^{-1} \circ \mathrm{res}_{F_U}$ from the subgroup of $H^2(F)$ consisting of images of elements of $H^2(\hat{R}_U)$ under the composition $s' \circ f$. This map is a restriction; and s' splits this composition by commutativity of the diagram. This proves that s splits the restriction on the unramified part $H^2(k(X))$.

Now we will show that s splits the restriction on $H^1(k(X))$ as well. By construction of s, we know that $\mathrm{res} \circ s((\chi_0, t)) = (\chi', t)$ for some $\chi' \in H^1(\hat{F})$. Therefore in order to show that $\mathrm{res}_{\hat{F}} \circ s((\chi_0, t)) = (\chi_0, t)$, it suffices to show that $\mathrm{ram}(\mathrm{res}_{\hat{F}} \circ s((\chi_0, t))) = \chi_0$, where $\mathrm{ram} : H^2(\hat{F}) \to H^1(k(X))$ denotes the ramification map on $H^2(\hat{F})$ with respect to the valuation determined by the closed fibre X. Since $\chi_0 \in H^1(k[U])$, we have $\mathrm{ram}(\mathrm{res}_{\hat{F}} \circ s((\chi_0, t))) = \mathrm{ram}((\tilde{\chi}_0, t))$ where $\tilde{\chi}_0$ denotes the lift of χ_0 to $H^1(\hat{R}_U)$, as we did in Section 4.1 (Since $H^1(\hat{R}_U) \cong H^1(k[U])$, $\tilde{\chi}_0$ can be viewed as as element of $H^1(k[U])$, and hence element of $H^1(k(X))$ via the injection $H^1(k[U]) \hookrightarrow H^1(k(X))$, and finally element of $H^1(\hat{F})$ via the injection $H^1(k(X)) \hookrightarrow H^1(\hat{F})$). Therefore the image in $H^1(\hat{F})$ of $\tilde{\chi}_0$ under the composition of these maps is in fact χ_0, since all these maps are injective. Then it is easy to see that $\mathrm{ram}((\tilde{\chi}_0, t)) = \tilde{\chi}_0 = \chi_0 \in H^1(k[X])$, as desired. □

The following corollary is immediate:

Corollary 4.5. *Index of Brauer classes cannot go down under the map* s.

Proof. Take $\hat{\gamma} \in H^2(\hat{F})$ and let $\gamma = s(\hat{\gamma})$. By Proposition 4.4 we must have that $\hat{\gamma} = \mathrm{res}_{\hat{F}}(\gamma)$, therefore $\mathrm{ind}(\hat{\gamma}) | \mathrm{ind}(\gamma)$. This proves that s can never lower index of Brauer classes.

□

5

THE SPLITTING MAP PRESERVES INDEX

In this section, we will show that the splitting map s that we defined in Chapter 4 has one more property that is crucial to the construction of indecomposable and non-crossed product division algebras over p-adic curves, that is, s preserves index of Brauer classes. In other words, $\mathrm{ind}(\hat{\gamma}) = \mathrm{ind}(\gamma) = \mathrm{ind}(s(\hat{\gamma}))$. We make the following elementary observation, which is true for Brauer classes over an arbitrary field.

Proposition 5.1. *Let k be an arbitrary field. Let $\gamma \in H^2(k)$ be a Brauer class with the following decomposition: $\gamma = \gamma_0 + (\chi, t)$, where $\gamma_0 \in H^2(k)$, $\chi \in H^1(k)$ and t is an arbitrary element of k. Then $\mathrm{ind}(\gamma)|\mathrm{ind}(\gamma_{0,l}) \cdot \exp(\chi)$, where $\gamma_{0,l}$ denotes the base extension of γ_0 to l/k, where l is the field extension determined by χ.*

