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An abstract of
a thesis submitted to the Faculty of Emory College of Arts and Sciences
of Emory University in partial fulfillment
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Abstract

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Möbius transformations make up a very special class of conformal mappings. They are differentiable, continuous, and bijective in the entirety of their domain and make up the group of homeomorphisms of the extended complex plane. And with their conformal and homeomorphic properties they preserve the essence of a shape or space, and thus have many applications in physics, engineering, and complex analysis. But while Möbius transformations are very beautiful mathematical objects, their rigidity can cause problems when attempting to apply them to more complicated domains and structures. Thus the concept of a quasiconformal or "almost conformal" map was developed. Quasiconformal maps are generalizations of conformal maps and Möbius transformations that are both flexible enough to be applied to more difficult problems and yet have enough structure to be useful and interesting. The purpose of this paper is to gather and prove the key characterizations of Möbius transformations in a clear and succinct manner as well as to generalize some of them to properties for quasiconformal mappings.

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1 Introduction

1.1 Conformal Mappings

The study of conformal mappings has a long mathematical history which goes back as early as the sixteenth century. One example of their use during this time is the stereographic projection, which was widely popular in cartography and had been used at least since the time of Ptolemy for astronomy [7]. Another is the Mercator projection, which Flemish mapmaker Gerard Mercator devised in 1541 to draw loxodromes on a globe and to transfer them to lines on a sheet of paper. Loxodromes are logarithmic spirals toward the north and south poles which result from following a constant compass bearing and the fact that the earth is spherical and not flat like a map. The Mercator projection was used to plot these spirals and aided greatly in naval navigation [7].

Now before we get to the definition of a conformal map I wish to explain a few terms for the sake of the reader. First, a complex-valued function $w = f(z)$ is said to be *analytic* in a given domain if it is differentiable at every point in the domain. A function is *injective* if it assigns to every point in the domain exactly one point in the codomain. A function is *surjective* if for every point in the codomain there exists a point in the domain that is mapped to it. A function is *bijective* if it is both injective and surjective. A *Jordan curve* is the image of a circle under an injective continuous map. A function is said to be *sense preserving* if it is continuous and preserves the orientation of a Jordan curve. If a function is analytic then it must be sense preserving or else there will exist a point for which the derivative does not exist [3]. For example $f(z) = \bar{z}$ is not sense preserving and its derivative does not exist except at the origin, thus it is not analytic. A function is said to be *homeomorphic* (or a *homeomorphism*) if it is a continuous bijection whose inverse is also continuous.

Lastly, a brief note on the notation used. Throughout the paper we denote the complex plane by $\mathbb{C}$, Euclidean n-space by $\mathbb{R}^n$, the extended complex plane by $\mathbb{C}_\infty \equiv \mathbb{C} \cup \{\infty\}$ (also known as the one-point compactification of the complex plane), extended Euclidean space by $\mathbb{R}^n_\infty \equiv \mathbb{R} \cup \{\infty\}$, and the restriction of a map f to a set D by $f|_D$. Note that $\mathbb{C}_\infty$ and

$\mathbb{R}^2_\infty$ are both homeomorphic to the Riemann sphere. Thus the real number line (or x-axis) is equivalent to a circle and the upper half plane is equivalent to a disk in these two topologies.

Definition 1.1. A mapping $w = f(z)$ is *conformal* if it preserves the angle between two differentiable arcs. Equivalently, a mapping is conformal if and only if it is analytic and homeomorphic in some domain $D \subseteq \mathbb{C}_\infty$ [3].

Conformal maps are helpful because they can be used to transform complicated shapes and spaces into simpler ones while preserving some of the defining properties of the shape or space. Thus they are extremely important in complex analysis and many areas of physics and engineering. One example often studied in physics is a type of problem known as a boundary value problem. Because conformal maps are analytic and homeomorphic, they preserve certain values along the boundary of a given domain. Specifically we have the following theorem from [3].

Theorem 1.2. Suppose that there is a domain C whose boundary, ∂C, is a smooth arc on which the transformation $w = f(z) = u(x, y) + i(v(x, y))$ is conformal and let Γ be the image of C under the transformation. If a function $h(u, v)$ satisfies either of the conditions $h = h_0$ or $\frac{dh}{dn} = 0$ along $\partial\Gamma$, where h_0 is a real constant and $\frac{dh}{dn}$ denotes derivatives normal to $\partial\Gamma$, then the function $H(x, y) = h[u(x, y), v(x, y)]$ satisfies the corresponding condition $H = h_o$ or $\frac{dH}{dN} = 0$ along ∂C, where $\frac{dH}{dN}$ denotes derivatives normal to ∂C.

The following example illustrates one such boundary value problem and comes from [3]. Suppose we want to find an expression for the steady temperatures $T(x, y)$ in a thin, semi-infinite plate $y \geq 0$ whose faces are insulated and whose edge $y = 0$ is kept at temperature $T = 0$ except for the segment $-1 < x < 1$, where it is kept at temperature $T = 1$. We will call this domain C. This problem can be written as: find a function $T(x, y)$ such that

$$T_{xx}(x, y) + T_{yy}(x, y) = 0 \qquad (-\infty < x < \infty, y > 0), \tag{1.1}$$

and

$$T(x, 0) = \begin{cases} 1 & \text{when } |x| < 1, \\ 0 & \text{when } |x| > 1. \end{cases} \tag{1.2}$$

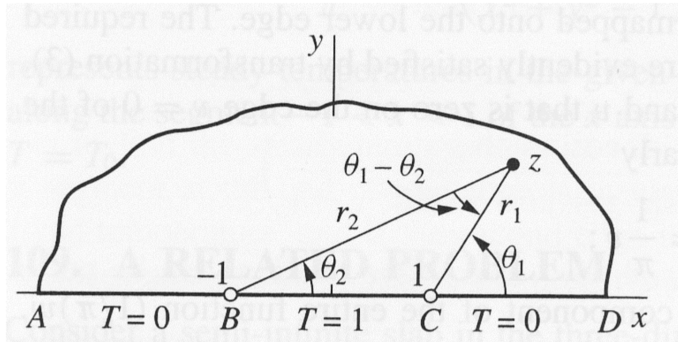

Figure 1.1: The domain C representing a thin, semi-infinite plate.

This problem does not have an obvious answer because of the singular points and restrictions on the domain. But we could make it easier if we were able to use a conformal map to transform this problem into a related problem in a simpler domain. For this example we will use the conformal map

$$f = \ln \frac{z-1}{z+1} = \ln \frac{r_1}{r_2} + i(\theta_1 - \theta_2)$$

to map our domain C in the x-y plane into a new domain Γ the u-v plane, where $z = u(x,y) + i(v(x,y)) = u(r,\theta) + i(v(r,\theta))$.

