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On Chorded Cycles

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An abstract of
A dissertation submitted to the Faculty of the
James T. Laney School of Graduate Studies of Emory University
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy
in Mathematics
2015

Abstract

On Chorded Cycles
By Megan Cream

Historically, there have been many results concerning sufficient conditions for implying certain sets of cycles in graphs. My thesis aims to extend many of these well known results to similar results on sets of *chorded* (and sometimes even *doubly chorded*) cycles. In particular, we consider the minimum degree, $\delta(G)$ and a Ore-type degree sum condition, $\sigma_2(G)$ of a graph G, sufficient to guarantee the existence of k vertex disjoint chorded cycles, often containing specified elements of the graph, such as certain vertices or edges. Further, we extend a result on vertex disjoint cycles and chorded cycles to an analogous result on vertex disjoint cycles and *doubly* chorded cycles. We define a new graph property called chorded pancyclicity, and investigate a density condition and forbidden subgraphs in claw-free graphs that imply this new property. Specifically, we forbid certain paths and triangles with pendant paths. This is joint work with Dongqin Cheng, Ralph Faudree, Ron Gould, and Kazuhide Hirohata.

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Contents

Chapter 1

Introduction

1.1 History

In the field of extremal graph theory, we are interested in the way that local graph properties can affect global properties of the graph. We often investigate how far we can push certain properties of a graph before we can guarantee the existence of some other property, or even some specific structure in the graph. In the past, there have been many results concerning sufficient conditions for graphs containing certain types of cycles, such as hamiltonian cycles, or certain sets of cycles. In particular, there has been a large focus on sets of vertex disjoint cycles that contain specified graph elements including certain sets of vertices, or sets of independent edges, for example. Arguably one of the most famous such results is due to Corrádi and Hajnal [6].

Theorem 1.1. [6] *Let G be a graph of order $n \geq 3k$ for an integer $k \geq 1$ and suppose the minimum degree of G, $\delta(G) \geq 2k$. Then G contains a set of k vertex disjoint cycles.*

Enomoto proved a stronger result in [10] for the Ore-type degree sum condition, $\sigma_2(G)$ as follows.

Theorem 1.2. [10] *Suppose G is a graph of order $n \geq 3k$ for an integer $k \geq 1$. If $\sigma_2(G) \geq 4k-1$, then G contains a set of k vertex disjoint cycles.*

A type of cycle that was largely ignored until relatively recently is a chorded cycle, that is a cycle with an edge between two vertices that are nonadjacent on the cycle. In this thesis, we turn our attention to chorded cycles and we extend many well-known results on cycles or vertex disjoint sets of cycles to analogous results on chorded cycles and sometimes even doubly chorded cycles. Admittedly, we were not the first to focus on chorded cycles. Back in 1960, Posa [25] asked the question: which graph properties imply the existence of a cycle with a chord? The first answer was due to Czipzer (see Problem 10.2 in [25]) in 1963; he found that any graph with minimum degree at least three must contain a chorded cycle. However, it was not until 45 years later in 2008, that the interest in chorded cycles began to pick up steam. That year, Finkel [16] extended the Corrádi-Hajnal theorem to a chorded cycle version.

Theorem 1.3. [16] *Let G be a graph of order $n \geq 4k$ for an integer $k \geq 1$ and suppose the minimum degree of G, $\delta(G) \geq 3k$. Then G contains a set of k vertex disjoint chorded cycles.*

Further, this result is sharp as can be easily seen from the complete bipartite graph $K_{3k-1,n-3k+1}$. (Note that the smallest chorded cycle in a complete bipartite graph is a 6-cycle.) In 2005, Faudree and Gould [12] found the best lower bound for the order of neighborhood unions of nonadjacent vertices that implies the existence of k disjoint cycles in a graph. This result was extended to chorded cycles in 2012 by Gould, Hirohata, and Horn [21] and independently Gao, Li, and Yan [17]. They proved the following result.

Theorem 1.4. [21], [17] *For an integer $k \geq 1$, le G be a graph of order $n \geq 4k$ such that for any pair of nonadjacent vertices x and y, $|N(x) \cup N(y)| \geq 4k+1$. Then G contains at least k vertex disjoint chorded cycles.*

Inspired by these extensions of theorems on sets of cycles to theorems on sets of chorded cycles, we consider other cycle results to extend similarly. These results are stated in §2.2, 3.1, and 4.2.

1.2 Some Basics (Definitions and Notation)

In this thesis, we assume a working knowledge of graph theory. For a thorough background on the subject, see [20]. We consider only finite simple graphs, that is, graphs with no loops or multiple edges. Let $G = (V(G), E(G))$ be a finite simple graph. We denote the *degree* of a vertex $v \in V(G)$ as $deg_G(v)$ which is the number of edges incident to v in G. For a set of vertices $\{u_1, u_2, \ldots, u_k\}$ and a subgraph H of G, let $\deg_H(u_1, u_2, \ldots, u_k)$ be the sum of the degrees of the vertices u_i (for $1 \leq i \leq k$) to the subgraph H. The set of vertices adjacent to v in the graph G is called the *neighborhood* of v in G and is denoted $N_G(v)$. The *minimum degree* of a graph is the smallest degree over all the vertices in G and is denoted $\delta(G)$. Similarly, the *maximum degree* of G is denoted $\Delta(G)$. The *independence number*, $\alpha(G)$ of a graph is the largest number of vertices in a mutually nonadjacent set of vertices in G. For a noncompete graph G, we define the following Ore-type degree sum condition

$$\sigma_2(G) = \min\{\deg_G(x) + \deg_G(y) | xy \notin E(G)\}.$$

If G is a complete graph, then by convention $\sigma_2(G) := \infty$.

Let G be a graph and P_t be a path on t vertices. In Chapter 3, a path P is denoted $P = \langle u_1, u_2, \ldots, u_m \rangle$ for $u_1, u_2, \ldots, u_m \in V(G)$, otherwise it is denoted as simply $u_1, u_2, \ldots, u_m$. Next we define two terms crucial to this thesis on chorded cycles.

Definition 1.5. *Given a cycle C, an edge $e = v_1v_2$ is called a* **chord** *of C*

if its end-vertices, v_1 and v_2, are not adjacent on C. We then say that C is a **chorded cycle**.

Further, a *doubly chorded cycle* is a cycle with at least two chords. Next we define two important concepts for the study of forbidden subgraphs, which we will investigate in Chapter 4.

Definition 1.6. *A vertex-induced subgraph, which we will simply refer to as an* **induced subgraph**, *is a subset of the vertices of a graph G together with any edges whose end-vertices are both in the subset of $V(G)$. For a subset $S \subseteq V(G)$, we denote the subgraph of G induced by the vertices of S as $G[S]$.*

Definition 1.7. *Given a subgraph H of a graph G, we say G is* H-**free** *if G does not contain a subgraph isomorphic to H.*

In Chapter 4, we consider only $K_{1,3}$-free (or *claw*-free) graphs, and we will forbid certain paths and triangles with pendant paths. Let Z_i be a triangle with a pendant P_i adjacent to one of the vertices of the triangle. In particular, we will focus on the graphs Z_1 and Z_2, shown below (Figure 1a).

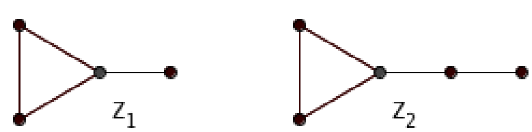

Figure 1a.

We will also use the following terms in Chapter 4.

Definition 1.8. *A graph G is called* **traceable** *if it contains a path consisting of every vertex in $V(G)$.*

Definition 1.9. *The* **diameter** *of a graph G, denoted $d(G)$, is the longest shortest path between any two vertices in $V(G)$.*

For any terms not defined here, see [20].

1.3 Known Results for Sets of Cycles Containing Specified Graph Elements

In Chapter 2, we consider minimum degree conditions $\delta(G)$ and degree sum conditions $\sigma_2(G)$. Our goal is to find sufficient such conditions to imply the existence of vertex disjoint sets of chorded cycles, often containing specific graph elements. In particular, we extend some well-known results for sets of cycles much like Gould, Hirohata, and Horn's [21] extension of Gould's previous result with Faudree on neighborhood unions in [12]. One source of our inspiration came about in 2000, when Egawa, Faudree, Gyori, Ishigami, Schelp, and Wang [9] proved the following result for vertex disjoint cycles.

Theorem 1.10. [9] *Let G be a graph on $n \geq 4k$ vertices for an integer $k \geq 1$ and let $\{e_1, e_2, \ldots, e_k\}$ be a set of k independent edges in G. If the minimum degree*

$$\delta(G) \geq \frac{n + 2k - 2}{2},$$

then G contains k vertex disjoint cycles $\{C_1, C_2, \ldots, C_k\}$ such that $e_i \in E(C_i)$ and $3 \leq |V(C_i)| \leq 4$ for all $1 \leq i \leq k$.

This result can be extended in multiple ways, as shown in Chapter 2, §2.2.2. Next, we note a similar known result about placing specified vertices on vertex disjoint cycles.

Theorem 1.11. [8] *For an integer $k \geq 1$, let G be a graph of order $n \geq 6k-3$. If the minimum degree*

$$\delta(G) \geq \frac{n}{2},$$

then for any set of k vertices $\{v_1, v_2, \ldots, v_k\}$ in G, there exists a set of k vertex disjoint cycles $\{C_1, C_2, \ldots, C_k\}$ such that $v_i \in V(C_i)$ and $3 \leq |V(C_i)| \leq 5$ for all $1 \leq i \leq k$.

In Chapter 2, we extend this result to chorded cycles. Further, we consider the basic question of when a set of k independent edges can be chords for k vertex disjoint cycles, one per cycle. We also investigate under which conditions the k vertex disjoint chorded cycles can be extended to span the entire vertex set of a graph.

1.4 Known Results for Vertex Disjoint Cycles and Chorded Cycles

The motivation for the results in Chapter 3 comes from a natural conjecture based on Theorems 1.1 and 1.2 proposed by Bialostocki, Finkel, and Gyárfás, in [3].

Conjecture 1.12. [3] Let $r, s \geq 0$ be two integers and let G with a graph of order $n \geq 3r + 4s$ and minimum degree

$$\delta(G) \geq 2r + 3s.$$

Then G contains a collection of $r + s$ vertex disjoint cycles such that s of them are chorded cycles.

This minimum degree bound is sharp, and its sharpness is shown in [17]. The $r = 0, s = 2$ case and the $s = 1$ cases were shown to be true in [3]. The $r = 0$ case was completed by Finkel [16] in the aforementioned Theorem 1.3.

The general conjecture was proved to be true by Babu and Diwan [1], and Chiba and Fujita [5], independently. In particular, the authors in [1, 5] generalized the minimum degree condition to a degree sum condition, and proved the following theorem.

Theorem 1.13 ([1, 5]). *Let $r \geq 0$, $s \geq 0$ be two integers and G be a graph with $|V(G)| \geq 3r + 4s$ and*

$$\sigma_2(G) \geq 4r + 6s - 1,$$

then G contains $r+s$ vertex disjoint cycles such that s of them are chorded cycles.

Balister, Li, and Schelp [2] improved on Theorem 1.13 under a minimum degree condition in the following result.

Theorem 1.14 ([2])**.** *If G is a simple graph on $|V(G)| \geq 3r+4s$ vertices with minimum degree*

$$\delta(G) \geq 2r+3s,$$

then G contains $r+s$ vertex disjoint cycles, each of s of them with two chords, or a C_4 with one chord.

Qiao and Zhang [26] proved that $|V(G)| \geq 4k$ and $\delta(G) \geq \lceil 7k/2 \rceil$ can ensure k vertex disjoint doubly chorded cycles in any graph G. Gould, Hirohata and Horn [21] improved the degree condition in [26] to a degree sum condition and proved the following result.

Theorem 1.15 ([21])**.** *If G is a graph with $|V(G)| \geq 6k$ and*

$$\sigma_2(G) \geq 6k-1,$$

then G contains k vertex disjoint doubly chorded cycles.

Motivated by the above theorems, in Chapter 3 our goal is to guarantee a collection of $r+s$ vertex disjoint cycles, such that s of them are doubly chorded cycles.

1.5 Known Results for Pancyclic Graphs

Forbidden subgraphs for hamiltonian properties in graphs have been widely studied (for an overview, see [13]). A property even stronger than hamiltonicity is pancyclicity in a graph G, which is the property of containing a cycle of every possible length $i = 3, 4, \ldots, |V(G)|$. Pancyclicity only arises in 2-connected graphs and is one of the well-studied cycle properties in graphs. In Chapter 4, we extend this property and study the notion of *chorded pancyclicity*, that is, containing a chorded cycle of every possible length $i = 4, 5, \ldots, |V(G)|$. We study the density sufficient to guarantee this property in a graph. There are also many known forbidden subgraph results for pancyclicity. We take the natural step of extending some of these results from pancyclicity to chorded pancyclicity.

In Chapter 4, we extend the following theorem to a similar result on chorded pancyclicity, with slight variations in some conditions.

Theorem 1.16. *Let R, S be connected graphs and let G $(G \neq C_n)$ be a 2-connected graph of order $n \geq 10$. Then if G is $\{R, S\}$-free then G is pancyclic for $R = K_{1,3}$ and S is either $P_4, P_5, P_6, Z_1,$ or Z_2.*

The proof of a theorem in [18] yields the following result.

Theorem 1.17. [18] *If G is a 2-connected graph of order $n \geq 10$ that contains no induced subgraph isomorphic to $K_{1,3}$ or Z_1, then G is either a cycle or G is pancyclic.*

Gould and Jacobson proved a similar result for Z_2 in [24].

Theorem 1.18. [24] *If G is a 2-connected graph of order $n \geq 10$ that contains no induced subgraph isomorphic to $K_{1,3}$ or Z_2, then G is either a cycle or G is pancyclic.*

Faudree, Ryjacek, and Schiermeyer proved a similar result for certain paths in [15].

Theorem 1.19. [15] *If G is a 2-connected graph of order $n \geq 6$ that is $\{K_{1,3}, P_5\}$-free, then G is either a cycle or G is pancyclic.*

Theorem 1.20. [15] *If G is a 2-connected graph of order $n \geq 10$ that is $\{K_{1,3}, P_6\}$-free, then G is either a cycle or G is pancyclic.*

Theorem 1.19 implies the following result for P_4.

Theorem 1.21. *If G is a 2-connected graph of order $n \geq 6$ that is $\{K_{1,3}, P_4\}$-free, then G is either a cycle or G is pancyclic.*

In Chapter 4, §4.2.3 of this thesis, we extend each of these results to similar results on chorded pancyclicity.

Chapter 2

Vertex Disjoint Chorded Cycles Containing Specified Graph Elements

This chapter explores minimum degree and degree sum conditions for graphs sufficient to imply that the graph contains k vertex disjoint chorded (or sometimes doubly chorded) cycles which contain specific elements of the graph. In particular, sets of specific chords, sets of specific edges, and sets of specific vertices are all considered. We also investigate degree sum and minimum degree conditions sufficient to imply that a set of independent edges are the chords of vertex disjoint cycles. The results in this chapter are joint with Ralph Faudree, Ron Gould, and Kazuhide Hirohata.

2.1 Introduction

The study of cycles and systems of vertex disjoint cycles in graphs is well established. Recently, there have been numerous papers considering cycles with additional properties such as containing a specific set of vertices, or containing a specific set of edges, or even containing a set of vertex disjoint paths (for a good overview, see the survey [19]). Another natural property for cycles is that of containing at least one chord or at least some number $t \geq 1$ of chords. The study of chorded cycles has been increasing recently (see for example [3], [16], [22] and [23]). In this chapter, we extend several well-known results on sets of vertex disjoint cycles containing specific elements such as edges or vertices to results on sets of vertex disjoint chorded cycles containing these elements.

As described in §1.2, we consider only finite simple graphs. Again, we let G be a graph and P_t be a path on t vertices. In particular, we will focus on minimum degree and degree sum conditions in this chapter. Recall that $\delta(G)$ is the minimum degree of a graph G, and for a non-complete graph G, we define the following degree sum condition:

$$\sigma_2(G) = \min \{deg_G(u) + deg_G(v) \mid u \text{ and } v \text{ are nonadjacent}\}.$$

By convention we say $\sigma_2(G) = 2$ if G is a complete graph. For terms not defined here see [20].

As one source of inspiration, we note the following well-known result on cycles containing independent, i.e. vertex disjoint, edges from a given set.

Theorem 2.1. [9] *Let $\{e_1, e_2, \ldots, e_k\}$ be a set of independent edges in G, a graph of order $n \geq 4k$ such that*

$$\delta(G) \geq \left\lceil \frac{n}{2} \right\rceil + k - 1,$$

for $2 \leq k \leq \frac{n}{3}$, then G contains a set of k vertex disjoint cycles $\{C_1, C_2, \ldots, C_k\}$, such that $3 \leq |V(C_i)| \leq 4$ and $e_i \in E(C_i)$ for all $1 \leq i \leq k$. Further, there exists a set of k vertex disjoint cycles $\{D_1, D_2, \ldots, D_k\}$ such that $e_i \in E(D_i)$ for all $1 \leq i \leq k$, and $V(G) = \bigcup_{i=1}^{k} V(D_i)$.

We also note a similar known result about placing vertices on cycles.

Theorem 2.2. [8] *Let G be a graph of order $n \geq 6k - 3$ for some integer $k \geq 1$. If*

$$\delta(G) \geq \frac{n}{2},$$

then for any set of k vertices $\{v_1, v_2, \ldots, v_k\}$ there exists a set of k vertex disjoint cycles $\{C_1, C_2, \ldots, C_k\}$ such that $v_i \in V(C_i)$ and $|V(C_i)| \leq 5$ for all $1 \leq i \leq k$.

In this chapter, we extend each of the last two results to chorded cycles, and in one case, doubly chorded cycles. In addition, we consider the basic question of when a set of k independent edges can be chords for k vertex disjoint cycles, one per cycle. We also show when the k vertex disjoint cycles can be extended to span the entire vertex set of the graph.

2.2 Results

2.2.1 Placing Edges as Chords on Cycles

We begin with a natural question: When can a set of k independent edges be the set of chords for k vertex disjoint cycles of a graph, one per cycle?

Theorem 2.3. *Let G be a graph of order $n \geq 6k+1$ for an integer $k \geq 2$. If*

$$\sigma_2(G) \geq n + 3k - 2 \text{ and } \delta(G) \geq 6k - 3,$$

then for any k independent edges $\{e_1, e_2, \ldots, e_k\}$ there exists a set of k vertex disjoint cycles $\{C_1, C_2, \ldots, C_k\}$ such that e_i is a chord of C_i and $4 \leq |V(C_i)| \leq 6$ for all $1 \leq i \leq k$.