Proof. Let E/l be a minimal extension that splits $\gamma_{0,l}$. Then $[E : l] = \mathrm{ind}(\gamma_{0,l})$. Also there is some E'/k with $[E' : k] = \exp(\chi)$ which splits χ and hence (χ, t); therefore EE' will split γ, furthermore it is not hard to see that $[EE' : k]|\mathrm{ind}(\gamma_{0,l}) \cdot \exp(\chi)$ and hence $\mathrm{ind}(\gamma)|\mathrm{ind}(\gamma_{0,l}) \cdot \exp(\chi)$. □

We will apply Harbater et al. [19, Theorem 5.1], which states that $\operatorname{ind}(\gamma) = \operatorname{lcm}(\operatorname{ind}(\gamma_U), \operatorname{ind}(\gamma_P))$ for each $P \in \mathfrak{P}$. Since we already showed that s can never lower index of Brauer classes as in Chapter 4, we will be done if we could show that $\operatorname{ind}(\gamma)|\operatorname{ind}(\hat{\gamma})$; therefore it suffices to show that $\operatorname{ind}(\gamma_U)|\operatorname{ind}(\hat{\gamma})$ and $\operatorname{ind}(\gamma_P)|\operatorname{ind}(\hat{\gamma})$ for each $P \in \mathfrak{P}$, respectively. We will deal with them in order.

We start by recalling the notion of *Azumaya algebras* and their generalized Severi-Brauer varieties. The notion of a central simple algebra over a field can be generalized to the notion of an *Azumaya algebra* over a domain R (cf. Saltman [31, Chapter 2], or Grothendieck [11, Part I, Section 1]). The degree of an Azumaya algebra A over R is the degree of $A \otimes_R F$ as a central simple algebra over the fraction field F over R. The *Brauer group* of a domain R is defined as the set of equivalence classes of Azumaya algebras with the analogous operations, where one replaces the vector spaces V_i with projective modules in the definition of Brauer equivalences. If A is an Azumaya algebra of degree n over a domain R, and $1 \leqslant i < n$, there is a functorially associated smooth projective R-scheme $\mathrm{SB}_i(A)$, called the i-th *generalized Severi-Brauer variety of* A (cf. Van den Bergh [35, p. 334]). For each R-algebra S, the S-points of $\mathrm{SB}_i(A)$ are in bijection with the right ideals of $A_S = A \otimes_R S$ that are direct summands of the S-module A_S having dimension (i.e. S-rank) ni. If R is a field F, so that A is a central simple F-

algebra, and if E/F is a field extension, then $SB_i(A)(E) \neq \phi$ if and only if $\mathrm{ind}(A_E)$ divides i (cf. Knus et al. [24, Proposition 1.17]). Here $A_E \cong \mathrm{Mat}_m(D_E)$ for some E-division algebra D_E and some $m \geqslant 1$, and the right ideals of E-dimension ni are in natural bijection with the subspaces of D_E^m of D_E-dimension $i/\mathrm{ind}(A_E)$ (cf. Knus et al. [24, Proposition 1.12, Definition 1.9]). Thus the F-linear algebraic group $GL_1(A) = GL_m(D_F)$ acts transitively on the points of the F-scheme $SB_i(A)$. We record Knus et al. [24, Proposition 1.17] here since we will be using it over and over again in the sequel.

Proposition 5.2. *Let A be a central simple algebra over a field F. The Severi-Brauer variety $SB_r(A)$ has a rational point over an extension K/F if and only if the index $\mathrm{ind}(A_K)$ divides r. In particular, $SB(A)$ has a rational point over K if and only if K splits A.*

5.1 INDEX COMPUTATION OVER AFFINE OPEN SET

We compute $\mathrm{ind}(\gamma_U)$ in this section; in particular, we show that $\mathrm{ind}(\gamma_U)|\mathrm{ind}(\hat{\gamma})$. Thanks to Lemma 4.3, it suffices to show that there exists an open affine subset $V \subset X$ so that $\mathrm{ind}(\gamma_V)|\mathrm{ind}(\hat{\gamma})$ since we could replace U by V if necessary in the construction we outlined in section 4.1 and this would not change $\gamma \in H^2(K(\hat{X})$ by Lemma 4.3. Therefore we will prove the following proposition, which shows that there exists such an open affine subset V.

