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Doctor of Philosophy

Mathematics

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An abstract of

A dissertation submitted to the Faculty of the

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Abstract

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Let K be a complete discretely valued field with the residue field κ. Let F be the function field of a smooth, projective, geometrically integral curve over K and $\mathscr{X}$ be a regular proper model of F such that the reduced special fibre X is a union of regular curves with normal crossings. Suppose that the graph associated to $\mathscr{X}$ is a tree (e.g. $F = K(t)$). Let L/F be a Galois extension of degree n such that n is coprime to $\text{char}(\kappa)$. Suppose that κ is an algebraically closed field or a finite field containing a primitive n^{th} root of unity. Then we show that the local-global principle holds for the norm one torus associated to the extension L/F with respect to discrete valuations on F, i.e., an element in $F^{\times}$ is a norm from the extension L/F if and only if it is a norm from the extensions $L \otimes_F F_\nu/F_\nu$ for all discrete valuations ν of F. We also prove that such a local-global principle holds for multinorm tori over F associated to two cyclic extensions each of degree p for a prime p if the residue field κ is algebraically closed or a finite field. We prove that for finitely many quadratic cyclic extensions, the local-global principle holds for the associated multinorm tori if the residue field κ is algebraically closed, $\text{char}(\kappa) \neq 2$ and the graph associated to a regular proper model $\mathscr{X}$ is a tree.

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Contents

Chapter 1

Introduction

Let F be a field and Ω_F be the set of all discrete valuations on F. For $\nu \in \Omega_F$, let F_ν denote the completion of F at ν. Let G be a linear algebraic group over F. One says that the *local-global principle* holds for G if for any G-torsor X, X has a rational point over F if and only if it has a rational point over F_ν for all $\nu \in \Omega_F$. If F is a number field, we also consider the completions at archimedean places while discussing local-global principles for algebraic groups. If F is a number field, then it is known that the local-global principle holds for various classes of linear algebraic groups ([PRR93, Chapter 6]), including semisimple simply connected groups. In particular, it is well-known that if $T_{L/F}$ is the norm one torus associated to a cyclic extension L/F, then the local-global principle holds for $T_{L/F}$, i.e., an element $\lambda \in F^\times$ is a norm from the extension L/F if and only if λ is a norm from $F \otimes_F F_\nu/F_\nu$ for all $\nu \in \Omega_F$ ([CF67, Chapter 11]). However, very little is known for general fields.

Let K be a complete discretely valued field with residue field κ. Let F be the function field of a smooth, projective, geometrically integral curve over K. Such a field F is called semi-global field. Let G be a linear algebraic group over F. Harbater, Hartmann and Krashen ([HHK15a]) developed patching techniques to study G-torsors over F and proved that if G is connected and F-rational, then a G-torsor over F has a rational point over F if and only if it has a rational point over certain overfields of F which are defined using patching (see Section 3.1). As a consequence of this result, Colliot-Thélène, Parimala and Suresh ([CTPS08, Theorem 4.3.]) showed that if G is reductive, F-rational and defined over the ring of integers of K, then the local-global principle holds for G. Similar local-global principles are proved for various linear algebraic groups G over F if the residue field of K is either finite or algebraically closed field; e.g. see [CTPS08], [CTPS16], [Hu12], [Pre13], and [PPS18].

The first example of a linear algebraic group G over F where such a local-global principle fails was given by Colliot-Thélène, Parimala and Suresh ([CTPS16, Section 3.1. & Proposition 5.9.]). In their example, the residue field of K is the field of complex numbers, G is the norm one torus of a Galois extension L/F with Galois group $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and the field F has a regular proper model with the associated graph not a tree. Suppose that F has a regular proper model with the associated graph a tree. If L/F is a Galois extension with Galois group $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and κ is algebraically closed, then they also proved that the local-global principle holds for the norm one torus $T_{L/F}$ ([CTPS16, Section 3.1. & Corollary 6.2.]).

The main aim of this thesis is to prove the following theorem (see Corollary 4.4.4):

Theorem 1.0.1. *Let K be a complete discretely valued field with residue field κ and F be the function field of a smooth, projective, geometrically integral curve over K. Let $\mathscr{X}$ be a regular proper model of F with reduced special fibre X a union of regular curves with normal crossings. Let L/F be a Galois extension over F of degree n with Galois group G. Suppose that the graph associated to $\mathscr{X}$ is a tree and κ is one of the following:*

- *κ is an algebraically closed field of characteristic coprime to n, or*

- *κ is a finite field of characteristic coprime to n and contains a primitive n^{th} root of unity.*

Then the local-global principle holds for the norm one torus $T_{L/F}$, i.e., an element $\lambda \in F$ is a norm from the extension L/F if and only if λ is a norm from the extensions $L \otimes_F F_\nu / F_\nu$ for all $\nu \in \Omega_F$.

For a finite separable extension L/F, let $T_{L/F}$ denote the norm one torus associated to L/F. For any extension N/F, let $RT_{L/F}(N)$ be the subgroup of $T_{L/F}(N)$ consisting of R-trivial elements (see Section 2.4). The above theorem follows from the following more general theorem (see Theorem 4.4.3), where we allow more general residue fields κ.

Theorem 1.0.2. *Let K be a complete discretely valued field with residue field κ and F be the function field of a smooth, projective, geometrically integral curve over K. Let $\mathscr{X}_0$ be a regular proper model of F with reduced special fibre X_0 a union of regular curves with normal crossings. Let*

L/F be a Galois extension over F of degree n. Suppose that the graph associated to $\mathscr{X}_0$ is a tree, and:

- *$char(\kappa)$ is coprime to n,*

- *κ contains a primitive n^{th} root of unity ρ, and*

- *for all finite extensions κ'/κ and for all finite Galois extensions l/κ' of degree d dividing n,*

$$T_{l/\kappa'}(\kappa') = RT_{l/\kappa'}(\kappa')\langle \rho^{n/d} \rangle.$$

Then the local-global principle holds for the norm one torus $T_{L/F}$.

Remark 1.0.3. In fact one can restrict to divisorial discrete valuations in the above theorems.

We now briefly describe the strategy of the proof of Theorem 1.0.2. Let K be a complete discretely valued field with residue field κ and F be the function field of a smooth, projective, geometrically integral curve over K. Let $\mathscr{X}$ be a regular proper model of F with reduced special fibre X a union of regular curves with normal crossings. For any point $P \in X$, let F_P be the fraction field of the completion of the local ring at P on $\mathscr{X}$.

For a linear algebraic group G over F, let us define:

$$\bullet Ш_X(F,G) := \ker\left(H^1(F,G) \to \prod_{P\in X} H^1(F_P,G)\right)$$

$$\bullet Ш(F,G) := \ker\left(H^1(F,G) \to \prod_{\nu\in\Omega_F} H^1(F_\nu,G)\right).$$

It is known that $Ш_X(F,G) \subseteq Ш(F,G)$ ([HHK15a, Proposition 8.2.]).

In general, $Ш_X(F,G)$ and $Ш(F,G)$ are just pointed sets, but they are abelian groups if G is abelian. The pointed sets $Ш_X(F,G)$ and $Ш(F,G)$ measure the obstruction to the local-global principle for the group G with respect to points on X and with respect to discrete valuations on F respectively.

First we prove the following (see Corollary 4.4.2) in which the notion of R-equivalence and R-trivial elements play an important role.

Theorem 1.0.4. *Let K be a complete discretely valued field with residue field κ and F be the function field of a smooth, projective, geometrically integral curve over K. Let $\mathscr{X}$ be a regular proper model of F with reduced special fibre X a union of regular curves with normal crossings. Let L/F be a Galois extension over F of degree n. Suppose that*

- *n is coprime to char(κ),*

- *κ contains a primitive n^{th} root of unity ρ,*

- *for all finite extensions κ'/κ and for all finite Galois extensions l/κ' of degree d dividing n,*

$$T_{l/\kappa'}(\kappa') = RT_{l/\kappa'}(\kappa')\langle\rho^{n/d}\rangle,$$

- *the graph associated to $\mathscr{X}$ is a tree.*

Then $Ш_X(F, T_{L/F}) = 0$.

We conclude our main theorem (Theorem 1.0.2), by proving that for K, F and L as in Theorem 1.0.4, $Ш(F, T_{L/F}) = \bigcup_X Ш_X(F, T_{L/F})$ (Theorem 4.3.1), where X runs over the reduced special fibres of regular proper models $\mathscr{X}$ of F which are obtained as a sequence of blow-ups of $\mathscr{X}_0$ centered at closed points of $\mathscr{X}_0$.

We also study local-global principle for multinorm tori associated to finite Galois extensions over semi-global fields with respect to discrete valuations.

Over number fields, this has been studied more generally for separable extensions quite extensively, in particular by Hürlimann ([Hür84]), Colliot-Thélène and Sansuc (unpublished), Platonov and Rapinchuk ([PRR93, Section 6.3.]), Prasad and Rapinchuk ([PR08, Section 4]), Pollio and Rapinchuk ([PR12]), Demarche and Wei ([DW14]), Pollio ([Pol14]), Bayer-Fluckiger, Lee and Parimala([BFLP19]). Let F be a number field. Let L be a product of m ($m \geq 2$) many non-isomorphic quadratic field extensions. Let $T_{L/F}$ be the associated multinorm torus. Then the local-global principle holds for $T_{L/F}$ if $m = 2$ ([Hür84, Prop. 3.3.]). However, the local-global principle does not hold in general for $m = 3$ ([CT14]). It is a recent result of Bayer-Fluckiger, Lee and Parimala ([BFLP19, Thm 8.3.]) that the local-global principle holds if $m \geq 4$. In the same paper, they have results for more general multinorm tori which includes product of non-isomorphic degree p extensions, where p is any prime number.

Over semi-gobal fields, again using the patching technique of Harbater, Hartmann and Krashen ([HHK15a]), we prove the following theorems (see Theorem 5.2.8 and Theorem 5.2.24):

Theorem 1.0.5. *Let K be a complete discretely valued field with residue field κ. Let F be the function field of a smooth, projective, geometrically integral curve over K. Assume that κ is either algebraically closed or a finite field. Let L_1, L_2 be two degree p cyclic extensions of F. Assume that $p \neq char(\kappa)$. Let $L = L_1 \times L_2$. Then $Ш(F, T_{L/F}) = 0$.*

Remark 1.0.6. It is important to note here that we do not need any condition on the associated graph in Theorem 1.0.5.

Theorem 1.0.7. *Let K be a complete discretely valued field with residue field κ algebraically closed. Let F be the function field of a smooth, projective, geometrically integral curve over K. Let $L_i/F, 1 \leq i \leq m$, be quadratic cyclic extensions. Assume that $char(\kappa) \neq 2$. Let $L = \prod_{i=1}^{m} L_i$. If the graph associated to F is a tree then $Ш(F, T_{L/F}) = 0$.*

We begin by first proving the following result (see Theorem 5.1.3):

Theorem 1.0.8. *Let K be a complete discretely valued field with residue field κ. Let F be the function field of a smooth, projective, geometrically integral curve over K. Let L_i/F be Galois field extensions of degrees coprime to $char(\kappa)$ for $i = 1, 2, \ldots, m$. Let $L = \prod_{i=1}^{m} L_i$. Let $\mathscr{X}_0$ be a regular proper model of F. Then*

$$Ш(F, T_{L/F}) = \bigcup_{X} Ш_X(F, T_{L/F}),$$

where X runs over the reduced special fibres of regular proper models $\mathscr{X}$ of F which are obtained as a sequence of blow-ups of $\mathscr{X}_0$ centered at closed points of $\mathscr{X}_0$.

We conclude Theorem 1.0.5 and Theorem 1.0.7 by proving that for these multinorm tori $T_{L/F}$, $\text{Ш}_X(F, T_{L/F}) = 0$ for appropriate choices of $\mathscr{X}$ (see Corollary 5.2.7 and Theorem 5.2.23). Here again, the notion of R-equivalence and R-trivial elements play an important role.

In the last chapter (Chapter 6), we also give counterexamples to the local-global principles for certain norm one tori and multinorm tori over a semi-global field.

Some parts of Chapter 2 and Chapter 3 and the whole of Chapter 4 and Chapter 6 are excerpted from the author's paper titled 'Local-global principle for norm one tori' ([Mis19]).

Chapter 2

Prerequisites

2.1 Linear algebraic groups

We recall the notion of a linear algebraic group G over a field F below. We refer the reader to [Hum75], [Mil17], [MT11], and [Sza06] for thorough expositions.

Let F be a field. A linear algebraic group is an affine algebraic variety with a group structure which is compatible with the variety structure. More precisely, we define:

Definition 2.1.1. *(Linear algebraic group) A linear algebraic group defined over F is an affine algebraic variety G over F endowed with the structue of a group such that the following two maps:*

$$\mu \colon G \times G \to G; \quad \mu(x, y) = xy$$

and

$$\iota \colon G \to G; \quad \iota(x) = x^{-1}$$

are morphisms of varieties.

Examples:

- The additive group $\mathbb{G}_a$ over F is the affine line $\mathbb{A}^1_F$ endowed with the group law $\mu(x, y) = x + y$. Here 0 is the identity element and the inverse map is given by $\iota(x) = -x$.

- The multipilcative group $\mathbb{G}_m$ over F is the affine open set $F^\times \subset \mathbb{A}^1$ with the group law $\mu(x, y) = xy$. Here 1 is the identity element and the inverse map is given by $\iota(x) = x^{-1}$.

- The general linear group GL_n over F consists of invertible n by n matrices over F with the usual matrix multiplication as group law. Here the n by n identity matrix serves as the identity and the inverse of an element A is the inverse A^{-1} (the inverse matrix of the matrix A).

- The n^{th} roots of unity μ_n over F for $(n, \text{char}(F)) = 1$ with the group law $\mu(x, y) = xy$. Here 1 is the identity and the inverse map is given by $\iota(x) = x^{-1}$.

A different but equivalent definition of linear algebraic group is as follows:

Definition 2.1.2. *A linear algebraic group over F is a functor G from the category of F-algebras to the category of groups such that there exists a reduced finitely generated F-algebra A with $F \simeq Hom(A, \_)$ as functors to category of sets.*

Let G_1 and G_2 be two linear algebraic groups.

Definition 2.1.3. *A map $\phi\colon G_1 \to G_2$ is called a morphism of linear algebraic groups if it is a group homomorphism as well as a morphism of varieties.*

Example: The determinant map

$$\det\colon GL_n \to G_m; \quad A \mapsto det(A)$$

is a morphism of linear algebraic groups.

Lemma 2.1.4. *A closed subgroup of a (linear) algebraic group is again a (linear) algebraic group.*

Proposition 2.1.5. *([Hum75, Prop 1.5.]) Kernels and images of morphisms of (linear) algebraic groups are closed.*

Hence we see that kernels and images of morphisms of linear algebraic groups is again a linear algebraic group.

Short exact sequence of algebraic groups
Let G, G', G'' be linear algebraic groups defined over a field F. Let F_{sep}/F be a fixed separable closure of F.

We say that we have a *short exact sequence of linear algebraic groups*

$$1 \to G' \to G \to G'' \to 1$$

if we have the following short exact sequence of $\mathrm{Gal}(F_{sep}/F)$-modules:

$$1 \to G'(F_{sep}) \to G(F_{sep}) \to G''(F_{sep}) \to 1.$$

Definition 2.1.6. *(Torus) An algebraic group G defined over F is called a torus if $G(F_{sep}) \simeq \mathbb{G}_m^n$ for some positive integer n.*

Some examples of tori:

• $\mathbb{G}_m^n$ for any positive integer n.

• Weil restriction of $\mathbb{G}_m$ correponding to finite separable extensions as defined below (see Definition 2.1.7).

• Norm one tori correponding to finite separable extensions as defined below (see Definition 2.1.8).

Let F be any field and L/F be a finite separable extension. Let X be a variety defined over F. For any ring R, we denote by $R^\times$ the set of units in R.

Definition 2.1.7. *(Weil restriction) The Weil restriction of X, denoted by $Res_{L/F}X$ is a variety such that for any F-algebra A, the A-points of $Res_{L/F}X$ are given by*

$$[Res_{L/F}X](A) = X(A \otimes_F L).$$

Actually, $Res_{L/F}$ is a functor from the category of varieties over F to the category of varieties over L sending a variety X over F to the variety $Res_{L/F}X$. This functor is right adjoint to the base-change functor $\_ \times_F L$

from the category of varieties over L to the category of varieties over F. Also, for a linear algebraic group G, $R_{L/F}G$ is again a linear algebraic group.

Next we define the norm one torus associated to the extension L/F. The norm map $N_{L/F} \colon L^\times \to F^\times$ extends to a morphism from $R_{L/F}\mathbb{G}_m$ to $\mathbb{G}_m$, which we also denote by $N_{L/F}$.