Sharpness: To see that $\sigma_2(G) \geq n + 3k - 2$ is needed, consider the following graph, G: place the chosen set of edges $E = \{e_1, e_2, \ldots, e_k\}$ in a clique, which is completely adjacent to a clique on $2k-1$ vertices, which is completely adjacent to a clique on the remaining $n - 4k + 1$ vertices. In addition, exactly one end-vertex from each edge of E is adjacent to every vertex in the K_{n-4k+1} (see Figure 2a below).

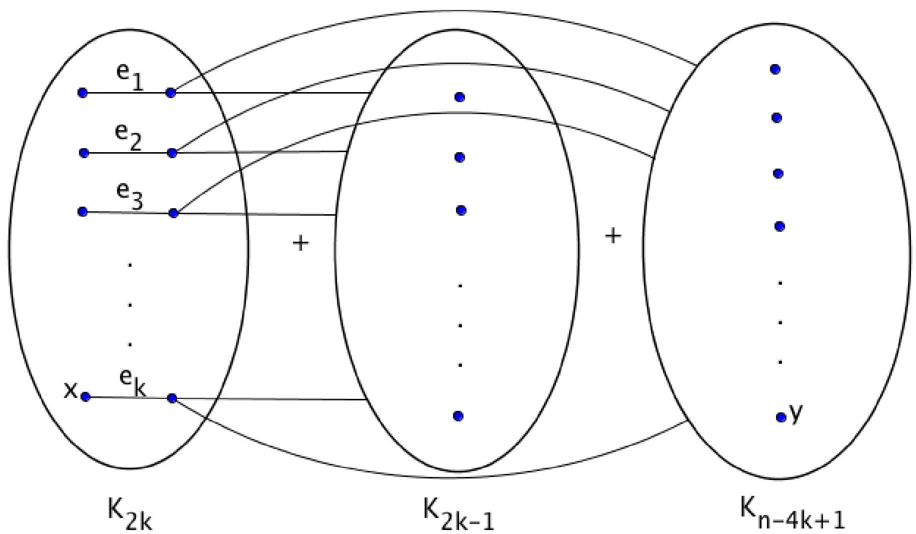

Figure 2a.

To find $\sigma_2(G)$ we must consider all pairs of non-adjacent vertices in G. The only type of non-adjacent pair of vertices in G is an end-vertex of an edge in E, say x, and a vertex in the K_{n-4k+1}, say y, as illustrated above in Figure 2a. Then

$$\begin{aligned}\sigma_2(G) &= deg_G(x) + deg_G(y)\\ &= (2k-1+2k-1) + (n-4k+2k-1+k)\\ &= n+3k-3.\end{aligned}$$

For any $e_i \in E$ to be a chord of a cycle requires using two neighbors in the K_{2k-1}, so there are at most $k-1$ vertex disjoint cycles with the desired edges as chords. This proves the necessity of $\sigma_2(G) \geq n+3k-2$.

Before proving Theorem 2.3, we first prove the following lemma and theorem.

Lemma 2.4. *Let G be a graph of order $n \geq 4$ with $\sigma_2(G) \geq n+1$. Then any edge of G lies either on a triangle or on a 4-cycle.*

Proof. Let $e = ab$ be any edge of G. If e lies on a triangle we are done, so assume e does not lie on any triangle. Since $\sigma_2(G) \geq n+1$, we know G is 2-connected. Let x be any vertex adjacent to an end-vertex of e. Without loss of generality, say x is adjacent to a. Then b is not adjacent to x, otherwise e would lie on a triangle. Now $\deg_G(x) + \deg_G(b) \geq n+1$, which implies that b and x have at least three common neighbors. Thus there is a vertex $y \neq a$ such that by and $xy \in E(G)$. Then the cycle a, x, y, b, a is a 4-cycle containing e as an edge. □

Example: Consider the graph G on n vertices that is a copy of K_{n-2} together with another edge $e = xz$. Let x be adjacent to exactly two vertices of K_{n-2} and let z be adjacent to all of the vertices in $V(G) - N_G(x)$. Then $\sigma_2(G) \geq \deg_G(x) + \deg_G(y) = 3 + n - 2 = n+1$, and the shortest cycle to have e as a chord in G is a 6-cycle. Thus, 6-cycles are necessary in the following result: the $k = 1$ case of Theorem 2.3.

Theorem 2.5. *Let G be a graph of order $n \geq 6$ with $\sigma_2(G) \geq n+1$, Then any edge of G is the chord of a cycle C with $4 \leq |V(C)| \leq 6$.*

Proof. Let e be any edge in the graph G. Then by Lemma 2.4, we know that e lies on either a triangle or a 4-cycle.

Case 1. Suppose e lies on a triangle, say $T = \{a, b, c\}$.
Let $e = ab$. Note that the degree-sum condition implies that G is at least 2-connected, thus at least two of the vertices in the set $\{a, b, c\}$ must have neighbors outside of T. Without loss of generality, assume a has such a neighbor. Let $R = G - \{a, b, c\}$ and consider some $x \in V(R)$ such that $ax \in E(G)$. If $xb \in E(G)$, then c, a, x, b, c is a 4-cycle with e as a chord, so suppose $xb \notin E(G)$. Then $\deg_T(b) = 2$ and $\deg_T(x) \leq 2$, so from the

degree-sum condition, x and b share at least three neighbors. Thus, there exists some $y \in V(R)$ with $y \neq x$ such that both b and x are adjacent to y. Now c, a, x, y, b, c is a 5-cycle with e as a chord.

Case 2. Suppose e lies on a 4-cycle, $F = a, b, c, d, a$, but lies on no triangle. Let $e = ab$. Since no vertex in $R = G - F$ can be adjacent to both a and b, consider the following disjoint sets of vertices: R_a, the neighborhood of a in R and R_b, the neighborhood of b in R. If $\delta(G) = 2$ then by the degree-sum condition, there is a vertex in R of degree $n - 1$ which implies that e is on a triangle. Since $\delta(G) \geq 3$, the neighborhoods R_a and R_b are both non-empty. If there are no edges between R_a and R_b then for any $x \in R_a$, the degree sum of the non-adjacent pair of vertices x and b is at most

$$\deg_G(x)+\deg_G(b) \leq |V(R)|-|R_b|-1+3+|R_b|+2 = |V(R)|+4 = (n-4)+4 = n$$

a contradiction of the assumed $\sigma_2(G)$. Thus there is an edge from some $x \in R_a$ to some $y \in R_b$ and hence d, a, x, y, b, c, d is a 6-cycle containing e as a chord, as desired.

□

Proof. (of Theorem 2.3). Assume the result fails and let G be an edge maximal counterexample. Then adding any new edge e to G will yield the desired set of k chorded cycles with e_i a chord of C_i for $i = 1, 2, \ldots, k$. Therefore, there must be at least one set of $k - 1$ cycles $\mathcal{C} = \{C_1, C_2, \ldots, C_{k-1}\}$ with e_i as a chord of C_i and $4 \leq |V(C_i)| \leq 6$ for $1 \leq i \leq k - 1$. Further, note that in $G - \mathcal{C}$ there is a 3, 4, or 5-cycle, C, containing the edge e_k and with an adjacency to a vertex not in C or $\mathcal{C}$ from at least one end-vertex of e_k. This follows as $G + e$ contains the desired cycle system of chorded 4, 5, and two different types of chorded 6-cycles, which are defined and shown below in Figure 2b. No matter where e falls on these cycles, we obtain C as described.

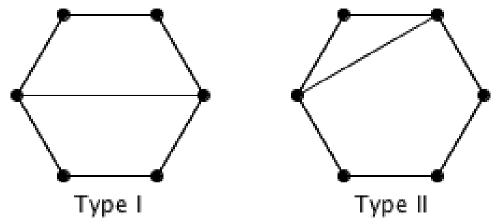

Figure 2b.

Over all possible edge maximal counterexamples and over all possible such cycles sets $\mathcal{C}$, choose $\mathcal{C}$ such that:

(1) $\cup_{i=1}^{k-1}|V(C_i)|$ is a minimum.

(2) Subject to (1), $|V(C)|$ is a minimum.

(3) Subject to (1) and (2), the number of Type I chorded 6-cycles is a maximum.

In $\mathcal{C}$ assume there are r 4-cycles, s 5-cycles, and t 6-cycles as shown in Figure 2c. Let $e_k = ab$ be an edge of C and let $R = G - \mathcal{C} - C$. Now we place some bounds on degrees of nonadjacent vertices. Without loss of generality, we assume a is the end-vertex of e_k with an adjacency, say x, off of C and not in $\mathcal{C}$ (i.e. $x \in V(R)$). Note that b is not adjacent to x, otherwise we would complete the desired cycle system, a contradiction.

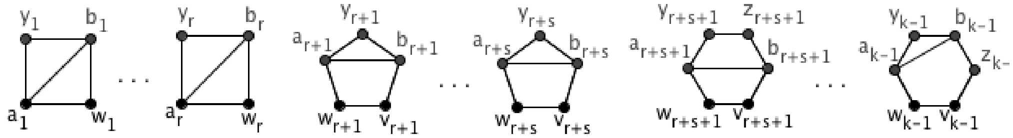

Figure 2c.

Note that b has at most two adjacencies in the cycle C, otherwise there would be a contradiction to (2). Also, since b has at most $6(k-1)$ adjacencies to $\mathcal{C}$, by the minimum degree condition b must have at least one neighbor in R. Thus, there is some $y \in V(R)$ such that b is adjacent to y and a is not adjacent to y (or the desired cycle system would be completed). Consider the nonadjacent pairs of vertices (a, y) and (b, x). Once again, we examine the adjacencies from these pairs to the cycles of $\mathcal{C}$.

Claim 1. For a 3-cycle $C = a, b, c, a$, the nonadjacent pairs (a, y) and (b, x) may have at most 14 edges to any chorded 4-cycle $C_i \in \mathcal{C}$, 16 edges to any chorded 5-cycle $C_i \in \mathcal{C}$, and 18 edges to any chorded 6-cycle $C_i \in \mathcal{C}$.

Proof. If a and b have a common neighbor on C_i, a 4-cycle, other than the end-vertices of the chord e_i, then both x and y cannot be adjacent to both ends of the chord e_i otherwise there are two cycles containing the desired chords, a contradiction. Thus, in this case the total number of edges to C_i is at most 8 from a and b, and at most 6 from x and y. If a and b have no such common neighbor in C_i, then they send at most 6 edges to C_i and x and y can both be adjacent to every vertex in C_i. Therefore, in either case we have $deg_{C_i}(a, b, x, y) \leq 14$.

If a and b have a common neighbor in C_i, a 5-cycle, that is not an end-vertex of the chord e_i, then there exists a chorded 4-cycle containing the chord e_k, contradicting the minimality of $\mathcal{C}$. Therefore, a and b have at most 7 edges to C_i. If either x or y is adjacent to both end-vertices of the chord of C_i, then there exists a chorded 4-cycle containing that same chord, which contradicts the minimality of the cycle set $\mathcal{C}$. Thus, x and y can have at most 8 edges to the chorded 5-cycle, C_i. In total, we have $deg_{C_i}(a, b, x, y) \leq 7 + 8 \leq 16$.

If a and b have a common neighbor in C_i, a 6-cycle, other than the end-vertices of e_i, then there exists a 4-cycle with e_k as a chord, a contradiction of the minimality of $\mathcal{C}$. Thus, together a and b can send at most 8 edges to C_i. If x or y are adjacent to both end-vertices of the chord e_i then there exists a 4 or 5-cycle (depending on where the chord of C_i is) containing e_i as a chord, again contradicting the minimality of $\mathcal{C}$. Thus, together x and y can send at most 10 edges to C_i. So in total, we have $deg_{C_i}(a,b,x,y) \leq 18$. □

Suppose C is a 3-cycle. Note that a and b cannot share any neighbors in R, otherwise there would be a 4-cycle containing e_k as a chord, which would complete the desired cycle set, a contradiction. Thus, the maximum possible number of edges from a and b together to R is $|V(R)|$. Note that x cannot be adjacent to y or any neighbor of y in R, otherwise there exists a 5-cycle or a 6-cycle, respectively, containing e_k as a chord, again a contradiction. So together x and y can have at most $|V(R)| - 2$ adjacencies in R. By Claim 1, we can bound the number of adjacencies from $\{a,b,x,y\}$ to $\mathcal{C}$. These bounds together with the maximum number of possible adjacencies from $\{a,b,x,y\}$ to the vertices of the triangle $\{a,b,c\}$ yield

$$\begin{aligned}
2(n+3k-2) \leq 2\sigma_2(G) &\leq deg_G(a) + deg_G(b) + deg_G(x) + deg_G(y) \\
&\leq |V(R)| + (|V(R)| - 2) + 14r + 16s + 18t + 8 \\
&= 2(n - 4r - 5s - 6t - 3 - 1) + 14r + 16s + 18t + 8 \\
&= 2n + 6r + 6s + 6t \\
\implies 6k - 4 &\leq 6r + 6s + 6t = 6(k-1) = 6k - 6, \text{ a contradiction.}
\end{aligned}$$

Claim 2. For a 4-cycle $C = a, b, c, d, a$, the nonadjacent pairs (a, y) and (b, x) have at most 14 edges to any chorded 4-cycle $C_i \in \mathcal{C}$, 16 edges to any chorded 5-cycle $C_i \in \mathcal{C}$, and 18 edges to any chorded 6-cycle $C_i \in \mathcal{C}$.

Proof. First suppose C_i is a chorded 4-cycle. Without loss of generality, if either $\deg_{C_i}(x) \leq 2$ or $\deg_{C_i}(y) \leq 2$, or if $\deg_{C_i}(x) \leq 3$ and $\deg_{C_i}(y) \leq 3$ then we are done. So assume x (or y) is adjacent to all of C_i. If a and b have a common neighbor on C_i other than an end-vertex of the chord e_i, say v_i, then v_i completes a 5-cycle with e_k as a chord and x can replace v_i on C_i, which completes the desired cycle set, a contradiction. Therefore, $\deg_{C_i}(a,b) \leq 6$ and $\deg_{C_i}(a,b,x,y) \leq 14$.

If C_i is a 5-cycle, note that x (and similarly, y) is not adjacent to both a_i and b_i otherwise a 4-cycle with e_i as a chord would exist, contradicting (1). Thus, $\deg_{C_i}(x,y) \leq 8$. Also note that if a and b have a common neighbor w_i (or v_i), then there exists a 5-cycle with e_k as a chord and e_i is left on a triangle, contradicting (2). Thus $\deg_{C_i}(a,b) \leq 8$, which implies $\deg_{C_i}(a,b,x,y) \leq 16$.

If C_i is a 6-cycle (of either type), then again x (and similarly, y) cannot be adjacent to both a_i and b_i otherwise there would exist a 4 or 5-cycle (depending on the Type of C_i) containing e_i as a chord, contradicting (1). Therefore we have $\deg_{C_i}(x,y) \leq 10$. Also, if a and b have a common neighbor in $\{v_i, w_i, y_i, z_i\}$ then a chorded 5-cycle with e_k as a chord exists, contradicting (1). Thus, $\deg_{C_i}(a,b,x,y) \leq 18$.

□

Now suppose C is a 4-cycle. Note that a and b cannot share any neighbors in R, otherwise there would be a 5-cycle containing e_k as a chord, which would complete the desired cycle set, a contradiction. So the neighborhoods $N_R(a)$ and $N_R(b)$ must be disjoint sets. Further, note that no neighbor of a in R can be adjacent to any neighbor b in R, otherwise there exists a 6-cycle with e_k as a chord, which is a contradiction as it completes the desired cycle set. Therefore, since x cannot be adjacent to itself nor any vertex in $N_R(b)$, we have $\deg_R(x) \leq |V(R)| - 1 - |N_R(b)|$. Similarly,

$\deg_R(y) \leq |V(R)| - 1 - |N_R(a)|$. Now we have

$$\begin{aligned}\deg_R(a,b,x,y) &\leq |N_R(a)| + |N_R(b)| + (|V(R)| - 1 - |N_R(b)|) \\ &\quad + (|V(R)| - 1 - |N_R(a)|) \\ &= 2|V(R)| - 2.\end{aligned}$$

By Claim 2, we can bound the number of adjacencies from $\{a, b, x, y\}$ to $\mathcal{C}$. These bounds together with the maximum number of possible adjacencies from $\{a, b, x, y\}$ to the vertices of the 4-cycle $\{a, b, c, d\}$ yield

$$\begin{aligned}2(n + 3k - 2) &\leq 2\sigma_2(G) \\ &\leq deg_G(a) + deg_G(b) + deg_G(x) + deg_G(y) \\ &\leq 2|V(R)| - 2 + 14r + 16s + 18t + 10 \\ &= 2(n - 4r - 5s - 6t - 4 - 1) + 14r + 16s + 18t + 10 \\ &= 2n + 6r + 6s + 6t \\ \implies 6k - 4 &\leq 6r + 6s + 6t = 6(k-1) = 6k - 6, \text{ a contradiction.}\end{aligned}$$

Claim 3. For a 5-cycle $C = a, b, c, d, e, a$, the nonadjacent pairs (a, y) and (b, x) have at most 14 edges to any chorded 4-cycle in $\mathcal{C}$, 16 edges to any chorded 5-cycle in $\mathcal{C}$, and 18 edges to any chorded 6-cycle in $\mathcal{C}$.

Proof. First assume C_i is a 4-cycle. If either x or y is adjacent to both ends of e_i, then we can rebuild C_i, again a 4-cycle, by replacing either w_i or y_i with either x or y. Further, if a and b have a common neighbor in $\{w_i, y_i\}$, then there exists a triangle with e_k as an edge, contradicting (2). Therefore either $\deg_{C_i}(a, b) \leq 6$ or $\deg_{C_i}(x, y) \leq 6$, and in either case, we have $\deg_{C_i}(a, b, x, y) \leq 14$.