Proposition 5.3. *Let T be a complete discrete valuation ring. Let $\hat{X}$ be a smooth projective T-curve with closed fibre X. Let F be the function field of $\hat{X}$ and $\hat{F}$ the completion of F with respect to the discrete valuation determined by X. Then for every $\hat{\gamma} \in H^2(\hat{F})$, there exists an affine open subset $V \subset X$ such that $\operatorname{ind}(\gamma_V) | \operatorname{ind}(\hat{\gamma})$, where γ_V is the lift of $\hat{\gamma}$ to F_V as defined in section 4.1.*

Proof. Recall that $\hat{\gamma} = \gamma_0 + (\chi_0, t) \in H^2(\hat{F})$ where $\gamma_0 \in H^2(k(X))$ and $\chi_0 \in H^1(k(X))$. Therefore $\operatorname{ind}(\hat{\gamma}) = \operatorname{ind}(\gamma_{0,l}) \cdot \exp(\chi_0)$, where $l/k(X)$ is the field extension determined by χ_0, by Jacob and Wadsworth [22, Theorem 5.15], since $\hat{F}$ is a complete discretely valued field.

Let U be an open affine subset of X such that neither γ_0 nor χ_0 ramifies on any point of U. Recall that $\gamma_U = \tilde{\gamma}_0 + (\tilde{\chi}_0, t)$ where $\tilde{\gamma}_0 \in H^2(\hat{R}_U)$ and $\tilde{\chi}_0 \in H^1(\hat{R}_U)$. Note that $\exp(\tilde{\chi}_0) = \exp(\chi_0)$ since $H^1(\hat{R}_U) \cong H^1(k(X))$. By Proposition 5.1, we have $\operatorname{ind}(\gamma_U) | \operatorname{ind}(\tilde{\gamma}_{0,S}) \cdot \exp(\tilde{\chi}_0)$, where $S/\hat{R}_U$ denotes the Galois cyclic extension determined by $\tilde{\chi}_0$. Note when $V \subseteq U$, we have $H^r(k[U]) \subseteq H^r(k[V])$ by purity, and hence $H^r(\hat{R}_U) \subseteq H^r(\hat{R}_V)$; so we have $\tilde{\gamma}_0 \in H^2(\hat{R}_V)$ and $\tilde{\chi}_0 \in H^1(\hat{R}_V)$. Therefore it suffices to find some affine open subset $V \subset U$ such that $\operatorname{ind}(\tilde{\gamma}_{0,S'}) | \operatorname{ind}(\gamma_{0,l})$, where $S'/\hat{R}_V$ denotes the Galois cyclic extension determined by $\tilde{\chi}_0$.

Let $i = \mathrm{ind}(\gamma_{0,l})$ be the index of the restriction of γ_0 to l. Then Proposition 5.2 implies that $SB_i(\gamma_0)(l) \neq \phi$; in other words, there is an l-rational point in the i-th generalized Severi-Brauer variety of γ_0. Therefore the morphism $\pi : SB_i(\gamma_0) \times_{k(X)} l \to \mathrm{Spec}(l)$ has a section $\mathrm{Spec}(l) \to SB_i(\gamma_0) \times_{k(X)} l$ over $\mathrm{Spec}(k(X))$, the generic point of the closed fibre U of $\mathrm{Spec}(\hat{R}_U)$. Choose a Zariski dense open subset $V \subseteq U$ such that this section over $\mathrm{Spec}(k(X))$ extends to a section over V, and such that the image of this latter section lies in an open subset of $SB_i(\gamma_0) \times_{k(X)} l$ that is affine. Then by Lemma 3.3, the section over V lifts to a section over $\mathrm{Spec}(\hat{R}_V)$, thus we obtain an L-rational point of $SB_i(\tilde{\gamma}_0) \times_{\hat{R}_V} S'$, where L/F_V is the Galois cyclic extension determined by $\tilde{\chi}_0$; or equivalently, L is the fraction field of S'. This implies that $\mathrm{ind}(\tilde{\gamma}_{0,S'})|i = \mathrm{ind}(\gamma_{0,l})$ by Proposition 5.2 again. □