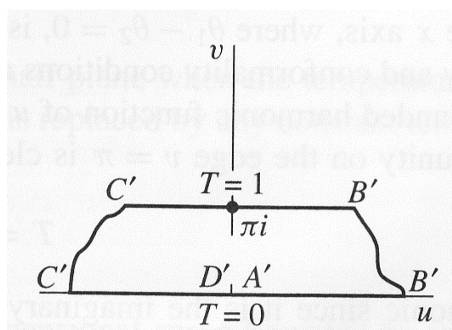

Figure 1.2: The codomain $\Gamma = f(C)$.

In the above figures the transformation can be visualized by taking point A to $A^{'}$, B to $B^{'}$, C to $C^{'}$, and D to $D^{'}$. With this new domain condition (1.2) is transformed to

$$T(u,v) = \begin{cases} 0 & \text{when } v = 0, \\ 1 & \text{when } v = \pi. \end{cases}$$

Clearly if we let $T = (\frac{1}{\pi})v$ then this condition is satisfied, and when we write T in terms of x and y it will satisfy the boundary conditions in our original problem thanks to Theorem 1.2. Thus,

$$T(x,y) = \frac{1}{\pi}\arctan\left(\frac{2y}{x^2+y^2-1}\right)$$

is the solution to our problem. Theorem 1.2 also guarantees that this function accurately models the temperature on the inside of our plate even though we only used the boundary values in order to construct it. This result has even more significance when attempting to model the temperature of a three dimensional object of which the inside temperature is not directly measurable.

The implications of the above example are even more important if we take the Riemann Mapping Theorem into account. This theorem states that if $U \subset \mathbb{C}$ is a nonempty, simply connected open set which is not all of $\mathbb{C}$ then there exists a conformal map f from U to the open unit disk $\mathbb{D}$. This f may be hard to find, but the fact that it always exists implies that any boundary value problem in a domain satisfying these conditions can be greatly simplified.

1.2 Möbius Transformations

Möbius transformations are a special class of conformal mappings. They are more rigidly defined but also have many useful properties that conformal maps do not necessarily have. They arise naturally in relation to conformal mappings because, as we shall see in Section 2, a conformal map defined in the entire extended complex plane can only be a Möbius transformation. Thus Möbius transformations make up the group of automorphisms of $\mathbb{C}_\infty$. Furthermore they are the only conformal mappings in higher dimensions by application of Liouville's Theorem [11].

Definition 1.3. Let $a, b, c, d \in \mathbb{C}$. A map $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ of the form

$$f(z) = \frac{az+b}{cz+d} \tag{1.3}$$

such that $ad - bc \neq 0$ is called a *Möbius transformation* (also known as a *linear fractional transformation*). Furthermore we define $f(\frac{-d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$ in order to prevent a possible zero in the denominator.

Möbius transformations are conformal because they are analytic and homeomorphic (which we will prove later). The fact that they are homeomorphisms is a particularly nice property because they preserve the topological properties of the complex plane. In the study of topology homeomorphisms are used to compare different spaces and give us a concrete definition of what it means for two spaces to be topologically "the same," as well as giving a method of determining what properties two spaces may share. For example, $\mathbb{R}^2_\infty$ is homeomorphic to $\mathbb{C}_\infty$ and as such mathematicians are able to apply many properties from one plane to the other with only slight modifications (if any are needed at all). Another result of this property is that the inverse of a Möbius transformation always exists and is itself a Möbius transformation. Specifically, given some Möbius transformation $f(z) = w$, we can always define its inverse to be

$$f^{-1}(z) = \frac{dw - b}{-cw + a}.$$

Note that if we multiply the coefficients a, b, c, d of a Möbius transformation by a nonzero constant $\lambda \in \mathbb{C}$ we see that

$$\frac{az + b}{cz + d} = \frac{\lambda az + \lambda b}{\lambda cz + \lambda d}.$$

Furthermore if two sets of coefficients a, b, c, d and a', b', c', d' give the same Möbius transformation then there exists a λ such that [9]

$$a' = \lambda a, \qquad b' = \lambda b, \qquad c' = \lambda c, \qquad d' = \lambda d.$$

We will explore more properties of Möbius transformations in Section 2.

1.3 Quasiconformal Mappings

Although conformal maps and Möbius transformations are wonderful mathematical tools, they are not always helpful when applied to more difficult problems. Just as a freshman physics class begins by assuming ideal conditions and later adds the forces of friction and drag so we too must modify conformal maps in order to apply them to problems with more complex domains. This brings us to the concept of a quasiconformal or "almost conformal" map. While conformal maps preserve angles, quasiconformal maps may deform them, but only within a limited, finite amount. But to be rigorous we need to specify exactly what this means. First, a few preliminary definitions. Suppose that D and $D^{'}$ are domains in $\mathbb{C}_\infty$ and that $f : D \to D^{'}$ is a homeomorphism. For $z \in D\backslash\{\infty, f^{-1}(\infty)\}$ and $0 < r < dist(z, \partial D)$, we define

$$l_f(z,r) = \min_{|z-w|=r} |f(z) - f(w)|, \quad L_f(z,r) = \max_{|z-w|=r} |f(z) - f(w)|,$$

and

$$H_f(z) = \limsup_{r \to 0} \frac{L_f(z,r)}{l_f(z,r)} \tag{1.4}$$

and call H_f the *linear dilation* of f at z. This H_f gives us a tool to measure the extent of the deformation of f caused at a point z and brings us to the metric definition of a quasiconformal map.

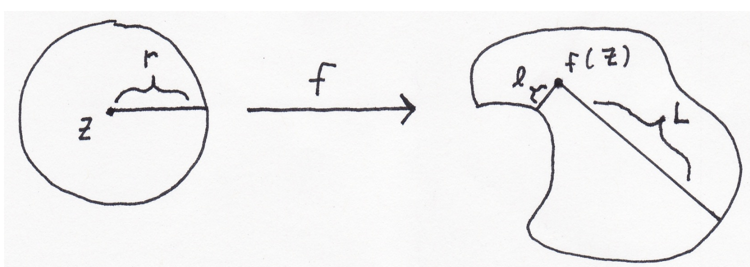

Figure 1.3: Example of computing L_f, l_f, and the linear dilation.