Now suppose C_i is a 5-cycle. Neither x nor y can be adjacent to both end-vertices of e_i, otherwise there exists a 4-cycle containing e_i as a chord, contradicting (1), so $\deg_{C_i}(x) \leq 4$ and $\deg_{C_i}(y) \leq 4$. If $\deg_{C_i}(x) \leq 3$ and $\deg_{C_i}(y) \leq 3$, then we are done. So without loss of generality, suppose $\deg_{C_i}(x) = 4$ and $\deg_{C_i}(y) \leq 3$. Again without loss of generality, assume x is not adjacent to a_i. If a and b are both adjacent to v_i then v_i can be used to complete a 6-cycle with e_k as a chord, and x can replace v_i on the 5-cycle C_i with chord e_i, thus completing the desired cycle set, a contradiction. Therefore $\deg_{C_i}(a, b) \leq 9$, so $\deg_{C_i}(a, b, x, y) \leq 16$. Now suppose $\deg_{C_i}(x) = 4 = \deg_{C_i}(y)$. If x and y each are adjacent to a different end-vertex of e_i, then either w_i or v_i can be replaced (by x or y) on C_i and thus a and b cannot both be adjacent to either w_i or v_i. Therefore $\deg_{C_i}(a, b) \leq 8$, so $\deg_{C_i}(a, b, x, y) \leq 16$. Instead suppose x and y are adjacent to the same end-vertex of e_i, without loss of generality, say b_i. Then v_i can be replaced by either x or y on the 5-cycle C_i with chord e_i. Note that if both a and b are adjacent to v_i, then there exists a 6-cycle containing e_k as a chord, thus completing the desired cycle set, a contradiction. If both a and b are not adjacent to v_i then we are done. So suppose, without loss of generality, that a is adjacent to v_i. Now if a and b are both adjacent to w_i then a, v_i, y, b, w_i, a is a 5-cycle containing e_k as a chord and e_i is left on a triangle, contradicting (2). Therefore, a and b cannot both be adjacent to w_i, so $\deg_{C_i}(a, b) \leq 8$ and thus $\deg_{C_i}(a, b, x, y) \leq 16$.

Now assume C_i is a Type I 6-cycle. If a and b have a common neighbor in $\{v_i, w_i, y_i, z_i\}$, then a new chorded 6-cycle with chord e_k is formed, which leaves e_i on a 4-cycle, contradicting (2). Thus, $\deg_{C_i}(a, b) \leq 8$. As above, x and y each must be nonadjacent to one of $\{a_i, b_i\}$, otherwise there is a shorter cycle with e_i as a chord (which contradicts (1)), so $\deg_{C_i}(x, y) \leq 10$ and therefore $\deg_{C_i}(a, b, x, y) \leq 18$ holds.

Lastly, instead suppose C_i is a Type II 6-cycle. If a and b have a common neighbor in $\{v_i, w_i, z_i\}$ then there exists a new 6-cycle with e_k as a chord, and e_i is left on a 3-cycle, contradicting (2). Therefore, $\deg_{C_i}(a, b) \leq 9$. Neither x nor y can be adjacent to both end-vertices of e_i, otherwise there exists a 4-cycle with e_i as a chord, contradicting (1). If $\deg_{C_i}(x, y) = 10$ then either x and y are adjacent to the same end-vertex of e_i, or x and y are adjacent to different end-vertices of e_i. First, without loss of generality, suppose both x and y are adjacent to all of C_i except for b_i. Then $a_i, y, y_i, b_i, z_i, x, a_i$ is a Type I 6-cycle with e_i as a chord, which contradicts (3). Now, without loss of generality, assume x is adjacent to all of C_i except b_i and y is adjacent to all of C_i except a_i. Then $b_i, y_i, x, a_i, w_i, y, b_i$ is a Type II 6-cycle with e_i as a chord, again contradicting (3). Thus $\deg_{C_i}(a, b, x, y) \leq 9 + 9 = 18$. □

Now suppose C is a 5-cycle. Note that a and b cannot share any neighbors in R, otherwise there would be a 6-cycle containing e_k as a chord, which would complete the desired cycle set, a contradiction. So the neighborhoods $N_R(a)$ and $N_R(b)$ must be disjoint sets. Further, note that no neighbor of a in R can be adjacent to any neighbor b in R, otherwise there exists a 4-cycle containing e_k, contradicting (2). Therefore, since x cannot be adjacent to itself nor any vertex in $N_R(b)$, we have $\deg_R(x) \leq |V(R)| - 1 - |N_R(b)|$. Similarly, $\deg_R(y) \leq |V(R)| - 1 - |N_R(a)|$. Now we have

$$\begin{aligned}\deg_R(a, b, x, y) \leq |N_R(a)| + |N_R(b)| &+ (|V(R)| - 1 - |N_R(b)|) \\ &+ (|V(R)| - 1 - |N_R(a)|) \\ = 2|V(R)| - 2. &\end{aligned}$$

By Claim 3, we can bound the number of adjacencies from $\{a, b, x, y\}$ to $\mathcal{C}$. These bounds together with the maximum number of possible adjacencies from $\{a, b, x, y\}$ to the vertices of the 5-cycle $\{a, b, c, d, e\}$ yield

$$\begin{aligned}
2(n+3k-2) &\le 2\sigma_2(G) \\
&\le deg_G(a) + deg_G(b) + deg_G(x) + deg_G(y) \\
&\le 2|V(R)| - 2 + 14r + 16s + 18t + 12 \\
&= 2(n - 4r - 5s - 6t - 5 - 1) + 14r + 16s + 18t + 12 \\
&= 2n + 6r + 6s + 6t \\
\implies 6k - 4 &\le 6r + 6s + 6t = 6(k-1) = 6k - 6, \text{ a contradiction.}
\end{aligned}$$

This completes the proof. □

Corollary 2.6. *Let G be a graph of order $n \ge 9k - 4$ for an integer $k \ge 2$. If*

$$\delta(G) \ge \frac{n + 3k - 2}{2}$$

then for any k independent edges $\{e_1, e_2, \ldots, e_k\}$ there exists a set of k vertex disjoint cycles $\{C_1, C_2, \ldots, C_k\}$ such that e_i is a chord of C_i and $4 \le |V(C_i)| \le 6$ for all $1 \le i \le k$.

Based on Theorem 2.3 and Corollary 2.6, we make the following stronger conjecture.

Conjecture 2.7. Let G be a graph of order $n \ge 6k + 1$ for some integer $k \ge 1$. If

$$\sigma_2(G) \ge n + 3k - 2$$

then for any k independent edges $\{e_1, e_2, \ldots, e_k\}$ there exists a set of k vertex disjoint cycles $\{C_1, C_2, \ldots, C_k\}$ such that e_i is a chord of C_i and $4 \le |V(C_i)| \le 6$ for all $1 \le i \le k$.

Now we will focus on a minimum degree condition that will allow us to place edges along chorded cycles.

2.2.2 Placing Edges on Chorded Cycles

Before stating our next result, we describe notation used in upcoming theorems.

Notation: If c is a chord of the cycle C_i then we denote this cycle with a chord as C_i^c. Further, note that $E(C_i^c) = E(C_i) \cup \{c\}$. We will also use the notation C_i^* to denote a cycle C_i with two chords.

Next we establish a useful lemma.

Lemma 2.8. *Let G be a graph of order $n \geq 18k - 2$ (or $n \geq 18k + 1$) for an integer $k \geq 1$ with minimum degree*

$$\delta(G) \geq \frac{n + 2k - 2}{2}.$$

If G contains a set of k vertex disjoint cycles $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ containing a total of at most $6k - 1$ (or $6k$) vertices, then for any triple of vertices $\{v_1, v_2, v_3\}$ on any cycle $C \in \mathcal{C}$, at least one pair of these vertices has a common neighbor in $A = G - \mathcal{C}$.

Proof. We have that $b = |V(\mathcal{C})| \leq 6k - 1$. Suppose for the vertices v_1, v_2 and v_3 in some cycle C of $\mathcal{C}$, no pair of them has a common neighbor in A. Then

$$\begin{aligned} 3\left(\frac{n + 2k - 2}{2}\right) - 3(b - 1) &\leq 3\delta(G) - 3(b - 1) \\ &\leq \deg_A(v_1) + \deg_A(v_2) + \deg_A(v_3) \leq n - b, \\ n &\leq 4b - 6k \\ &\leq 4(6k - 1) - 6k \\ &= 18k - 4, \text{ a contradiction.} \end{aligned}$$

Hence, some pair of the set $\{v_1, v_2, v_3\}$ has a common neighbor in A. A similar argument holds when $n \geq 18k + 1$ and $|V(\mathcal{C})| \leq 6k$.

$\square$

We are now ready to extend Theorem 2.1 to chorded cycles. In our first such result, the specified edges are either chords or cycle-edges on the vertex disjoint chorded cycles.

Theorem 2.9. *Let G be a graph of order $n \geq 18k - 2$ for an integer $k \geq 2$ and let $\{e_1, e_2, \ldots, e_k\}$ be a set of k independent edges in G. If*

$$\delta(G) \geq \frac{n + 2k - 2}{2},$$

then there exists a system of k vertex disjoint chorded cycles $\{C_1^{c_1}, C_2^{c_2}, \ldots, C_k^{c_k}\}$ such that $e_i \in E(C_i^{c_i})$ and $|V(C_i^{c_i})| \leq 6$ for all $1 \leq i \leq k$.

Proof. Theorem 2.1 implies that in G there is a set of k vertex disjoint cycles $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$, such that $e_i \in E(C_i)$ and $|C_i| \leq 4$ for all $1 \leq i \leq k$. Let $A = G - \mathcal{C}$. Note that for any three vertices of C_i, some pair of these vertices must have a common neighbor in A by Lemma 2.8.

Case 1: Suppose $e_i = v_1v_3$ is an edge of a 3-cycle $C_i = v_1, v_2, v_3, v_1$. If v_2 and v_3 or v_1 and v_2 share a common neighbor in A then there exists a chorded 4-cycle C_i^* with $e_i \in E(C_i^{c_i})$ such that $C_i^{c_i}$ has a chord, $c_i = v_2v_3$ or v_1v_2. If v_1 and v_3 have a common neighbor in A then e_i is the chord of a 4-cycle.

Case 2: Now suppose $e_j = v_1v_2$ is an edge of a 4-cycle $C_j = v_1, v_2, v_3, v_4, v_1$. Consider the vertex set $\{v_1, v_3, v_4\}$. By Lemma 2.8, the set of vertices $\{v_1, v_3, v_4\}$ contains a pair of vertices with a common neighbor in A. If v_1 and v_4 or v_3 and v_4 have a common neighbor in A then there is a chorded 5-cycle C_j^* with $e_j \in E(C_j^*)$. If v_1, v_3 is the pair with a common neighbor in A, say u, then we repeat the argument using the set of vertices $\{v_2, v_3, v_4\}$:

if either v_2, v_3 or v_3, v_4 is the pair that has a common adjacency in A then we have a chorded 5-cycle with e_j as an edge of the cycle. If v_2, v_4 is the pair with the common neighbor in A, say v, then $v_1, v_4, v, v_2, v_3, u, v_1$ is a 6-cycle with e_j as a chord.

By applying Lemma 2.8 repeatedly for each C_i, we obtain the desired system of k vertex disjoint chorded cycles with e_i as an edge or chord of C_i for all $1 \leq i \leq k$.

□

Using a similar technique, we obtain a stronger result where each e_i is a cycle-edge, not a chord.

Theorem 2.10. *Let G be a graph of order $n \geq 18k - 2$ for an integer $k \geq 2$ and let $\{e_1, e_2, \ldots, e_k\}$ be a set of independent edges in G. If*

$$\delta(G) \geq \frac{n + 2k - 2}{2},$$

then there exists a set of k vertex disjoint chorded cycles $\{C_1^{c_1}, C_2^{c_2}, \ldots, C_k^{c_k}\}$ with $e_i \in E(C_i)$, $e_i \neq c_i$, and $|V(C_i)| \leq 6$ for all $1 \leq i \leq k$.

Proof. Again Theorem 2.1 implies that there exists a set of k vertex disjoint cycles $\{C_1, C_2, \ldots, C_k\}$ in G with $|V(C_i)| \leq 4$ and $e_i \in E(C_i)$ for all $1 \leq i \leq k$. Let $A = G - \bigcup_{i=1}^{k} V(C_i)$.

Case 1: Suppose $C_i = v_1, v_2, v_3, v_1$ and $e_i = v_1v_3$. By Lemma 2.8, we know that either the pair $\{v_1, v_2\}$,$\{v_2, v_3\}$, or $\{v_1, v_3\}$ must share a common neighbor in A, say x. If v_1 and v_2 or v_2 and v_3 have a common neighbor in A then there is a chorded 4-cycle with e_i as an edge on the cycle. If v_1 and v_3 share the common neighbor $x \in A$, then x, v_1, v_2, v_3, x is a chorded 4-cycle with e_i as a chord. Now consider the triple $\{x, v_2, v_3\}$. Again by Lemma 2.8, either x and v_2, x and v_3, or v_2 and v_3 must share a common neighbor, say $y \in A$. If x and v_2 are both adjacent to $y \in A$ then

x, v_1, v_3, v_2, y, x is a chorded 5-cycle with e_i as an edge of the cycle and v_1v_2 as a chord. Either of the remaining pairs yields the same case, so without loss of generality if x and v_3 are both adjacent to some $y \in A$ then x, y, v_3, v_1, x is a chorded 4-cycle with e_i as an edge of the cycle and xv_3 as a chord. So if C_i is a 3-cycle, we can always find a chorded 4 or 5-cycle with e_i as an edge of the cycle, provided the common neighbor exists in A.

Case 2: Suppose $C_j = v_1, v_2, v_3, v_4, v_1$ and $e_j = v_1v_2$. Consider the triple $\{v_2, v_3, v_4\}$. By Lemma 2.8, one of the pairs in the triple shares a common neighbor, say $x \in A$. If v_2 and v_3 are both adjacent to x then $v_1, v_2, x, v_3, v_4, v_1$ is a chorded 5-cycle with e_j as an edge of the cycle and v_2v_3 as a chord. If v_3 and v_4 are both adjacent to $x \in A$ then $v_1, v_2, v_3, x, v_4, v_1$ is a chorded 5-cycle with e_j as a cycle edge and v_3v_4 as a chord. If v_2 and v_4 share a neighbor $x \in A$ then we need to consider the pairs of another triple: $\{x, v_2, v_3\}$. We have already checked the case where v_2 and v_3 share a neighbor in A. If x and v_2 are both adjacent to some $y \in A$ then v_1, v_2, y, x, v_4, v_1 is a chorded 5-cycle with e_j as a cycle edge and xv_2 as a chord. If instead x and v_3 share a neighbor $y \in A$ then $v_1, v_2, v_3, y, x, v_4, v_1$ is a chorded 6-cycle with e_j as a cycle edge and v_3v_4 as a chord. So if C_j is a 4-cycle then there exists a chorded 5 or 6-cycle with e_j as a cycle edge, provided the common neighbor exists in A.

By applying Lemma 2.8 repeatedly to each C_i, we have the desired system of vertex disjoint chorded cycles.

$\square$

Note: If n is even, the two preceding results are sharp. To see this, consider the graph G, of the set of edges $E = \{e_1, \ldots, e_{k-1}\}$ and two $K_{\frac{n-2k+2}{2}}$ cliques, each of which is completely adjacent to all vertices $V(E)$ and the edge e_k has one end-vertex in each of the cliques. In this graph, the minimum degree $\delta(G) = \frac{n-2k+2}{2} - 1 + 2(k-1) = \frac{n}{2} + k - 2$. However, e_k lies on no chorded cycle disjoint from the edges of E.

We can extend the process used in the proofs of Theorems 2.9 and 2.10 to prove the following result.

Theorem 2.11. *Let G be a graph of order $n \geq 18k-2$ for an integer $k \geq 2$ and let $\{e_1, e_2, \ldots, e_k\}$ be a set of independent edges in G. If*

$$\delta(G) \geq \frac{n+2k-2}{2},$$

then there exists a set of k vertex disjoint doubly chorded cycles $C_1^*, C_2^*, \ldots, C_k^*$ with $e_i \in E(C_i)$ and $|V(C_i)| \leq 6$ for all i.

Proof. From Theorem 2.1, we know we can place each edge e_i on a 3-cycle or 4-cycle. By the degree condition, for any set of three vertices we know from Lemma 2.8 that at least one pair of vertices in that set must have a common neighbor not in any other cycle. We must consider the following three cases: e_i is a cycle-edge of a chorded 4-cycle, a chord of a 4-cycle, or an edge of a chordless 4-cycle. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ be this collection of cycles, and let $A = G - \mathcal{C}$.

Case 1: Suppose $e_i \in E(G)$ is a non-chord of the chorded 4-cycle v_1, v_2, v_3, v_4, v_1. Without loss of generality, assume $e_i = v_1v_2$ and v_1v_3 is a chord. Now consider the set of vertices $\{v_2, v_3, v_4\}$. Some pair in this set must have a common neighbor in A, say x. If x is a common neighbor of v_2 and v_3 then $v_1, v_2, x, v_3, v_4, v_1$ is a doubly chorded 5-cycle with e_i as an edge and v_1v_3 and v_2v_3 as chords. If instead v_3 and v_4 have a common neighbor x, then $v_1, v_2, v_3, x, v_4, v_1$ is a 5-cycle with e_i as an edge and v_1v_3 and v_3v_4 as chords. If v_2 and v_4 have the common neighbor x, then x, v_2, v_1, v_3, v_4, x is a 5-cycle containing e_i as a cycle edge and v_2v_3 and v_1v_4 as chords.

Case 2: Suppose $e_i = v_1v_3$ is the chord of a 4-cycle, v_1, v_2, v_3, v_4, v_1. Again consider the set of vertices $\{v_2, v_3, v_4\}$. One pair of this set must share a common neighbor in A, say x.

Case 2a: If x is a common neighbor of v_2 and v_3 (or, by symmetry, v_3 and v_4), then $v_1, v_2, x, v_3, v_4, v_1$ is a C_5 containing $e_i = v_1v_3$ and v_2v_3 as chords. Since there is no doubly chorded cycle with e_i as a cycle edge, consider the set $\{v_1, v_2, x\}$. If v_1 and v_2 have a common neighbor y, then v_1, y, v_2, x, v_3, v_1 is a 5-cycle containing e_i as a cycle edge and v_1v_2 and v_2v_3 as chords. If v_2 and x share y as a neighbor, then v_1, v_2, y, x, v_3, v_1 is a 5-cycle with e_i as a cycle edge and v_2x and v_2v_3 as chords. If v_1 and x share y as a neighbor, then v_1, y, x, v_2, v_3, v_1 is a 5-cycle with e_i as an edge and v_1v_2 and v_3x as chords.

Case 2b: If instead v_2 and v_4 have a common neighbor x, then x, v_2, v_3, v_1, v_4, x is a doubly chorded 5-cycle with e_i as an edge and v_1v_2 and v_3v_4 as chords. This completes the cases when e_i is the chord of a chorded 4-cycle.

Next we suppose we start with e_i as an edge of a 4-cycle without a chord.

Case 3: Suppose $e_i = v_1v_2$ is initially an edge of the chordless 4-cycle v_1, v_2, v_3, v_4, v_1. Consider the set $\{v_2, v_3, v_4\}$. Then one of the following must occur: v_2 and v_3 share a common neighbor x in A or v_3 and v_4 share a common neighbor x, or v_2 and v_4 share x as a neighbor.