Definition 2.1.8. *(Norm one torus) We define the* ***norm one torus*** *for the extension L/F, denoted by $T_{L/F}$, to be the kernel of the map $R_{L/F}\mathbb{G}_m \xrightarrow{N_{L/F}} \mathbb{G}_m$, i.e.,*

$$T_{L/F} = ker\left(R_{L/F}\mathbb{G}_m \xrightarrow{N_{L/F}} \mathbb{G}_m\right).$$

Thus, by definition, we have the following short exact sequence of linear algebraic groups:

$$1 \to T_{L/F} \to R_{L/F}\mathbb{G}_m \to \mathbb{G}_m \to 1.$$

For any F-algebra A, the A-points of $T_{L/F}$ are given by

$$T_{L/F}(A) = \ker\left((A \otimes_F L)^\times \xrightarrow{N_{L/F} \otimes_F A} A^\times\right),$$

where $N_{L/F} \otimes_F A$ is induced from the norm map $N_{L/F}$.

In particular, the F-points of $T_{L/F}$ are given by

$$T_{L/F}(F) = \{a \in L^\times \mid N_{L/F}(a) = 1\}.$$

Next, we define multinorm tori over a field F.

Let F be a field and $L_1, L_2, \ldots, L_m$ be finite Galois extensions of F. Let $L = \prod_{i=1}^{m} L_i$.

We denote by $T_{L/F}$ the multinorm torus corresponding to the extensions L_i/F, $1 \le i \le m$, and define it as

$$T_{L/F} = \ker\left(\prod_{i=1}^{m} R_{L_i/F}\mathbb{G}_m \xrightarrow{\prod_{i=1}^{m} N_{L_i/F}} \mathbb{G}_m\right),$$

where $N_{L_i/F}$ are the maps induced from the usual norm maps from L_i to F.

2.2 Galois cohomology and torsors

We refer the readers to [Ser97] for detailed expositions on these topics.

We start with the definition of inverse systems of groups and inverse limits.

Definition 2.2.1. *A (filtered) inverse system of groups* $(\Gamma_i, \phi_{ij})_I$ *consists of the following data:*

- *a partially ordered set* $(I, \le)$ *which is directed, i.e., for all* $i, j \in I$, *there exists a* $k \in I$ *with* $i \le k$ *and* $j \le k$;

- *for each* $i \in I$, *there is a group* Γ_i;

- *for each* $i \le j$, *there is a homomorphism* $\phi_{ij} \colon \Gamma_j \to \Gamma_i$ *such that we have* $\phi_{ik} = \phi_{ij} \circ \phi_{jk}$ *for* $i \le j \le k$.

For an inverse system, we define the inverse limit as follows:

Definition 2.2.2. *For an inverse system* $(\Gamma_i, \phi_{ij})_I$, *we define the inverse system, denoted by* $\varprojlim \Gamma_i$, *to be the subgroup of the direct product* $\prod_i \Gamma_i$ *consisting of sequences* (g_i) *such that* $\phi_{ij}(g_j) = g_i$ *whenever* $i \le j$.

Definition 2.2.3. *A profinite group is a topological group which is the inverse limit of finite groups, where the finite groups are considered with the discrete topology.*

Example: (Galois groups)

Let L/F be a finite or infinite Galois extension of fields. Then the Galois group $\mathrm{Gal}(L/F)$ is the inverse limit of the finite Galois groups $\mathrm{Gal}(K/F)$, where K varies over all the finite Glaois extensions K/F which are contained in L/F. Hence $\mathrm{Gal}(L/F)$ is a profinite group.

2.2.1 Profinite Cohomology

Let Γ be a profinite group.

Definition 2.2.4. *We say a group G (not necessarily abelian) is a Γ-group if*:

• Γ *acts continuously on G, which is equivalent to saying that $G = \cup G^U$ (the subgroup of G consisiting of elements fixed under Γ), where U runs over all the open subgroups of Γ; and*

• *the action of Γ is compatible with the group structure of G, i.e., ${}^\gamma(gh) = {}^\gamma a {}^\gamma b$ for all $\gamma \in \Gamma$ and $g, h \in G$.*

Definition of H^0 and H^1

For a profinite group Γ and Γ-group G, one defines one defines $H^0(\Gamma, G) = G^\Gamma$, the set of elements of G fixed under Γ. This is a subgroup of Γ.

A 1*-cocycle of Γ in G is defined as a continuous map from Γ to G sending γ to g_γ such that $g_{\gamma\gamma'} = g_\gamma {}^\gamma g_{\gamma'}$ for all $\gamma, \gamma' \in \Gamma$. Let 1_G be the identity element in G. The set of* 1*-cocycles is denoted by $Z^1(\Gamma, G)$. The map from Γ to G given by sending every γ in Γ to 1_A is a* 1*-cocycle, called the trivial cocycle.*

We say two cocycles g and g' are cohomologous if there exists $h \in G$ such that $g'_\gamma = h^{-1} g_\gamma {}^\gamma h$. Being cohomologous is an equivalence relation in $Z^1(\Gamma, G)$. We define $H^1(\Gamma, G)$ to be the quotient of $Z^1(\Gamma, G)$ under this equivalence relation and call it the first cohomology set of Γ in G. The set $H^1(\Gamma, G)$ is a pointed set, i.e., a set equipped with a distinguished element coming from the trivial cocycle.

The cohomology sets $H^0(\Gamma, G)$ and $H^1(\Gamma, G)$ are functorial in G.

2.2.2 The Long exact sequence associated to a short exact sequence of Γ-groups

If $f\colon H \to G$ is a morphism of Γ-groups then it induces a canonical map from $H^i(\Gamma, H)$ to $H^i(\Gamma, G)$ for $i = 0, 1$.

For a short exact sequence of Γ-groups, we get a long exact sequence:

Proposition 2.2.5. *[Ser97, Chapter I, Proposition 38]*
Let $1 \to H \to G \to J \to 1$ be a short exact sequence of Γ-groups. Then there exists a connecting morphism $\delta\colon J^\Gamma \to H^1(\Gamma, H)$ such that the sequence of pointed sets

$$1 \to H^\Gamma \to G^\Gamma \to J^\Gamma \xrightarrow{\delta} H^1(\Gamma, H) \to H^1(\Gamma, G) \to H^1(\Gamma, J)$$

is exact.

Galois cohomology

Let F be a field. We fix a separable closure F_{sep} of F. Let $\Gamma = Gal(F_{sep}/F)$ be the absolute Galois group of F. Then Γ is a profinite group. For a Γ-group G, the profinite cohomology is also called the Galois cohomology.

Galois cohomology of the additive and the multiplicative groups

Proposition 2.2.6 (Hilbert 90)**.** *[GS06, Lemma 4.3.7]*
The Galois cohomology group $H^1(\Gamma, F_{sep}^\times)$ is trivial.

Proposition 2.2.7. *[GS06, Lemma 4.3.11]*
The Galois cohomology group $H^1(\Gamma, F_{sep})$ is trivial.

In this thesis, we are interested in the case when G is a linear algebraic group defined over F.

Next we define torsors under linear algebraic groups.

Let F be any field and G be a linear algebraic group over F.

Definition 2.2.8. *We say a F-variety X is a G-torsor over F if there is an action $G \times X \to X$ such that for every field extension L/F, the induced action of $G(L)$ on $X(L)$ is simply transitive.*

Let X, Y be G-torsors over F.

Definition 2.2.9. *(Morphism of G-torsors) A morphism of G-torsors over F is a morphism $f\colon X \to Y$ of varieties compatible with the action of G.*

Definition 2.2.10. *(Trivial torsor) A G-torsor X over F is called a trivial G-torsor if it is isomorphic to G considered as a G-torsor under left multiplication. A G-torsor X over F is trivial if and only if $X(F) \neq \emptyset$.*

A G-torsor is trivial if and only if it has a rational point over F.

Proposition 2.2.11. *There is a canonical bijection between the $H^1(\Gamma, G)$ and the set of isomorphism classes of G-torsors over F.*

Proof. Let X be a G-torsor over F. Let us fix a $x_0 \in X(F_{sep})$. Then for any $\gamma \in \Gamma$, there exists a unique c_γ in $G(F_{sep})$ such that ${}^\gamma x_0 = x_0 \cdot c_\gamma$. We have ${}^{\gamma_1\gamma_2}x_0 = x_0 \cdot c_{\gamma_1\gamma_2}$. Also, ${}^{\gamma_1\gamma_2}x_0 = {}^{\gamma_1}({}^{\gamma_2}x_0) = {}^{\gamma_1}(x_0 \cdot c_{\gamma_2}) = {}^{\gamma_1}x_0 \cdot {}^{\gamma_1}c_{\gamma_2} = x_0 \cdot (c_{\gamma_1}{}^{\gamma_1}c_{\gamma_2})$. Hence $c_{\gamma_1\gamma_2} = c_{\gamma_1}{}^{\gamma_1}c_{\gamma_2}$. So the map $\gamma \mapsto c_\gamma$ defines a cocycle. Also, the class of cocycle c does not choice of $x_0 \in X$. Let $y_0 \in X$. Then there exists a unique $g_o \in G(F_{sep})$ with $y_o = x_o \cdot g_0$. We have: ${}^\gamma y_o = {}^\gamma(x_o \cdot g_0) = {}^\gamma x_o \cdot {}^\gamma g_o = x_0 \cdot c_\gamma \cdot {}^\gamma g_0 = y_o \cdot (g_0^{-1}c_\gamma)^\gamma g_0)$. As $(g_0^{-1}c_\gamma)^\gamma g_0)$ and c_γ represent the same class in $H^1(\Gamma, G)$, we actually get a well-defined map from the set of isomorphism classes of G-torsors over F to $H^1(\Gamma, G)$. Clearly, this map takes the equivalence class of trivial G-torsors to the identity element in $H^1(\Gamma, G)$.

Conversely, let $c \in Z^1(\Gamma, G)$ be a cocycle. We consider $G \times_F F_{sep}$ and define a Γ action by $\gamma(x) = c_\gamma{}^\gamma x$. This defines a variety over F, which is isomorphic to G over F_{sep}. Hence X_c is a G-torsor over F. If we choose some other cocycle representative for the equivalence class of c in $H^1(\Gamma, G)$, say d with $d_\gamma = g^{-1}c_\gamma{}^\gamma g$ for some $g \in G(F_{sep})$. Then X_d is isomorphic to X_c as a G-torsor via left translation by g since $g(d_\gamma{}^\gamma x) = g(g^{-1}c_\gamma{}^\gamma g^\gamma x) = c_\gamma{}^\gamma(gx)$. Hence we get a map from $H^1(\Gamma, G)$ to the set of equivalence classes of G-torsors over F.

One checks that the two maps defined above are inverses of each other. Hence we get the desired bijection.

□

Example- *(Torsors under norm one tori and multinorm tori)*

Let L/F be a Galois extension. By definition of norm one torus $T_{L/F}$, we have the short exact sequence:

$$1 \to T_{L/F} \to R_{L/F}\mathbb{G}_m \xrightarrow{N_{L/F}} \mathbb{G}_m \to 1.$$

By taking Galois cohomology, we get:

$$1 \to T_{L/F}(F) \to L^\times \xrightarrow{N_{L/F}} F^\times \to H^1(F, T_{L/F}) \to 1$$

since $H^1(F, R_{L/F}\mathbb{G}_m) = \{1\}$ (by Hilbert 90 and Shapiro's lemma).

Hence $H^1(F, T_{L/F}) \simeq F^\times/N_{L/F}(L^\times)$. Thus, the isomorphism classes of $T_{L/F}$-torsors correspond to the equivalence classes of $F^\times/N_{L/F}(L^\times)$, with the trivial torsor corresponding to the equivalence class of 1*. For any $\lambda \in F^\times$, we can define X_λ to be the variety defined by $N_{L/F}(Z) = \lambda$. The varieties X_λ are $T_{L/F}$-torsors over F. And, these are all $T_{L/F}$-torsors over F upto isomorphism.*

Similarly, let $T_{L/F}$ be the multinorm torus corresponding to the extensions L_i/F, $1 \le i \le m$. Then we have:

$$1 \to T_{L/F} \to \left(\prod_{i=1}^{m} R_{L_i/F}\mathbb{G}_m\right) \xrightarrow{\prod_{i=1}^{m} N_{L_i/F}} \mathbb{G}_m \to 1.$$

By taking Galois cohomology, we get:

$$1 \to T_{L/F}(F) \to \left(\prod_{i=1}^{m}(L_i^\times)\right) \xrightarrow{\prod_{i=1}^{m} N_{L_i/F}} F^\times \to H^1(F, T_{L/F}) \to 1$$

since $H^1(F, R_{L/F}\mathbb{G}_m) = \{1\}$ (by Hilbert 90 and Shapiro's lemma).

Hence $H^1(F, T_{L/F}) \simeq F^\times/N_{L/F}(L^\times)$. Thus, the isomorphism classes of $T_{L/F}$-torsors correspond to the equivalence classes of $F^\times/\prod_{i=1}^{m} N_{L_i/F}(L_i^\times)$, with the trivial torsor corresponding to the equivalence class of 1*. For any*

$\lambda \in F^\times$, we can define X_λ to be the variety defined by $\prod_{i=1}^{m} N_{L_i/F}(Z_i) = \lambda$. The varieties X_λ are $T_{L/F}$-torsors over F. And, these are all $T_{L/F}$-torsors over F upto isomorphism.

2.3 Local-global principles for linear algebraic groups

Let F be any field and Ω_F be the set of discrete valuations on F. Let G be a linear algebraic group defined over F.

We say that the local-global principle holds for G if for all G-torsors X, we have: X has a rational point over F if and only if X has a rational point over F_ν for all $\nu \in \Omega_F$.

We define the group Ш(F, G) as

$$Ш(F, G) := ker\left(H^1(F, G) \to \prod_{\nu \in \Omega_F} H^1(F_\nu, G)\right).$$

Since a G-torsor is trivial if and only if it has a rational point over F, the local-global principle holds for G if and only if Ш(F, G) is trivial.

When F is a number field, the local-global principle holds for various classes of linear algebraic groups ([PRR93, Chapter6]), including semisimple simply connected algebraic groups. Here we also consider the completions at archimedean places while discussing local-global principles. Sansuc ([San81]) proved that the failure of the local-global principle for connected linear algebraic groups over number fields can be explained by the Brauer-Manin obstruction. However, not much is known for general fields. In this thesis, we are exploring local-global principles for norm one tori over semi-global fields.

2.4 R-equivalence and flasque tori

We refer the reader to [CTS77] and [CTS87] for more details about R-equivalence and flasque tori.

Notation 2.4.1. *Let F be a field and L be an étale algeba over F. Throughout this thesis, we will denote the norm 1 torus $R^1_{L/F}\mathbb{G}_m$ by $T_{L/F}$.*

Let X be a variety over a field F. For a field extension L of F, let $X(L)$ be the set of L-points of X. We say that two points $x_0, x_1 \in X(L)$ are elementary R-equivalent, denoted by $x_0 \sim x_1$, if there is a rational map $f\colon \mathbb{P}^1(L) \dashrightarrow X(L)$ such that $f(0) = x_0$ and $f(1) = x_1$. The equivalence relation generated by $\sim$ is called R-equivalence. When $X = G$ is an algebraic group defined over F with the identity element e, we define $RG(L) = \{\ x \in G(L) \mid x \text{ is } R\text{-equivalent to } e\}$. The elements of $RG(L)$ are called R-trivial elements. It is well-known that $RG(L)$ is a normal subgroup of $G(L)$ (cf. [Gil10, p-1]) Sometimes, we denote $G(L)/RG(L)$ by $G(L)/R$. Let L/F be a Galois extension with Galois group G, and $T_{L/F}$ be the norm 1 torus associated to the extension L/F. Then for any extension N/F, $RT_{L/F}(N)$ is the subgroup generated by the set $\{a^{-1}\sigma(a) \mid a \in (L \otimes_F N)^\times, \sigma \in G\}$ ([CTS77, Proposition 15]).

We note a well-known fact here:

Lemma 2.4.2. *Let F be a field and L/F be a finite cyclic extension. Then $T_{L/F}(F) = RT_{L/F}(F)$.*

Proof. By [CTS77, Proposition 15], we know that $RT_{L/F}(F)$ is generated by the set $\{a^{-1}\sigma(a) \mid a \in (L)^\times, \sigma \in \mathrm{Gal}(L/F)\}$. Hence by Hilbert 90 theorem, we conclude that $T_{L/F}(F) = RT_{L/F}(F)$. □

Now we discuss few basic results about R-equivalence on norm one tori and multinorm tori.