5.2 INDEX COMPUTATION OVER CLOSED POINTS

It remains to show $\mathrm{ind}(\gamma_P)|\mathrm{ind}(\hat{\gamma})$. This is what we are going to do in this section. Note that γ_P is defined as $s_P \circ \mathrm{res}_{F_\mathfrak{p}}(\gamma_U)$, where $\mathrm{res}_{F_\mathfrak{p}}$ can only lower index of γ_U. Since we have already shown that $\mathrm{ind}(\gamma_U)|\mathrm{ind}(\hat{\gamma})$, we have that $\mathrm{ind}(\gamma)$ will be completely determined by $\mathrm{ind}(\gamma_U)$ if we could show that $\mathrm{ind}(\gamma_\mathfrak{p})$ does not go up under the map

s_P. Therefore we just need to show that s_P cannot increase index of Brauer classes, or, $\mathrm{ind}(\gamma_P) = \mathrm{ind}(s_P(\gamma_{\mathfrak{p}}))|\mathrm{ind}(\gamma_{\mathfrak{p}})$.

We compute $\mathrm{ind}(\gamma_{\mathfrak{p}})$ first. Since $F_{\mathfrak{p}}$ is a complete discretely valued field, we have $\mathrm{ind}(\gamma_{\mathfrak{p}}) = \mathrm{ind}((\gamma_{0,0} + (\chi_1, \pi))_M) \cdot \exp(\chi_2 + (\pi) \cup \xi_r)$, where $M/\kappa(\mathfrak{p})$ is the Galois cyclic extension determined by $\chi_2 + (\pi) \cup \xi_r \in H^1(\kappa(\mathfrak{p}))$ by Jacob and Wadsworth [22, Theorem 5.15]. It is not hard to compute $\mathrm{ind}((\gamma_{0,0} + (\chi_1, \pi))_M)$: Since M is a finite extension of $\kappa(\mathfrak{p})$, which is a complete discretely valued field, we have that M is a complete discretely valued field as well. Let e be the ramification index of $M/\kappa(\mathfrak{p})$ and $\bar{M}$ the residue field of M. Then by Serre [33, Exercise XII.3.2], $(\gamma_{0,0} + (\chi_1,\pi))_M = (\gamma_{0,0})_{\bar{M}} + (e \cdot \chi_1, \pi')$, where π' is some uniformizer of M. Let $L/\kappa(\mathfrak{p})$ be the field extension determined by $e \cdot \chi_1$ and $\bar{L}$ the residue field of L. Then $\mathrm{ind}((\gamma_{0,0} + (\chi_1,\pi))_M) = \mathrm{ind}((\gamma_{0,0})_{\bar{M}} + (e \cdot \chi_1, \pi')) = \mathrm{ind}((\gamma_{0,0})_{\bar{M}\bar{L}}) \cdot \exp(e \cdot \chi_1)$.

Now that we have an index formula for Brauer classes over $F_{\mathfrak{p}}$, we are ready to show the following

Proposition 5.4. *Let* T *be a complete discrete valuation ring. Let* $\hat{X}$ *be a smooth projective* T*-curve with closed fibre* X*. Suppose that* U *is an open affine subset of* X *and* $P \in X \backslash U$ *is a closed point. Let* $\mathfrak{p} = (U, P)$ *be the unique branch of* U *at* P *and let* γ_P *and* $\gamma_{\mathfrak{p}}$ *be defined as above. Then we have* $\mathrm{ind}(\gamma_P)|\mathrm{ind}(\gamma_{\mathfrak{p}})$.