Case 3a: Suppose v_2 and v_3 share x as a neighbor. There is no doubly chorded cycle, so consider the set $\{v_3, v_4, x\}$. At least one pair in that set must have a common neighbor in A, say y. If v_3 and v_4 share y as a common neighbor, then $v_1, v_2, x, v_3, y, v_4, v_1$ is a 6-cycle containing e_i as an edge and v_2v_3 and v_3v_4 as chords. If v_4 and x share y as a neighbor, then $v_1, v_2, v_3, x, y, v_4, v_1$ is a 6-cycle containing e_i as an edge and v_2x and v_3v_4 as chords. If v_3 and x have y as a common neighbor, then $v_1, v_2, x, y, v_3, v_4, v_1$ is a 6-cycle containing e_i as an edge and v_2v_3 and v_3x as chords.

Case 3b: Suppose v_3 and v_4 share x as a neighbor. Again there is no doubly chorded cycle yet. Consider the set of vertices $\{v_2, v_3, x\}$. By Lemma 2.8, at least one pair in the set must have a common neighbor in A, say y. If v_2 and v_3 have y as a common neighbor, then $v_1, v_2, y, v_3, x, v_4, v_1$ is a 6-cycle with e_i as an edge and v_2v_3 and v_3v_4 as chords. If v_3 and x share y as a common neighbor, then $v_1, v_2, v_3, y, x, v_4, v_1$ is a 6-cycle containing e_i as an edge and v_3v_4 and v_3x as chords. If v_2 and x share y as a neighbor, then $v_1, v_2, y, x, v_3, v_4, v_1$ is a 6-cycle containing e_i as an edge and v_2v_3 and v_4x as chords.

Case 3c: Suppose v_2 and v_4 share x as a neighbor. Since there are no doubly chorded cycles, consider the vertex set $\{v_2, v_3, x\}$. Again, at least one pair in this triple must have a common neighbor in A, say y.

If v_2 and v_3 share y as a neighbor, there are still no doubly chorded cycles, so consider the set $\{v_3, v_4, y\}$. If v_3 and v_4 have a common neighbor z in A, then, $v_1, v_2, y, v_3, z, v_4, v_1$ is a 6-cycle with e_i as an edge and v_2v_3 and v_3v_4 as chords. If v_4 and y share z as a neighbor, then $v_1, v_2, v_3, y, z, v_4, v_1$ is a 6-cycle containing e_i as an edge and v_2y and v_3v_4 as chords. If z is a common neighbor of v_3 and y then $v_1, v_2, y, z, v_3, v_4, v_1$ is a 6-cycle with e_i as an edge and v_2v_3 and v_3y as chords. This completes the case when v_2 and v_3 are adjacent to y.

If v_3 and x share y as a neighbor, then $v_1, v_2, x, y, v_3, v_4, v_1$ is a 6-cycle with e_i as en edge and v_2v_3 and v_4x as chords.

If instead v_2 and x share y as a neighbor, then there are still no doubly chorded cycles containing e_i as an edge, so consider the set of vertices $\{v_1, v_4, x\}$. At least one pair of these vertices must have a common neighbor, say z in A. If z is a common neighbor of v_1 and v_4, then $v_1, v_2, y, x, v_4, z, v_1$ is a 6-cycle with e_i as an edge and v_1v_4 and v_2x as chords. If v_4 and x have z as a common neighbor, then $v_1, v_2, y, x, z, v_4, v_1$ is a 6-cycle containing e_i as an edge and v_2x and v_4x as chords. If v_1 and x share z as a neighbor, then $v_1, v_2, v_3, v_4, x, z, v_1$ is a 6-cycle containing e_i as a cycle edge and v_1v_4 and v_2x as chords.

By applying Lemma 2.8 repeatedly to each C_i, we have the desired system of vertex disjoint doubly chorded cycles. This completes the proof. □

This concludes our results on placing edges on chorded cycles. In the following section, we place vertices on chorded cycles instead of edges.

2.2.3 Placing Vertices on Chorded Cycles

Next, we extend Theorem 2.2 to chorded cycles.

Theorem 2.12. *Let G be a graph of order $n \geq 16k - 12$ for an integer $k \geq 1$. If*

$$\delta(G) \geq \frac{n}{2},$$

then for any set of k vertices in G, $\{v_1, v_2, \dots, v_k\}$, there exists a collection of k vertex disjoint chorded cycles $\{C_1, C_2, \dots, C_k\}$ such that $v_i \in V(C_i)$ and $|V(C_i)| \leq 6$ for $1 \leq i \leq k$.

Proof. For the sake of contradiction, suppose not. Let $\{v_1, v_2, \dots, v_k\}$ be a set of vertices in the graph G, a maximal counterexample. Choose an admissible collection of $k-1$ vertex disjoint chorded cycles in G, $\mathcal{C} = \{C_1, C_2, \dots, C_{k-1}\}$, such that $v_k \notin \cup_{i=1}^{k-1} V(C_i)$ with:

(1) $\displaystyle\sum_{i=1}^{k-1} |V(C_i)|$ is a minimum, and

(2) $v_i \in V(C_i)$, for $i = 1, 2, \dots, k-1$.

Let $H = G - \mathcal{C}$.

Note:

$$\begin{aligned} |H| &= |G| - |\mathcal{C}| \\ &\geq n - 6(k-1), \text{ since each cycle is at most a 6-cycle} \\ &\geq (16k - 12) - 6k + 6 \\ &= 10k - 6 \\ &\geq 4, \text{ since } k \geq 1. \end{aligned}$$

Thus there are at least 4 vertices in H.

Lemma 2.13. *For any $h \in V(H)$, $deg_{C_i}(h) \leq 4$ for all i.*

Proof. For the sake of contradiction, suppose $deg_{C_i}(h) = 5$ for some i and some $h \in V(H)$.

Case 1: Suppose $C_i = w_1, w_2, w_3, w_4, w_5, w_6, w_1$ is a 6-cycle and $h \in V(H)$ such that $h \neq v_k$. Without loss of generality, let h be adjacent to w_1, w_2, w_3, w_4, and w_5. First suppose $v_i = w_j$ for some $j \in \{1, 2, 3, 4, 5\}$. If $j = 1$, then the 5-cycle h, w_1, w_2, w_3, w_4, h contains $v_i = w_1$ and has the chord hw_2, contradicting the choice of $\mathcal{C}$, specifically the minimality of $\sum_{i=1}^{k-1} |V(C_i)|$. Similarly, if $j = 2, 3, 4$, or 5 then the 5-cycle h, w_2, w_3, w_4, w_5, h contains $v_i = w_j$ and has the chord hw_3, again contradicting the choice of $\mathcal{C}$. Now suppose $j = 6$, that is, $v_i = w_6$. Then h, w_5, w_6, w_1, w_2, h is a 5-cycle containing v_i with the chord hw_1, contradicting the choice of $\mathcal{C}$.

Case 2: Again suppose $C_i = w_1, w_2, w_3, w_4, w_5, w_6, w_1$ is a 6-cycle, but now suppose $h = v_k$. Without loss of generality, assume $h = v_k$ is adjacent to w_1, w_2, w_3, w_4, and w_5. If $v_i = w_1, w_2$, or w_6, then h, w_3, w_4, w_5, h is a 4-cycle containing v_k with w_4h as a chord. This contradicts the minimality of the size of the cycles in $\mathcal{C}$. If $v_i = w_4$ or w_5 then h, w_1, w_2, w_3, h is a 4-cycle containing v_k with w_2h as a chord. Again, this contradicts the choice of $\mathcal{C}$.

If $v_i = w_3$, then h, w_4, w_5, w_6, w_1, h is a 5-cycle containing v_k and not containing v_i, with w_5h as a chord, again contradicting the choice of $\mathcal{C}$.

Case 3: Suppose $C_i = w_1, w_2, w_3, w_4, w_5, w_1$ is a 5-cycle and $h \neq v_k$. Then h must be adjacent to all of $V(C_i)$. Without loss of generality, assume $v_i = w_1$. Then h, w_5, w_1, w_2, h is a 4-cycle containing v_i that has w_1h as a chord. This chorded 4-cycle contradicts the minimality of the order of the chosen collection of cycles, $\mathcal{C}$.

Case 4: Suppose $C_i = w_1, w_2, w_3, w_4, w_5, w_1$ is a 5-cycle and $h = v_k$. Again, h must be adjacent to all of $V(C_i)$. Without loss of generality, assume $v_i = w_1$. Then h, w_2, w_3, w_4, h is a 4-cycle containing the vertex v_k and the chord w_3h. This chorded 4-cycle contradicts the minimality of the order of the chosen collection of cycles, $\mathcal{C}$.

This completes the proof of the lemma.

□

Now, $\delta(G) \geq \frac{n}{2} \geq 8k - 1$ and $\deg_{\mathcal{C}}(v_k) \leq 4(k-1)$, so $\deg_H(v_k) \geq 4k + 3 > 3$ for all $k \geq 1$. Consider $u_1, u_2, u_3 \in N_H(v_k)$. Then Lemma 2.13 implies

$$\sum_{m=1}^{3} \deg_{C_i}(u_m) \leq 12, \text{ for } 1 \leq i \leq k-1,$$

which implies

$$\sum_{m=1}^{3} \deg_L(u_m) \leq 12(k-1).$$

Let r_j be the number of cycles, C_i, such that $\deg_{C_i}(v_k) = j$. Then $r_0 + r_1 + r_2 + r_3 + r_4 = k - 1$.

$$\begin{aligned} \deg_L(v_k) &= 4r_4 + 3r_3 + 2r_2 + r_1 \\ &= (k-1) + 3r_4 + 2r_3 + r_2 - r_0. \end{aligned}$$

We know that

$$\deg_H(v_k) \geq \delta(G) - [(k-1) + 3r_4 + 2r_3 + r_2 - r_0]. \tag{2.2.1}$$

For any $u \in V(H)$, $\deg_L(u) \leq 4(k-1)$. So on average,

$$\bar{d}_H(u) \geq \delta(G) - 4(k-1). \tag{2.2.2}$$

Now consider $u_1, u_2, u_3 \in N_H(v_k)$. There is at most one edge among these vertices, otherwise there is a chorded 4-cycle containing v_k in H and we're done. So we will consider the different possible neighborhoods of $\{u_1, u_2, u_3\}$ in $H - \{v_k\}$.

Case 1: Suppose v_k has at least two neighbors in $\{u_1, u_2, u_3\}$ with disjoint neighborhoods in $H - \{v_k\}$. Therefore Equations (1) and (2) yield that the order of H is

$$\begin{aligned}
|V(H)| &\geq \deg_H(v_k) + 1 + 2(\delta(G) - 4(k-1)) - 2 \\
&\geq \delta(G) - k + 1 - 3r_4 + 2r_3 - r_2 + r_0 + 1 + 2\delta(G) - 8k + 8 - 2 \\
&= 3\delta(G) - 9k + 8 - 3r_4 - 2r_3 - r_2 + r_0 \\
&\geq 3\delta(G) - 9k + 8 - 3r_4 - 2r_3 - r_2 \\
&\geq 3\delta(G) - 9k + 8 - 3(r_4 + r_3 + r_2 + r_1 + r_0) \\
&= 3\delta(G) - 9k + 8 - 3(k-1).
\end{aligned}$$

Hence, $|V(H)| \geq 3\delta(G) - 12k + 11$.

Similarly, we have that the order of G is

$$\begin{aligned} n &= |V(H)| + |V(L)| \\ &\geq 3\delta(G) - 12k + 11 + 4(k-1) \\ &\geq 3\left(\frac{n}{2}\right) + 12k + 11 + 4k - 4 \\ &= \left(\frac{3}{2}\right) n - 8k + 7. \end{aligned}$$

$$\text{Thus, } 0 \geq \left(\frac{1}{2}\right) n - 8k + 7, \text{ and hence,}$$

$$n \leq 16k - 14, \text{ a contradiction of the order of } G.$$

Case 2: Suppose there is one edge in the graph induced by $\{u_1, u_2, u_3\}$ and at least two pairs from $\{u_1, u_2\}$, $\{u_2, u_3\}$, and $\{u_1, u_3\}$ share a distinct common neighbor in $H - \{v_k\}$. Without loss of generality, one of the following scenarios must occur. There is either a chorded 4-cycle or a chorded 5-cycle in H containing v_k, and we're done.

Case 3: Suppose the pairs $\{u_1, u_2\}$, $\{u_2, u_3\}$, and $\{u_1, u_3\}$ each have at least one distinct common neighbor in $H - \{v_k\}$. Then we have
There is a chorded 6-cycle in H containing v_k, and we're done.

Case 4: Suppose the vertices u_1, u_2, and u_3 all share exactly one common neighbor in $H - \{v_k\}$. Then the order of H is

$$\begin{aligned} |V(H)| &\geq \deg_H(v_k) + 1 + 3(\delta(G) - 4(k-1)) - 1 - 3 \\ &= \delta(G) - k + 1 - 3r_4 - 2r_3 - r_2 + r_0 + 3\delta(G) - 12k + 12 - 1 + 3 \\ &= 4\delta(G) - 13k + 9 - 3r_4 - 2r_3 - r_2 + r_0 \\ &\geq 4\delta(G) - 13k + 9 - 3r_4 - 2r_3 - r_2 \\ &\geq 4\delta(G) - 13k + 9 - 3(r_4 + r_3 + r_2 + r_1 + r_0) \\ &= 4\delta(G) - 13k + 9 - 3(k-1) \\ &= 4\delta(G) - 16k + 12. \end{aligned}$$

Similarly, the order of G is

$$\begin{aligned} n &= |V(H)| + |V(L)| \\ &\geq 4\delta(G) - 16k + 12 + 4(k-1) \\ &\geq 4\left(\frac{n}{2}\right) - 16k + 12 + 4k - 4. \\ \text{Hence } n &= 2n - 12k + 8, \text{ and thus,} \\ n &\leq 12k - 8, \text{ a contradiction of the order of } G. \end{aligned}$$

This completes the proof of the theorem.

□

Next we will show that under sufficient minimum degree conditions, the disjoint cycle systems in some of the previous theorems can be extended to a 2-factor of the graph, that is, a set of vertex disjoint cycles that spans $V(G)$. First, we need the following lemma.

Lemma 2.14. *If G is a graph of order $n \geq 2k+2$ for an integer $k \geq 1$, and $\delta(G) \geq \dfrac{n+2k-2}{2}$, then G is at least $2k$-connected.*

Proof. Consider two vertices x and $y \in V(G)$ and let $H = G - \{x, y\}$. If x and y are nonadjacent, then by the minimum degree condition we have

$$|N_H(x) \cap N_H(y)| \geq \left(\frac{n+2k-2}{2}\right) + \left(\frac{n+2k-2}{2}\right) - (n-2) = 2k.$$

So there are at least $2k$ paths of length two between x and y in G. If instead, x and y are adjacent, then

$$\begin{aligned} |N_H(x) \cap N_H(y)| &\geq \left(\frac{n+2k-2}{2} - 1\right) + \left(\frac{n+2k-2}{2} - 1\right) - (n-2) \\ &= 2k - 2. \end{aligned}$$

So there are at least $2k-2$ paths of length two between x and y in G. Consider the following vertex sets in H

$$X = \{v \in N_H(x) | v \notin v_H(y)\}$$

$$Y = \{v \in N_H(y) | v \notin v_H(x)\}.$$

If X or $Y = \emptyset$ then $|N_H(x) \cap N_H(y)| \geq \frac{n+2k-2}{2} \geq \frac{(2k+2)+2k-2}{2} = 2k$ and we are done. So consider $x' \in X$ and $y' \in Y$. If $x'y' \in E(G)$ then $\kappa(G) \geq (2k-2)+2 = 2k$ and if $x'y' \notin E(G)$ then we are in the same case as the case where $xy \notin E(G)$. Therefore, $\kappa(G) \geq 2k$. $\square$

Theorem 2.15. *Let G be a graph of order $n \geq 18k-2$ for an integer $k \geq 2$, such that*

$$\delta(G) \geq \frac{n+2k-2}{2},$$

and G contains a set of k vertex disjoint chorded cycles $\mathcal{C} = \{C_1, C_2, \dots, C_k\}$. Then the cycle system can be extended to span the vertices of G.

Proof. Let $\mathcal{C}$ be the set of k vertex disjoint cycles and let $R = G - \mathcal{C}$. Assume $\mathcal{C}$ has been chosen with $|V(\mathcal{C})|$ as large as possible. If $|V(\mathcal{C})| = n$, we are done, so assume not.

First suppose $|V(R)| < k$. Then for any $v \in R$, $deg_{\mathcal{C}}\ v \geq \frac{n+2k-2}{2} - (k-2) = \frac{n+2}{2}$. Also note that $|\mathcal{C}| = n - |V(R)| \geq n-(k-1) = n-k+1$. However, since $k \geq 2$, $n-k+1 < n$. This implies that $deg_{\mathcal{C}}\ v \geq \frac{|\mathcal{C}|}{2}$. Thus, on some cycle $C \in \mathcal{C}$ there must be at least two consecutive vertices that are both neighbors of v. By adding v between its two such neighbors, we can extend C, a contradiction of the maximality of $\mathcal{C}$.

Now suppose $|V(R)| \geq k$. Since $\mathcal{C}$ is as large as possible, no vertex of R can be adjacent to any two consecutive vertices in any cycle of $\mathcal{C}$. Thus, for any vertex $v \in R$, we know $deg_{\mathcal{C}}\ v \leq \frac{|\mathcal{C}|}{2}$. Since $n = |\mathcal{C}| + |V(R)|$ and for $k \geq 2$, $\delta(G) > \frac{n}{2}$, $deg_R\ v > \frac{|V(R)|}{2}$ and hence R is hamiltonian connected. Thus, for any adjacency of a vertex in R, to a vertex in a cycle $C_i \in \mathcal{C}$ there cannot be any adjacencies to the next $|V(R)|$ vertices from any other vertex in R, otherwise we could insert all of R and extend the cycle, thus extending $\mathcal{C}$, a contradiction.