Proposition 2.4.3. *Let F be a field and L_0/F be a finite separable extension. Let L be the product of m copies of L_0. Then the homomorphism $T_{L/F} \to T_{L_0/F}$ given by $(a_1, \ldots, a_m) \mapsto a_1 \cdots a_m$ induces an isomorphism*

$$T_{L/F}(F)/RT_{L/F}(F) \to T_{L_0/F}(F)/RT_{L_0/F}(F).$$

Proof. In fact the isomorphism $(R_{L_0/F}(\mathbb{G}_m))^m \to (R_{L_0/F}(\mathbb{G}_m))^m$ given by sending $(b_1, \ldots, b_m)$ to $(b_1, \ldots, b_{m-1}, b_1 b_2 \cdots b_m)$ induces an isomorphism of algebraic groups $T_{L/F} \to (R_{L_0/F}(\mathbb{G}_m))^{m-1} \times T_{L_0/F}$ ([BFLP19, Lemma 1.1.]). Since $R_{L_0/F}(\mathbb{G}_m)$ is rational, $R_{L_0/F}(\mathbb{G}_m)/R = \{1\}$ ([Gil10, Corollary 1.6.]). Hence the homomorphism $T_{L/F} \to T_{L_0/F}$ given by $(a_1, \ldots, a_m) \mapsto a_1 \cdots a_m$ induces an isomorphism $T_{L/F}(F)/RT_{L/F}(F) \to T_{L_0/F}(F)/RT_{L_0/F}(F)$. □

Corollary 2.4.4. *Let F be a field and L_0/F be a finite separable extension of degree d and L be the product of m copies of L_0. Suppose that F contains ρ, a primitive $(dm)^{\text{th}}$ root of unity. If*

$$T_{L_0/F}(F) = RT_{L_0/F}(F)\langle\rho^m\rangle,$$

then

$$T_{L/F}(F) = RT_{L/F}(F)\langle\rho\rangle.$$

Proof. Since $(\rho, \rho, \dots, \rho) \in T_{L/F}$ maps to ρ^m under the isomorphism given in Proposition 2.4.3, the corollary follows from Proposition 2.4.3). □

The following proposition is a generalisation of Proposition 2.4.3.

Proposition 2.4.5. *Let F be a field and let $L_1, L_2, \dots, L_m$ be finite separable extensions of F where m is a natural number. Let $L = \prod_{i=1}^{m} L_i$. Let r_i be positive integers for i, $1 \le i \le m$. Let $L' = \prod_{i=1}^{m}\left(\prod^{r_i} L_i\right)$. Then $T_{L/F}(F)/R \simeq T_{L'/F}(F)/R$.*

Proof. By induction, it is enough to consider the case when $r_1 = 2$ and $r_i = 1$ for $2 \le i \le m$. Let us consider the map

$$f\colon R_{L_1/F}\mathbb{G}_m \times \prod_{i=1}^{m} R_{L_i/F}\mathbb{G}_m \to R_{L_1/F}\mathbb{G}_m \times \prod_{i=1}^{m} R_{L_i/F}\mathbb{G}_m$$

which sends $(a_0, a_1, \dots, a_m)$ to $(a_o, a_0a_1, a_2, \dots, a_m)$. Then f is an isomorphism of algebraic groups. This induces an isomorphism from $T_{L'/F} \to R_{L_1/F}\mathbb{G}_m \times T_{L/F}$. Since $R_{L_1/F}(\mathbb{G}_m)$ is rational, $R_{L_1/F}(\mathbb{G}_m)/R = \{1\}$ ([Gil10, Corollary 1.6.]). Hence by ([Gil10, p-1]), the isomorphism above induces an isomorphism $T_{L'/F}(F)/R \to T_{L/F}(F)/R$.

□

Lemma 2.4.6. *Let F be a field and $L_i, 1 \le i \le m$, be finite separable extensions of F and let $L = F \times \prod_{i=1}^{m} L_i$. Then $T_{L/F}(F) = RT_{L/F}(F)$.*

Proof. Let us consider the map $f\colon \prod_{i=1}^{m} R_{L_i/F}\mathbb{G}_m \to T_{L/F}$ which sends $(a_1, \dots, a_m)$ to $(\prod_{i=1}^{m} N_{L_i/F}(a_i)^{-1}, a_1, \dots, a_m)$. Then f is an isomorphism of algebraic groups. Since $R_{L_i/F}(\mathbb{G}_m)$ are rational, $R_{L_i/F}(\mathbb{G}_m)(F)/R = \{1\}$ ([Gil10, Corollary 1.6.]) and consequently, by ([Gil10, p-1]),

$$\left(\prod_{i=1}^{m} R_{L_i/F}\mathbb{G}_m(F)\right) \Big/ R = \{1\} = T_{L/F}(F)/R.$$

□

Lemma 2.4.7. *Let L/F be a finite Galois extension of degree n and N/F be any field extension. If $\alpha \in (L \otimes_F N)^\times$, then $N_{L\otimes_F N/N}(\alpha)^{-1}\alpha^n \in RT_{L/F}(N)$.*

Proof. Let G be the Galois group of L/F. Since $N_{L\otimes_F N/N}(\alpha) = \prod_{\sigma\in G} \sigma(\alpha)$, we have

$$N_{L\otimes_F N/N}(\alpha)^{-1}\alpha^n = [\prod_{\sigma\in G} \sigma(\alpha)]^{-1}\alpha^n = \prod_{\sigma\in G}\left([\sigma(\alpha)]^{-1}\alpha\right).$$

Hence $N_{L\otimes_F N/N}(\alpha)^{-1}\alpha^n \in RT_{L/F}(N)$. □

Next, we define character group of a torus, quasitrivial torus, flasque torus and flasque resolution of a torus.

Let F be a field and F_{sep} be a separable closure of F. For a torus T over F, we define its character group $\hat{T}$ as the group of homomorphisms of F_{sep}-algebraic groups from $T \times_F F_{sep}$ to $\mathbb{G}_{m,F_{sep}}$. The group $\hat{T}$ has a natural action of the Galois group Gal(F_{sep}/F) which makes it into a Gal(F_{sep}/F)-lattice. In other words, $\hat{T}$ is free and finitely generated abelian group with a continuous, discrete action of Gal(F_{sep}/F). This lattice determines the F-torus T and there is anti-equivalence between the category of F-tori and the category of Gal(F_{sep}/F)-lattices.

Definition 2.4.8. *A quasitrivial F-torus E is a F-torus which is F-isomorphic to a finite product $\prod_{i=1}^{m} R_{L_i/F}\mathbb{G}_m$, where L_i/F are finite separable extensions and $R_{L_i/F}$ denote Weil restrictions.*

The group $\prod_{i=1}^{m} R_{L_i/F}\mathbb{G}_m$ is an open subset of $\prod_{i=1}^{m} R_{L_i/F}\mathbb{G}_a$, which is F-isomorphic to $\mathbb{A}^n_F$ for some n. Thus quasitrivial tori are rational linear algebraic groups.

Definition 2.4.9. *A flasque F-torus S is a F-torus for which*

$$H^1(H, Hom_{\mathbb{Z}}(\hat{S}, \mathbb{Z})) = 0$$

for all closed subgroups H of $Gal(F_{sep}/F)$.

Definition 2.4.10. *A flasque resolution of a F-torus T is a short exact sequence of F-tori*

$$1 \to S \xrightarrow{f} E \xrightarrow{g} T \to 1$$

with E quasitrivial and S flasque.

The important fact here, due to Colliot-Thélène and Sansuc ([CTS77]), is the following:

For any F-torus T, there always exists a flasque resolution

$$1 \to S \xrightarrow{f} E \xrightarrow{g} T \to 1$$

and it is unique upto taking a direct product of S and E with the same quasitrivial torus, respectively.

Lemma 2.4.11. *Let T be a torus defined over a semi-global field F. Then for any $\alpha_{P,U} \in RT(F_{P,U})$, there exists $\alpha_P \in RT(F_P)$ and $\alpha_U \in RT(F_U)$ such that $\alpha_{P,U} = \alpha_P \alpha_U$ in $RT(F_{P,U})$.*

Proof. We consider a flasque resolution of T over F given by

$$1 \to S \xrightarrow{f} E \xrightarrow{g} T \to 1,$$

where E is a quasitrivial torus and S is a flasque torus. Considering the above sequence over $F_{P,U}$ and by [CTS77, Théorème 2, p-199], $g(E(F_{P,U})) = RT(F_{P,U})$. Let $\beta_{P,U} \in E(F_{P,U})$ with $g(\beta_{P,U}) = \alpha_{P,U}$. Then since E is quasitrivial, it is also rational. Hence by [HHK09, Theorem 3.7.], there exists $\beta_P \in E(F_P)$ and $\beta_U \in E(F_U)$ with $\beta_{P,U} = \beta_P \beta_U$ in $E(F_{P,U})$. Thus, applying g, we get that $\alpha_{P,U} = g(\beta_P) g(\beta_U)$, where $g(\beta_P) \in RT(F_P) \subseteq RT(F_{P,U})$ and $g(\beta_U) \in RT(F_U) \subseteq RT(F_{P,U})$. Hence we are done.

□

Chapter 3

Semi-global fields and Patching

Let F be a field and Ω_F be the set of all equivalence classes of discrete valuations ν on F. For $\nu \in \Omega_F$, let F_ν denote the completion of F at ν and $\kappa(\nu)$ the residue field at ν. For an algebraic group G over F, let

$$Ш(F, G) := ker\left(H^1(F, G) \to \prod_{\nu \in \Omega_F} H^1(F_\nu, G)\right).$$

In this thesis, we are concerned with a special class of fields called semi-global fields.

Definition 3.0.1. *(Semi-global field) A semi-global field is the function field of a smooth, projective, geometrically integral curve over a complete discretely valued field.*

Example: *Some examples of semi-global fields are: $\mathbb{C}((t))(x)$, $\mathbb{Q}_p(x)$, $\mathbb{F}_p((t))(x)$ and $\dfrac{\mathbb{C}((t))(x)[y]}{\langle xy(x+y-1)-t\rangle}$.*

Let T be a complete discretely valued ring with fraction field K and residue field κ. Let $t \in T$ be a parameter. Let F be a function field of a smooth, projective, geometrically integral curve over K. Thus F is a semi-global field. Then there exists a regular 2-dimensional integral scheme $\mathscr{X}$ which is proper over T with function field F. We call such a scheme $\mathscr{X}$ a regular proper model of F. Further there exists a regular proper model of F with the reduced special fibre a union of regular curves with only normal crossings. Let $\mathscr{X}$ be a regular proper model of F with the reduced special fibre X a union of regular curves with only normal crossings.

3.1 Overfields of a semi-global field

For a semi-global field F and a regular proper model $\mathscr{X}$ of F, one can associate three different kinds of overfields of F. We describe them below and discuss how they are related to each other.

For any point x of $\mathscr{X}$, let R_x be the local ring at x on $\mathscr{X}$, $\widehat{R}_x$ the completion of the local ring R_x, F_x the fraction field of $\widehat{R}_x$ and $\kappa(x)$ the residue field at x.

For any subset U of X that is contained in an irreducible component of X, let R_U be the subring of F consisting of the rational functions which are regular at every point of U. Let $\widehat{R_U}$ be the t-adic completion of R_U and F_U the fraction field of $\widehat{R_U}$.

Let $\eta \in X$ be a codimension zero point and $P \in X$ a closed point such that P is in the closure X_η of η. Such a pair (P, η) is called a branch. For a branch (P, η), we define $F_{P,\eta}$ to be the completion of F_P at the discrete valuation of F_P associated to η. We call such fields branch fields. If η is a codimension zero point of X, $U \subset X_\eta$ an open subset and $P \in X_\eta$ a closed point, then we will use (P, U) to denote the branch (P, η) and $F_{P,U}$ to denote the field $F_{P,\eta}$.

With P, U, η as above, there are natural inclusions of F_P, F_U and F_η into $F_{P,\eta} = F_{P,U}$. Also, there is a natural inclusion of F_U into F_η.

Let $\mathcal{P}$ be a nonempty finite set of closed points of X that contains all the closed points of X, where distinct irreducible components of X meet. Let $\mathcal{U}$ be the set of connected components of the complement of $\mathcal{P}$ in X and let $\mathcal{B}$ be the set of branches (P, U) with $P \in \mathcal{P}$ and $U \in \mathcal{U}$ with P in the closure of U.

3.2 Tate-Shafarevich groups

Let G be a linear algebraic group over F. Let us define

$$Ш_{\mathscr{X},\mathcal{P}}(F, G) := ker\left(H^1(F, G) \to \prod_{\xi \in \mathcal{P} \cup \mathcal{U}} H^1(F_\xi, G)\right).$$

If $\mathscr{X}$ is understood, we will just use the notation $Ш_{\mathcal{P}}(F, G)$.

Similarly, let us define

$$Ш_{\mathscr{X},X}(F,G) := ker\left(H^1(F,G) \to \prod_{P\in X} H^1(F_P,G)\right).$$

Again, if $\mathscr{X}$ is understood, we will just use the notation $Ш_X(F,G)$.

We have a bijection([HHK15a, Corollary 3.6.]) :

$$\prod_{U\in\mathcal{U}} G(F_U)\Big\backslash \prod_{(P,U)\in\mathcal{B}} G(F_{P,U})\Big/ \prod_{P\in\mathcal{P}} G(F_P) \to Ш_{\mathcal{P}}(F,G)$$

This is a very useful result since the double coset mentioned above is usually more manageable to work with.

By [HHK15a, Corollary 5.9.], we have $Ш_X(F,G) = \bigcup Ш_{\mathcal{P}}(F,G)$, where union ranges over all finite subsets $\mathcal{P}$ of closed points of $\mathscr{X}$ which contain all the singular points of X. We also have $Ш_X(F,G) \subseteq Ш(F,G)$ ([HHK15a, Proposition 8.2.]).

3.3 The associated graph

We start with basic facts about finite bipartite trees.

Lemma 3.3.1. *Let Γ be a finite bipartite graph and G be an abstract group. Let V be the set of vertices with parts V_1 and V_2. For each edge θ of Γ, let $g_\theta \in G$. If Γ is a tree, then for every $v \in V$, there exists $g_v \in G$ such that if e is an edge joining two vertices $v_i \in V_i$, then $g_e = g_{v_1}g_{v_2}$.*

Proof. Suppose that Γ is a tree. Without loss of generality, we may assume that Γ is a connected graph. We prove the lemma by the induction on number of vertices. Suppose that Γ has one one vertex. Then there is nothing to prove.

Suppose that Γ has more than one vertex. Since Γ is a connected tree, there exists a vertex $v_0 \in V$ with exactly one edge θ at v_0. Without loss of generality, we may assume $v_0 \in V_1$. Let Γ' be the graph obtained from Γ by removing the vertex v_0 and the edge θ. Then Γ' is again a finite bipartite graph which is a tree. Then by induction hypothesis, for every vertex v of Γ', there exists $g_v \in G$ such that if e is an edge in Γ' joining $v_1 \in V_1 \setminus \{v_0\}$ and $v_2 \in V_2$, then $g_e = g_{v_1}g_{v_2}$. Let $v_0' \in V_2$ be the other vertex of the edge θ. Let $g_{v_0} = g_\theta g_{v_0'}^{-1}$. Then it follows that g_v have the required property. □

We can generalize the above lemma as follows:

Lemma 3.3.2. *Let Γ be a finite bipartite tree. Let V be the set of vertices with parts V_1 and V_2 and E be the set of edges. Suppose that for every edge $e \in E$, we have an abstract group G_e and for every vertex $v \in V$, we have an abstract group G_v such that for all $v \in V$ and edges e with v as one of the vertex, we have a surjective group homomorphism $f_{v,e}\colon G_v \to G_e$. Then for every tuple $(g_e)_{e\in E} \in \prod_{e\in E} G_e$, there exists a tuple $(g_v)_{v\in V} \in \prod_{v\in V} G_v$ such that if e is the edge joining the vertices v_1 and v_2,*

$$g_e = f_{v_1,e}(g_{v_1}) \cdot f_{v_2,e}(g_{v_2}).$$

Proof. Without loss of generality, we may assume that Γ is a connected tree. We prove the lemma by induction on the number of vertices of Γ. Suppose Γ has only one vertex. Then there is nothing to prove.

Suppose Γ has n vertices, where $n > 1$. Let $(g_e)_{e\in E} \in \prod_{e\in E} G_e$. Since Γ is a finite connected tree, there exists a vertex v_0 with exactly one edge θ at v_0. Without loss of generality, we may assume that $v_0 \in V_1$. Let Γ' denote the graph obtained by deleting the vertex v_0 and the edge θ. Then Γ' is again a finite bipartite tree with $n-1$ vertices. Thus, by induction hypothesis, there exists a tuple $(g_v)_{v\in V\setminus\{v_0\}} \in \prod_{v\in V\setminus\{v_0\}} G_v$ such that whenever $e \in E \setminus \{\theta\}$ is an edge between some vertices v_1 and v_2 in $V \setminus \{v_0\}$, we have $g_e = f_{v_1,e}(g_{v_1}) \cdot f_{v_2,e}(g_{v_2})$.