Definition 1.4 (Metric Definition)**.** A sense preserving homeomorphism $f : D \to D^{'}$ is a *K-quasiconformal mapping* where $1 \leq K < \infty$ if both

1. H_f is bounded in $D\backslash\{\infty, f^{-1}(\infty)\}$,

2. $H_f \leq K$ for almost every point in D.

Notice that this definition of a quasiconformal map is a natural extension of a conformal map, or in other words a given map is conformal if and only if it is 1-quasiconformal [5]. Thus the K value gives us a concrete method of measuring the deformation caused by a given quasiconformal mapping. For example, the map $f(z = x+yi) = 2x+yi$ is a 2-quasiconformal map. Thus the distortion caused by f is limited by a factor of 2. This can be seen visually since f turns the unit circle into an oval whose width is twice the height.

Quasiconformal mappings have only been studied since the 1930's, a relatively short time compared with the study of conformal mappings. Surely they were known and experimented with before then, but it was in 1935 that Lars Ahlfors coined the term "quasi-conformal" in his work on *Überlagerungsflächen* that earned him a Fields Medal [8]. Oswald Teichmüller also used quasiconformal mappings during this time to measure a distance between two conformally inequivalent compact Riemann surfaces, starting what is now called Teichmüller theory [8]. Today quasiconformal maps are an important study in complex analysis and have ties to studies of elliptic partial differential equations and geometric group theory, to name just a few examples. They also have important applications in medical image analysis and computer vision and graphics [8]. Thus we wish to study them further and will examine more characterizations in Section 3.

1.4 Quasicircles

One characterization of Möbius transformations which we will prove in the next section is that they preserve circles in the complex plane, or in other words the image of a circle is again a circle. Thus, as one might expect, we are interested in studying the image of the unit circle under a quasiconformal map in order to provide more tools for classifying such maps. We call such an image a *quasicircle*, and similarly we call the image of the unit disk under a quasiconformal map a *quasidisk*. All quasicircles are Jordan curves but as we shall see later not every Jordan curve is a quasicircle. We will explore this further in Section 3.

2 Characterizations of Möbius Transformations

2.1 Known Characterizations

Recall from Section 1.2 that a Möbius Transformation is a map $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ of the form

$$f(z) = \frac{az+b}{cz+d}$$

such that $ad - bc \neq 0$. The following are some important properties of Möbius transformations, some of which can be found in a good textbook on complex analysis such as [4]. These properties present some of the beauty and usefulness of Möbius transformations in various scientific fields. They are also the building blocks to further theorems and characterizations, and as such I will call upon them on occasion in later sections.

Proposition 2.1. If f is a Möbius transformation then f is a composition of translations, dilations, and/or an inversion.

Proof. Case I: $c = 0$. Then $d \neq 0$ and $f(z) = (\frac{a}{d})z + \frac{b}{d}$. If $f_1 = (\frac{a}{d})z$ (a dilation) and $f_2 = z + \frac{b}{d}$ (a translation) then $f = f_2 \circ f_1$.

Case II: $c \neq 0$. Then let $f_1(z) = z + \frac{d}{c}$, $f_2(z) = \frac{1}{z}$, $f_3(z) = \frac{(bc-ad)}{c^2}z$, and $f_4(z) = z + \frac{a}{c}$. Then $f = f_4 \circ f_3 \circ f_2 \circ f_1$. $\square$

Proposition 2.2. If f is a Möbius transformation then f is analytic.

Proof. From Proposition 2.1 we know f is the composition of translations, dilations, and/or an inversion. All of these can easily be shown to be analytic, and thanks to the chain rule we know that compositions of analytic functions are also analytic. The only possible exception is $\frac{1}{z}$ which is normally not analytic in $\mathbb{C}$ because the limit approaching 0 from the right and left is ∞ and $-\infty$ respectively. However, with the additional topology that comes from adding the point at infinity these limits are now equal and so $\frac{1}{z}$ is analytic in $\mathbb{C}_\infty$. Thus f is analytic. $\square$

Proposition 2.3. If f is a Möbius transformation then f is a homeomorphism of the extended complex plane, that is f is a continuous bijection whose inverse is also continuous.

Proof. Part I (Injectivity): Suppose $f(z) = f(w)$ for some $z, w \in \mathbb{C}_\infty$. Then

$$\frac{az+b}{cz+d} = \frac{aw+b}{cw+d}$$
$$(az+b)(cw+d) = (aw+b)(cz+d)$$
$$aczw + adz + bcw + bd = aczw + adw + bcz + bd$$
$$adz - bcz = adw - bcw$$
$$z(ad-bc) = w(ad-bc)$$

and this last line implies that $z = w$ since $ad - bc \neq 0$. Therefore f is injective.

Part II (Surjectivity): Suppose $w \in \mathbb{C}_\infty$. If $w = \frac{a}{c}$ then let $z = \infty$, else let $z = \frac{dw-b}{-cw+a}$. Then $f(z) = w$. Therefore f is surjective.

Part III (Continuity): From Proposition 2.2 we know f is analytic and thus differentiable everywhere in its domain. This then implies that f is continuous and since f^{-1} is also a Möbius transformation then it is also continuous. □

Proposition 2.4. The composition of two Möbius transformations is itself a Möbius transformation.

Proof. Let f, g be Möbius transformations such that

$$f(z) = \frac{az+b}{cz+d}, (ad-bc \neq 0) \quad \textit{and} \quad g(z) = \frac{\alpha z+\beta}{\gamma z+\delta}, (\alpha\delta - \beta\gamma \neq 0).$$

Then

$$(f \circ g)(z) = \frac{a(\frac{\alpha z+\beta}{\gamma z+\delta}) + b}{c(\frac{\alpha z+\beta}{\gamma z+\delta}) + d} = \frac{(\dfrac{a\alpha z + a\beta + b\gamma z + b\delta}{\gamma z+\delta})}{(\dfrac{c\alpha z + c\beta + d\gamma z + d\delta}{\gamma z+\delta})} = \frac{(a\alpha + b\gamma)z + (a\beta + b\delta)}{(c\alpha + d\gamma)z + (c\beta + d\delta)}$$

since $\alpha\delta - \beta\gamma \neq 0 \Rightarrow \gamma z + \delta \neq 0$. And

$$(a\alpha + b\gamma)(c\beta + d\delta) - (a\beta + b\delta)(c\alpha + d\gamma)$$

$$= a\alpha c\beta + a\alpha d\delta + b\gamma c\beta + b\gamma d\delta - a\beta c\alpha - a\beta d\gamma - b\delta c\alpha - b\delta d\gamma$$

$$= ad(\alpha\delta - \beta\gamma) - bc(\alpha\delta - \beta\gamma) = (\alpha\delta - \beta\gamma)(ad - bc) \neq 0.$$

Therefore $f \circ g$ is a Möbius transformation. $\square$

Theorem 2.5. If f is a conformal map defined in all of $\mathbb{C}_\infty$ then f can only be a Möbius transformation.