By Lemma 2.2.3, $\kappa(G) \geq 2k$, there is a $2k$ matching between R and $\mathcal{C}$. Therefore, the cycles can be broken up into at least $2k$ intervals with separate end-vertices. Out of all of these intervals, k of them could contain the specified edges e_i for $1 \leq i \leq k$. Therefore, there are at least k other intervals, each as large as R, otherwise we could insert R and get a larger cycle system, contradicting the maximality of $|V(C)|$. Hence

$$|V(R)|(k+1) \leq n, \text{ thus,}$$
$$|V(R)| \leq \frac{n}{k+1}.$$

So each vertex in R has at least $\frac{n+2k-2}{2} - \frac{n}{k+1}$ adjacencies into $\mathcal{C}$. However,

$$\frac{n+2k-2}{2} - \frac{n}{k+1} = \frac{n}{2} - \frac{n}{k+1} + k - 1 = \frac{k-1}{2(k+1)}n + k - 1.$$

Hence, a pair of vertices in $V(R)$ will form at least $\left(\frac{k-1}{2k+2}\right)n + k$ intervals in the cycles of $\mathcal{C}$. Therefore, there are $\frac{k-1}{2k+2}$ intervals that do not contain any edge e_i for $1 \leq i \leq k$. Thus we have

$$|V(R)|\left(\frac{k-1}{2k+2}\right)n \leq |\mathcal{C}| = n - |V(R)|.$$
$$\text{Therefore, } |V(R)| \leq \frac{2k+2}{k-1}.$$

So we have $k \leq |V(R)| \leq \frac{2k+2}{k-1}$, and this inequality holds for $2 \leq k \leq 3$. Now suppose $k = 3$ and assume we cannot extend $\mathcal{C}$. Then $\delta(G) \geq \frac{n+4}{2}$ and

$|V(R)| \leq 4$. If $|V(R)| = 1$, then $|\mathcal{C}| = n-1$ and the vertex $v \in R$ must have $\frac{n+4}{2}$ edges to $\mathcal{C}$. Since v is adjacent to more than half of vertices in $\mathcal{C}$, it must be adjacent to at least two consecutive vertices on some cycle $C \in \mathcal{C}$. The cycle set $\mathcal{C}$ can be extended by adding v between its two neighbors on C, a contradiction.

If $R = 2, 3$, or 4, then for any $v \in R$ $\deg_{\mathcal{C}}\ v \geq \frac{n-2}{2}$ and $|\mathcal{C}| \geq n-4$. Thus for $1 \leq |V(R)| \leq 4$, we have $n-4 \leq |\mathcal{C}| \leq n-1$ and $\delta(G) \geq \frac{n+4}{2} - (|V(R)|-1) = \frac{n+6}{2} - |V(R)| > \frac{|\mathcal{C}|}{2}$, so we can always extend $\mathcal{C}$ by inserting a vertex, a contradiction.

Suppose $k = 2$. Then $\delta(G) \geq \frac{n+2}{2}$ and $|V(R)| \leq 6$. If $|V(R)| = 1$, $|\mathcal{C}| = n-1$ and $\delta(G) \geq \frac{n+2}{2} > \frac{|\mathcal{C}|}{2}$, and we are done. If $|V(R)| = 2$, then $|\mathcal{C}| = n-2$ and for any $v \in R$, $\deg_G\ v \geq 1 + \frac{n+2}{2} = \frac{n}{2} > \frac{\mathcal{C}}{2}$. If $|V(R)| = 3$, then $|\mathcal{C}| = n-3$ and for any $v \in R$, $\deg_G\ v \geq \frac{n+2}{2} - 2 = \frac{n-2}{2} > \frac{\mathcal{C}}{2} = \frac{n-3}{2}$, a contradiction. When $|V(R)| \geq 4$, $\deg_R\ v \geq \frac{n+2}{2} - (|V(R)|-1)$, and we can no longer guarantee this is strictly larger than $\frac{|\mathcal{C}|}{2}$. However, $\deg_G\ v = \deg_{\mathcal{C}}\ v + \deg_R\ v > \frac{n}{2}$, so if $\deg_G\ v \leq \frac{\mathcal{C}}{2}$ then $\deg_R\ v > \frac{|V(R)|}{2}$. Therefore, R is hamiltonian connected.

Let $v_1, v_2, \ldots, v_m, v_1$ be a cycle in R, where $m = |V(R)|$. Further, every vertex in R is adjacent to at most every other vertex on each cycle of $\mathcal{C}$, otherwise we could extend at least one of the cycles, a contradiction. If v_1 and v_2 are adjacent vertices in R that are adjacent to different sets of alternating vertices on some C_i, then if v_1 is adjacent to x on C_i and v_2 is adjacent to y on C_i such that x is adjacent to y, then clearly C_i may be extended, a contradiction. Therefore, v_1 and v_2 must have essentially the same neighbors in $\mathcal{C}$. But then, by the degree condition, on some C_i there will exist vertices x, a, and y such that a is between x and y on the cycle, with v_1 adjacent to x and v_2 adjacent to y. Then by deleting a and adding v_1 and v_2 we can extend C_i, a contradiction.

This completes the proof.

□

Corollary 2.16. *The sets of vertex disjoint cycles in Theorems 2.3 and 2.12 can be extended to span $V(G)$.*

Proof. Theorem 2.3 follows directly. For Theorem 2.12, when $k = 1$ note that $\delta(G) \geq \frac{n}{2}$ so G is hamiltonian and the hamiltonian cycle must be chorded and must include the specified vertex, as desired. □

Definition 2.17. *A* **linear forest** *is a set of vertex disjoint paths.*

Given a set of vertex disjoint paths $\{P_{r_1}, P_{r_2}, \ldots, P_{r_k}\}$ where $r_i \geq 2$ is the order of the path P_{r_i}, let $r = \sum_{i=1}^{k} r_i$. Then the number of interior vertices in the path system is given by $r - 2k$.

Theorem 2.18. *Let $P_{r_1}, P_{r_2}, \ldots, P_{r_k}$ be a linear forest in a G of order $n \geq 16k + r - 2$. If the minimum degree*

$$\delta(G) \geq \frac{n}{2} + r - 1 - k,$$

then there exists a system of k chorded cycles $\{C_1, C_2, \ldots, C_k\}$ such that the path P_{r_i} lies on the cycle C_i and $|V(C_i)| \leq r_i + 4$ for all $1 \leq i \leq k$.

Proof. Consider the graph G' produced by replacing each path P_{r_i} in G by an edge e_i. Then the order of G' is

$$|V(G')| = n - (r - 2k) \geq 16k + r - 2 - r + 2k = 18k - 2.$$

Also, the minimum degree

$$\delta(G') \geq \frac{n}{2} + r - 1 - k - (r - 2k) = \frac{n}{2} + k - 1.$$

Then by Theorem 2.10, there exists a set of k vertex disjoint chorded cycles $C_1^{c_1}, C_2^{c_2}, \ldots, C_k^{c_k}$ such that $e_i \in E(C_i)$ and $e_i \neq c_i$ for all i. Now replace the edge e_i by the path P_{r_i} for all i, i.e. insert the set of interior vertices of each path back into G' to form the original graph G. Then the path P_{r_i} now lies on the cycle C_i and each C_i is still a chorded cycle since the addition of the interior vertices of P_{r_i} does not affect the end-vertices of any chord. $\square$

Chapter 3

A Result on Vertex Disjoint Cycles and Doubly Chorded Cycles

In this chapter, we prove a result on vertex disjoint cycles and doubly chorded cycles. Recall that a cycle is doubly chorded if it contains at least two chords (which may share at most one end-vertex). The results in this chapter differ from the results in the Chapter 2, in that the cycles in these results are not required to contain any specific elements of the graph. First, let r and s be nonnegative integers and let G be a graph with $|V(G)| \geq 4r + 6s$ and $\sigma_2(G) \geq 4r + 6s - 1$. In this chapter, we prove that such a graph G contains a collection of $r + s$ vertex disjoint cycles such that s of them are doubly chorded. This is an extension of earlier results. This chapter is joint work with Dongqin Cheng and Ron Gould.

3.1 Introduction

As always, we let G be a simple graph and P_t (respectively, C_t) be a path (respectively, cycle) with t vertices. In this chapter, a path P is denoted $P = \langle u_1, u_2, \ldots, u_n \rangle$. Recall that the set of neighbors of x in the subgraph

$H \subseteq G$ is denoted $N_H(x)$. Again, we will focus on minimum degree and degree sum conditions. Let $\delta(G)$ denote the minimum degree of $V(G)$ and let

$$\sigma_2(G) = \min \{\deg_G(u) + \deg_G(v) \mid u \text{ and } v \text{ are nonadjacent}\}.$$

For terms not defined here, see [20].

As seen in Chapters 1 and 2, the study of vertex disjoint cycles and chorded cycles in graphs has attracted much attention over the past few decades. In 1963, Corrádi and Hajnal [6] proved that for an integer $r \geq 1$, any graph G with $|V(G)| \geq 3r$ and minimum degree $\delta(G) \geq 2r$ contains r vertex disjoint cycles. In 2008, Finkel [16] proved the following extension of the Corrádi-Hajnal theorem for chorded cycles.

Theorem 3.1 ([16])**.** *For an integer $k \geq 1$, let G be graph with $|V(G)| \geq 4k$ and $\delta(G) \geq 3k$. Then G contains k vertex disjoint chorded cycles.*

Using Theorem 3.1 as inspiration, Bialostocki, Finkel, and Gyárfás [3] proposed the following conjecture.

Conjecture 3.2 ([3])*.* Let $s \geq 0$, $t \geq 0$ be two integers and G be a graph with at least $3r + 4s$ vertices. If

$$\delta(G) \geq 2r + 3s,$$

then G contains $r + s$ vertex disjoint cycles such that s of them are chorded cycles.

They also proved that the conjecture is true for $r = 0$, $s = 2$ and for $s = 1$. Their conjecture was proved to be true by Babu and Diwan [1], and Chiba and Fujita [5]. In particular, the authors in [1, 5] generalized the minimum degree condition to a degree sum condition, and proved the following theorem.

Theorem 3.3 ([1, 5]). *Let $r \geq 0$, $s \geq 0$ be two integers and let G be a graph of order $n \geq 3r + 4s$. If*

$$\sigma_2(G) \geq 4r + 6s - 1,$$

then G contains $r + s$ vertex disjoint cycles such that s of them are chorded cycles.

Balister, Li and Schelp [2] improved on Theorem 3.3 under a minimum degree condition.

Theorem 3.4 ([2]). *For integers $r, s \geq 0$, let G be a graph of order $n \geq 3r + 4s$. If the minimum degree of G*

$$\delta(G) \geq 2r + 3s,$$

then G contains $r + s$ vertex disjoint cycles, such that s of them are either doubly chorded or 4-cycles with one chord.

Qiao and Zhang [26] proved that $|V(G)| \geq 4k$ and $\delta(G) \geq \lceil 7k/2 \rceil$ can ensure k vertex disjoint doubly chorded cycles in any graph G. Gould, Hirohata and Horn [21] improved the degree condition in [26] to a degree sum condition and proved the following result.

Theorem 3.5 ([21]). *For an integer $k \geq 1$, let G is a graph of order $|V(G)| \geq 6k$. If*

$$\sigma_2(G) \geq 6k - 1,$$

then G contains k vertex disjoint doubly chorded cycles.

Using the aforementioned theorems as motivation, our goal for this chapter is to guarantee a collection of $r + s$ vertex disjoint cycles, such that s of them are doubly chorded cycles. Specifically, we prove the following theorem.

Theorem 3.6. *Let $r \geq 0$, $s \geq 0$ be two integers and let G be a graph with $|V(G)| \geq 4r + 6s$. If*

$$\sigma_2(G) \geq 4r + 6s - 1,$$

then G contains $r + s$ vertex disjoint cycles such that s of them are doubly chorded cycles.

3.2 Some Lemmas

The following lemmas are useful to our main proof.

In [21], Gould, Hirohata, and Horn proved the following *Exchange Lemma.*

Lemma 3.7 ([21])**.** *Let G be a graph containing a doubly chorded cycle Q. Suppose $x, y \in V(G - Q)$ are nonadjacent vertices with $\deg_Q(x) \geq 4$ and $\deg_Q(y) \geq 3$. Then there exist vertices $z_x, z_y \in V(Q)$ such that z_x is adjacent to x and z_y is adjacent to y and both $(Q - z_x) \cup \{y\}$ and $(Q - z_y) \cup \{x\}$ induce doubly chorded cycles.*

Chiba et al. [5] proved the following lemma.

Lemma 3.8 ([5])**.** *For an integer $p \geq 4$, let $P = \langle x_1, x_2, \ldots, x_p \rangle$ be a path and let C be a cycle of order at least four in G such that P and C are vertex disjoint. Consider the vertex set $S = \{x_1, x_2, x_{p-1}, x_p\}$. If $\sum_{x \in S} \deg_C(x) \geq 8$, then either $G[V(C) \cup V(P)]$ contains a cycle of length less than $|V(C)|$ or $G[V(C) \cup V(P)]$ contains two disjoint cycles.*

There are two kinds of doubly chorded 5-cycles, denoted by $C_5^{\wedge}$ and $C_5^{\times}$, see Figure 3a. Qiao and Zhang [26] proved the following lemma.

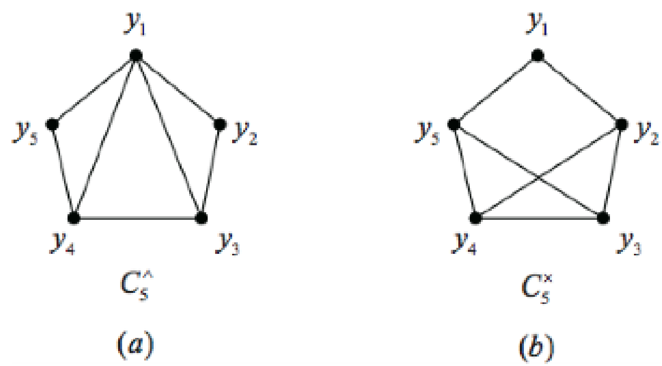

Figure 3a: The two doubly chorded cycles of C_5.

Lemma 3.9 ([26])**.** *Let C be a cycle of a graph G with at least two chords, and let u be a vertex in $G - C$. If $\deg_C(u) \geq 4$, then either $G[V(C) \cup u]$ contains a cycle C' with at least two chords and $|V(C')| < |V(C)|$, or $|V(C)| \leq 5$ and $G[V(C) \cup u]$ contains K_4 or $C_5^{\wedge}$ as a subgraph.*

We prove the following lemma.

Lemma 3.10. *Let C be a cycle. If $u, v \in V(G - C)$ and $\deg_C(u) + \deg_C(v) \geq 5$ and each vertex has at least one adjacent vertex on C, then there exists a vertex $z_u \in V(C)$ such that $z_u u \in E(G)$ and $G[V(C - z_u) \cup v]$ contains a cycle.*

Proof. We only need to consider $\deg_C(u) + \deg_C(v) = 5$. Let $C = \langle x_1, x_2, x_3, \ldots, x_{n-1}, x_n, x_1 \rangle$. Consider the following cases according to the value of n.

Case 1: Suppose $n = 3$.

Since $\deg_C(u) + \deg_C(v) = 5$, we see that $\deg_C(u) = 3$ or $\deg_C(v) = 3$. Without loss of generality, we may assume that $\deg_C(u) = 3$, and so $N_C(u) = \{x_1, x_2, x_3\}$. By symmetry, we only need to consider $N_C(v) = \{x_2, x_3\}$. Then $z_u = x_1$ and $C' = \langle v, x_2, x_3, v \rangle$ is a cycle in $G[V(C - z_u) \cup v]$.

Case 2: Suppose $n = 4$.

Case 2.1: Suppose $\deg_C(v) = 4$ or $\deg_C(u) = 4$.

Without loss of generality, we may assume that $\deg_C(v) = 4$, then $\deg_C(u) = 1$. By symmetry, we only need to consider $N_C(u) = \{x_2\}$. Then $z_u = x_2$ and $C' = \langle v, x_1, x_4, v\rangle$ is a cycle in $G[V(C - z_u) \cup v]$.

Case 2.2: Suppose $\deg_C(u) = 3$ or $\deg_C(v) = 3$.

Without loss of generality, assume that $\deg_C(u) = 3$, then $\deg_C(v) = 2$. By symmetry, we only need to consider $N_C(u) = \{x_1, x_3, x_4\}$. Note that $N_C(v) = \{x_3, x_4\}$ is symmetric to $N_C(v) = \{x_1, x_4\}$; $N_C(v) = \{x_2, x_3\}$ is symmetric to $N_C(v) = \{x_1, x_2\}$. In both cases, we only need to consider the former case. The desired vertices and cycles can be found in Table A.

Table A: The desired edges and cycles in C_4.

$N_C(v)$	z_u	C'
$\{x_3, x_4\}$	x_1	$\langle v, x_3, x_4, v\rangle$
$\{x_1, x_3\}$	x_4	$\langle v, x_1, x_2, x_3, v\rangle$
$\{x_2, x_3\}$	x_1	$\langle v, x_2, x_3, v\rangle$
$\{x_2, x_4\}$	x_3	$\langle v, x_2, x_1, x_4, v\rangle$

Case 3: Suppose $n \geq 5$.

If $\deg_C(u) = 4$ or $\deg_C(v) = 4$, we may assume that $\deg_C(v) = 4$, then $\deg_C(u) = 1$. Assume that $N_C(u) = \{x\}$. Then $z_u = x$ and $G[V(C - z_u) \cup v]$ contains a cycle. Hence, we only need to consider $\deg_C(u) = 3$ or $\deg_C(v) = 3$. Without loss of generality, we may assume that $\deg_C(u) = 3$. Let $N_C(u) = \{x_a, x_b, x_c\}$ and $N_C(v) = \{x_m, x_n\}$, where $x_i \in V(C)$ for $i \in \{a, b, c, m, n\}$. We consider the following cases according to the distribution of the vertices in $N_C(u)$ and $N_C(v)$.

Let the path between x_m and x_n on C be denoted by P_1, and the path $C - P_1$ be denoted by P_2, where $m < n$. Now $\deg_C(u) = 3$, so u has an adjacent vertex, say x_u, on C other than x_m and x_n. It does not matter which path this adjacent vertex lies on, so without loss of generality, say $x_u \in P_2$, then $x_u = z_u$ and v along with P_1 induces a cycle.

By the above cases, the proof is complete. $\square$

3.3 Proof of Theorem 3.6

In this section, we will prove our main result, Theorem 3.6.

Proof. We prove the following stronger inductive statement. Assume there exist $k + s$ vertex disjoint cycles, s of them doubly chorded, $0 \leq k < r$ and with at least $4(r - k) \geq 4$ vertices remaining off the cycles. If $r = 0$ and $s \geq 1$, then by Theorem 3.5 the result is true. If $s = 0$ and $r \geq 1$, then by Theorem 3.1 the result is also true.