Let $v_0' \in V_2$ be the other vertex of the edge θ. Choosing g_{v_0} to be an element in $f_{v_0,\theta}^{-1}(g_\theta \cdot f_{v_0',\theta}(g_{v_0'}))$, we see that $(g_v)_{v\in V}$ satisfies the required property. Hence we are done.

□

Let T be a complete discretely valued ring with fraction field K, and residue field κ. Let $t \in T$ be a parameter. Let F be the function field of a smooth, projective, geometrically integral curve over K and $\mathscr{X}$ be a regular proper model of F with the reduced special fibre X a union of regular curves with only normal crossings. Let $\mathcal{P}$ be a nonempty finite *set of closed points of X that contains all the closed points of X, where distinct irreducible components of X meet. Let $\mathcal{U}$ be the set of connected components of the complement of $\mathcal{P}$ in X and let $\mathcal{B}$ be the set of branches (P, U) with $P \in \mathcal{P}$ and $U \in \mathcal{U}$ with P in the closure of U.*

We have a graph $\Gamma(\mathscr{X}, \mathcal{P})$ associated to $\mathscr{X}$ and $\mathcal{P}$ whose vertices are elements of $\mathcal{P} \cup \mathcal{U}$ and edges are elements of $\mathcal{B}$. Since there are no edges between any vertices which are in $\mathcal{P}$ (resp. $\mathcal{U}$), $\Gamma(\mathscr{X}, \mathcal{P})$ is a finite bipartite graph with parts $\mathcal{P}$ and $\mathcal{U}$. If $\mathcal{P}'$ is another finite set of closed points of X containing all the closed points of X where distinct irreducible components of X meet, then $\Gamma(\mathscr{X}, \mathcal{P})$ is a tree is and only if $\Gamma(\mathscr{X}, \mathcal{P}')$ is a tree ([HHK15a, Remark 6.1(b)]). Hence if $\Gamma(\mathscr{X}, \mathcal{P})$ is a tree for some $\mathcal{P}$ as above, then we say that the graph $\Gamma(\mathscr{X})$ associated to $\mathscr{X}$ is a tree.

Now we have the following result as a corollary to Lemma 3.3.1 and Lemma 3.3.2.

Corollary 3.3.3. *Let F, $\mathscr{X}$, X, $\mathcal{P}$, $\mathcal{U}$ and $\mathcal{B}$ be as above. Let G be an abstract group and for each branch $b \in \mathcal{B}$, let $g_b \in G$. Suppose that the graph $\Gamma(\mathscr{X})$ associated to $\mathscr{X}$ is a tree. Then for every $\zeta \in \mathcal{U} \cup \mathcal{P}$, there exists $g_\zeta \in G$ such that if $b = (P, U) \in \mathcal{B}$, then $g_b = g_P g_U$.*

Proof. The proof immediately follows from Lemma 3.3.1.

□

Corollary 3.3.4. *Let K be a complete discretely valued field with residue field κ and F be the function field of a smooth, projective, geometrically integral curve over K. Let $\mathscr{X}$ be a regular proper model of F and X be the reduced special fibre. Let T be a torus defined over F. Let $\mathcal{P}$ be a finite set of closed points of X containing all the nodal points .*
Assume that:

- *the graph associated to F is a tree,*

- *the natural map $T(F_U) \to T(F_{P,U})/R$ is surjective for all possible U and branches (P, U), where U is one of the components of $X \setminus \mathcal{P}$*

- *the natural map $T(F_P) \to T(F_{P,U})/R$ is surjective for all possible P and branches (P, U).*

Then $Ш_{\mathcal{P}}(F, T) = 0$.

Proof. We have a finite bipartite tree with $V_1 = \mathcal{P}$, $V_2 = \mathcal{U}$ and $E = \mathcal{B}$, where $(P, U) \in \mathcal{B}$ is the edge joining $P \in \mathcal{P}$ and $U \in V_2$. We get the result by considering $G_{P,U} = T(F_{P,U})/R$, $G_P = T(F_P)$ and $G_U = T(F_U)$ and applying Lemma 3.3.2.

□

Chapter 4

Local-global principle for norm one tori over semi-global fields

4.1 Norm one elements - complete discretely valued fields

Let F be a complete discretely valued field with residue field κ. Let L/F be a Galois extension. Let l denote the residue field of L. In this section, we investigate the relationship between the groups $T_{L/F}(F)/RT_{L/F}(F)$ and $T_{l/\kappa}(\kappa)/RT_{l/\kappa}(\kappa)$. This allows us to transfer our assumptions from the residue fields to the branch fields (see Section 3.1), which is crucial for proving local-global principles for norm one tori for semi-global fields (see Lemma 4.4.1).

Lemma 4.1.1. *Let F be a complete discretely valued field with residue field κ and L/F be a finite Galois extension of degree n with residue field l. Suppose that n is coprime to char(κ). Let $z \in T_{L/F}(F)$. If the image of z in l is 1, then $z \in RT_{L/F}(F)$.*

Proof. Let S be the integral closure of R in L. Then S is a complete discrete valuation ring with residue field l. Let $z \in T_{L/F}(F)$ with the image of z in l is 1. Since n is coprime to char(κ), by Hensel's lemma, there is a $w \in S$ with $\overline{w} = 1$ and $z = w^n$. Since $N_{L/F}(z) = 1$, $N_{L/F}(w)^n = 1$ and hence $\rho = N_{L/F}(w)$ is an n^{th} root of unity. Since $\overline{w} = 1$, $\overline{N_{L/F}(w)} = N_{l/\kappa}(\overline{w})^e = 1$, where e is the ramification index of the extension L/F. Hence $\overline{\rho} = 1$. Since n is coprime to char(κ), by Hensel's lemma, the quotient map $S \to l$ induces a bijection from the set of n^{th} roots of unity in S to the set of n^{th} roots of unity in l. Hence $\rho = 1$ and $w \in T_{L/F}(F)$. Since $z = w^n$, $z \in RT_{L/F}(F)$ by Lemma 2.4.7. □

Lemma 4.1.2. *Let F be a complete discretely valued field with residue field κ. Let L/F be a Galois extension of degree n. Suppose that $(n, char(\kappa)) = 1$. Suppose that F contains a primitive n^{th} root of unity ρ_n. Let l be the residue field of L and $f = [l : \kappa]$. If*

$$T_{l/\kappa}(\kappa) = RT_{l/\kappa}(\kappa)\langle \rho_n^{n/f} \rangle,$$

then

$$T_{L/F}(F) = RT_{L/F}(F)\langle \rho_n \rangle.$$

Proof. Let R be the discrete valuation ring of F and S be the integral closure of R in L. Let e be the ramification index of the extension L/F. Then $n = ef$. For any element $y \in S$(resp. R), we will use $\overline{y}$ to denote its image in the residue field l(resp. κ).

Let $x \in L$ with $N_{L/F}(x) = 1$. Then $N_{l/\kappa}(\overline{x})^e = \overline{N_{L/F}(x)} = 1$. Hence $N_{l/\kappa}(\overline{x}) = \rho_n^{fi}$ for some i with $0 \le i < e$. Let $y = \rho_n^{-i}x$. Then $N_{L/F}(y) = 1$ and $N_{l/\kappa}(\overline{y}) = N_{l/\kappa}(\rho_n^{-i})N_{l/\kappa}(\overline{x}) = \rho_n^{-fi}\rho_n^{fi} = 1$. Thus $\overline{y} \in T_{l/\kappa}(\kappa)$ and hence, by the assumption, $\overline{y} = \theta\rho_n^{ej}$ for some $\theta \in RT_{l/\kappa}(\kappa)$ and j an integer. We write

$$\theta = \prod_{\sigma \in \mathrm{Gal}(l/\kappa)} (a_\sigma)^{-1}\sigma(a_\sigma)$$

for some $a_\sigma \in l^\times$. Since $\mathrm{Gal}(l/\kappa)$ is a quotient of $Gal(L/F)$, for every $\sigma \in \mathrm{Gal}(l/\kappa)$ we choose a lift $\tilde{\sigma} \in \mathrm{Gal}(L/F)$ of σ. Let $b_\sigma \in S$ with $\overline{b}_\sigma = a_\sigma$ and

$$z = y^{-1}\rho_n^{ej} \prod_{\sigma \in \mathrm{Gal}(l/\kappa)} (b_\sigma)^{-1}\tilde{\sigma}(b_\sigma).$$

Then $z \in T_{L/F}(F)$ and $\overline{z} = 1$. Thus, by Lemma 4.1.1, $z \in RT_{L/F}(F)$. Therefore $y \in RT_{L/F}(F)\langle \rho_n \rangle$ and hence $x \in RT_{L/F}(F)\langle \rho_n \rangle$. □

Definition 4.1.3. *A complete discretely valued field K with finite residue field is called a 1-local field. For $m \ge 1$, a complete discretely valued field K with m-local residue field k is called a $(m+1)$-local field. If K is a 1-local field, the residue field of K is called the first residue field of K. If K is a $(m+1)$-local field with residue field k, then the first residue field of k is called the first residue field of K.*

Corollary 4.1.4. *Let K be an m-local field with first residue field κ or an iterated Laurent series in m variables over an algebraically closed field κ. Let L/K be a finite Galois extension of degree n. If n is coprime to char(κ) and K contains a primitive n^{th} root of unity ρ_n, then*

$$T_{L/K}(K) = RT_{L/K}(K)\langle \rho_n \rangle.$$

Proof. Every finite extension l/κ is cyclic and by Hilbert 90, $T_{l/\kappa}(\kappa) = RT_{l/\kappa}(\kappa)$. Thus, by Lemma 4.1.2, $T_{L/K}(F) = RT_{L/K}(K)\langle \rho_n \rangle$. The corollary follows by induction on m and by Lemma 4.1.2. □

4.2 Two dimensional complete fields

Let F be a field with a discrete valuation v. Let $\kappa(v)$ be the residue field of v. Let L/F be a finite separable extension and w be a discrete valuation on L extending v. Let $e(w/v)$ be the ramification index of w over v. For any field E, $a \in E^\times$ and $n \geq 1$, let $E(\sqrt[n]{a})$ denote the field generated by E and $\sqrt[n]{a}$ in a fixed algebraic closure of E.

Lemma 4.2.1. *Let F be a field with a discrete valuation v, $\pi \in F^\times$ with $v(\pi) = 1$. Let L/F be a finite separable extension of degree coprime to char$(\kappa(v))$ and w be a discrete valuation of L extending v. Let ℓ be a prime not equal to char$(\kappa(v))$. Then there is a unique discrete valuation $\tilde{v}$ on $F(\sqrt[\ell]{\pi})$ extending v. Let $\tilde{w}$ on $L(\sqrt[\ell]{\pi})$ be a discrete valuation extending w. If ℓ divides $e(w/v)$, then $e(\tilde{w}/\tilde{v}) = e(w/v)/\ell$.*

Proof. Since $v(\pi) = 1$, v is totally ramified in $F(\sqrt[\ell]{\pi})$ and hence there is a unique extension $\tilde{v}$ of v to $F(\sqrt[\ell]{\pi})$.

For the ramification index calculations, we can replace F by F_v, the completion of F with respect to the valuation v and hence may assume that F is complete. Let L^{nr} be the maximal unramified subextension of L/F. Since the ramification index of L/L^{nr} is same as the ramification index of L/F, replacing F by L^{nr}, we may assume that L/F is totally ramified. Since $n = e = [L\colon F]$ is coprime to char$(\kappa(v))$, we have $L = F(\sqrt[n]{u\pi})$ for some $u \in F$ with $v(u) = 0$ ([CF67, Proposition 1, p-32]).

By hypothesis, we have that ℓ divides n. Suppose that $u \in F^{\times \ell}$. Then $F(\sqrt[\ell]{\pi}) \subseteq L$ and hence $L/F(\sqrt[\ell]{\pi})$ is a totally ramified extension of degree n/ℓ. Now suppose that $u \notin F^{\times\ell}$. Then $L(\sqrt[\ell]{\pi}) = F(\sqrt[\ell]{\pi})(\sqrt[\ell]{u})(\sqrt[n/\ell]{\sqrt[\ell]{u\pi}})$. Hence the ramification index of the extension $L(\sqrt[\ell]{\pi})/F(\sqrt[\ell]{\pi})$ is n/ℓ. □

Notation 4.2.2. *Let A be a complete regular local ring of dimension 2 with residue field κ and fraction field F. Let $\mathfrak{m} = (\pi_1, \pi_2) \subset A$ be the maximal ideal of A. For $i = 1, 2$, we denote by $\widehat{A}_{(\pi_i)}$ be the completion of the local ring $A_{(\pi_i)}$ with respect to the ideal (π_i) and by F_{π_i} the fraction field of $\widehat{A}_{(\pi_i)}$.*

We are studying these fields since the fields F_P appearing in the patching setup (see Section 3.1) are fraction fields of complete regular local rings of dimension 2 and the branch fields are obtained as completions of the fields F_P as discussed above.

4.2.1 Structure of extensions of two dimensional complete fields

Let F be as in Notation 4.2.2. In this subsection, we study finite Galois extensions of F.

In the following lemma, we prove that under some assumptions, a field extension L over F remains a field after base change to the completion F_{π_i} for $i = 1, 2$:

Lemma 4.2.3. *Let A be a complete regular local ring of dimension 2 with residue field κ and fraction field F. Let L/F be a separable field extension of degree n, where n is coprime to $char(\kappa)$. Let $\mathfrak{m} = (\pi_1, \pi_2) \subset A$ be the maximal ideal of A. Suppose that L/F is unramified on A except possibly at π_1 and $\pi_2 \in A$. Then $L \otimes_F F_{\pi_i}$ is a field for $i = 1, 2$.*

Proof. Let v_i be the discrete valuation of F given by π_i, for $i = 1, 2$. To show that $L \otimes_F F_{\pi_i}$ is a field, it is enough to show that there is a unique extension of v_i to a discrete valuation on L.

Let w_i^j be the extensions of the valuations v_i to L. Let m be the maximum of $e(w_i^{j_i}/v_i)$ for $i = 1, 2$, where $1 \leq j_i \leq n_i$ for some positive integers n_1 and n_2. Since each $e(w_i^{j_i}/v_i) \geq 1$, $m \geq 1$. We prove the result by induction on m.

Suppose that $m = 1$. Then $e(w_i^{j_i}/v_i) = 1$ for for $i = 1, 2$ and $1 \leq j_i \leq n_i$. Hence L/F is unramified at π_i for $i = 1, 2$. Since L/F is unramified on A except possibly at π_1, π_2, L/F is unramified on A. Let $\tilde{A}$ be the integral closure of A in L. Then $\tilde{A}$ is again a complete regular local ring of dimension 2 with (π_1, π_2) as maximal ideal and fraction field L. Thus

π_i remains a prime over $\tilde{A}$. Hence there is a unique extension of v_i to a discrete valuation of L. Hence $L \otimes_F F_{\pi_i} \cong L_{\pi_i}$ is a field.

Now suppose that $m > 1$. Let ℓ be a prime which divides m. Let $E = F(\sqrt[\ell]{\pi_1}, \sqrt[\ell]{\pi_2})$ and $M = L(\sqrt[\ell]{\pi_1}, \sqrt[\ell]{\pi_2})$. Let B be the integral closure of A in E. Then by [PS14, Corollary 3.3.], B is a regular local ring with maximal ideal (π_1', π_2'), where $\pi_1' = \sqrt[\ell]{\pi_1}$ and $\pi_2' = \sqrt[\ell]{\pi_2}$. Then M/E is unramified on B except possibly at π_1' and π_2'. Since (π_1, π_2) is the maximal ideal of A, it follows that there is a unique extension of v_i to E, which we denote by $\tilde{v}_i$. Let ω be a discrete valuation of M extending $\tilde{v}_i$ for some i. Then the restriction of ω to L is equal to $w_i^{j_i}$ for some j_i. Let $E_i = F(\sqrt[\ell]{\pi_i})$ and $M_i = L(\sqrt[\ell]{\pi_i})$. Let ω' and v_i' be the restrictions of ω and $\tilde{v}_i$ to M_i and E_i respectively. Suppose that $e(w_i^{j_i}/v_i) = m$. Then by Lemma 4.2.1, $e(\omega'/v_i') = e(w_i^{j_i}/v_i)/\ell$ and $e(\omega/\tilde{v}_i) \leq e(\omega'/v_i') = e(w_i^{j_i}/v_i)/\ell$. Hence, by induction hypothesis, for each $i = 1, 2$, there is a unique extension of $\tilde{v}_i$ to M. Since L is a subfield of M, there is a unique extension of v_i to L. Hence $L \otimes_F F_{\pi_i}$ is a field. □

The next lemma describes the structure of finite Galois extensions L/F which are unramified on A except possibly at π_1 and $\pi_2 \in A$ and totally ramified at π_2.