Proof. Suppose f is a conformal map defined in all of $\mathbb{C}_\infty$.

Case I: Assume $f(\infty) = \infty$. First, notice that $f \neq \frac{1}{z}$ else $f(0) = \infty$ and f is not injective, contradicting our assumption that f is conformal and thus homeomorphic. Next, consider $h(z) = f(\frac{1}{z})$. Since $h(0) = \infty$ then 0 is a pole of h, and thus ∞ is a pole of f [4]. Next, we can apply Taylor's Theorem to f because f is analytic in $\mathbb{C}_\infty$. Then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

But a contradiction arises if the power of this series is greater than 1. This is because f is defined in $\mathbb{C}_\infty$ and by the Fundamental Theorem of Algebra a polynomial of degree n in $\mathbb{C}_\infty$ has exactly n zeros. Thus if $n > 1$ then f is not injective. Therefore the order of the pole at ∞ is 1 and f must be a polynomial of degree 1, or in other words f has the form $cz + d$ for some $c, d \in \mathbb{C}$, $c \neq 0$. Thus f is Möbius.

Case II: Assume $f(\infty) \neq \infty$. Then define

$$g(z) = \frac{1}{f(z) - f(\infty)}$$

and notice that g is also a conformal map defined in all of $\mathbb{C}_\infty$ with $g(\infty) = \infty$. Then by Case I there exist $c, d \in \mathbb{C}$, $c \neq 0$ such that

$$cz + d = g(z) = \frac{1}{f(z) - f(\infty)}.$$

Rewriting this equation we get that

$$f(z) = \frac{f(\infty)cz + df(\infty) + 1}{cz + d}.$$

If we let $a = f(\infty)c$ and $b = df(\infty) + 1$ then it is clear that f is Möbius since $ad - bc \neq 0$ because $ad \neq bc$ and $c \neq 0$. $\square$

Theorem 2.6. For any distinct $z_2, z_3, z_4 \in \mathbb{C}_\infty$ the function

$$f(z) = \frac{(z - z_3)(z_2 - z_4)}{(z - z_4)(z_2 - z_3)}$$

is the unique Möbius function such that $f(z_2) = 1$, $f(z_3) = 0$, and $f(z_4) = \infty$. In addition if $f(z) = w$ is a Möbius transformation such that $f(z_i) = w_i$ for $i = 2, 3, 4$ then w and z are related by the formula

$$\frac{(w - w_3)(w_2 - w_4)}{(w - w_4)(w_2 - w_3)} = \frac{(z - z_3)(z_2 - z_4)}{(z - z_4)(z_2 - z_3)}$$

which can be used to find f explicitly in special cases. Thus Möbius transformations are uniquely determined by their action on three points [9].

Corollary 2.7. Let f be a Möbius transformation. If f has three fixed points then f is the identity map.

Definition 2.8. If $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ then $[z_1, z_2, z_3, z_4]$ (the *cross-ratio* of z_1, z_2, z_3, and z_4) is the image of z_1 under the unique Möbius transformation which takes z_2 to 1, z_3 to 0, and z_4 to ∞. Equivalently,

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}. \tag{2.1}$$

This definition is extended by continuity to include the case when one of the z_i is ∞ [2]. For example,

$$[z_1, z_2, z_3, \infty] = \frac{(z_1 - z_3)}{(z_2 - z_3)}.$$

We also define the *absolute cross-ratio* to be the value obtained by taking the absolute value of each of the differences in the cross-ratio.

Theorem 2.9. Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{C}_\infty$. Then $[z_1, z_2, z_3, z_4] \in \mathbb{R}$ if and only if all four points lie on a circle [4].

2.2 Key Results

Two of the most important characterizations of Möbius transformations is that they are both cross-ratio and circle preserving. But what is not always stated is that the converse is also true: if a function preserves either cross-ratios or circles then it must be a Möbius transformation. Thus if we have a function which does either then we can apply to it all the properties of Möbius transformations. In the two following theorems f is assumed to be sense preserving, a necessary condition in order for f to be analytic.

Theorem 2.10. Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a homeomorphism. f is Möbius if and only if f preserves cross-ratios.

Proof. Part I: Assume f is Möbius. Pick $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$. Define $g(z) = [z, z_2, z_3, z_4]$, i.e. the value of the cross-ratio of z, z_2, z_3, z_4 as z varies. Then g is Möbius by the definition of a cross-ratio. Let $h = g \circ f^{-1}$, then h is Möbius by Proposition 2.4. Notice that $h(f(z_2)) = 1$, $h(f(z_3)) = 0$, and $h(f(z_4)) = \infty$. Thus $h(z) = [z, f(z_2), f(z_3), f(z_4)]$ by the definition of a cross-ratio and because a Möbius transformation is uniquely determined by three points using Theorem 2.6. Therefore

$$[z_1, z_2, z_3, z_4] = g(z_1) = (g \circ f^{-1})(f(z_1)) = h(f(z_1)) = [f(z_1), f(z_2), f(z_3), f(z_4)].$$

Part II: Assume f preserves cross-ratios.

Case I: Assume $f(\infty) = \infty$. Pick distinct $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ all not equal to ∞. Now:

$$\begin{aligned}\frac{z_1 - z_4}{z_2 - z_3} = \frac{(z_2 - z_4)/(z_2 - z_3)}{(z_2 - z_4)/(z_1 - z_4)} = \frac{[\infty, z_2, z_3, z_4]}{[z_1, z_2, \infty, z_4]} = \frac{[\infty, f(z_2), f(z_3), f(z_4)]}{[f(z_1), f(z_2), \infty, f(z_4)]} \\ = \frac{(f(z_2) - f(z_4))/(f(z_2) - f(z_3))}{(f(z_2) - f(z_4))/(f(z_1) - f(z_4))} = \frac{f(z_1) - f(z_4)}{f(z_2) - f(z_3)}\end{aligned}$$

because f preserves cross-ratios. Thus

$$\frac{f(z_1) - f(z_4)}{z_1 - z_4} = \frac{f(z_2) - f(z_3)}{z_2 - z_3}.$$

Notice that this ratio is independent of the points chosen. So if we take the absolute value of both sides of this equation then there exists a positive nonzero constant $K \in \mathbb{R}$ such that

$$\frac{|f(z_1) - f(z_4)|}{|z_1 - z_4|} = K.$$

Then if we rewrite $z = z_1$ and $z_0 = z_4$ we get that $|f(z) - f(z_0)| = K|(z - z_0)|$. If $K = 1$ then f can only be a rotation and/or a translation since this equality holds for any two points in the domain. If $K \neq 1$ then f is also a dilation. Thus $f = az + b$ and is Möbius.