Now suppose that $r \geq 1$ and $s > 0$ and the result holds for $k < r$. Since $|V(G)| \geq 4r + 6s > 4k + 6s$ and $\sigma_2(G) \geq 4r + 6s - 1 > 4k + 6s - 1$, by the induction hypothesis, G contains $k + s$ vertex disjoint cycles such that s of them are doubly chorded cycles. Without loss of generality, we let $\{C_1, C_2, \ldots, C_k\}$ be the k vertex disjoint cycles and $\{Q_1, Q_2, \ldots, Q_s\}$ be the s vertex disjoint doubly chorded cycles. Let

$$H = G - \bigcup_{i=1}^{k} C_i - \bigcup_{j=1}^{s} Q_j.$$

Let $\mathcal{H}_1 = \{C_1, C_2, \ldots, C_k\}$, $\mathcal{H}_2 = \{Q_1, Q_2, \ldots, Q_s\}$. Obviously, $|H| \geq 4(r - k) \geq 4$ for $k < r$ and H contains no cycles, or else we are done. Hence, H is a forest. We consider a tree $R \subseteq H$ as follows. Let

$P = \langle u_1, u_2, u_3, \ldots, u_p \rangle$ be a longest path in R. We assume that the cycles $C_1, C_2, \ldots, C_k, Q_1, Q_2, \ldots, Q_s$ are chosen in G such that

(*i*) $\sum_{i=1}^{k} |V(C_i)| + \sum_{j=1}^{s} |V(Q_j)|$ is as small as possible;

(*ii*) the number of $C_5^{\wedge}$ in $\mathcal{H}_2$ is as large as possible, subject to (*i*);

(*iii*) P is as long as possible, subject to (*i*) and (*ii*).

First, we need the following lemma and claim.

Lemma 3.11. *Let $P = \langle x_1, x_2, \ldots, x_{p-1}, x_p \rangle$ be a path and $Q \in \mathcal{H}_2$ such that P and Q are vertex disjoint. Consider the vertex set $S = \{x_1, x_2, x_{p-1}, x_p\}$. If $\sum_{x \in S} \deg_Q(x) \geq 13$, then either there are two vertex disjoint cycles such that one is a cycle and the other is a doubly chorded cycle in $G[V(P) \cup V(Q)]$, or there exists a doubly chorded cycle that is shorter than Q.*

We postpone the proof of Lemma 3.11 until after the proof of Theorem 3.6. First we will prove the following claim.

Claim 1. $|V(P)| \geq 4$.

Proof. For the sake of contradiction, assume that $|V(P)| < 3$. Let x and y be the end-vertices of P, then $|N_H(x)| = 1$ and $|N_H(y)| = 1$. Hence,

$$\deg_{G-H}(x) + \deg_{G-H}(y) \geq (4r + 6s - 1) - 2 = 4r + 6s - 3 = 4(r-1) + 6s + 1.$$

Then there exists a cycle C_i with $1 \leq i \leq k$ such that either $\deg_{C_i}(x) + \deg_{C_i}(y) \geq 5$, or there exists a doubly chorded cycle Q_j with $1 \leq j \leq s$ such that $\deg_{Q_j}(x) + \deg_{Q_j}(y) \geq 7$.

We first consider the case where $\deg_{C_i}(x) + \deg_{C_i}(y) \geq 5$. If $\deg_{C_i}(x) \geq 4$ or $\deg_{C_i}(y) \geq 4$ (without loss of generality, we may assume that the former is true), then $|V(C_i)| \geq 4$ and the neighbors of x, say x_1, x_2, x_3 and x_4 in order, partition C_i into four paths, , say P_1', P_2', P_3' and P_4', such that each path has one end-vertex which is a neighbor of x. Obviously, $G[V(P_j') \cup \{x, x_{j+1}\}]$ contains a cycle C_j' for each $j \in \{1, 2, 3, 4\}$ such that $|V(C_j')| = |V(P_j')| + 2$. Since there is a cycle C_k' for some $k \in \{1, 2, 3, 4\}$ such that $|V(C_k')| = |V(P_k')| + 2 \leq \left\lceil \frac{|V(C_i)|}{4} \right\rceil + 2 < |V(C_i)|$ for $|V(C_i)| \geq 4$, we can find a cycle C_k' shorter than C_i in $G[V(P) \cup V(C)]$, which is a contradiction to the choice of (i). Hence $\deg_{C_i}(x) \geq 3$ or $\deg_{C_i}(y) \geq 3$. Without loss of generality, we may assume that $\deg_{C_i}(x) \geq 3$, then $\deg_{C_i}(y) \geq 2$. By Lemma 3.10, there exists a vertex $z_x \in V(C_i)$ such that $z_x x \in E(G)$ and $G[(C - z_x) \cup y]$ contains a cycle. Then we can find a path longer than P, which is a contradiction to the choice of P.

We next consider the case where $\deg_{Q_j}(x) + deg\ (y, Q_j) \geq 7$. If $\deg_{Q_j}(x) \geq 5$ or $\deg_{Q_j}(y) \geq 5$ (without loss of generality, we may assume that the former is true), then $|V(Q_j)| \geq 5$ and, similar to above, the neighbors of x partition Q_j into five paths, say P_1'', P_2'', P_3'', P_4'' and P_5'' in order. Then every four consecutive adjacent vertices of x on Q_j along with x induce a doubly chorded cycle Q_t' such that $|V(Q_t')| = |V(P_j'')| + |V(P_k'')| + |V(P_l'')| + 2$ for three consecutive integers $j, k, l \in \{1, 2, 3, 4, 5\}$, where $t \in \{1, 2, 3, 4, 5\}$. If $|V(Q_j)| > 5$, then there exists a doubly chorded cycle Q_t' for some $t \in \{1, 2, 3, 4, 5\}$ such that $|V(Q_t')| \leq \left\lceil \frac{3|V(Q_j)|}{5} \right\rceil + 2 < |V(Q_j)|$; if $|V(Q_j)| = 5$, then we can find a K_4 in $G[V(Q_j) \cup \{x\}]$. In both cases, we can find a doubly chorded cycle shorter than Q_j, which is a contradiction to (i). Hence, either $\deg_{Q_j}(x) \geq 3$ and $\deg_{Q_j}(y) \geq 4$, or $\deg_{Q_j}(x) \geq 4$ and $\deg_{Q_j}(y) \geq 3$. Without loss of generality, we may assume that the former is true. By Lemma 3.7, there exist vertices $z_x, z_y \in V(Q_j)$ such that z_x is adjacent to x and z_y is adjacent to y and $(Q_j - z_x) \cup y$ induces a doubly chorded cycle. Then $P \cup \{z_x x\}$ is a path longer than P, which is a contradiction to the choice of P. Thus, we conclude that $|V(P)| \geq 4$.

This completes the proof of the Claim. $\square$

By Claim 1, $|V(P)| = p \geq 4$. Obviously, $\deg_P(u_1) = \deg_P(u_p) = 1$ and $u_1u_{p-1}, u_2u_p \notin E(G)$, otherwise a cycle would result. We then have that

$$\sum_{(x,y)\in\{(1,p-1),(2,p)\}} deg\ (u_x, H) + deg\ (u_y, H) \leq 6.$$

By the degree sum condition, we have

$$\begin{aligned}
&\sum_{(x,y)\in\{(1,p-1),(2,p)\}} \deg_{G-H}(u_x) + \deg_{G-H}(u_y) \\
&\geq 2(4r + 6s - 1) - 6 \\
&= 8r + 12s - 8 \\
&= 8(r - 1) + 12s.
\end{aligned}$$

Recall that $s > 0$ and $r > 1$. If any cycle C_i with $1 \leq i \leq k$ has

$$\sum_{(x,y)\in\{(1,p-1),(2,p)\}} \deg_{C_i}(u_x) + \deg_{C_i}(u_y) \geq 8,$$

then by Lemma 3.8 either there exists a cycle in $G[V(C) \cup V(P)]$ that is shorter than C, which is a contradiction; or $G[V(C) \cup V(P)]$ contains two disjoint cycles, in which case there are $(k + 1) + s$ vertex disjoint cycles in G such that s of them are doubly chorded cycles. Then by induction, the result follows.

Therefore, $\{x_1, x_2, x_{p-1}, x_p\}$ must send at most seven edges to each cycle C_i for $1 \leq i \leq k$. However, now there must be a doubly chorded cycle Q_j with $1 \leq j \leq s$ such that

$$\sum_{(x,y)\in\{(1,p-1),(2,p)\}} \deg_{Q_j}(u_x) + \deg_{Q_j}(u_y) \geq 13.$$

Note that in this case, there is a vertex $x \in \{x_1, x_2, x_{p-1}, x_p\}$ such that $\deg_{Q_j}(x) \geq 4$. By Lemma 3.9 and the choices of (i) and (ii), we must have $Q = K_4$ or $Q = C_5^{\wedge}$. Then by Lemma 3.11, there exist two vertex disjoint cycles such that one is a cycle and the other is a doubly chorded cycle in $G[V(Q_j) \cup V(P)]$. Hence, there exist $(k+1)+s$ vertex disjoint cycles in G such that s of them are doubly chorded cycles, and again by induction, the result follows. □

Proof of Lemma 3.11

Proof. Since $\sum_{x \in S} \deg_Q(x) \geq 13$, there exists a vertex $x \in S$ such that $\deg_Q(x) \geq 4$. By Lemma 3.9, either $G[V(C) \cup u]$ contains a doubly chorded cycle Q' shorter than Q, then we are done; or we may assume that $Q = K_4$ or $Q = C_5^{\wedge}$. We need to examine the following two cases.

Case 1: Suppose $Q = K_4$ with 4-cycle $\langle y_1, y_2, y_3, y_4, y_1 \rangle$ and two chords y_1y_3 and y_2y_4. By symmetry, we need to consider the following two cases.

Case 1.1: Suppose $\deg_Q(x_1) = 4$. (Note that the case where $\deg_Q(x_p) = 4$ is symmetric to the $\deg_Q(x_1) = 4$ case.)

Then $N_Q(x_1) = \{y_1, y_2, y_3, y_4\}$. Note that there is at most one vertex, say $x \in \{x_2, x_{p-1}, x_p\}$, such that $|N_Q(x)| \leq 2$. For $x', x'' \in \{x_2, x_{p-1}, x_p\} \backslash \{x\}$, $|N_Q(x')| \geq 3$ and $|N_Q(x'')| \geq 3$. Let P^* be the sub-path of P from x' to x''. (Note this path may be just one edge.) Without loss of generality, we may assume that $y_3 \in N_Q(x') \cap N_Q(x'')$. Then $C' = \langle y_3, x', P^*, x'', y_3 \rangle$ and $Q' = \langle x_1, y_2, y_1, y_4, x_1 \rangle$ with two chords y_2y_4 and x_1y_1 are the desired cycles.

Case 1.2: Suppose $\deg_Q(x_2) = 4$. (Note that the case where $\deg_Q(x_{p-1}) = 4$ is symmetric to the $\deg_Q(x_2) = 4$ case.)

Then $N_Q(x_2) = \{y_1, y_2, y_3, y_4\}$. Note that there is at most one vertex, say $x \in \{x_1, x_{p-1}, x_p\}$, such that $|N(x, Q)| \leq 2$. For $x', x'' \in \{x_1, x_{p-1}, x_p\} \backslash \{x\}$, $|N_Q(x')| \geq 3$ and $|N_Q(x'')| \geq 3$. Hence, there exists a vertex $y \in V(Q)$ such that $y \in N_Q(x_{p-1}) \cap N_Q(x_p)$. It can be easily seen that $\langle y, x_{p-1}, x_p, y\rangle$ is a cycle and $G[V(Q-y) \cup x_2]$ is a K_4.

Case 2: Suppose $Q = C_5^{\wedge}$ with cycle $\langle y_1, y_2, y_3, y_4, y_5, y_1\rangle$ and with two chords y_1y_3 and y_1y_4.

For any vertex $x_i \in S$, if $\{y_1, y_4, y_5\} \subset N_Q(x_i)$ (respectively, $\{y_1, y_2, y_3\} \subset N_Q(x_i)$), then $G[\{x_i, y_1, y_4, y_5\}]$ (respectively, $G[\{x_i, y_1, y_2, y_3\}]$) induces a K_4, which is a contradiction to the choice of (i). Hence, we only need to consider $N_Q(x_i) = \{y_2, y_3, y_4, y_5\}$. By symmetry, we need only to consider the following two cases.

Case 2.1: Suppose $\deg_Q(x_1) = 4$. (Note that the case where $\deg_Q(x_p) = 4$ is symmetric to the $\deg_Q(x_1) = 4$ case.) For any vertex $x \in \{x_2, x_{p-1}, x_p\}$, we need to consider the following three cases.

Case 2.1.1: Suppose $N_Q(x) = N_Q(x_1) = \{y_2, y_3, y_4, y_5\}$.

If $x = x_2$, then $G[\{x_1, x, y_4, y_5\}]$ induces a K_4, which is a contradiction to (i). Suppose $x = x_{p-1}$. Now if x_p has a common neighbor, say y, with x_{p-1} on $C_5^{\wedge}$, then these three vertices form a K_3 and $G[(C_5^{\wedge} - y) \cup x_1]$ contains a doubly chorded cycle. If x_p is only adjacent to y_1 on $C_5^{\wedge}$, then x_2 must have 4 neighbors and we are done by an argument similar to the case where $x = x_2$.

Now assume $x = x_p$. Now since x_2 or x_{p-1} has a common neighbor, say y, with x_p, then y and the path between x_2 (or x_{p-1}) and x_p induce a cycle and again $G[(C_5^{\wedge} - y) \cup x_1]$ contains a doubly chorded cycle.

Case 2.1.2: Suppose $\deg_Q(x) = 4$ and $N_Q(x) \neq N_Q(x_1)$ for $x \in \{x_2, x_p, x_{p-1}\}$.

In this case, $y_1 \in N_Q(x)$ and three other vertices from $\{y_2, y_3, y_4, y_5\}$. No matter which three vertices are chosen, a K_4 exists, which is a contradiction.

Case 2.1.3: Suppose $|N_Q(x)| = 3$.

In this case, any two vertices of $\{x_2, x_{p-1}, x_p\}$, say x' and x'', have a common neighbor, say y. If $y = y_1$, then $C' = \langle y_1, x', x'', y_1\rangle$ and $Q' = \langle x_1, y_2, y_3, y_4, y_5, x_1\rangle$ with two chords x_1y_4 and x_1y_3 are the desired cycles. If $y = y_2$, then $C' = \langle y_2, x', x'', y_2\rangle$ and $Q' = \langle x_1, y_4, y_3, y_1, y_5, x_1\rangle$ with two chords y_1y_4 and y_5y_4 are the desired cycles. If $y = y_3$, then $C' = \langle y_3, x', x'', y_3\rangle$ and $Q' = \langle x_1, y_5, y_4, y_1, y_2, x_1\rangle$ with two chords y_1y_5 and x_1y_4 are the desired cycles. If $y = y_4$, then $C' = \langle y_1, y_2, y_3, y_1\rangle$ and $Q' = \langle x_1, y_5, y_4, x'', x', x_1\rangle$ with two chords x_1y_4 and $x'y_4$ are the desired cycles. If $y = y_5$, then $C' = \langle y_1, y_2, y_3, y_1\rangle$ and $Q' = \langle x_1, y_4, y_5, x'', x', x_1\rangle$ with two chords x_1y_5 and $x'y_5$ are the desired cycles. See Figure 3b.

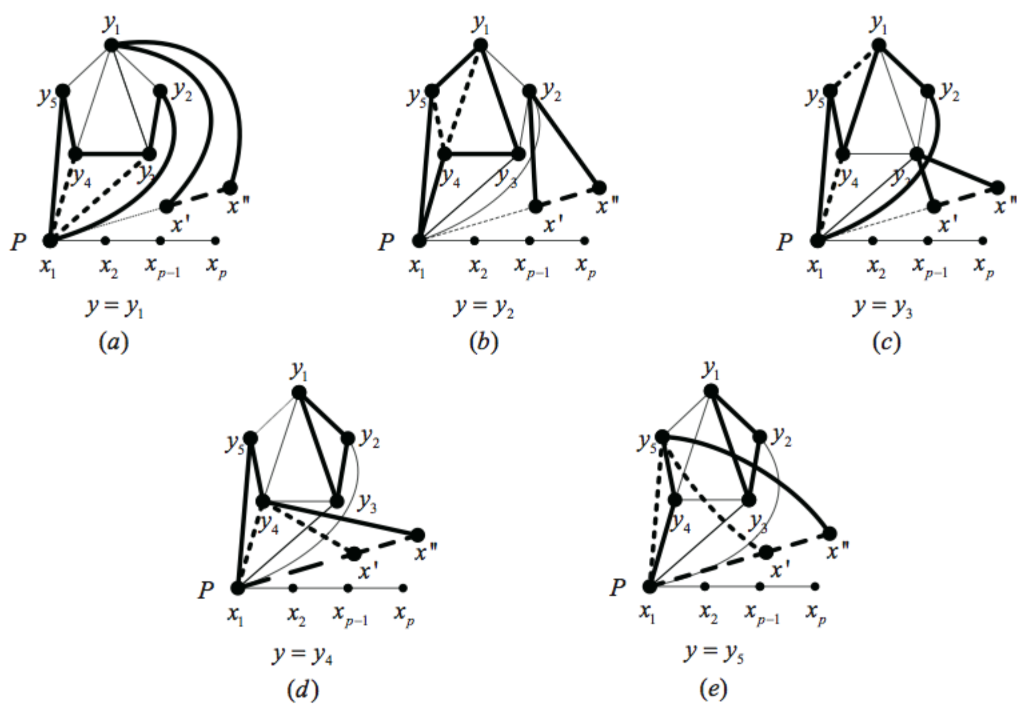

Figure 3b: $|N(x, Q)| = 3$

Case 2.2: Suppose $\deg_Q(x_2) = 4$. (Note that the case where $\deg_Q(x_{p-1}) = 4$ is symmetric to the $\deg_Q(x_2) = 4$ case.)

By substituting x_2 for x_1 in Case 2.1, it can be checked that this case also follows.

This completes the proof.

$\square$

By Lemma 3.11, we directly obtain the following corollary.

Corollary 3.12. *Let P be a path with $|V(P)| = 3$ and $Q \in \mathcal{H}_2$ such that P and Q are vertex disjoint. If $\sum_{x \in V(P)} \deg_Q(x) \geq 10$, then there are two vertex disjoint cycles such that one is a cycle and the other is a doubly chorded cycle in $G[V(P) \cup V(Q)]$.*

Chapter 4

Forbidden Subgraphs for Chorded Pancyclicity

We call a graph G *pancyclic* if it contains at least one cycle of every possible length m, for $3 \leq m \leq |V(G)|$. In this chapter, we focus on a new property we call *chorded pancyclicity*. We explore both an edge-density condition and forbidden subgraphs in claw-free graphs sufficient to imply that the graph contains at least one chorded cycle of every possible length $4, 5, \ldots, |V(G)|$. In particular, certain paths and triangles with pendant paths are forbidden along with $K_{1,3}$. The results in this chapter are joint work with Ron Gould.