Proof. By Proposition 5.1 we have that $\mathrm{ind}(\gamma_P)|\mathrm{ind}((\tilde{\gamma}_{0,0} + (\tilde{\chi}_1, \pi))_{\tilde{M}}) \cdot \exp(\tilde{\chi}_2)$, where $\tilde{M}/F_P$ is the Galois cyclic extension determined by $\tilde{\chi}_2 + \tilde{\chi}'$. We claim that $\exp(\tilde{\chi}_2 + \tilde{\chi}') = \exp(\chi_2 + (\pi) \cup \xi_r)$: we have that

$$\exp(\tilde{\chi}_2 + \tilde{\chi}') = \operatorname{lcm}(\exp(\tilde{\chi}_2), \exp(\tilde{\chi}'))$$

and $\exp(\chi_2 + (\pi) \cup \xi_r) = \operatorname{lcm}(\exp(\chi_2), \exp((\pi) \cup \xi_r))$. Since $\exp(\tilde{\chi}_2) = \exp(\chi_2)$ and $\exp(\tilde{\chi}') = \exp((\pi) \cup \xi_r)$, we have that $\exp(\tilde{\chi}_2 + \tilde{\chi}') = \exp(\chi_2 + (\pi) \cup \xi_r)$. Therefore this proposition will follow if we can show that

$$\operatorname{ind}((\tilde{\gamma}_{0,0} + (\tilde{\chi}_1, \pi))_{\tilde{M}}) | \operatorname{ind}((\gamma_{0,0})_{\tilde{M}} + (e \cdot \chi_1, \pi')).$$

Next we compute

$$\begin{aligned} (\tilde{\gamma}_{0,0} + (\tilde{\chi}_1, \pi))_{\tilde{M}} &= (\tilde{\gamma}_{0,0})_{\tilde{M}} + (\tilde{\chi}_1, \pi)_{\tilde{M}} \\ &= (\tilde{\gamma}_{0,0})_{\tilde{M}} + ((\tilde{\chi}_1)_{\tilde{M}}, \pi) \\ &= (\tilde{\gamma}_{0,0})_{\tilde{M}} + ((\tilde{\chi}_1)_{\tilde{M}}, (\pi')^e) \\ &= (\tilde{\gamma}_{0,0})_{\tilde{M}} + (e \cdot (\tilde{\chi}_1)_{\tilde{M}}, \pi') \end{aligned}$$

By Proposition 5.1 again we immediately see $\operatorname{ind}((\tilde{\gamma}_{0,0} + (\tilde{\chi}_1, \pi))_{\tilde{M}}) | \operatorname{ind}((\tilde{\gamma}_{0,0})_{\tilde{M}\tilde{L}}) \cdot \exp(e \cdot (\tilde{\chi}_1)_{\tilde{M}})$, where $\tilde{L}/F_P$ denotes the Galois cyclic extension determined by $e \cdot \tilde{\chi}_1$. Clearly $\exp(e \cdot (\tilde{\chi}_1)_{\tilde{M}}) | \exp(e \cdot (\chi_1))$, so we will be done if we can show that $\operatorname{ind}((\tilde{\gamma}_{0,0})_{\tilde{M}\tilde{L}}) | \operatorname{ind}((\gamma_{0,0})_{\tilde{M}\tilde{L}})$, which we will do in the following Lemma 5.5. □

Lemma 5.5. *In line with the notation in 5.4, we have that* $\operatorname{ind}((\tilde{\gamma}_{0,0})_{\tilde{M}\tilde{L}}) | \operatorname{ind}((\gamma_{0,0})_{\tilde{M}\tilde{L}})$.