Case II: Suppose $f(\infty) \neq \infty$. Compose f with a Möbius map g such that $g(f(\infty)) = \infty$ by using Theorem 2.6. Using the results from Case I we then know that $g \circ f$ is Möbius, g^{-1} exists and is Möbius, and $g^{-1} \circ g \circ f = f$. Thus f is Möbius by Proposition 2.4. $\square$

Theorem 2.11. Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a homeomorphism. f is Möbius if and only if f is circle-preserving.

Proof. Part I: Assume f is Möbius. Pick some circle $\Gamma \in \mathbb{C}_\infty$, produce distinct $z_2, z_3, z_4 \in \Gamma$. Then $f(z_2), f(z_3), f(z_4)$ determine a circle $\Gamma' \in \mathbb{C}_\infty$. And for any $z \in \Gamma$, $[z, z_2, z_3, z_4] = [f(z), f(z_2), f(z_3), f(z_4)]$ by Theorem 2.10 and also $[z, z_2, z_3, z_4] \in \mathbb{R}$ by Theorem 2.9. Therefore if $z \in \Gamma$ then the above cross-ratios are equal and are both real numbers, hence $f(z) \in \Gamma'$, again using Theorem 2.9.

Part II: Assume f is circle-preserving. Pick a domain $D \subseteq \mathbb{C}_\infty$ and pick $a \in D$, produce an open ball $B_a \ni a$ a subset of D such that $f|_{B_a}$ is circle-preserving and injective. Then produce γ_1 Möbius such that $\gamma_1(E) = B_a$, where E is the area outside the unit circle with center (0, 1). Such a γ_1 exists because of the Riemann Mapping Theorem and Theorem 2.6, we simply need to invert E and then move it using three points in B_a. γ_1 Möbius implies that $\gamma_1(\mathbb{R})$ is a circle in B_a, call this C. Thus $f \circ \gamma_1$ maps E into $\mathbb{C}_\infty$ and $(f \circ \gamma_1)(\mathbb{R}) = f(C)$ is also a circle in $\mathbb{C}_\infty$ because of our hypothesis. Furthermore, by Lemma 2.3 from [11], $f \circ \gamma_1$ maps the upper half plane in E into the inside of $f(C)$. Also, can produce γ_2 Möbius such that $\gamma_2(f(C)) = \mathbb{R}$ and the inside of $f(C)$ maps to the upper half plane in $\mathbb{C}_\infty$. Let $F = \gamma_2 \circ f \circ \gamma_1$, which leaves $\mathbb{R}$ invariant and maps the upper half plane to itself. $\mathbb{R}$ is invariant, but if we need to we can also compose F with another Möbius map that fixes 0, 1, and ∞. Then $F = Id_E$ by Theorem 2.2 from [11]. Therefore $\gamma_2 \circ f \circ \gamma_1 = Id_E \Rightarrow f = \gamma_2^{-1} \circ \gamma_1^{-1}$ and f is the restriction of a Möbius map to B_a using Proposition 2.4. By Lemma 1.7 from [11], f is the restriction of this Möbius map to the plane domain D. $\square$

3 Characterizations of Quasiconformal Mappings and Quasicircles

3.1 Characterizations of Quasiconformal Mappings

In Section 1.3 we gave the metric definition of a quasiconformal map. However, by now the reader has probably realized that the defined computation of the linear dilation can be difficult to use because of the limits and supremums involved. Fortunately, we can use Wirtinger derivatives to simplify the linear dilation when the map is differentiable and acting on the extended complex plane. For a complex function where $z = x + yi$ we normally define derivatives in terms of z and partial derivatives in terms of x and y. But we can also rewrite the function to be in terms of z and $\bar{z}$, where $\bar{z} = x - yi$ is the complex conjugate of z, in which case we get the Wirtinger (partial) derivatives

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \quad and \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}).$$

Thus if f is differentiable in a domain $D \subseteq \mathbb{C}_\infty$ then H_f can be rewritten as

$$H_f(z) = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}, \tag{3.1}$$

where f is assumed to be orientation preserving and f_z is the partial derivative of f with respect to z using Wirtinger derivatives [11]. And if f is conformal then $f_{\bar{z}} = 0$ which implies that $H_f(z) = 1$ for any z, a fact that agrees with our previous statement that a conformal map is 1-quasiconformal.

There exist two other definitions of a quasiconformal map besides the metric definition: the modulus (or geometric) definition and the analytic definition. I will not state the analytic definition here because I do not use it in this paper but the reader will be able to find it in [5] if so desired. The modulus definition is stated as follows.

Definition 3.1 (Modulus Definition)**.** A sense preserving homeomorphism $f : D \to D'$ is *K-quasiconformal* if and only if

$$\frac{1}{K} mod(\Gamma) \leq mod(f(\Gamma)) \leq K mod(\Gamma) \tag{3.2}$$

for each family Γ of curves in D, where $f(\Gamma)$ is the image of Γ under f and $mod(\Gamma)$ is the *modulus* of Γ, the definition of which we omit here but can be found in [5].

Fortunately, all three definitions are proven to be equivalent in [5] and so they may be used interchangeably when proving theorems about quasiconformal maps. This is extremely helpful because it gives more tools with which to solve a problem and formulate conjectures. For example, the following Proposition would be difficult to prove with the metric definition, but is quite easy when the modulus definition is used.

Proposition 3.2. If $f : D \to D'$ is K-quasiconformal and $g : D' \to D''$ is K'-quasiconformal then $g \circ f$ is KK'-quasiconformal and f^{-1} is K-quasiconformal (note that the inverse always exists because f is a homeomorphism).