Lemma 4.2.4. *Let A be a complete regular local ring of dimension 2 with residue field κ and fraction field F. Let L/F be a Galois extension of degree n, where n is coprime to char(κ). Let $\mathfrak{m} = (\pi_1, \pi_2) \subset A$ be the maximal ideal of A. Suppose that L/F is unramified on A except possibly at π_1 and $\pi_2 \in A$ and totally ramified at π_2. Then $L = F(\sqrt[n]{u\pi_1^m\pi_2})$ for some $u \in A$ a unit and some integer m.*

Proof. Let G be the Galois group of L/F. Since the degree of L/F is coprime to char(κ) and L/F is unramified on A except possibly at π_1 and π_2, by Lemma 4.2.3, $L \otimes_F F_{\pi_2}$ is a field. Since $L \otimes_F F_{\pi_2}/F_{\pi_2}$ is a totally tamely ramified extension, F_{π_2} contains a primitive n^{th} root of unity and we have $L \otimes_F F_{\pi_2} = F_{\pi_2}(\sqrt[n]{\theta\pi_2})$ for some $\theta \in F_{\pi_2}$ which is a unit in the discrete valuation ring of F_{π_2} by [CF67, Proposition 1, p-32]. In particular G is a cyclic group. Since F_{π_2} contains a primitive n^{th} root of unity, the residue field $\kappa(\pi_2)$ of F_{π_2} contains a primitive n^{th} root of unity. Since κ is the residue field of $\kappa(\pi_2)$, κ also contains a primitive n^{th} root of unity. Since A is complete, by Hensel's lemma, F contains a primitive n^{th} root of unity.

Hence $L = F(\sqrt[n]{a})$ for some $a \in F$. Since L/F is unramified on A except possibly at π_1, π_2, we can choose $a = u\pi_1^m\pi_2^d$ for some $u \in A$ a unit and integers m, d. Since L/F is totally ramified at π_2, d is coprime to n and hence we can assume that $d = 1$. $\square$

Next we consider finite Galois extensions L/F which are unramified on A except possibly at π_1.

Lemma 4.2.5. *Let A be a complete regular local ring of dimension 2 with residue field κ and fraction field F. Let L/F be a Galois extension of degree coprime to char(κ). Let $\mathfrak{m} = (\pi_1, \pi_2) \subset A$ be the maximal ideal of A. Suppose that L/F is unramified on A except possibly at π_1. Then there exists a subextension L_1/F of L/F such that:*

- *L_1/F is unramified on A, and*
- *$L = L_1(\sqrt[e]{u\pi_1})$ for some unit u in the integral closure of A in L_1.*

Proof. Let G be the Galois group of L/F. Since $L \otimes_F F_{\pi_1}/F_{\pi_1}$ is a field extension by Lemma 4.2.3, the Galois group $\mathrm{Gal}(L \otimes_F F_{\pi_1}/F_{\pi_1})$ is isomorphic to G. We will identify these two groups. We consider the inertia group H of the extension $L \otimes_F F_{\pi_1}/F_{\pi_1}$ which is a subgroup of G. Let $L_1 = L^H$. Then $(L \otimes_F F_{\pi_1})^H = L_1 \otimes_F F_{\pi_1}$ is unramified over F_{π_1} by [CF67, Theorem 2, p-27]. Hence $(L_1)_{\pi_1} \cong L_1 \otimes_F F_{\pi_1}$ is unramified over F_{π_1} and L_1/F is unramified at π_1. Since L/F is unramified on A except possibly at π_1, L_1/F is unramified on A. Then the integral closure B of A in L_1 is a regular local ring with maximal ideal (π_1, π_2). Let $e = [L\colon L_1]$. Since L/F is unramified on A except possibly at π_1, L/L_1 is unramified on B except possibly at π_1. Hence by Lemma 4.2.4, with the roles of π_1 and π_2 interchanged, we have $L = L_1(\sqrt[e]{u\pi_2^m\pi_1})$ for some $u \in B$ a unit. Since L/L_1 is unramified on B except possibly at π_1, m is divisibly by e and hence $L = L_1(\sqrt[e]{u\pi_1})$. $\square$

The next lemma describes the structure of finite Galois extensions L/F unramified on A except possibly at π_1 and $\pi_2 \in A$, hence generalising Lemma 4.2.4 and Lemma 4.2.5.

Theorem 4.2.6. *Let A be a complete regular local ring of dimension 2 with residue field κ and fraction field F. Let L/F be a Galois extension of degree coprime to char(κ). Let $\mathfrak{m} = (\pi_1, \pi_2) \subset A$ be the maximal ideal of A. Suppose that L/F is unramified on A except possibly at π_1 and $\pi_2 \in A$. Then there exists subfields L_1 and L_2 of L such that :*

- *$F \subseteq L_1 \subseteq L_2 \subseteq L$,*

- *L_1/F is unramified on A,*

- *$L_2 = L_1(\sqrt[d_1]{u\pi_1})$ for some unit u in the integral closure of A in L_1, and*

- *$L = L_2(\sqrt[d_2]{v(\sqrt[d_1]{u\pi_1})^i\pi_2})$ for some unit v in the integral closure of A in L_2.*

Proof. Let G be the Galois group of L/F. Since $L \otimes_F F_{\pi_2}/F_{\pi_2}$ is a field extension by Lemma 4.2.3, the Galois group $\mathrm{Gal}(L \otimes_F F_{\pi_2}/F_{\pi_2})$ is isomorphic to G. We identify these two groups. We consider the inertia group H of the extension $L \otimes_F F_{\pi_2}/F_{\pi_2}$ which is a subgroup of G. Let $L_2 = L^H$. Then as in Lemma 4.2.5, L_2/F is unramified on A except possibly at π_1. Hence, by Lemma 4.2.5, there exists a sub extension L_1/F of L_2/F such that L_1/F is unramified on A and $L_2 = L_1(\sqrt[d_1]{u\pi_1})$ for some unit u in the integral closure of A in L_1. Let B be the integral closure of A in L_2. Then B is a regular local ring with maximal ideal $(\sqrt[d_1]{u\pi_1}, \pi_2)$ by [PS14, Lemma 3.2.]. Since L/L_2 is unramified on B except possibly at $\sqrt[d_1]{u\pi_1}$, π_2 and totally ramified at π_2, by Lemma 4.2.4, $L = L_2(\sqrt[d_2]{v(\sqrt[d_1]{u\pi_1})^i\pi_2})$ for some unit $v \in B$. □

4.2.2 Norms over two dimensional complete fields

Let A, F, π_1, π_2 be as in Notation 4.2.2. Let $\lambda = u\pi_1^r\pi_2^s$ for some unit $u \in A$ and integers r, s. In this subsection, we show that if λ is a norm from the extension $L\otimes_F F_{\pi_1}/F_{\pi_1}$, then λ is a norm from the extension L/F.

We begin by describing the elements of $F_{\pi_i}^{\times}/F_{\pi_i}^{\times n}$.

Lemma 4.2.7. *Let A be a complete regular local ring of dimension 2 with residue field κ and fraction field F. Let $\mathfrak{m} = (\pi_1, \pi_2)$ be the maximal ideal of A. Let $m \geq 1$ be an integer coprime to char(κ). Let F_{π_1} be the completion of F at the discrete valuation of F given by π_1. Then every element in F_{π_1} can be written as $u\pi_1^s\pi_2^t a^m$ for some $u \in A$ a unit, $a \in F_{\pi_1}$ and integers s, t.*

Proof. Let $\widehat{A}_{(\pi_1)}$ be the completion of the local ring $A_{(\pi_1)}$. Then F_{π_1} is the fraction field of $\widehat{A}_{(\pi_1)}$. Let $x \in F_{\pi_1}$. Then $x = v{\pi_1}^s$ for some unit $v \in \widehat{A}_{(\pi_1)}$ and integer s. Let $\overline{v}$ be the image of v in the residue field $\kappa(\pi_1)$ of F_{π_1}. Since $\kappa(\pi_1)$ is the fraction field of $A/(\pi_1)$ and $A/(\pi_1)$ is a discrete valuation ring with the image $\overline{\pi}_2$ as a parameter, we can write $\overline{v} = z\overline{\pi_2}^t$ for some unit $z \in A/(\pi_1)$ and integer t. Let $u \in A$ with $\overline{u} = z \in A/(\pi_1)$. Since z is a unit in $A/(\pi_1)$, u is a unit in A. Hence $x^{-1}u\pi_1^s\pi_2^t$ maps to 1 in $\kappa(\pi_1)$. Since m is coprime to char(κ), by Hensel's lemma, $x = u{\pi_1}^s{\pi_2}^t a^m$ for some $a \in F_{\pi_1}$. □

Lemma 4.2.8. *Let A be a complete regular local ring of dimension 2 with residue field κ and fraction field F. Let L/F be a Galois field extension of degree n, where n is coprime to char(κ). Let $\mathfrak{m} = (\pi_1, \pi_2)$ be the maximal ideal of A. Suppose that L/F is unramified on A except possibly at π_1. Let $\lambda = u\pi_1^r\pi_2^s$ for some unit $u \in A$ and integers r, s. If λ is a norm from the extension $L \otimes_F F_{\pi_1}/F_{\pi_1}$, then λ is a norm from the extension L/F.*

Proof. Let $\mu \in L\otimes_F F_{\pi_1}$ be such that $N_{L\otimes_F F_{\pi_1}/F_{\pi_1}}(\mu) = \lambda$. Since L/F is a Galois extension which is unramified on A except possibly at π_1, by Lemma 4.2.5, we have a subfield $F \subseteq L_1 \subseteq L$ such that L_1/F is unramified on A and $L = L_1(\sqrt[e]{v\pi_1})$, where $e = [L\colon L_1]$ and v is a unit in the integral closure of A in L_1. Let B be the integral closure of A in L. Then B is a regular local ring with maximal ideal $(\sqrt[e]{v\pi_1}, \pi_2)$ by [PS14, Lemma 3.2.]. Hence, by Lemma 4.2.7, $\mu = w{\sqrt[e]{v\pi_1}}^i\pi_2^j b^n$ for some integers i, j, $b \in L \otimes_F F_{\pi_1}$ and w a unit in B. Let $\theta = w{\sqrt[e]{v\pi_1}}^i\pi_2^j \in L$. Since $N_{L\otimes_F F_{\pi_1}/F_{\pi_1}}(\mu) = \lambda$, we have $N_{L/F}(\theta^{-1})\lambda = N_{L\otimes_F F_{\pi_1}/F_{\pi_1}}(b^n) \in F_{\pi_1}^{\times n}$. Since $N_{L/F}(\theta^{-1})\lambda = [N_{L/F}(w)v{\pi_1}^{ni/e}\pi_2^{nj}]^{-1}u\pi_1^r\pi_2^s = [u(N_{L/F}(w))^{-1}]\pi_1^{r-ni/e}\pi_2^{s-nj}$ and $u(N_{L/F}(w))^{-1}$ is a unit in A, by [PPS18, Corollary 5.5.], $N_{L/F}(\theta^{-1})\lambda \in F^{\times n}$. In particular $N_{L/F}(\theta^{-1})\lambda$ is a norm from the extension L/F and hence λ is a norm from L/F. □

Theorem 4.2.9. *Let A be a complete regular local ring of dimension 2 with residue field κ and fraction field F. Let L/F be a Galois field extension of degree n, where n is coprime to char(κ). Let $\mathfrak{m} = (\pi_1, \pi_2)$ be the maximal ideal of A. Suppose that L/F is unramified on A except possibly at π_1, π_2. Let $\lambda = u\pi_1^r\pi_2^s$ for some unit $u \in A$ and integers r, s. If λ is a norm from the extension $L \otimes_F F_{\pi_1}/F_{\pi_1}$, then λ is a norm from the extension L/F.*

Proof. Let $\mu \in L \otimes_F F_{\pi_1}$ be such that $N_{L\otimes_F F_{\pi_1}/F_{\pi_1}}(\mu) = \lambda$. We show by induction on the degree of the field extension L/F that λ is a norm from the extension L/F.

Since L/F is a Galois extension which is unramified on A except possibly at $\pi_1$and π_2, we have subfields L_1 and L_2 as in Theorem 4.2.6. Let B

be the integral closure of A in L_2. Then B is a complete regular local ring with maximal ideal $(\sqrt[d_1]{v\pi_1}, \pi_2)$ by ([PS14, Lemma 3.2.]). By Lemma 4.2.7, we have $N_{L\otimes_F F_{\pi_1}/L_2\otimes_F F_{\pi_1}}(\mu) = w\sqrt[d_1]{v\pi_1}^i\pi_2^j b^n$ for some integers i, j, $b \in L_2 \otimes_F F_{\pi_1}$ and w a unit in B. Then $\theta = w\sqrt[d_1]{v\pi_1}^i\pi_2^j$ is a norm from $L \otimes_F F_{\pi_1}/L_2 \otimes_F F_{\pi_1}$.

Suppose that $F \neq L_2$. Then $[L\colon L_2] < [L\colon F]$ and by induction, θ is a norm from L/L_2. Write $\theta = N_{L/L_2}(\theta')$. Then

$$\begin{aligned}
\lambda &= N_{L\otimes_F F_{\pi_1}/F_{\pi_1}}(\mu)\\
&= N_{L_2\otimes_F F_{\pi_1}/F_{\pi_1}}(N_{L\otimes_F F_{\pi_1}/L_2\otimes_F F_{\pi_1}}(\mu))\\
&= N_{L_2\otimes_F F_{\pi_1}/F_{\pi_1}}(\theta b^n)\\
&= N_{L_2\otimes_F F_{\pi_1}/F_{\pi_1}}(\theta) N_{L_2\otimes_F F_{\pi_1}/F_{\pi_1}}(b^n)\\
&= N_{L_2/F}(\theta) N_{L_2\otimes_F F_{\pi_1}/F_{\pi_1}}(b)^n\\
&= N_{L_2/F}(N_{L/L_2}(\theta') N_{L_2\otimes_F F_{\pi_1}/F_{\pi_1}}(b)^n\\
&= N_{L/F}(\theta') N_{L_2\otimes_F F_{\pi_1}/F_{\pi_1}}(b)^n
\end{aligned}$$

Since $N_{L/F}(\theta') = N_{L_2/F}(\theta) = N_{L_2/F}(w\sqrt[d_1]{v\pi_1}^i\pi_1^j)$, $N_{L/F}(\theta')^{-1}\lambda$ is a product of a unit in A with a power of π_1 and a power of π_2. Since $N_{L/F}(\theta')^{-1}\lambda = N_{L_2\otimes_F F_{\pi_1}/F_{\pi_1}}(b)^n \in F_{\pi_1}^n$, by [PPS18, Corollary 5.5.], we conclude that $N_{L/F}(\theta')^{-1}\lambda$ is a n^{th} power in F and hence a norm from L to F. Hence λ is also a norm from L to F.

Now suppose $F = L_2$. Then $L = F(\sqrt[n]{v\pi_2})$, where v is a unit in A. Hence L/F is a cyclic extension of degree n. Let σ be a generator of the Galois group of L/F and C be the cyclic algebra (L, σ, λ). Since L/F is unramified on A except at π_2, C is a unramified on A except possibly at π_1 and π_2. Since λ is a norm from $L \otimes_F F_{\pi_1}$, $C \otimes_F F_{\pi_1}$ is a split algebra. Thus, by [PPS18, Corollary 5.5.], C is a split algebra and hence λ is a norm from the extension L/F by [Alb39, Theorem 6, p-95]. □

4.3 Ш vs Ш_X

In this section, we compare the groups $\text{Ш}_X(F, T_{L/F})$ and $\text{Ш}(F, T_{L/F})$ for a semi-global field F and a finite Galois extension L/F of degree coprime to the characteristic of the residue field. The proof uses Theorem 4.2.9. This allows us to use patching techniques to prove the local-global principle for norm one tori $T_{L/F}$ over F with respect to discrete valuations.