4.1 Introduction

In the past, forbidden subgraphs for hamiltonian properties in graphs have been widely studied (for an overview, see [13]). A graph containing a cycle of every possible length from three to the order of the graph is called pancyclic and it is one of the well-studied cycle properties in graphs. In this chapter, we define the notion of chorded pancyclicity, and study the density necessary to guarantee this property in a graph. We also extend some forbidden subgraph results for pancyclicity to analogous results for chorded pancyclicity. We consider only $K_{1,3}$-free (or *claw*-free) graphs, and we forbid certain paths and triangles with pendant paths.

In this chapter, we extend the following theorem to analogous results on chorded pancyclicity. Recall these results that were stated earlier in §1.5.

Theorem 4.1. *Let R, S be connected graphs and let G ($G \neq C_n$) be a 2-connected graph of order $n \geq 10$. Then if G is $\{R, S\}$-free then G is pancyclic for $R = K_{1,3}$ and S is either $P_4, P_5, P_6, Z_1,$ or Z_2.*

The proof of a theorem in [18] yields the following result.

Theorem 4.2. [18] *If G is a 2-connected graph of order $n \geq 10$ that contains no induced subgraph isomorphic to $K_{1,3}$ or Z_1, then G is either a cycle or G is pancyclic.*

Gould and Jacobson proved a similar result for Z_2 in [24].

Theorem 4.3. [24] *If G is a 2-connected graph of order $n \geq 10$ that contains no induced subgraph isomorphic to $K_{1,3}$ or Z_2, then G is either a cycle or G is pancyclic.*

Faudree, Gould, Ryjacek, and Schiermeyer proved a similar result for certain paths in [15].

Theorem 4.4. [15] *If G is a 2-connected graph of order $n \geq 6$ that is $\{K_{1,3}, P_5\}$-free, then G is either a cycle or G is pancyclic.*

Theorem 4.5. [15] *If G is a 2-connected graph of order $n \geq 10$ that is $\{K_{1,3}, P_6\}$-free, then G is either a cycle or G is pancyclic.*

Theorem 4.4 implies the following result for P_4.

Theorem 4.6. [15] *If G is a 2-connected graph of order $n \geq 6$ that is $\{K_{1,3}, P_4\}$-free, then G is either a cycle or G is pancyclic.*

In this chapter, we extend each of these results to chorded pancyclicity, which sometimes requires added or altered conditions.

4.2 Results

4.2.1 Introduction to Chorded Pancyclicity

Definition 4.7. *A graph G of order n is called* **chorded pancyclic** *if it contains a chorded cycle of every length i, $4 \leq i \leq n$.*

Note: not all pancyclic graphs are chorded pancyclic. The graph in Figure 4a below is pancyclic, but contains no chorded 5-cycle. Further, the graph in Figure 4b represents an infinite family of pancyclic graphs that do not contain a chorded 4-cycle.

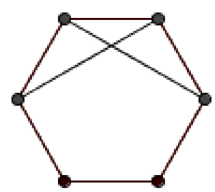

Figure 4a.

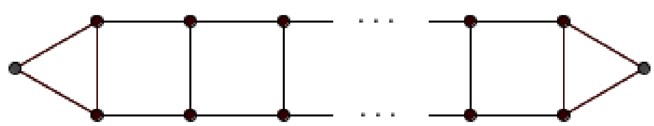

Figure 4b.

Before stating our results, we note particular notation that will be used throughout the proofs. For vertices v_1, v_2, v_3, v_4, the notation v_1–$v_2v_3v_4$ represents the $K_{1,3}$ graph where v_1 is adjacent to the three other vertices and the set $\{v_2, v_3, v_4\}$ is an independent set. To represent Z_1, we will use the notation $v_1v_2v_3$–v_4, where v_1, v_2, v_3 is a triangle and v_4 is adjacent to v_3. Similarly, $v_1v_2v_3$–v_4v_5 will be used to represent Z_2. The graphs $K_{1,3}$, Z_1, and Z_2 with the aforementioned notation are shown below (Figure 4c).

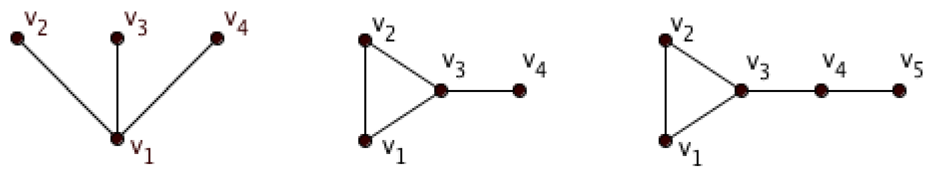

Figure 4c.

4.2.2 Edge Density and Chorded Pancyclicity

Following a similar idea from Chapters 2 and 3, it is natural to expect enough density in a graph will imply chorded pancyclicity. We show that this is true in the following theorem.

Theorem 4.8. *Let G is a graph on $n \geq 5$ vertices. If $\sigma_2(G) \geq n+1$, then G is chorded pancyclic and further, this degree sum condition is best possible.*

Example: The complete bipartite graph on n vertices $G = K_{\frac{n}{2},\frac{n}{2}}$, has $\sigma_2(G) = n$ and only contains cycles of even length. Therefore G is not pancyclic and hence clearly not chorded pancyclic either. Thus, the bound in Theorem 4.8 is sharp.

Proof. Consider a graph G of order $n \geq 4$ with $\sigma_2(G) \geq n+1$. By Ore's Theorem, G is hamiltonian and by $\sigma_2(G)$, any cycle of order n in G must be chorded. According to a result of Bondy in [4], if $|E(G)| \geq n^2/4$ then G is either pancyclic or G is the balanced complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$. Since $\sigma_2(G) \geq n+1$, $|E(G)| > n^2/4$ and thus G must be pancyclic. Let m be the largest number with $4 \leq m < n$ such that all of the m-cycles in G are nonchorded.

Claim 1. G contains a chorded 4-cycle.

Proof. Let x and y be a pair of nonadjacent vertices chosen to have the smallest number $m \geq 3$ of common neighbors in G. Partition $V(G)$ into the following sets:

$$\begin{aligned} N &= \{\text{common neighbors of } x \text{ and } y\}, \\ N^*(x) &= \{\text{vertices adjacent to } x \text{ but not } y\}, \\ N^*(y) &= \{\text{vertices adjacent to } y \text{ but not } x\}, \\ D &= \{\text{vertices not adjacent to both } x \text{ and } y\}. \end{aligned}$$

Let $|N^*(x)| = X$, $|N^*(y)| = Y$, and $|D| = d$.

Claim 1.1. If $|N| = m \geq 3 + r$, then $|D| \leq r$.

Proof. Suppose not, say $|D| = r + t$ for some $t \geq 1$. Then from the partition of $V(G)$, we have

$$\begin{aligned} X + Y + m + d = n - 2, \text{ or} \\ X + Y + m = n - 2 - d. \\ \text{Then } \sigma_2(G) \leq n - 2 - (r + t) + r + 3. \\ \text{Hence, } \sigma_2(G) \leq n + 1 - t < n + 1, \text{ a contradiction.} \end{aligned}$$

□

Now consider $a, b \in N$. If $ab \in E(G)$, then x, a, y, b, x is a chorded 4-cycle with chord ab. If $ab \notin E(G)$, then a and b must have at least m common neighbors (with x and y being two of them). Now a and b must send no edges into N, otherwise we would have a chorded 4-cycle. But now the common neighbors of a and b (other than x and y) cannot all lie in D, otherwise a and b would have at most $2 + r < m$ common neighbors, a contradiction. Hence, a and b have at least one common neighbor in either $N^*(x)$ or $N^*(y)$. Without loss of generality, suppose a and b share a neighbor $z \in N^*(X)$. Then a, z, b, x, a is a 4-cycle with chord xz. Thus G must contain a chorded 4-cycle.

□

Claim 2. If G contains a chorded 4-cycle, then G also contains a chorded 5-cycle.

Proof. Consider a chorded 4-cycle in G, say $C = v_1, v_2, v_3, v_4, v_1$ with chord v_1v_3. If $v_2v_4 \notin E(G)$ then since $\sigma_2(G) \geq n+1$, v_2 and v_4 must have a common neighbor $x \notin V(C)$ and then $v_1, v_3, v_2, x, v_4, v_1$ is a 5-cycle with chord v_1v_2. Now suppose C is the complete graph K_4. Without loss of generality, if $x \notin V(C)$ is adjacent to v_1 but not to v_2, then x and v_2 must share a common neighbor, $y \neq v_1$. If y is any vertex in C then there clearly exists a chorded 5-cycle. If $y \notin V(C)$, then v_1, x, y, v_2, v_4, v_1 is a 5-cycle with chord v_1v_2. Thus, when $n \geq 4$ the existence of a chorded 4-cycle implies the existence of a chorded 5-cycle in G. □

Claim 3. G contains a chorded k-cycle, for all $k \geq 6$.

Proof. Suppose $m \geq 6$. Consider a chordless m-cycle $C = v_1, v_2, \ldots, v_m, v_1$ in G. Since C is chordless, v_1 and v_3 are nonadjacent and must have a common neighbor $x \notin V(C)$. Similarly v_2 and v_6 are nonadjacent and must have a neighbor $y \notin V(C)$ such that $x \neq y$. Now $v_1, x, v_3, v_2, y, v_6, \ldots, v_m, v_1$ is an m-cycle with chord v_1v_2, a contradiction. □

Claims 1, 2, and 3 imply that there must be at least one chorded 4-cycle in G, and thus at least one chorded 5-cycle in G, as well as at least one chorded m-cycle in G for $6 \leq m \leq n$. Therefore, G is chorded pancyclic.

□

4.2.3 Forbidden Subgraphs

We now turn our attention to forbidden subgraphs for chorded pancyclicity. We will use Theorem 4.1 as a guide for the following results.

Theorem 4.9. *Let G be a 2-connected graph of order $n \geq 10$. If G is $\{K_{1,3}, Z_1\}$-free, then if $G \neq C_n$, G is chorded pancyclic.*

Proof. From Theorem 4.2, we know that G must be pancyclic. Now suppose G is not chorded pancyclic. Then there exists at least one i with $4 \leq i < n$, such that every i-cycle in G does not contain a chord. Consider the largest such i, and let G be such a pancyclic graph with no chorded i-cycles. Note that $i \neq n$, otherwise $G = C_n$, a contradiction. Consider an i-cycle, $C = v_1, v_2, \ldots, v_{i-1}, v_i, v_1$. Since $i < n$, there exists some vertex $x \notin V(C)$. By our assumption, G is 2-connected, so there must be an edge xy in $E(G)$ such that $y \in V(C)$. Without loss of generality, suppose $v_1x \in E(G)$. Then $v_1 - xv_2v_i$ is a claw, so one of $\{v_iv_2, v_ix, v_2x\}$ must be an edge in G. If $v_iv_2 \in E(G)$ then v_iv_2 is a chord of C and thus G contains a chorded i-cycle, a contradiction. By symmetry, the case where $v_ix \in E(G)$ and the case where $v_2x \in E(G)$ are equivalent, so without loss of generality, suppose v_2x is an edge in G. Then xv_1v_2–v_3 is a Z_1 subgraph of G, therefore either v_1v_3 or xv_3 must be an edge in G. If $v_1v_3 \in E(G)$ then C is a chorded i-cycle with chord v_1v_3, a contradiction. If $xv_3 \in E(G)$ then x can replace v_1 on C and C becomes a chorded i-cycle with chord xv_3, again a contradiction. Therefore, G always contains a chorded i-cycle, so G must be chorded pancyclic. □

To prove our next lemma, we will use the following theorem from Duffus, Gould, and Jacobson, on net-free graphs. The net graph is shown below.

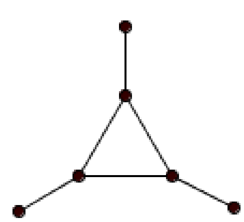

Figure 4d.

Theorem 4.10. [7] *Any connected, net-free, claw-free graph G contains a Hamiltonian path (i.e. is traceable).*

Lemma 4.11. *Let G be claw-free. For any $x \in V(G)$, the neighborhood of x, $N_G(x)$ is either connected and traceable, or two disjoint cliques.*

Proof. First note that the independence number of the neighborhood of any vertex in a claw-free graph must be at most two, otherwise G contains an induced claw. Let G be a claw-free graph and consider a vertex $x \in V(G)$. If $N_G(x)$ is disconnected, then it must be two cliques in order to have independence number two. If the neighborhood of a vertex contains an induced net, then it has independence number at least three (the independence number of the net graph). Therefore, the neighborhood of any vertex in a claw-free graph cannot contain an induced net subgraph. So if $N_G(x)$ is connected, then it is also net-free, claw-free and therefore, by Theorem 4.10, must be traceable. □

Example: The graph shown below is 2-connected, $\{K_{1,3}, Z_2\}$-free, pancyclic, and has maximum degree $\Delta(G) = 4$, but it does not contain a chorded 4-cycle so it is not chorded pancyclic. However, adding any edge yields a chorded 4-cycle, thus making the graph chorded pancyclic.

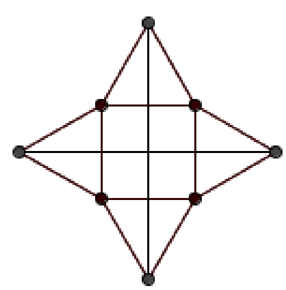

Figure 4e.

Theorem 4.12. *Let G be a 2-connected graph of order $n \geq 10$ with $\Delta(G) \geq 5$. If G is $\{K_{1,3}, Z_2\}$-free, then if $G \neq C_n$, G is chorded pancyclic.*

Proof. By Theorem 4.3, we know such a graph G must be pancyclic. For the sake of contradiction, suppose there is some m with $4 \leq m < n$ such that every m-cycle in G does not contain a chord.

Case 1. Suppose $m \geq 5$.

Consider an m-cycle $C = v_1, v_2, v_3, \ldots, v_m, v_1$ in G. Since G is 2-connected and of order at least 10, there exists a vertex $x \in V(G) - V(C)$ such that $xv \in E(G)$ for some $v \in V(C)$. Without loss of generality, suppose $xv_1 \in E(G)$. Then v_1–xv_2v_m is an induced claw in G. To eliminate this induced claw, either xv_2, xv_m or v_2v_m must be added as an edge. If v_2v_m is added then C is a chorded m-cycle and we are done. By symmetry, adding either xv_2 or xv_m as an edge is equivalent, so without loss of generality, suppose $xv_2 \in E(G)$. Then xv_1v_2–v_3v_4 is an induced Z_2 in G. The only two edges that would eliminate this induced Z_2 and would not add a chord to the m-cycle C are xv_3 and xv_4.

First suppose $xv_3 \in E(G)$. Then xv_2v_3–v_4v_5 is an induced Z_2 in G. The only edges that will eliminate the induced Z_2, but will not yield a chorded m-cycle are xv_4 and xv_5. If $xv_4 \in E(G)$ then $v_1, v_2, x, v_4, v_5, \ldots, v_m, v_1$ is an m-cycle with chord xv_1. If instead $xv_5 \in E(G)$, then $v_1, v_2, v_3, x, v_5, \ldots, v_m, v_1$ is an m-cycle with chord xv_1. This completes the case where $xv_3 \in E(G)$. Now suppose instead that $xv_4 \in E(G)$. Then $v_1, v_2, x, v_4, v_5, \ldots, v_m, v_1$ is an m-cycle with chord v_1x. This completes the case where $xv_4 \in E(G)$ and thus completes the proof of Case 1.

Case 2. Suppose $m = 4$.

Since $\Delta(G) \geq 5$ there must be some vertex $x \in V(G)$ such that $|N_G(x)| \geq 5$. Consider such a vertex x and let $|N_G(x)| = l \geq 5$. By Lemma 4.11, $N_G(x)$ is either connected and traceable, or two disjoint cliques. If $N_G(x)$ is connected and traceable, say $v_1, v_2, \ldots, v_l$ is a hamiltonian path in $N_G(x)$. Then x, v_1, v_2, v_3, x is a 4-cycle with xv_2 as a chord. If $N_G(x)$ is two disjoint cliques, then the one of the cliques is of order at least three. Three vertices from this clique together with the vertex x yield a chorded 4-cycle. So no matter what the neighborhood of x looks like, there is always a chorded 4-cycle in G. This completes the proof of Case 2 and we are done. □

Conjecture 4.13. Let G be a 2-connected graph of order $n \geq 10$. If G is $\{K_{1,3}, Z_2\}$-free, then if $G \neq C_n$, G is chorded pancyclic.

A proof of this conjecture was done using a straightforward case analysis, similar to the proof of Theorem 4.9, but resulted in very, very many cases. Consequently, the proof has been omitted here.

Theorem 4.14. *Let G be a 2-connected graph of order $n \geq 10$. If G is $\{K_{1,3}, P_4\}$-free, then if $G \neq C_n$, G is chorded pancyclic.*

Proof. By Theorem 4.6, we know that G is pancyclic. Suppose, for the sake of contradiction, that G is not chorded pancyclic. Let m be the largest number with $4 \leq m \leq n$ such that every m-cycle in G is chordless.

Case 1. Suppose $m = 4$.

Consider a 4-cycle $C = v_1, v_2, v_3, v_4, v_1$ in G. Since $n \geq 10$ and G is 2-connected, there exists a vertex $x \notin V(C)$ such that $xv \in E(G)$ for some $v \in V(G)$. Without loss of generality suppose $v_1x \in E(G)$. Then x, v_1, v_2, v_3 is an induced P_4 in G. Since G must be P_4-free, v_1v_3, v_2x, or v_3x must be in $E(G)$. If $v_1v_3 \in E(G)$ then the cycle C is a chorded 4-cycle, a contradiction.

If $v_2x \in E(G)$ then x, v_1, v_4, v_3 is an induced P_4 subgraph of G, therefore G must contain either v_1v_3, v_3x, or v_4x as an edge. The edge v_1v_3 creates a chord of the 4-cycle C, a contradiction. The edge v_3x yields the 4-cycle v_1, x, v_3, v_2, v_1 with chord v_2x, a contradiction. Similarly, the edge v_4x yields the 4-cycle v_1, v_2, x, v_4, v_1 with chord v_1x, again a contradiction.

If $v_3x \in E(G)$ then v_1–xv_2v_4 is an induced claw in G, therefore G must contain either xv_2, xv_4, or v_2v_4 as an edge. The edge xv_2 forms the 4-cycle

v_1, v_2, v_3, x, v_1 with chord xv_2, a contradiction. The edge xv_4 yields the 4-cycle v_1, v_4, v_3, x, v_1 with chord xv_4, a contradiction. The edge v_2v_4 is a chord of the 4-cycle C, again a contradiction. Thus, every 4-cycle in G must be chorded.

Case 2. Suppose $m > 4$.