Proof. Let f and g be as defined and let Γ be a family of curves in D. Then

$$mod(g \circ f)(\Gamma) = mod(g(f(\Gamma)) \leq K' mod(f(\Gamma)) \leq KK' mod(\Gamma)$$

since $f(\Gamma)$ is a family of curves in D'. Similarly we can show that

$$mod(g \circ f)(\Gamma) \geq \frac{1}{KK'} mod(\Gamma)$$

thus $g \circ f$ is KK'-quasiconformal. Second,

$$mod(\Gamma) = mod((f \circ f^{-1})(\Gamma)) \leq K mod(f^{-1})(\Gamma)$$

which implies that

$$\frac{1}{K} mod(\Gamma) \leq mod(f^{-1}(\Gamma))$$

and similarly we can show that

$$mod(f^{-1}(\Gamma)) \leq K mod(\Gamma)$$

Therefore $g \circ f$ is KK'-quasiconformal and f^{-1} is K-quasiconformal by the modulus definition. □

In Section 2.2 we proved that a Möbius transformation preserves cross-ratios and we have shown that quasiconformal maps develop naturally from conformal maps. Thus a natural question which arose in my research was if quasiconformal maps had a natural extension of the cross-ratio preserving property as well. This lead me to the definition of what is known as a quasimöbius map. For ease of notation whenever we are dealing with an absolute cross-ratio $\tau = |[z_1, z_2, z_3, z_4]|$ we will denote the absolute cross-ratio of the image as $\tau' = |[f(z_1), f(z_2), f(z_3), f(z_4)]|$.

Definition 3.3. Suppose that D and D' are domains in $\mathbb{C}_\infty$ and that $f : D \to D'$ is a sense preserving homeomorphism. We say that f is *θ-quasimöbius* if there is a homeomorphism $\theta : [0, \infty) \to [0, \infty)$ such that $\tau' \leq \theta(\tau)$ for any τ.

Notice that if $\theta(\tau) = \tau$ for any quadruple of points then f is Möbius by Theorem 2.10. It turns out that every quasimöbius map is also a quasiconformal map. However, not every quasiconformal map is necessarily quasimöbius, but there are certain conditions we can add to make it quasimöbius. First, if a quasiconformal map f is defined in the entire extended complex plane then it is quasimöbius [10]. If $f : D \to D'$ is K-quasiconformal and if D is $\lambda - QED$ and D' is $c - LLC$ then f is θ-quasimöbius with θ depending only on K, λ, and c [10]. However, the proof of this and the characterization of what it means for D to be $\lambda - QED$ and D' to be $c - LLC$ is a paper unto itself, but if the reader wishes they can learn more in [10]. Thus we have the following theorem which directly parallels Theorem 2.10 in Section 2.2.

Theorem 3.4. Suppose that $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a sense preserving homeomorphism. Then f is quasiconformal if and only if f is quasimöbius [10].

Proof. Part I: Assume f is quasiconformal in the extended complex plane. Then f is quasimöbius via the discussion in the previous paragraph.

Part II: Assume f is quasimöbius. Pick a point $z \in \mathbb{C}_\infty \backslash \{\infty, f^{-1}(\infty)\}$. Fix another point $y \neq z$ and let $0 < r < \min\{1, |z - y|\}$ such that $L_f(z, r) < \infty$. Next, choose points a and b

such that $|a-z| \le r$, $|b-z| \ge r$,

$$|f(a)-f(z)| \ge (1-r)L_f(z,r), \quad and \quad |f(b)-f(z)| \le (1+r)l_f(z,r).$$

Now let us investigate what happens to a and b as $r \to 0$. First, $a \to z$ since $|a-z| \le r$. Also, $b \to z$ since $|f(b)-f(z)| \le (1+r)l_f(z,r)$ and because f is continuous. Hence, keeping y fixed,

$$\tau = |[a,b,z,y]| = \frac{|a-z||b-y|}{|a-y||b-z|} \to \frac{|z-z||z-y|}{|z-z||z-y|} = 1$$

and

$$\frac{|f(b)-f(y)|}{|f(a)-f(y)|} \to 1$$

as $r \to 0$. On the other hand,

$$\tau^{'} \ge \frac{(1-r)L_f(z,r)|f(b)-f(y)|}{(1+r)l_f(z,r)|f(a)-f(y)|}.$$

But f is quasimöbius, so can produce a θ such that $\tau^{'} \le \theta(\tau)$ for any τ. Thus, as $r \to 0$ we get that $H_f(z) \le \theta(1)$. Therefore f is quasiconformal by the metric definition since this is true for any z. □

In summary, while a homeomorphism of $\mathbb{C}_\infty$ is a Möbius transformation if and only if it preserves cross-ratios, a homeomorphism of $\mathbb{C}_\infty$ is quasiconformal if and only if it does not distort cross-ratios by more than a functional factor θ.

3.2 Characterizations of Quasicircles

As mentioned at the end of Section 1 the study of quasicircles is also a very useful tool for classifying quasiconformal maps. Recall that a quasicircle is the image of the unit circle under a quasiconformal map. In the study of quasicircles two important characterizations arise: the two point inequality and the reversed triangle inequality. For both inequalities we suppose that Γ is a Jordan curve in the complex plane.

Definition 3.5. Γ satisfies the two point inequality if there exists a constant $a \ge 1$ such that for each pair of points $z_1, z_2 \in \Gamma$

$$\min_{j=1,2} dia(\gamma_j) \le a\,|z_1 - z_2| \tag{3.3}$$

where *dia* denotes the euclidean diameter and γ_1, γ_2 are the components of Γ formed from deleting z_1 and z_2 from the curve.

Theorem 3.6. Γ is a quasicircle if and only if it satisfies the two point inequality [5].

Proof. Part I: The first part of this proof deals with a specific type of quasiconformal map called a quasiconformal reflection, and was originally postulated by Lars Ahlfors in [1]. Suppose Γ is a Jordan curve which divides the extended complex plane into two components, Ω and Ω^*. A quasiconformal reflection is a quasiconformal map f which maps Ω to Ω^* such that $f(\Gamma) = \Gamma$ and whose restriction to Γ is sense reversing. Ahlfors proved that such an f exists if and only if there exists a C such that

$$|P_1 - P_3| \leq C|P_1 - P_2| \tag{3.4}$$

for any three ordered points P_1, P_2, P_3 in Γ. Ahlfors called this the three point property, a name which the two point inequality is often called because of their similarity.

Part II: On the other hand, Frederick Gehring proved in [6] that a Jordan curve Γ is a quasicircle if and only if it admits a quasiconformal reflection. Thus, Γ is a quasicircle if and only if it satisfies (3.4). But Gehring also took this one step further. He picked two of the three points from (3.4) and let the third be variable. Thus (3.3) must also hold with $a = C$ because the C value in (3.4) must hold for any P_3 chosen. Conversely, (3.3) implies (3.4) as long as the third point chosen is within the component of minimum diameter. □

Definition 3.7. Γ satisfies the reversed triangle inequality if there exists a constant $b \geq 1$ such that

$$|z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_4 - z_1| \leq b|z_1 - z_3||z_2 - z_4| \tag{3.5}$$

for each ordered quadruple of points $z_1, z_2, z_3, z_4 \in \Gamma \backslash \{\infty\}$. Note that (3.5) can be rewritten in terms of absolute cross-ratios:

$$|[z_2, z_3, z_1, z_4]| + |[z_2, z_1, z_3, z_4]| \leq b$$

Theorem 3.8. Γ satisfies the reversed triangle inequality if and only if it satisfies the two point inequality [5].