Theorem 4.3.1. *Let K be a complete discretely valued field with residue field κ. Let F be the function field of a smooth, projective, geometrically integral curve over K and $\mathscr{X}_0$ a regular proper model of F with reduced special fibre X_0. Let L/F be a Galois field extension of degree coprime to char(κ). Then $Ш(F, T_{L/F}) = \bigcup_X Ш_X(F, T_{L/F})$, where X runs over the reduced special fibres of regular proper models $\mathscr{X}$ of F which are obtained as a sequence of blow-ups of $\mathscr{X}_0$ centered at closed points of $\mathscr{X}_0$.*

Proof. Let x be an element in $Ш(F, T_{L/F}) \subseteq H^1(F, T_{L/F})$. Since we have $H^1(F, T_{L/F}) \simeq F^\times/N_{L/F}(L^\times)$, we can choose $\lambda \in F^\times$ a lift of x. For a regular proper model $\mathscr{X}$ of F, let $supp_{\mathscr{X}}(\lambda)$ denote the support of λ in $\mathscr{X}$ and $ram_{\mathscr{X}}(L/F)$ denote the ramification locus for the extension L/F with respect to $\mathscr{X}$. By [Lip75, p-193], there exists a sequence of blow-ups $\mathscr{X} \to \mathscr{X}_0$ centered at closed points of $\mathscr{X}_0$ such that the union of $supp_{\mathscr{X}}(\lambda)$, $ram_{\mathscr{X}}(L/F)$ and the reduced special fibre X of $\mathscr{X}$ is a union of regular curves with normal crossings. We show that $x \in Ш_X(F, T_{L/F})$.

Let $P \in X$. First suppose P is a generic point of X. Then P gives a discrete valuation ν of F with $F_\nu = F_P$. Since $x \in Ш(F, T_{L/F})$, x maps to 0 in $H^1(F_P, T_{L/F})$.

Next suppose that P is a closed point. Let η_1 be the generic point of an irreducible component of X containing P. Let $\mathcal{O}_{\mathscr{X},P}$ be the local ring at P and $\mathfrak{m}_{\mathscr{X},P}$ be its maximal ideal. Then by our choice of $\mathscr{X}$, $\mathfrak{m}_{\mathscr{X},P} = (\pi_1, \pi_2)$, where π_1 is a prime defining η_1 at P, $\lambda = u\pi_1^r\pi_2^s$ for some unit $u \in \mathcal{O}_{\mathscr{X},P}$ and integers r, s, and $L \otimes_F F_P/F_P$ is unramified on $\mathcal{O}_{\mathscr{X},P}$ except possibly at π_1, π_2. Since L/F is a Galois extension, $L \otimes_F F_P = \prod L_P$ for some Galois extension L_P/F_P. Since $L \otimes_F F_P/F_P$ is unramified on $\mathcal{O}_{\mathscr{X},P}$ except possibly at π_1, π_2, L_P/F_P is unramified on $\mathcal{O}_{\mathscr{X},P}$ except possibly at π_1, π_2. Since λ is a lift of $x \in Ш_X(F, T_{L/F})$, λ is a norm from $L \otimes_F F_{\eta_1}/F_{\eta_1}$. Since $F_{\eta_1} \subset F_{P,\eta_1}$, λ is a norm from $L \otimes_F F_{P,\eta_1}/F_{P,\eta_1}$. Hence λ is a norm from $L_P \otimes_{F_P} F_{P,\eta_1}/F_{P,\eta_1}$. Thus, by Theorem (4.2.9), λ is a norm from L_P/F_P and x maps to 0 in $H^1(F_P, T_{L/F})$. Therefore $x \in Ш_X(F, T_{L/F})$. By [HHK15a, Proposition 8.2.], we have $\bigcup_X Ш_X(F, T_{L/F}) = Ш(F, T_{L/F})$, where X runs over the reduced special fibres of regular proper models $\mathscr{X}$ of F which are obtained as a sequence of blow-ups of $\mathscr{X}_0$ centered at closed points of $\mathscr{X}_0$. □

Remark 4.3.2. The proof of Theorem 4.3.1 also works if we just consider divisorial discrete valuations instead of considering all discrete valuations on F.

4.4 Local-Global Principle

In this section, we prove the local-global principle for norm one tori over semi-global fields with respect to points on the reduced special fibre under some assumptions on the semi-global field and the residue field. This, combined with Theorem 4.3.1, allows us to conclude that under the same assumptions, the local-global principle for norm one tori also holds with respect to discrete valuations.

Lemma 4.4.1. *Let K be a complete discretely valued field with residue field κ and F be the function field of a smooth, projective, geometrically integral curve over K. Let $\mathscr{X}$ be a regular proper model of F with the reduced special fibre X a union of regular curves with normal crossings. Let L/F be a finite Galois extension over F of degree n. Let $P \in X$ be a closed point and U an irreducible open subset of X with P in the closure of U. Suppose that:*

- *n is coprime to char(κ),*
- *K contains a primitive n^{th} root of unity ρ, and*
- *for all finite Galois extensions $l/\kappa(P)$ of degree d dividing n,*

$$T_{l/\kappa(P)}(\kappa(P)) = RT_{l/\kappa(P)}(\kappa(P))\langle \rho^{n/d} \rangle.$$

Then

$$T_{L\otimes_F F_{P,U}/F_{P,U}}(F_{P,U}) = RT_{L\otimes_F F_{P,U}/F_{P,U}}(F_{P,U})\langle \rho \rangle.$$

Proof. Let $\kappa(U)$ be the function field of U. Since X is a union of regular curves, P gives a discrete valuation on $\kappa(U)$. Let $\kappa(U)_P$ be the completion of $\kappa(U)$ at P. Then by definition, $F_{P,U}$ is a complete discretely valued field with residue field $\kappa(U)_P$. Since L/F is a Galois extension of degree n, $L \otimes_F F_{P,U} \simeq \prod L_0$ for some finite Galois extension $L_0/F_{P,U}$ of degree d dividing n. Since $F_{P,U}$ is a complete discretely valued field with residue field $\kappa(U)_P$, L_0 is a complete discretely valued field with residue field M_0 a finite extension of $\kappa(U)_P$ of degree d_1 dividing d. Since $\kappa(U)_P$ is a complete discretely valued field with residue field $\kappa(P)$, M_0 is a complete discretely valued field with residue field l_0 a finite Galois extension of $\kappa(P)$ of degree d_2 dividing d_1. Hence, by the assumption on $\kappa(P)$ and Lemma 4.1.2, we have

$$T_{M_0/\kappa(U)_P}(\kappa(U)_P) = RT_{M_0/\kappa(U)_P}(\kappa(U)_P)\langle \rho^{n/d_1} \rangle.$$

Hence, once again by Lemma 4.1.2, we have

$$T_{L_0/F_{P,U}}(F_{P,U}) = RT_{L_0/F_{P,U}}(F_{P,U})\langle \rho^{n/d} \rangle.$$

Since $L \otimes_F F_{P,U}$ is the product of n/d copies of L_0, by Corollary 2.4.4, we have

$$T_{L\otimes_F F_{P,U}/F_{P,U}}(F_{P,U}) = RT_{L\otimes_F F_{P,U}/F_{P,U}}(F_{P,U})\langle \rho \rangle.$$

□

We are now ready to prove the local-global principle for norm one tori over semi-global fields with respect to points on the reduced special fibre of the model under some assumptions on the given semi-global field F and the residue field κ.

Theorem 4.4.2. *Let K be a complete discretely valued field with residue field κ and F be the function field of a smooth, projective, geometrically integral curve over K. Let $\mathscr{X}$ be a regular proper model of F with reduced special fibre X a union of regular curves with normal crossings. Let L/F be a Galois extension over F of degree n. Suppose that*

- *n is coprime to char(κ),*
- *K contains a primitive n^{th} root of unity ρ,*
- *for all finite extensions κ'/κ and for all finite Galois extensions l/κ' of degree d dividing n,*

$$T_{l/\kappa'}(\kappa') = RT_{l/\kappa'}(\kappa')\langle \rho^{n/d} \rangle,$$

- *the graph associated to $\mathscr{X}$ is a tree.*

Then $Ш_X(F, T_{L/F}) = 0$.

Proof. Let $\mathcal{P}$ be a finite set of closed points of X containing all the nodal points of X. By [HHK15a, Corollary 5.9.], it is enough to show that $Ш_{\mathcal{P}}(F, T_{L/F}) = 0$. Let $X \setminus \mathcal{P} = \cup_i U_i$. Then each U_i is an irreducible open subset of X. By ([HHK15a, Corollary 3.6.]), it is enough to show that the product map

$$\psi : \prod_i T_{L/F}(F_{U_i}) \times \prod_{P\in\mathcal{P}} T_{L/F}(F_P) \to \prod_{(P,U_i)} T_{L/F}(F_{P,U_i})$$

is surjective, where the product on the right hand side is taken over all pairs (P, U_i) with $P \in \mathcal{P}$ and U_i such that P in the closure of U_i.

Let $(\lambda_{P,U_i}) \in \prod_{(P,U_i)} T_{L/F}(F_{P,U_i})$. We show that (λ_{P,U_i}) is in the image of ψ. By Lemma 4.4.1, for each pair (P, U_i) with P in the closure of U_i, we have $\lambda_{P,U_i} = \rho^{j_{P,U_i}} \mu_{P,U_i}$ for some integer j_{P,U_i} and $\mu_{P,U_i} \in RT_{L/F}(F_{P,U_i})$. Let G be the Galois group of L/F. For each $\sigma \in G$, there exists $a_{\sigma,P,U_i} \in (L \otimes_F F_{P,U_i})^\times$ such that

$$\mu_{P,U_i} = \prod_{\sigma \in G(L/F)} \sigma(a_{\sigma,P,U_i})(a_{\sigma,P,U_i})^{-1}.$$

Since the group $R_{L/F}(\mathbb{G}_m)$ is F-rational, by ([HHK09, Theorem 3.6.]),

$$\prod_i (L \otimes_F F_{U_i})^\times \times \prod_{P \in \mathcal{P}} (L \otimes_F F_P)^\times \to \prod_{(P,U_i)} (L \otimes_F F_{P,U_i})^\times$$

is surjective. Hence for each $\sigma \in G$, there exist $b_{\sigma,U_i} \in (L \otimes_F F_{U_i})^\times$ and $b_{\sigma,P} \in (L \otimes_F F_P)^\times$ such that $a_{\sigma,P,U_i} = b_{\sigma,U_i} b_{\sigma,P}$. We have

$$\begin{aligned}
\mu_{P,U_i} &= \prod_{\sigma \in G(L/F)} \sigma(a_{\sigma,P,U_i})(a_{\sigma,P,U_i})^{-1} \\
&= \prod_{\sigma \in G(L/F)} \sigma(b_{\sigma,U_i} b_{\sigma,P})(b_{\sigma,U_i} b_{\sigma,P})^{-1} \\
&= \prod_{\sigma \in G(L/F)} \sigma(b_{\sigma,U_i})(b_{\sigma,U_i})^{-1} \sigma(b_{\sigma,P})(b_{\sigma,P})^{-1}.
\end{aligned}$$

Since $\sigma(b_{\sigma,U_i})(b_{\sigma,U_i})^{-1} \in T_{L/F}(F_{U_i})$ and $\sigma(b_{\sigma,P})(b_{\sigma,P})^{-1} \in T_{L/F}(F_P)$, (μ_{P,U_i}) is in the image of ψ.

Since $T_{L/F}(F)$ is a group and $\rho \in T_{L/F}(F)$, by Corollary 3.3.3, $(\rho^{j_{P,U_i}})$ is in the image of ψ. Since ψ is a homomorphism, (λ_{P,U_i}) is in the image of ψ, hence proving that ψ is surjective. □

Using the above theorem and Theorem 4.3.1, we get that the under same assumptions, we have the local-global principle for norm one tori over semi-global fields with respect to discrete valuations:

Theorem 4.4.3. *With the hypothesis as in Theorem 4.4.2, we have* $Ш(F, T_{L/F}) = 0$.

Proof. Let $\mathscr{X}$ be a regular proper model of F which is obtained as a sequence of blow-ups of $\mathscr{X}_0$ at closed points. Since the graph $\Gamma(\mathscr{X}_0)$ is a tree, $\Gamma(\mathscr{X})$ is also a tree (see [HHK15a, Remark 6.1(b)]). Let X be the reduced special fibre of $\mathscr{X}$. Then $Ш_X(F, T_{L/F}) = 0$ by Theorem 4.4.2. Thus, by Theorem (4.3.1), we have $Ш(F, T_{L/F}) = 0$. □

Corollary 4.4.4. *Let K be an m-local field with first residue field κ or an iterated Laurent series in m variables over an algebraically closed field κ. Let F be the function field of a smooth, projective, geometrically integral curve over K and L/F be a finite Galois extension of degree n with $(n, char(\kappa)) = 1$. Let $\mathscr{X}$ be a regular proper model of F with reduced special fibre X a union of regular curves with normal crossings. Suppose that the graph associated to $\mathscr{X}$ is a tree. If K contains a primitive n^{th} root of unity, then $Ш(F, T_{L/F}) = 0$.*

Proof. This immediately follows from Corollary 4.1.4 and Theorem (4.4.3). □

Remark 4.4.5. By Remark 4.3.2, the results also hold true if we just consider divisorial discrete valuations instead of all discrete valuations on F in Theorem 4.4.3 and Corollary 4.4.4.

Chapter 5

Local-global principle for multinorm tori over semi-global fields

Let F be a field and $L_1, L_2, \ldots, L_m$ be finite Galois extensions of F. Let $L = \prod_{i=1}^{m} L_i$. We denote by $T_{L/F}$ the multinorm torus corresponding to the extensions L_i/F, $1 \leq i \leq m$. More precisely, $T_{L/F}$ is the torus defined as

$$T_{L/F} = ker\left(\prod_{i=1}^{m} R_{L_i/F}\mathbb{G}_m \xrightarrow{\prod_{i=1}^{m} N_{L_i/F}} \mathbb{G}_m\right),$$

where $N_{L_i/F}$ are the maps induced from the usual norm maps from L_i to F.

For any field F and finite separable extensions L_i/F, $1 \leq i \leq m$, if the gcd of the degrees $[L_i : F]$ is 1, then the local-global principle holds for the multinorm torus $T_{L/F}$. For any $\lambda \in F^{\times}$, $\lambda^{[L_i:F]}$ is in the image of the norm map $N_{L_i/F}$. Since the gcd of the degrees $[L_i : F]$ is 1, λ is a product of norm from the extensions L_i/F. So it is interesting only to consider the cases when the gcd of the degrees of the extensions is not 1.

For a semi-global field F and étale algebras L/F which are either product to two degree p cyclic extensions or product of finitely many quadratic cyclic extensions, we study local-global principle for multinorm tori $T_{L/F}$ with respect to discrete valuations on F.

5.1 Ш vs $Ш_X$

We follow the same strategy as we had in the norm one tori case and compare the groups Ш *and* $Ш_X$ *for multinorm tori.*

Notation 5.1.1. *For a regular proper model $\mathscr{X}$ of a semi-global field F and a field extension L/F, we use $ram_{\mathscr{X}}(L/F)$ to denote the ramification locus for the exxtension L/F with respect to $\mathscr{X}$. Also, for $\lambda \in F^\times$, we use $supp_{\mathscr{X}}(\lambda)$ to denote the support of λ in $\mathscr{X}$.*

Theorem 5.1.2. *Let A be a complete regular local ring of dimension 2 with the residue field κ and the fraction field F. Let $L_1, L_2, \ldots, L_m$ be finite Galois extensions of F with $[L_i\colon F] = n_i$. Assume that all n_i are coprime to char(κ). Let $\mathfrak{m} = (\pi_1, \pi_2)$ be the maximal ideal of A. Assume that L_i/F are unramified on A except possibly at π_1, π_2. Let $\lambda = u\pi_1^r\pi_2^s \in F$, where $u \in A$ is a unit and r, s are integers. Suppose that λ is a product of norms from the extensions $L_i \otimes_F F_{\pi_1}/F_{\pi_1}$. Then λ is a product of norms from the extensions L_i/F.*

Proof. Let us consider $n = \prod_{i=1}^{m} n_i$. By our assumption, n is coprime to char(κ). Let $\lambda = \prod_{i=1}^{m} \beta_i$, where $\beta_i = N_{L_i\otimes_F F_{\pi_1}/F_{\pi_1}}(\alpha_i)$ for some $\alpha_i \in (L_i \otimes_F F_{\pi_1})^\times$. By Lemma 4.2.7, we can write $\beta_i = u_i\pi_1^{r_i}\pi_2^{s_i}{b_i}^n$ for some $u_i \in A$ a unit, $b_i \in F_{\pi_1}$ and integers r_i, s_i.
Let $b = \prod_{i=1}^{m}(b_i)^n$. Then $b = \lambda\prod_{i=1}^{m}(u_i\pi_1^{r_i}\pi_2^{s_i})^{-1}$. Hence $b \in F$. Since b is a n^{th} power in F_{π_1}, it is a n^{th} power in F by [PPS18, Corollary 5.5.] and hence is a norm from L_i/F. Thus we can assume that $b_i = 1$. So let $\lambda = \prod_{i=1}^{m} \beta_i$ with $\beta_j = u_i\pi_1^{r_i}\pi_2^{s_i}$ such that β_i is a norm from $L_i \otimes_F F_{\pi_1}/F_{\pi_1}$. By Theorem 4.2.9, we conclude that β_i is a norm from L_i/F and hence λ is a product of norms from the extensions L_i/F.