Proof. Let $\tilde{M}'/F_P$ be the Galois cyclic extension determined by χ_2. Clearly it suffices to prove $\operatorname{ind}((\tilde{\gamma}_{0,0})_{\tilde{M}'\tilde{L}})|\operatorname{ind}((\gamma_{0,0})_{\bar{M}\bar{L}})$ since $\operatorname{ind}((\tilde{\gamma}_{0,0})_{\tilde{M}\tilde{L}})|\operatorname{ind}((\tilde{\gamma})_{\tilde{M}'\tilde{L}})$. Let $i = \operatorname{ind}((\gamma_{0,0})_{\bar{M}\bar{L}})$. By Proposition 5.2, we have that $SB_i(\gamma_{0,0})(\bar{M}\bar{L}) \neq \phi$, or equivalently, that the morphism $SB_i(\gamma_{0,0}) \times_{\kappa(P)} \bar{M}\bar{L}$ has a section $\operatorname{Spec}(\bar{M}\bar{L}) \to SB_i(\gamma_{0,0}) \times_{\kappa(P)} \bar{M}\bar{L}$. By Lemma 3.3, this section lifts to a section over $\operatorname{Spec}(\hat{R}_P)$; thus we obtain a $\tilde{M}'\tilde{L}$-rational point of $SB_i(\tilde{\gamma}_{0,0}) \times_{\hat{R}_P} S$ (note that $\gamma_{0,0} \in H^2(\hat{R}_P)$), where S is the integral closure of $\hat{R}_P$ in $\tilde{M}'\tilde{L}$; or equivalently, a $\tilde{M}'\tilde{L}$-rational point of $SB_i(\tilde{\gamma}_{0,0}) \times_{F_P} \tilde{M}'\tilde{L}$. Therefore $\operatorname{ind}((\tilde{\gamma}_{0,0})_{\tilde{M}'\tilde{L}})|i$ again by Proposition 5.2, which proves this lemma. □

The following Corollary is immediate:

Corollary 5.6. *The homomorphism $s : H^2(\hat{F}) \to H^2(F)$ preserves index of Brauer classes.*

Proof. This is simply Corollary 4.5 plus Proposition 5.4.

□

6

INDECOMPOSABLE AND NONCROSSED PRODUCT DIVISION ALGEBRAS OVER CURVES OVER COMPLETE DISCRETE VALUATION RINGS

Let T be a complete discrete valuation ring. Note that throughout this chapter, we will assume that T has finite residue field k. Let $\hat{X}$ be a smooth projective T-curve with closed fibre X. Let F be the function field of $\hat{X}$ and $\hat{F}$ the completion of F with respect to the discrete valuation determined by X. We construct indecomposable division algebras and noncrossed product division algebras over F of prime power index for all primes q where q is different from the characteristic of the residue field of T. Note that the existence of such algebras are already known when residue field of T is a finite filed, cf. Brussel et al. [6]. Our construction here is almost identical to Brussel et al. [6, Section 4], we list it here for the reader's convenience.

First we recall the construction of indecomposable division algebras over $\hat{F}$, this is done in Brussel et al. [6, Proposition 4.2].

Proposition 6.1. *Let T be a complete discrete valuation ring with finite residue field and let $\hat{X}$ be a smooth projective curve over* $\mathrm{Spec}(T)$ *with closed fibre X. Let F be the function field of $\hat{X}$ and $\hat{F}$ the completion of F with respect to the discrete valuation induced by X. Let e, i be integers satisfying $l \leqslant e \leqslant 2e-1$. For any prime $q \neq \mathrm{char}(k)$, there exists a Brauer class $\hat{\gamma} \in H^2(\hat{F})$ satisfying $\mathrm{ind}(\hat{\gamma}) = q^i, \exp(\hat{\gamma}) = q^e$ and whose underlying division algebra is indecomposable.*

Then we lift $\hat{\gamma}$ to F by using the splitting map s we defined in section 4, and show that the lift is in fact indecomposable.

Theorem 6.2. *In the notation of Theorem 6.1. Then there exists an indecomposable division algebra D over F such that* $\mathrm{ind}(D) = q^i$ *and* $\exp(D) = q^e$.