Proof. Part I: Suppose Γ satisfies the two point inequality with constant $a \geq 1$ and choose $z_1, z_2, z_3, z_4 \in \Gamma \backslash \{\infty\}$. By relabeling if necessary we may assume that

$$|z_1 - z_3| \leq |z_2 - z_4|.$$

Let γ_2 and γ_4 be the components of $\Gamma \backslash \{z_1, z_3\}$ which contain z_2 and z_4 respectively. Again by relabeling we may assume that

$$dia(\gamma_2) \leq dia(\gamma_4).$$

Then

$$|z_1 - z_2| \leq dia(\gamma_2) \leq a|z_1 - z_3|, \quad |z_2 - z_3| \leq dia(\gamma_2) \leq a|z_1 - z_3|,$$

$$|z_3 - z_4| \leq |z_2 - z_3| + |z_2 - z_4| \leq (a+1)|z_2 - z_4|,$$

and

$$|z_4 - z_1| \leq |z_1 - z_2| + |z_2 - z_4| \leq (a+1)|z_2 - z_4|.$$

Thus if we let $b = 2a(a+1) \geq 1$ then

$$|z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_4 - z_1| \leq b|z_1 - z_3||z_2 - z_4|.$$

Part II: Suppose Γ satisfies the reversed triangle inequality with constant $b \geq 1$. Pick $z_1, z_3 \in \Gamma \backslash \{\infty\}$ and let γ_2 and γ_4 be the components of $\Gamma \backslash \{z_1, z_3\}$. Suppose that

$$\min_{j=2,4} dia(\gamma_j) > 2b \, |z_1 - z_3| \, ,$$

then we can choose $z_2 \in \gamma_2$ and $z_4 \in \gamma_4$ such that

$$b|z_1 - z_3| < \frac{1}{2} dia(\gamma_2) \leq |z_1 - z_2| \ \ and \ \ b|z_1 - z_3| < \frac{1}{2} dia(\gamma_4) \leq |z_1 - z_4|,$$

in which case

$$\begin{aligned} b|z_1 - z_3||z_2 - z_4| &\leq b|z_1 - z_3|(|z_2 - z_3| + |z_3 - z_4|) \\ &= b|z_1 - z_3||z_3 - z_4| + b|z_1 - z_3||z_2 - z_3| \\ &< |z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_4 - z_1| \end{aligned}$$

which is a contradiction to our hypothesis. Therefore Γ satisfies the two point inequality with $a = 2b \geq 1$ $\square$

Corollary 3.9. Γ is a quasicircle if and only if it satisfies the reversed triangle inequality.

An interesting point to note is that these inequalities are natural extensions of similar properties for circles in the same way that the definition of a quasiconformal map is an extension of that of a conformal map. In other words Γ is a circle if and only if it satisfies the two point inequality with $a = 1$ and if and only if it satisfies the reversed triangle inequality with $b = 1$ [5]. Thus the constants a and b give us a measure of how close a given Jordan curve is to being a true circle. The closer the value is to 1 the more circular the curve and the larger the value the more deformed or elongated it may be.

3.3 Examples

Thanks to the above theorems we can use the two inequalities to decide if a given Jordan curve is a quasicircle or not. For example, we can use the two point inequality to show that the Jordan curves in Figure 3.1 are all quasicircles. Similarly, the two Jordan curves in Figure 3.2 are not quasicircles.

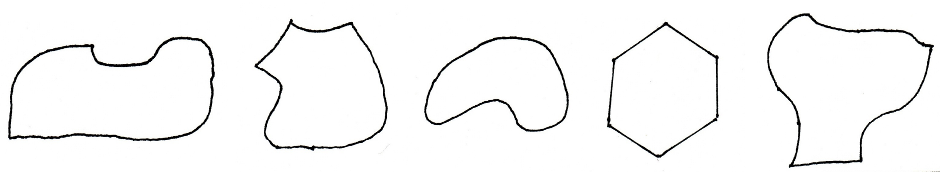

Figure 3.1: Examples of quasicircles.

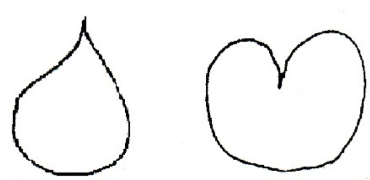

Figure 3.2: Examples of non-quasicircles.

The geometric reason why the examples from Figure 3.2 fail the two point inequality is the cusp at the top of the raindrop and in the center of the heart-shaped curve. It is possible to produce a value for a that satisfies the inequality for two specific points on opposite sides of the cusp. But as you travel along the cusp towards the tip the points get closer together much quicker than the diameter of the component cusp shrinks. For example, in Figure 3.3 we can approximate that $|z_1 - z_2| \approx 2|z_1^{'} - z_2^{'}|$ and $dia(\gamma) \approx \frac{4}{3}dia(\gamma^{'})$. Then given an a value that satisfies the two point inequality for z_1 and z_2 we see that we would need $a^{'} \approx \frac{3}{2}a$ in order to satisfy the inequality for $z_1^{'}$ and $z_2^{'}$. This effect continues as the points travel towards the cusp and thus the a value needed continues to grow and no one finite a value works for any two points. Therefore the two point inequality fails and these are not quasicircles.

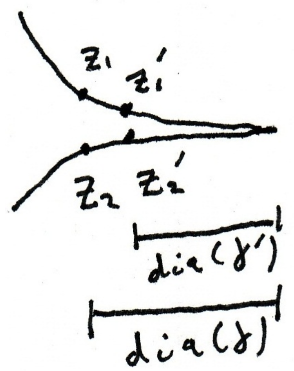

Figure 3.3: Example of a cusp failing the two point inequality.

Next, let us consider the rectangle for an example on quantifying the a value of the two point inequality. We call h the height and l the length of the rectangle and assume that $l \geq h$ by relabeling as necessary.

Proposition 3.10. If Γ is a rectangle then it is a quasicircle.

Proof. First, note that in order to satisfy the two point inequality we need a finite a value such that

$$a \geq \frac{\min_{j=1,2} dia(\gamma_j)}{|z_1 - z_2|} \tag{3.6}$$

for any two points on the rectangle. So pick z_1 and z_2 on the rectangle, and let γ be the component of minimum diameter when those points are removed.