□

Theorem 5.1.3. *Let K be a complete discretely valued field with residue field κ. Let F be the function field of a smooth, projective, geometrically integral curve over K. Let L_i/F be Galois field extensions of degrees* *coprime to char(κ) for $i = 1, 2, \ldots, m$. Let $L = \prod_{i=1}^{m} L_i$. Let $\mathscr{X}_0$ be a regular proper model of F. Then*

$$Ш(F, T_{L/F}) = \bigcup_X Ш_X(F, T_{L/F}),$$

where X runs over the reduced special fibres of regular proper models $\mathscr{X}$ of F which are obtained as a sequence of blow-ups of $\mathscr{X}_0$ centered at closed points of $\mathscr{X}_0$.

Proof. Let $x \in Ш(F, T_{L/F}) \subset H^1(F, T_{L/F})$. Since

$$H^1(F, T_{L/F}) \simeq F^\times / (\prod_{i=1}^{m} N_{L_j/F}(L_i^\times)),$$

let $\lambda \in F^\times$ be a lift of x. By [Lip75, p-193], there exists a sequence of blow-ups $\mathscr{X} \to \mathscr{X}_0$ centered at closed points of $\mathscr{X}_0$ such that the union of $supp_{\mathscr{X}}(\lambda)$, $ram_{\mathscr{X}}(L_i/F)$ and the reduced special fiber X of $\mathscr{X}$ is a union of regular curves with normal crossings. We show that $x \in Ш_X(F, T_{L/F})$.

Let $P \in X$. Suppose P is a generic point of X. Then P gives a discrete valuation ν of F with $F_\nu = F_P$. Since $x \in Ш(F, T_{L/F})$, x maps to 0 in $H^1(F_P, T_{L/F})$.

Suppose that P is a closed point. Let η_1 be the generic point of an irreducible component of X containing P. Let $\mathcal{O}_{\mathscr{X},P}$ be the local ring at P and $\mathfrak{m}_{\mathscr{X},P}$ be its maximal ideal. Then by our choice of $\mathscr{X}$, $\mathfrak{m}_{\mathscr{X},P} = (\pi_1, \pi_2)$, where π_1 is a prime defining η_1 at P, $\lambda = u\pi_1^r \pi_2^s$ for some unit $u \in \mathcal{O}_{\mathscr{X},P}$ and integers r, s, and all $L_i \otimes_F F_P/F_P$ are unramified on $\mathcal{O}_{\mathscr{X},P}$ except possibly at π_1, π_2. Since L_i/F are Galois extensions, $L_i \otimes_F F_P = \prod (L_i)_P$ for some Galois extensions $(L_i)_P/F_P$. Since $L_i \otimes_F F_P/F_P$ is unramified on $\mathcal{O}_{\mathscr{X},P}$ except possibly at π_1, π_2, $(L_i)_P/F_P$ is unramified on $\mathcal{O}_{\mathscr{X},P}$ except possibly at π_1, π_2. Since λ is a lift of $x \in Ш_X(F, T_{L/F})$, λ is a product of norms from $L_i \otimes_F F_{\eta_1}/F_{\eta_1}$. Since $F_{\eta_1} \subset F_{P,\eta_1}$, λ is a product of norms from $L_i \otimes_F F_{P,\eta_1}/F_{P,\eta_1}$. Hence λ is a product of norms from $(L_i)_P \otimes F_{P,\eta_1}/F_{P,\eta_1}$. Thus, by Theorem 5.1.2, λ is a product of norms from $(L_i)_P/F_P$ and x maps to 0 in $H^1(F_P, T_{L/F})$. Therefore $x \in Ш_X(F, T_{L/F})$. By ([HHK15a, Proposition 8.2.]), we have $Ш(F, T_{L/F}) = \bigcup_X Ш_X(F, T_{L/F})$, where X runs over the reduced special fibres of regular proper models of F.

□

Remark 5.1.4. The proof of Theorem 5.1.3 also works if we just consider divisorial discrete valuations instead of considering all discrete valuations on F.

5.2 Local-Global Principle

We will be using the following notation throughout this section.

Notation 5.2.1. *For a semi-global field F and field extensions L_i/F, we use $L_{i,\eta}$, $L_{i,P}$, $L_{i,P,\eta}$ and $L_{i,P,U}$ to denote $L_i \otimes_F F_\eta$, $L_i \otimes_F F_P$, $L_i \otimes_F F_{P,\eta}$, and $L_i \otimes_F F_{P,U}$ respectively.*

5.2.1 Multinorm tori associated to two degree p cyclic extensions

Let K be a complete discretely valued field with residue field κ. Let F be the function field of a smooth, projective, geometrically integral curve over K and let L_1 and L_2 be two cyclic extensions of F each of degree p for some prime p. Let $L = L_1 \times L_2$ and $T_{L/F}$ denote the associated multinorm torus. Assume that κ is either algebraically closed or a finite field. Assume that $p \neq char(\kappa)$. Let $\mathscr{X}$ be a regular proper model of F. We prove that for any branch $F_{P,\eta}$, $T_{L/F}(F_{P,\eta})/R = \{1\}$. Using this, we conclude that over semi-global fields, the local-global principle holds for such multinorm tori with respect to discrete valuations.

We start with a basic result about multinorm tori over arbitrary fields.

Lemma 5.2.2. *Let $L_i, 1 \leq i \leq m$, be finite separable extensions of a given field F with $[L_i \colon F] = n_i$. Let $n := lcm(n_i \mid 1 \leq i \leq m)$. Let $L = \prod_{i=1}^{m} L_i$. Let $\alpha_i \in F^\times$ for all $i, 1 \leq i \leq m-1$. Then the element*

$$(\alpha_1^{n/n_1}, \alpha_2^{n/n_2}, \ldots, \alpha_{m-1}^{n/n_{m-1}}, \prod_{i=1}^{m-1} \alpha_i^{-n/n_m}) \in T_{L/F}(F)$$

actually belongs to $RT_{L/F}(F)$.

Proof. Let us consider $f(t) \in T_{L/F}(F(t))$ given by

$$f(t) = \left(\left(\frac{t+\alpha_1}{t+1}\right)^{n/n_1}, \left(\frac{t+\alpha_2}{t+1}\right)^{n/n_2}, \dots, \left(\frac{t+\alpha_{m-1}}{t+1}\right)^{n/n_{m-1}}, \prod_{i=1}^{m-1} \left(\frac{t+1}{t+\alpha_i}\right)^{n/n_m} \right).$$

Then for $t = 0$, we get

$$f(0) = (\alpha_1^{n/n_1}, \alpha_2^{n/n_2}, \dots, \alpha_{m-1}^{n/n_{m-1}}, \prod_{i=1}^{m-1} \alpha_i^{-n/n_m})$$

and for $t = \infty$, we get $f(\infty) = (1, 1, \dots, 1)$.

Hence $(\alpha_1^{n/n_1}, \alpha_2^{n/n_2}, \dots, \alpha_{m-1}^{n/n_{m-1}}, \prod_{i=1}^{m-1} \alpha_i^{-n/n_m})$ belongs to $RT_{L/F}(F)$.

□

Now we study R-trivial elements of $T_{L/F}(F)$ for complete discretely valued fields F where L is product of two degree p cyclic extensions.

Lemma 5.2.3. *Let F be a complete discretely valued field with residue field κ. Let L_1/F be a degree p unramified cyclic extension and L_2/F be a degree p ramified cyclic extension. Assume that $p \neq char(\kappa)$. Let $L = L_1 \times L_2$. Then $T_{L/F}(F)/R = \{1\}$.*

Proof. We can write $L_2 = F(\sqrt[p]{\pi})$ where π is a parameter in F. Then π is also a parameter in L_1. Let $(\mu_1, \mu_2) \in T_{L/F}(F)$. We can write $\mu_1 = u_1\pi^{q_1}$ and $\mu_2 = u_2(\sqrt[p]{\pi})^{q_2}$ where $u_1 \in L_1$ and $u_2 \in L_2$ are units and q_1, q_2 are integers. We have $1 = N_{L_1/F}(\mu_1)N_{L_2/F}(\mu_2) = N_{L_1/F}(u_1)\pi^{pq_1}N_{L_2/F}(u_2)(-1)^{n_p q_2}\pi^{q_2}$ where $n_p = 1$ if $p = 2$ and $n_p = 2$ if $p > 2$. Then $q_2 = -pq_1$. Also, $(-1)^{n_p q_2} = 1$ for any prime p.

By Lemma 5.2.2, $(\pi^{q_1}, \pi^{-q_1}) \in RT_{L/F}(F)$. Thus it is enough to consider the case when μ_1 and μ_2 are units in L_1 and L_2 respectively. Furthermore, since L_2/F is totally ramified, we have $\mu_2 = \mu_3^p\mu_4$ where $\mu_3 \in L_2$ is a unit and $\mu_4 \in F$ is a unit. We have $1 = N_{L_1/F}(\mu_1)[N_{L_2/F}(\mu_3)\mu_4]^p$. Let $\alpha = N_{L_2/F}(\mu_3)\mu_4 \in F$. Then $(\alpha, \alpha^{-1}) \in RT_{L/F}(F)$. Hence $(\mu_1\alpha, \mu_3^p\mu_4\alpha^{-1})$ is R-equivalent to (μ_1, μ_2). We have $N_{L_1/F}(\mu_1\alpha) = 1$ and $N_{L_2/F}(\mu_3^p\mu_4\alpha^{-1}) = 1$. Now, since L_1/F and L_2/F are cyclic extensions, by Lemma 2.4.2, $T_{L_1/F}(F)/R = \{1\}$ and $T_{L_2/F}(F)/R = \{1\}$. Thus $(\mu_1\alpha, \mu_3^p\mu_4\alpha^{-1})$ is in $RT_{L/F}(F)$ and we are done.

□

Lemma 5.2.4. *Let F be a complete discretely valued field with residue field κ. Let L_1/F and L_2/F be two degree p ramified cyclic extensions. Assume that $p \neq char(\kappa)$. Let $L = L_1 \times L_2$. Then $T_{L/F}(F)/R = \{1\}$.*

Proof. Let π be a parameter in F. Then we can write $L_1 = F(\sqrt[p]{\pi})$ and $L_2 = F(\sqrt[p]{v\pi})$ for some unit $v \in F^{\times}$. We can assume that $v \notin F^{\times p}$ otherwise both extensions are isomorphic and by Proposition 2.4.3, $T_{L/F}(F)/R \simeq T_{L_1/F}(F)/R$. By Lemma 2.4.2, since L_1/F is cyclic, we have $T_{L_1/F}(F)/R = \{1\}$. Hence we are done.

Let $(\mu_1, \mu_2) \in T_{L/F}(F)$. We can write $\mu_1 = u_1(\sqrt[p]{\pi})^{q_1}$ and $\mu_2 = u_2(\sqrt[p]{v\pi})^{q_2}$ where $u_1 \in L_1$ and $u_2 \in L_2$ are units and q_1, q_2 are integers. We have $1 = N_{L_1/F}(u_1)(-1)^{n_p q_1}\pi^{q_1} N_{L_2/F}(u_2)(-1)^{n_p q_2} v^{q_2}\pi^{q_2}$. We get that $q_2 = -q_1$. Consequently, $(-1)^{n_p q_1}(-1)^{n_p q_2} = 1$. Also, since the extensions are totally ramified, $N_{L_1/F}(u_1), N_{L_2/F}(u_2) \in F^{\times p}$. Hence $v^{q_2} \in F^{\times p}$ and p divides q_2 since $v \notin F^{\times p}$. Let $q_1 = pq$ for some integer q.

By Lemma 5.2.2, $((v\pi)^q, (v\pi)^{-q}) \in RT_{L/F}(F)$. Thus it is enough to show that $(\mu_1(v\pi)^{-q}, \mu_2(v\pi)^q) = (u_1 v^q, u_2) \in RT_{L/F}(F)$. Thus we can assume that μ_1 and μ_2 are units in L_1 and L_2 respectively. Furthermore, since L_2/F is totally ramified, we have $\mu_2 = \mu_3^p \mu_4$ where $\mu_3 \in L_2$ is a unit and $\mu_4 \in F$ is a unit. Continuing as in the proof of Lemma 5.2.3, we get that $(\mu_1, \mu_2) \in RT_{L/F}(F)$.

□

For the next lemma, we assume that the field F is complete discretely valued with residue field again a complete discretely valued field. Thus, we can apply this lemma to the branch fields $F_{P,\eta}$.

Lemma 5.2.5. *Let F be a complete discretely valued field with residue field κ a complete discretely valued field. Assume that κ', the residue field of κ, is algebraically closed or a finite field . Let L_1/F and L_2/F be two unramified cyclic extensions of degree p. Assume that $p \neq char(\kappa')$. Let $L = L_1 \times L_2$. Then $T_{L/F}(F)/R = \{1\}$.*

Proof. We first consider the case when κ' is algebraically closed. The unramified cyclic extensions L/F of degree p are in one-to-one correspondence with cyclic extensions l/κ of degree p, where l is the residue field of L ([Mil, Proposition 7.50, p-126]). Since κ' is algebraically closed, there is a unique cyclic extension of κ of degree pThus, in this case, there is only one unramified cyclic extension of F of degree p upto isomorphism. Hence, by Proposition 2.4.3, $T_{L/F}(F)/R \simeq T_{L_1/F}(F)/R = \{1\}$.

Now assume that κ' is a finite field.Let $\pi \in F^\times$ be a parameter. Then π is also a parameter in L_1 and L_2. Let $(\mu_1, \mu_2) \in T_{L/F}(F)$. We can write $\mu_i = u_i \pi^{q_i}$ for some units $u_i \in L_i$ and some integers q_i for $i = 1, 2$. Then $1 = N_{L_1/F}(u_1)\pi^{pq_1} N_{L_2/F}(u_2)\pi^{pq_2}$. Then $pq_1 = -pq_2$, hence $q_1 = -q_2 = q(say)$. By Lemma 5.2.2, $(\pi^q, \pi^{-q}) \in RT_{L/F}(F)$. Hence, we can assume that μ_1 and μ_2 are units.

Let l_1 and l_2 denote the residue fields of L_1 and L_2 respectively. Let $l = l_1 \times l_2$. We want to show $(\mu_1, \mu_2) \in RT_{L/F}(F)$. By [Gil04, Proposition 2.2.], it is enough to show that $T_{l/\kappa}(\kappa)/R = \{1\}$.

Suppose at least one of the extensions l_1/κ or l_2/κ is totally ramified. Then by Lemma 5.2.3 and Lemma 5.2.4, we get that $T_{l/\kappa}(\kappa)/R = \{1\}$.

Suppose that l_1/κ and l_2/κ are both unramified, then they are both isomorphic since κ' is a finite field. Hence,again by Proposition 2.4.3, we conclude that $T_{l/\kappa}(\kappa)/R = \{1\}$.

□

Proposition 5.2.6. *Let F be a complete discretely valued field with residue field κ a complete discretely valued field. Assume that κ', the residue field of κ, is algebraically closed or a finite field . Let L_1/F and L_2/F be two cyclic extensions of degree p. Assume that $p \neq char(\kappa')$. Let $L = L_1 \times L_2$. Then $T_{L/F}(F)/R = \{1\}$.*

Proof. The proof follows from Lemma 5.2.3, Lemma 5.2.4 and Lemma 5.2.5.

□

Corollary 5.2.7. *Let K be a complete discretely valued field with residue field κ. Let F be the function field of a smooth, projective, geometrically integral curve over K. Assume that κ is either algebraically closed or a finite field. Let L_1, L_2 be two degree p cyclic extensions of F. Assume that $p \neq char(\kappa)$. Let $L = L_1 \times L_2$. Then for any regular proper model $\mathscr{X}$ and any choice of $\mathcal{P}$, $Ш_{\mathcal{P}}(F, T_{L/F}) = 0$ and $Ш_X(F, T_{L/F}) = 0$.*

Proof. Let $\mathscr{X}$ be a regular proper model of F and X be the special fibre. Let (P, η) be a branch. We can assume that both $L_{i,P,\eta}$ are fields for $i = 1, 2$. Otherwise, if $L_{i,P,\eta}$ is not a field for some i, then $L_{i,P,\eta}$ is a product of p copies of $F_{P,\eta}$. Thus, in this case, we get that $T_{L/F}(F_{P,\eta})/R = \{1\}$ by Lemma 2.4.6. Then by Proposition 5.2.6, we get that for any branch (P, η), $T_{L/F}(F_{P,\eta})/R = \{1\}$. Hence, for any choice of $\mathcal{P}$, $Ш_{\mathcal{P}}(F, T_{L/F}) = 0$

by [CTHH+19, Theorem 3.1.]. Thus $Ш_X(F, T_{L/F}) = 0$ by [HHK15a, Corollary 5.9.].