Any chordless m-cycle contains at least one induced P_4 subgraph, for $m > 4$. Therefore, all m-cycles must contain at least one chord to ensure that G is P_4-free, for $4 < m \leq n$. Now we can conclude that G is chorded pancyclic. □

Theorem 4.15. *Let G be a 2-connected graph of order $n \geq 10$. If $G \neq C_n$ is $\{K_{1,3}, P_5\}$-free, then G is chorded pancyclic.*

Proof. From Theorem 4.4 we know G must be pancyclic. Suppose, for the sake of contradiction, that G is not chorded pancyclic. Let m be the largest number with $4 \leq m \leq n$ such that every m-cycle in G is not chorded.

Case 1. Suppose $m = 4$.

Consider a 4-cycle $C = a, b, c, d, a$ in G. Since G is 2-connected and of order at least 10, there is a vertex $x_1 \notin V(C)$ such that $x_1v \in E(G)$ for some $v \in V(C)$. Without loss of generality, suppose $x_1a \in E(G)$. Then a–x_1bd is an induced claw in G, so either x_1b, x_1d, or bd must be an edge in G. Adding the edge bd makes C into a chorded 4-cycle. By symmetry, adding x_1b or x_1d is equivalent, so without loss of generality, suppose $x_1b \in E(G)$.

Now there are no induced P_5 or $K_{1,3}$ subgraphs in G, but again since G is 2-connected and of order at least 10, there must be some vertex $x_2 \notin V(C)$ such that $x_1 \neq x_2$ and $x_2v \in E(G)$ for some $v \in V(C)$.

If x_2a (or, by symmetry, x_2b) is an edge in G, then a–x_2bd is an induced claw. Then either x_2b, x_2d, or bd must be an edge. If either bd or x_2b is an edge, then there exists a chorded 4-cycle in G. So suppose $x_2d \in E(G)$.

There is some vertex $x_3 \notin V(C)$ such that $x_3 \neq x_2$ or x_1 and $x_3v \in E(G)$ for some $v \in V(C) \cup \{x_1\}$.

Now we must consider all of the possible adjacencies of x_3: a, b (same case as d), c, and x_1. If $x_3a \in E(G)$ then a–x_3bd is an induced claw. Any edge added to eliminate the claw will result in a chorded 4-cycle.

If x_3b (or similarly x_3d) $\in E(G)$ then b–x_3ac is a claw. Adding either one of the edges ac or x_3a to eliminate the claw, will result in a chorded 4-cycle. Adding x_3c to eliminate the claw does not yield a chorded 4-cycle or any induced P_5 or $K_{1,3}$ subgraphs. However, following an earlier argument, there is a vertex $x_4 \notin V(C) \cup \{x_1, x_2, x_3\}$ such that $x_4v \in E(G)$ for some $v \in V(C) \cup \{x_1, x_2\}$. Now we must consider all of the possible adjacencies of x_4.

If x_4a (or similarly x_4b) $\in E(G)$ then a-x_4bd is an induced claw. Adding any edge to eliminate the claw yields a chorded 4-cycle. If x_4c (or similarly x_4d) $\in E(G)$ then c–x_4bd is an induced claw. Adding either bd or x_4b to eliminate the claw, yields a chorded 4-cycle. Adding x_4d to eliminate the claw does not result in any chorded 4-cycles or induced P_5 or $K_{1,3}$ subgraphs. There must exist another vertex $x_5 \notin V(C) \cup \{x_1, x_2, x_3, x_4\}$ such that $x_5v \in E(G)$ for some $v \in V(C) \cup \{x_1, x_2, x_3\}$.

If x_5 is adjacent to any of the original vertices in C, then an induced claw in each of those cases will yield a chorded 4-cycle. The remaining possible adjacencies to x_5 are all symmetric: x_1, x_2, and x_3. So without loss of generality suppose $x_1x_5 \in E(G)$. Then x_5, x_1, a, d, c is an induced P_5. Adding any edge that eliminates the induced P_5 will either yield a chorded 4-cycle or an induced claw that will yield a chorded 4-cycle.

If $x_4x_1 \in E(G)$, then x_4, x_1, a, d, c is an induced P_5. Adding any one of four of the six edges that eliminate the induced P_5 results in a chorded

4-cycle. However, adding either x_4d or x_4c will not produce a chorded 4-cycle. By symmetry, adding either edge is equivalent so without loss of generality suppose $x_4d \in E(G)$. Then d–x_4ac is an induced claw and x_4c is the only claw-eliminating edge whose addition will not yield a chorded 4-cycle. In fact, adding x_4c does not yield any induced P_5 or $K_{1,3}$ subgraphs either. There exists a vertex $x_5 \notin V(C) \cup \{x_1, x_2, x_3, x_4\}$ such that $v_5v \in E(G)$ for some $v \in V(C) \cup \{x_1, x_2, x_3\}$. The addition of either x_5a or x_5b yields an induced claw that results in a chorded 4-cycle. The addition of x_5c or x_5d results in an induced claw that yields either a chorded 4-cycle or a previously considered case, depending on which edge is added to eliminate the induced claw.

If $x_5x_1 \in E(G)$, then x_1–x_5x_4a is an induced claw and adding either x_5a or x_4a to eliminate the claw, yields a chorded 4-cycle. If x_5x_4 is added to eliminate the claw then x_5, x_1, a, d, c is an induced P_5 and any edge added to eliminate the induced P_5 results in a chorded 4-cycle.

If $x_5x_2 \in E(G)$ then x_5, x_2, a, b, c is an induced P_5. To eliminate the induced P_5, adding either x_5a, x_1d, x_1c, or ac yields a chorded 4-cycle. Adding x_5b to eliminate the induced P_5 produces the induced b–x_5ac claw. The only claw-eliminating edge that will not yield a chorded 4-cycle is x_5c, so suppose $x_5c \in E(G)$. Now b–x_5x_3a is an induced claw and the addition of any claw-eliminating edge yields a chorded 4-cycle. If $x_5c \in E(G)$ (and $x_5b \notin E(G)$) then c–x_5bd is an induced claw and any added claw-eliminating edge results in either a chorded 4-cycle or a previously checked case. If $x_2x_4 \in E(G)$ (and $x_1x_4 \notin E(G)$) then x_4, x_2, a, b, c is an induced P_5 and any induced-P_5-eliminating edge yields a chorded 4-cycle, except the edge x_4c. If $x_4c \in E(G)$ then c–x_4bd is an induced claw and the addition of any induced-claw-eliminating edge will yield a chorded 4-cycle. If $x_1, x_3 \in E(G)$ (and $x_3c \notin E(G)$) then x_3, x_1, b, c, d is an induced P_5 and any edge that will eliminate the induced P_5 except for x_3c or x_3d will result in a chorded 4-cycle.

If $x_3c \in E(G)$ then c–x_3bd is an induced claw. To eliminate the induced claw, the addition of either x_3b or bd yields a chorded 4-cycle, but the addition of x_3d does not. There exists a vertex $x_4 \notin V(C) \cup \{v_1, x_2, x_3\}$ such that $x_4v \in E(G)$ for some $v \in V(C) \cup \{x_1, x_2\}$. Considering the possible adjacencies of x_4, if x_4a, x_4b, x_4c, or $x_4d \in E(G)$ then an induced claw forces the existence of a chorded 4-cycle or a previously checked case. If x_4x_1 is added to eliminate the induced P_5 then x_1–x_4x_3b is an induced claw and the only induced claw-eliminating edge that does not yield a chorded 4-cycle is x_3x_4. So if $x_3x_4 \in E(G)$, then x_4, x_1, a, d, c is an induced P_5 and adding any edge to eliminate this induced P_5 yields a chorded 4-cycle. If $x_4x_2 \in E(G)$ (and $x_1x_4 \notin E(G)$) then x_4, x_2, a, b, c is an induced P_5. Adding any edge to eliminate this induced P_5 results in a chorded 4-cycle, except for the addition of either x_4b or x_4c. If $x_4b \in E(G)$ then b–x_4ac is an induced claw. Adding any edge except for x_4c to eliminate this induced claw will result in a chorded 4-cycle. If x_4c is added to eliminate the claw, then $x_4, x_2, , x_1, x_3$ is an induced P_5 and adding any edge to eliminate this induced P_5 yields a chorded 4-cycle. If x_4c was added first instead of x_4b, then c–x_4x_3b is an induced claw. To eliminate the claw, adding the edge x_3b yields a chorded 4-cycle and adding the edge x_4b results in a previously checked case. Adding x_3x_4 results in a x_3–x_1x_4d induced claw. Any edge added to eliminate this claw yields a chorded 4-cycle. This completes the case where $x_2a \in E(G)$.

If x_2c (or, by symmetry, x_2d) is an edge in G, then c–x_2bd is an induced claw, which can be eliminated by adding x_2b, x_2d, or bd as an edge. If bd is added then C is a chorded 4-cycle. Adding x_2b as an edge yields a previously considered structure. So suppose $x_2d \in E(G)$. There are no chorded 4-cycles or induced claws or P_5 subgraphs. There exists a vertex $x_3 \notin V(C) \cup \{x_1, x_2\}$ such that $x_3 v \in E(G)$ for some $v \in V(C) \cup \{x_1\}$. Now consider all of the possible adjacencies of x_3. By symmetry, adding the edge $x_3 a$ is equivalent to adding any one of the edges $x_3 b$, $x_3 c$, or $x_3 d$. Without loss of generality, suppose $x_3 a \in E(G)$. Then a–$x_3 bd$ is an induced claw. The addition of any one of the edges needed to eliminate this induced claw results in either a previously checked case, or a chorded 4-cycle. So instead suppose x_3 is adjacent to x_1 (the only other possibly adjacency of x_3). Then x_3, x_1, a, d, c is an induced P_5 in G. If any one of the edges needed to eliminate the induced P_5 is added, then there is a chorded 4-cycle, except when $x_3 d$ (or by symmetry, $x_3 c$ is added. Without loss of generality, suppose $x_3 d \in E(G)$. Then d–$x_3 ac$ is an induced claw in G. Adding any one of the edges needed to eliminate the induced claw, results in a chorded 4-cycle. This completes the case where $x_2 c \in E(G)$.

Case 2. Suppose $m = 5$.

Consider a 5-cycle $C = a, b, c, d, e, a$ in G. Since $n \geq 10$ and G is 2-connected, there exists a vertex $x_1 \notin V(C)$ such that $x_1 v \in E(G)$ for some $v \in V(C)$. Without loss of generality, say $ax_1 \in E(G)$. Then a–x_1, b, e is an induced claw in G, thus either $x_1 b$, $x_1 e$, or be must be an edge in G. If be is an edge then C is a chorded 5-cycle and we are done with this case. By symmetry, adding $x_1 b$ or $x_1 e$ as an edge is equivalent, so without loss of generality assume $x_1 b \in E(G)$.

Now x_1, b, c, d, e is an induced P_5, so either $x_1 c$, $x_1 d$, $x_1 e$, bd, be, or ce must be an edge in G. The edges bd, be, and ce make C a chorded 5-cycle, and we are done. If $x_1 d \in E(G)$ then x_1, b, a, e, d, x_1 is a 5-cycle with chord $x_1 a$ and we are done. By symmetry, adding the edge $x_1 c$ or adding $x_1 e$ is equivalent, so assume $x_1 c \in E(G)$.There are no chorded C_5's nor induced P_5's or $K_{1,3}$'s, but since $n \geq 10$ and G is 2-connected there must be some vertex $x_2 \notin V(C)$ such that $x_2 \neq x_1$ and $x_2v \in E(G)$ for some $v \in V(C)$.

If x_2a (or x_2c, by symmetry) $\in E(G)$, then a–x_2be is an induced claw subgraph in G so x_2b, x_2e, or be must be an edge of G. Adding be as an edge makes C a chorded 5-cycle. If $x_2b \in E(G)$, then b, c, x_1, a, x_2, b is a 5-cycle with chord ab. If $x_2e \in E(G)$ then x_2, a, b, c, d is an induced P_5, so either x_2b, x_2c, x_2d, ac, ad, or bd must be an edge in G. Adding any of the edges except x_2d will yield a chorded 5-cycle. So suppose $x_2d \in E(G)$. There are no induced P_5 or $K_{1,3}$ subgraphs, but since G is 2-connected and of order at least 10, there must be some vertex $x_3 \notin V(C)$ such that $x_3 \neq x_1$ or x_2 and $x_3v \in E(G)$ for some $v \in V(C) \cup \{x_1\}$. If $x_3a \in E(G)$, then a–x_3be is an induced claw so x_3b, x_3e, or be must be an edge in G. Adding any of these edges yields a chorded 5-cycle. If x_3b (or, similarly x_3e) is an edge, then b–x_3ac is an induced claw so either x_3a, x_3c, or $ac \in E(G)$. Adding any of these edges yields a chorded 5-cycle. If x_3c (or, similarly x_3d) is an edge then c–x_3bd is an induced claw so either x_3b, x_3d, or $bd \in E(G)$. Adding x_3b or bd yields a chorded 5-cycle. If $x_3d \in E(G)$ then x_3, d, e, a, b is an induced P_5. Adding any of the edges to eliminate this P_5 either creates a chorded 5-cycle or results in a previously considered case (which all yielded chorded 5-cycles). If $x_1x_3 \in E(G)$, then x_1–x_3ac is an induced claw so either x_3a, x_3c, or $ac \in E(G)$. Adding any of these edges yields a chorded 5-cycle in G. This completes the case where $x_2a \in E(G)$.

If $x_2b \in E(G)$, then b–x_2ac is an induced claw so either x_2a, x_2c or $ac \in E(G)$. Adding any of these edge will create a chorded 5-cycle in G. This completes the case where $x_2b \in E(G)$.

If x_2d (or x_2e, by symmetry) $\in E(G)$, then d–x_2ce is an induced claw so either x_2c, x_2e, or $ce \in E(G)$. The edge ce is a chord of C. If $x_2c \in E(G)$, then x_2, c, b, a, e is an induced P_5. To eliminate this P_5, adding either x_2a, x_2b, ca, be, or ce creates a chorded 5-cycle. The final edge that would eliminate the P_5 is x_2e. Adding x_2e creates a structure symmetric to L, which has been previously considered. If x_2e is an edge instead of x_2c then x_2, e, a, b, c is an induced P_5 in G. Adding either edge x_2a or x_2c yields a structure symmetric to L. Adding any one of the remaining edges to eliminate the P_5 yields a chorded 5-cycle. This completes the case where $x_2d \in E(G)$ and thus concludes the $m = 5$ case.

Case 3. Suppose $m \geq 6$.

There is an induced P_5 subgraph in every non-chorded m-cycle for $m \geq 6$, so every such m-cycle must contain at least one chord. The three cases together show that there is a chorded cycle of every possible length m, for $4 \leq m \leq n$, in G. □

Unlike the straightforward case analysis method used to prove Theorems 4.14 and 4.15, we took a different approach when forbidding P_6 from claw-free graphs, as otherwise the number of cases was problematic. We use Lemma 4.11 along with the following results to prove the P_6 result for sufficiently large graphs.

Lemma 4.16. *Let G be a $K_{1,3}$-free graph. For any $x \in V(G)$ and any $k > 2 \in \mathbb{Z}$, if $deg_G(x) \geq 2k-1$ then there is a chorded $(k+1)$-cycle in G.*

Proof. Consider a vertex $x \in V(G)$ such that $deg_G(x) \geq 2k-1$. Lemma 4.11 implies that $N_G(x)$ is either connected and traceable, or two disjoint cliques.

Case 1: Suppose $N_G(x)$ is connected and traceable.
Let $v_1, v_2, \ldots, v_{2k-1}$ be a Hamiltonian path in $N_G(x)$. Then $x, v_1, v_2, \ldots, v_k, x$ is a chorded $(k+1)$-cycle in G with chord xv_2 (in fact, there are $k-2$ chords in this $(k+1)$-cycle).

Case 2: $N_G(x)$ is two disjoint cliques.

Partitioning the $2k-1$ vertices into two cliques, the smallest order that the larger of the two sets can be is k (the k and $k-1$ case). Say the vertices $\{v_1, v_2, \ldots, v_k\}$ form a clique in $N_G(x)$, then $x, v_1, v_2 \ldots, v_k, x$ is a chorded $(k+1)$ cycle in G with chord xv_2. $\square$

Theorem 4.17. [11] *Let $f = F_d(p, \Delta)$ be the smallest integer such that every graph on p vertices with at least f edges has maximum degree Δ and diameter at most d. Then:*

$$f = F_d(p, \Delta) \geq \binom{p}{2} \frac{\Delta - 2}{(\Delta - 1)^d - 1}.$$

Theorem 4.18. *Let G be a 2-connected graph of order $n > 3201$. If G is $\{K_{1,3}, P_6\}$-free and $G \neq C_n$, then G is chorded pancyclic.*

Proof. By Theorem 4.5, G must be pancyclic, so now we must show G contains a chorded cycle of every possible length m for $4 \leq m \leq n$. Note that for $m \geq 7$, every m-cycle must be chorded, otherwise it contains an induced P_6 subgraph. Now we need to show there exists a chorded 4-cycle, chorded 5-cycle, and chorded 6-cycle in G.

For some $x \in V(G)$, if $d_G(x) \geq 5$ then there exists a chorded 4-cycle by Lemma 4.16 ($k = 3$ case). If $d_G(x) \geq 7$ then there exists a chorded 5-cycle in G by Lemma 4.16 ($k = 4$ case). If $d_G(x) \geq 9$ then there exists a chorded 6-cycle in G by Lemma 4.16 ($k = 5$ case). Therefore, if G contains a vertex of degree at least nine, then G contains at least one chorded 4-cycle, chorded 5-cycle, and chorded 6-cycle by Lemma 4.16 and is thus chorded pancyclic.

Suppose that G is 8-regular (i.e. adding any edge will yield a vertex of degree nine and hence the desired result). Then G has maximum degree $\Delta = 8$ and G contains $\frac{8n}{2} = 4n$ edges. Note that d is the diameter of G if and only if G is P_l-free for $l > d+2$. So since G is P_6-free, the diameter of G is $d \geq 4$. By Theorem 4.17, we have

$$\begin{aligned} 4n &\geq \binom{n}{2}\frac{8-2}{(8-1)^4-1} \\ 4n &\geq \frac{6n(n-1)}{2(7^4-1)} \\ 4 &\geq \frac{3(n-1)}{2400}, \end{aligned}$$

Hence, $n \leq 3201$.

Therefore, if G is a 2-connected $\{K_{1,3}, P_6\}$-free graph of order $n > 3201$, then G contains a vertex of degree at least nine and thus is chorded pancyclic. □

We feel the bound given on n in Theorem 4.18 is sufficient but not necessary, so we make the following conjecture.

Conjecture 4.19. Let G be a 2-connected graph of order $n \geq 10$. If G is $\{K_{1,3}, P_6\}$-free and $G \neq C_n$, then G is chorded pancyclic.

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