Proof. By Proposition 6.1, there exists $\hat{\gamma} \in \mathrm{Br}(\hat{F})$ with $\mathrm{ind}(\hat{\gamma}) = q^i$ and $\exp(\hat{\gamma}) = q^e$ and whose underlying division algebra is indecomposable. By Corollary 5.6, $\gamma = s(\hat{\gamma})$ has index q^i too. Since s splits the restriction map, we have $\exp(\gamma) = q^e$. We show the division algebra underlying γ is indecomposable.

We proceed by contradiction. Assume $\gamma = \beta_1 + \beta_2$ represents a nontrivial decomposition, then $\hat{\gamma} = \text{res}_{\hat{F}}(\beta_1) + \text{res}_{\hat{F}}(\beta_2)$. Since the index can only go down under restriction, we have that $\text{ind}(\hat{\gamma}) = \text{ind}(\text{res}_{\hat{F}}(\beta_1)) \cdot \text{ind}(\text{res}_{\hat{F}}(\beta_2))$, which represents a nontrivial decomposition of the division algebra underlying $\hat{\gamma}$, a contradiction. □

6.2 NONCROSSED PRODUCTS OVER f

Again we will construct noncrossed product division algebras over $\hat{F}$ and use the splitting map s to lift it to F and show that the lift represents a noncrossed product division algebra over F.

The construction over $\hat{F}$ is in line with Brussel [5] where noncrossed products over $Q(t)$ and $Q((t))$ are constructed. In order to mimic the construction in Brussel [5], we need only note that both Chebotarev density theorem and the Gruwald-Wang theorem hold for global fields which are characteristic p function fields. Then the arguments in Brussel [5] apply directly to yield noncrossed products over $\hat{K}(\hat{X})$ of index and exponent given below:

The following is Brussel et al. [6, Theorem 4.7].

Theorem 6.3. *Let T be a complete discrete valuation ring with finite residue field k and let $\hat{X}$ be a smooth projective curve over* $\text{Spec}(T)$*. Let F be the function field of $\hat{X}$ and let $\hat{F}$ be the completion of F with respect to the discrete valuation induced by* *the closed fibre. For any positive integer* a*, let* ϵ_a *be a primitive* a*-th root of unity. Set* r *and* s *to be maximum integers such that* $\mu_{q^r} \subset k(X)^\times$ *and* $\mu_{q^s} \subset k(X)(\epsilon_{q^{r+1}})$*. Let* n, m *be integers such that* $n \geqslant 1, n \geqslant m$ *and* $n, m \in r \cup [s, \infty)$*. Let* a, l *be integers such that* $l \geqslant n + m + 1$ *and* $0 \leqslant a \leqslant 1 - n$*. (See Brussel [5, Page 384-385] for more information regarding these constraints.) Let* $q \neq \mathrm{char}(k)$ *be a prime number. Then there exists noncrossed product division algebras over* $\hat{F}$ *with index* q^{l+1} *and exponent* q^l*.*

Corollary 6.4. *Let* $R, k, \hat{X}, X, F, \hat{F}, q, a, l$ *be as in Theorem 6.3. Then there exists noncrossed product division algebras over* F *of index* q^{l+a} *and exponent* q^l*.*

Proof. Let $\hat{\gamma}$ be the Brauer class representing a noncrossed product over $\hat{F}$ of index q^{l+a} and exponent q^l. Let D be the division algebra underlying the Brauer class $s(\hat{\gamma})$. By Corollary 5.6, we know that $\mathrm{ind}(D) = \mathrm{ind}(\hat{\gamma})$.

Assume that D is a crossed product with maximal Galois subfield M/F. Then $M\hat{F}$ splits $\hat{\gamma}$, is of degree $\mathrm{ind}(\hat{\gamma})$ and is Galois. This contradicts the fact that $\hat{\gamma}$ is a noncrossed product. □

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COLOPHON

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```
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