Case I: z_1 and z_2 are on the same side of the rectangle. Then $|z_1 - z_2| = dia(\gamma)$.

Case II: z_1 and z_2 are on adjacent sides of the rectangle. Then $|z_1 - z_2| = dia(\gamma)$ since γ is two sides of a right triangle of which $|z_1 - z_2|$ is the hypotenuse.

Case III: z_1 and z_2 are on opposite sides of the rectangle. In order to find z_1 and z_2 such that the right-hand side of (3.6) is the greatest let us set up the following equation. First, without a loss in generality let us assume that z_1 and z_2 are on opposite sides of the height of the rectangle and are symmetrically spaced from the perpendicular bisector that cuts the rectangle in half. Notice that if z_1 and z_2 are not symmetrically spaced from the perpendicular bisector then we can always modify the positions of the points in order to create a case with a larger a value. This is done by shifting both of the points the same amount to a case where they are symmetrically spaced such that $|z_1 - z_2|$ stays the same but $dia(\gamma)$ increases, resulting in a larger a value for (3.6). So let $x/2$ be the distance from each point to the previously mentioned bisector (Figure 3.4). Then (3.6) becomes

$$a^2 \geq f(x) = \frac{x^2 + 2lx + 4h^2 + l^2}{4x^2 + 4h^2}. \tag{3.7}$$

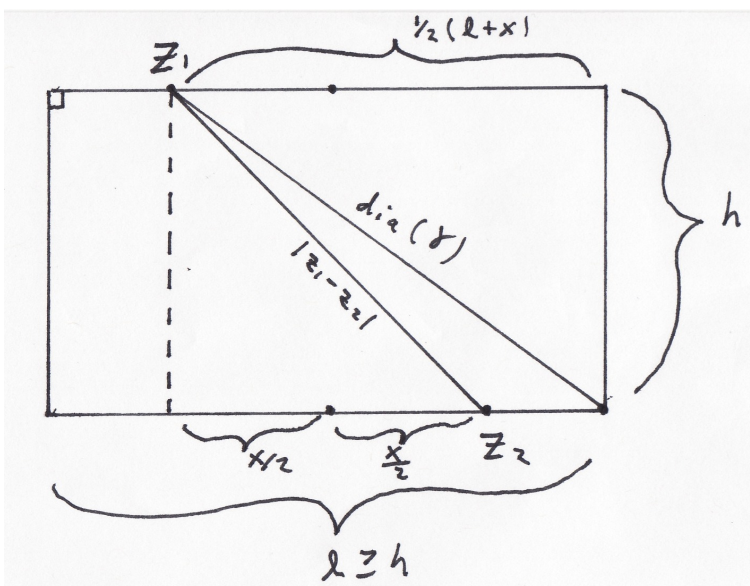

Figure 3.4: Finding the point of maximum value for (3.6) in a rectangle.

Notice that $f(x)$ is a fraction of two real polynomials, and thus it is differentiable using the quotient rule from calculus. Thus

$$f^{'}(x) = \frac{-lx^2 - (l^2 + 3h^2)x + lh^2}{2(x^2 + h^2)^2}.$$

$f'(x)$ will equal 0 when its numerator equals 0, thus we are able to use the quadratic equation to determine the values of x for which $f'(x) = 0$ and thus (3.7) has a maximum. Thus

$$x = \frac{l^2 + 3h^2 \pm \sqrt{(l^2 + 3h^2)^2 + 4l^2h^2}}{-2l}$$

and since we are only interested in positive x values we can rewrite this to be

$$x = \frac{\sqrt{(l^2 + 3h^2)^2 + 4l^2h^2} - (l^2 + 3h^2)}{2l} \tag{3.8}$$

which is always positive since

$$\sqrt{(l^2 + 3h^2)^2 + 4l^2h^2} \geq l^2 + 3h^2.$$

Note that in order to confirm that this x value is a local maximum we need to find the second derivative of f and ensure that its value at x is negative, but this is easy to show. Note also that when $x = l$ then the value of (3.7) is 1 and thus we see that our calculated x value lies between 0 and l and is realistic with the parameters of Figure 3.4 since the value of (3.7) is always greater than or equal to 1 when x is positive. We can now use this x value to calculate the position of z_1 and z_2 of maximum value for (3.6) and the a value that satisfies it by plugging x into (3.7). Lastly, if z_1 and z_2 are not on opposite sides of the height of the rectangle as we assumed but rather on opposite sides of the length then we can follow the same process and produce a new function f_2 and maximal x value simply by switching the values for h and l in (3.7) and (3.8). This does not affect our derivatives or other calculations because h and l are real constants. Then we simply need to compare the maximal values for f and f_2 and set a^2 to be the larger of the two. This will then satisfy (3.6) and thus Γ is a quasicircle. $\square$

Notice that if Γ is the unit square then $l = h = 1$ and using (3.8) $x = \sqrt{5} - 2$. Plugging this into (3.7) we calculate $a \approx 1.14$ which is very close to 1, and thus we see that a square is relatively close to being a circle. But the greater the difference between l and h the greater that a becomes and thus the more deformed the rectangle is compared to a circle.

4 Open Questions

In this paper we have explored many characterizations of conformal mappings and Möbius transformations and we have seen that quasiconformal maps and quasicircles are natural extensions of conformal maps and circles. Furthermore, many of the properties of Möbius transformations and circles also extend naturally to quasiconformal maps and quasicircles simply by multiplying by a finite constant greater than or equal to 1. This led me to question whether the same was true with our two key results about Möbius transformations from Section 2.2. We answered the first of these with the introduction of quasimöbius maps, and although not every quasiconformal map is necessarily quasimöbius there do exist sufficient conditions under which this is true. However, whether or not Theorem 2.11 can be extended to quasiconformal maps is still unknown.

Question 1. Suppose that $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a sense preserving homeomorphism. Is f quasiconformal if and only if f takes circles to quasicircles?

We do know that one direction of this question is true: if f is quasiconformal then it takes circles to quasicircles by the definition of a quasicircle. However, the converse of this is unknown. I would hypothesize that the converse is true globally because of the similar result with quasimöbius maps, but a rigorous proof still needs to be written. But if this is not true then like the quasimöbius maps there are probably conditions we could place upon f to make it true, and this results in the following question.

Question 2. If a quasicircle preserving map is not quasiconformal in general, then what conditions can we place upon either f or the domain to make it quasiconformal?

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