□

As a consequence, we have the following theorem:

Theorem 5.2.8. *Let K be a complete discretely valued field with residue field κ. Let F be the function field of a smooth, projective, geometrically integral curve over K. Assume that κ is either algebraically closed or a finite field. Let L_1, L_2 be two degree p cyclic extensions of F. Assume that $p \neq char(\kappa)$. Let $L = L_1 \times L_2$. Then $Ш(F, T_{L/F}) = 0$.*

Proof. The result follows from Corollary 5.2.7 and Theorem 5.1.3.

□

Remark 5.2.9. We note that we do not need any assumptions on the graph associated to the semi-global field F here.

5.2.2 More general multinorm tori

Let K be a complete discretely valued field with residue field κ algebraically closed. Let F be the function field of a smooth, projective, geometrically integral curve over K. Let $T_{L/F}$ be the multinorm torus associated to cyclic extensions L_i/F for $1 \leq i \leq m$ with degrees $[L_i\colon F]$ prime. We study $T_{L/F}(F_{P,\eta})/R$ for branches (P, η).

Notation 5.2.10. *For a discrete valuation ν on F, let $F_{\nu,h}$ be the henselization of F at ν.*

We start with some results concerning tori defined over semi-global fields.

Lemma 5.2.11. *Let T be a torus defined over a semi-global field F. Suppose that:*

- *a) the natural map $T(F_{\eta,h}) \to T(F_\eta)/R$ is surjective for all codimension zero points η of X, and,*
- *b) the natural map $T(F_P) \to T(F_{P,\eta})/R$ is surjective for all branches (P, η).*

Then for all proper open subsets U of X_η, the map $T(F_U) \to T(F_\eta)/R$ is surjective.

Proof. Let U be a proper open subset of X_η and $\mu \in T(F_\eta)$. Then by the assumption a), there exists $\mu_h \in T(F_{\eta,h})$ which maps to μ in $T(F_\eta)/R$. Thus, replacing μ by μ_h, we may assume that $\mu \in T(F_{\eta,h})$. Since, by [HHK14, Lemma 3.2.1.], the field $F_{\eta,h}$ is the filtered direct limit of the fields F_V, where V ranges over the nonempty open proper subsets of X_η, there exists a nonempty open proper subset V of X_η such that $\mu \in T(F_V)$. By taking intersection with U, if needed, we may assume that $V \subseteq U$.
If $V = U$, there is nothing to prove. Suppose that $V \neq U$. Let $P \in U \setminus V$. By assumption, the map $T(F_P) \to T(F_{P,\eta})/R$ is surjective. Let μ_P be an element in $T(F_P)$ mapping to the class of μ in $T(F_{P,\eta})/R$. Then $\mu = \mu_P \alpha_{P,\eta}$ for some $\alpha_{P,\eta} \in RT(F_{P,\eta})$. By Lemma 2.4.11, there exists $\alpha_V \in RT(F_V)$ and $\alpha_P \in RT(F_P)$ with $\alpha_{P,\eta} = \alpha_P \alpha_V$ in $RT(F_{P,\eta})$. Then $\mu = \mu_P \alpha_P \alpha_V$. Hence $\mu\alpha_V^{-1} = \mu_P \alpha_P \in T(F_V) \cap T(F_P)$. By [HHK15b, Lemma 2.12.] and [HHK15b, Proposition 3.9.], $\mu\alpha_V^{-1} \in T(F_{V\cup\{P\}})$. Also, $\mu\alpha_V^{-1}$ maps to the equivalence class of μ in $T(F_\eta)/R$ since $\alpha_V \in RT(F_V) \subseteq RT(F_\eta)$. Since $U \setminus V$ is a finite set, doing this process finitely many times, we get the result.

□

Lemma 5.2.12. *Let T be a torus defined over a semi-global field F. Let $\mathscr{X}$ be a regular proper model of F and X be the reduced special fiber. Let $\eta \in X$ be a codimension zero point. Suppose that :*

- *the natural map $T(F_{\eta,h}) \to T(F_\eta)/R$ is surjective, and,*
- *for every closed point P of X_η, the natural maps $T(F_\eta) \to T(F_{P,\eta})/R$ and $T(F_P) \to T(F_{P,\eta})/R$ are surjective.*

Then for any proper open subset U of X_η, the natural map

$$T(F_U) \to T(F_{P,\eta})/R$$

is surjective.

Proof. The proof follows immediately from Lemma 5.2.11 since the natural map $T(F_U) \to T(F_{P,\eta})/R$ factors through $T(F_\eta)/R$.

□

Proposition 5.2.13. *Let F be a semi-global field with a regular proper model $\mathscr{X}$ and the special fibre X. Let T be a torus defined over F. Assume that:*

- *the graph associated to F is a tree,*

- *the natural map $T(F_{\eta,h}) \to T(F_\eta)/R$ is surjective,*

- *the natural map $T(F_\eta) \to T(F_{P,\eta})/R$ is surjective for all possible η and branches (P,η) and all choices of $\mathcal{P}$, and*

- *the natural map $T(F_P) \to T(F_{P,\eta})/R$ is surjective for all possible P and branches (P,η) and all choices of $\mathcal{P}$.*

Then $Ш_X(F,T) = 0$.

Proof. By [HHK15a, Corollary 5.9.], we just need to show that for every choice of $\mathcal{P}$, $Ш_{\mathcal{P}}(F,T)$ is trivial. This immediately follows from Lemma 5.2.12 and Corollary 3.3.4. □

Lemma 5.2.14. *Let F be a field. Let $T_{L/F}$ be the multinorm torus associated to finite separable extensions L_i/F for $1 \leq i \leq m$ with $[L_i \colon F] = n_i$. Let $n := lcm(n_i \mid 1 \leq i \leq m)$.*
Assume that:

- *for all $(\mu_1, \dots, \mu_m) \in T_{L/F}(M)$, there exists $(\mu'_1, \dots, \mu'_m) \in T_{L/F}(F)$ such that $N_{L_i \otimes_F M/M}(\mu_i \mu_i^{'-1}) \in M^{\times n}$.*

- *The natural maps*

$$T_{L_i/F}(F) \to T_{L_i/F}(M)/R$$

are surjective for $1 \leq i \leq m$.

Then the natural map $T_{L/F}(F) \to T_{L/F}(M)/R$ is surjective.

Proof. Let $(\mu_1, \dots, \mu_m) \in T_{L/F}(M)$. Then by assumption, there exists $(\mu'_1, \dots, \mu'_m) \in T_{L/F}(F)$ with $N_{L_i \otimes_F M/M}(\mu_i \mu_i^{'-1}) = \alpha_i^n$ for some $\alpha_i \in M$ for $1 \leq i \leq m$.
Let us consider $x_i = \mu_i \mu_i^{'-1}$. Then it is enough to show that the class of $(x_1, \dots, x_m)$ is in the image of the map $T_{L/F}(F) \to T_{L/F}(M)/R$.

Now we consider

$$(\beta_1, \ldots, \beta_{m-1}, \beta_m) = (\alpha_1^{n/n_1}, \alpha_2^{n/n_2}, \ldots, \alpha_{m-1}^{n/n_{m-1}}, \prod_{i=1}^{m-1} \alpha_i^{-n/n_m})$$

in $T_{L/F}(M)$. Then by Lemma 5.2.2, $(\beta_1, \ldots, \beta_m) \in RT_{L/F}(M)$. Thus, it is enough to show that the class of $(x_1\beta_1^{-1}, \ldots, x_m\beta_m^{-1})$ lies in the image of the map $T_{L/F}(F) \to T_{L/F}(M)/R$. For $1 \leq i \leq m$, $x_i\beta_i^{-1} \in T_{L_i/F}(M)$. Now, by assumption, there exists $\gamma_i \in T_{L_i/F}(F)$ mapping to the class of $x_i\beta_i^{-1} \in T_{L_i/F}(M)/R$ for $1 \leq i \leq m$. Then $(\gamma_1, \ldots, \gamma_m)$ maps to the class of $(x_1\beta_1^{-1}, \ldots, x_m\beta_m^{-1})$ in $T_{L/F}(M)/R$. Hence we are done.

□

Lemma 5.2.15. *Let F be a Henselian discretely valued field with residue field κ and with completion $\hat{F}$. Assume that F has a primitive root of unity ρ_{n^2}. Let L/F be a finite separable field extension of degree n which is coprime to char(κ). Let $T_{L/F}$ be the associated norm one torus. Then the natural map*

$$T_{L/F}(F) \to T_{L/F}(\hat{F})/R$$

is surjective.

Proof. Since F is Henselian, L is also Henselian and $L \otimes_F \hat{F}$ is a field and completion of L. Let $u \in T_{L/F}(\hat{F})$. Then u is a unit. Let $v \in L$ be close to u in $L \otimes_F \hat{F}$. Since n is coprime to char(κ), we have $u = va^{n^2}$ for some $a \in L \otimes_F \hat{F}$. Since $N_{L/F}(v)[N_{L\otimes_F\hat{F}}(a)]^{n^2} = N_{L\otimes_F\hat{F}}(u) = 1$, $N_{L/F}(v) \in \hat{F}^{\times n^2}$. Since F is Henselian, there exists $b \in F$ such that $N_{L/F}(v) = b^{n^2}$. Then $N_{L/F}(vb^{-n}) = 1$. We have $u(vb^{-n})^{-1} = (a^n b^{-1})^n \in T_{L/F}(\hat{F})$. Since $N_{L\otimes_F\hat{F}}(a^n b^{-1})^n = 1$, we have $N_{L\otimes_F\hat{F}}(a^n b^{-1}) = \rho_{n^2}^{ni}$ for some i, $0 \leq i \leq n-1$. Then $N_{L\otimes_F\hat{F}}(a^n b^{-1}\rho_{n^2}^{-i}) = 1$. Let us consider $w = vb^{-n}\rho_{n^2}^{-i}$. Then $w \in T_{L/F}(F)$ and $uw^{-1} = (a^n b^{-1}\rho_{n^2}^{-i})^n$. Since $(a^n b^{-1}\rho_{n^2}^{-i})^n \in RT_{L/F}(\hat{F})$ [CTS77, Proposition 15], we conclude that $uw^{-1} \in RT_{L/F}(\hat{F})$ and hence we are done.

□

Lemma 5.2.16. *Let F be a Henselian discretely valued field and $\hat{F}$ be its completion. Let κ be the residue field of F. Let $L_1, L_2, \ldots, L_m$ be finite separable extensions of F with $[L_i : F] = n_i$. Assume that F has all primitive* n_i^2-th roots of unity for all i, $1 \leq i \leq m$. Assume that all n_i are coprime to char(κ). Let $L = \prod_{i=1}^{m} L_i$.

Then

$$T_{L/F}(F) \to T_{L/F}(\hat{F})/R$$

is surjective.

Proof. Let $\mu \in T_{L/F}(\hat{F})/R$. Then $\mu = (\mu_1, \ldots, \mu_m)$ with $\mu_i \in L_i \otimes_F \hat{F}$ and $\prod_{i=1}^{m} N_{L_i \otimes_F \hat{F}/\hat{F}}(\mu_i) = 1$. Let $n = \text{lcm}(n_i \mid 1 \leq i \leq m)$.

We first show that for all $(\mu_1, \ldots, \mu_m) \in T_{L/F}(\hat{F})$, there exists $(\mu'_1, \ldots, \mu'_m) \in T_{L/F}(F)$ such that $N_{L_i \otimes_F \hat{F}/\hat{F}}(\mu_i \mu_i'^{-1}) \in \hat{F}^{\times n}$. Since Ł$_i \otimes_F \hat{F}$ is the completion of L_i for $1 \leq i \leq m-1$, there exists $\mu'_i \in L_i$ with $\mu_i = \mu'_i \theta_i^n$ for some $\theta_i \in (L_i \otimes_F \hat{F})^{\times n}$. Let $\lambda = \prod_{i=1}^{m-1} N_{L_i/F}(\mu'_i) \in F$. Since $\lambda \prod_{i=1}^{m-1} N_{L_i \otimes_F \hat{F}/\hat{F}}(\theta_i)^n = \prod_{i=1}^{m-1} N_{L_i \otimes_F \hat{F}/\hat{F}}(\mu_i) = N_{L_m \otimes_F \hat{F}/\hat{F}}(\mu_m)^{-1}$, λ is a norm from $L_m \otimes_F \hat{F}/\hat{F}$. Since F is Henselian and $\hat{F}$ is the completion of F, λ is a norm from L_m/F by [Art69, Thm 2.2.1]. Let $\mu'_m \in L_m$ with $N_{L_m/\hat{F}}(\mu'_m) = \lambda$. Then $\mu' = (\mu'_1, \ldots, \mu'_m) \in T_{L/F}(F)$ and $N_{L_i \otimes_F \hat{F}/\hat{F}}(\mu_i \mu_i'^{-1}) \in \hat{F}^{\times n}$ for $1 \leq i \leq m$.

Now, by Lemma 5.2.15, we get that the maps $T_{L_i/F}(F) \to T_{L_i/F}(\hat{F})/R$ are surjective for all i, $1 \leq i \leq m$. Hence, by Lemma 5.2.14, $T_{L/F}(F) \to T_{L/F}(\hat{F})/R$ is surjective.

□

Remark 5.2.17. If all the extensions L_i/F, $1 \leq i \leq m$ are cyclic of degree n_i, then we do not need to assume that F has all primitive n_i^2-th roots of unity for all i, $1 \leq i \leq m$. Since, by Lemma 2.4.2, $T_{L_i/F}(\hat{F})/R = \{1\}$.

Lemma 5.2.18. *Let K be a complete discretely valued field with residue field κ. Let F be the function field of a smooth, projective, geometrically integral curve over K. Let $\mathscr{X}$ be a regular proper model of F and X be the reduced special fibre. Let $\mathcal{P} \in X$ be a nonempty finite set of closed points containing all the intersection points. Let η be the generic point of one of the components of X and let (P, η) be a branch. Let L_η/F_η be a field extension of degree n with n coprime to char(κ). Assume that $L_{P,\eta} = L_\eta \otimes_F F_{p,\eta}$ is a field. Then for any $\lambda \in N_{L_{P,\eta}/F_{P,\eta}}((L_{P,\eta})^\times)$ and an integer N coprime to char(κ), there exists a $\lambda' \in N_{L_\eta/F_\eta}((L_\eta)^\times)$ such that $\lambda\lambda'^{-1}$ is a N^{th} power in $F_{P,\eta}^\times$.*

Proof. Let $\pi \in F_\eta$ be a parameter. Since L_η/L_η^{nr} is totally ramified, we can write L_η as $L_\eta = L_\eta^{nr}(\sqrt[e]{u\pi})$, where u is a unit in L_η^{nr} ([CF67, Proposition 1, p-32]). Then $L_{P,\eta} = L_{P,\eta}^{nr}(\sqrt[e]{u\pi})$.

Let $L(\eta)$ and $F(\eta)$ denote the residue field of L_η and F_η respectively. Similarly, $L(P,\eta)$ and $F(P,\eta)$ denote the residue field of $L_{P,\eta}$ and $F_{P,\eta}$ respectively. Then $F(P,\eta)$ and $L(P,\eta)$ are completions of $F(\eta)$ and $L(\eta)$ respectively. Let L_η^{nr} and $L_{P,\eta}^{nr}$ be the maximal unramified subextensions of L_η/F_η and $L_{P,\eta}/F_{P,\eta}$ respectively. Since $[L_{P,\eta}\colon L_{P,\eta}^{nr}] = [L_\eta\colon L_\eta^{nr}] = e$ and $[L_{P,\eta}\colon F_{P,\eta}] = n = [L_\eta\colon F_\eta]$, we get that $[L(\eta)\colon F(\eta)] = [L_\eta^{nr}\colon F_\eta] = n/e = [L_{P,\eta}^{nr}\colon F_{P,\eta}] = [L(P,\eta)\colon F(P,\eta)]$.

Let $\mu \in L_{P,\eta}^\times$ with $\lambda = N_{L_{P,\eta}/F_{P,\eta}}(\mu)$. Then we can write $\mu = \theta(\sqrt[e]{u\pi})^i$, where $\theta \in L_{P,\eta}$ is a unit and i is an integer. Since $L_{P,\eta}$ and $L_{P,\eta}^{nr}$ have same residue field $L(P,\eta)$, and since N is coprime to char(κ), by Hensel's lemma, there exists θ' a unit in $L_{P,\eta}^{nr}$ and $\alpha \in L_{P,\eta}$ with $\theta = \theta'(\alpha)^N$. Hence without loss of generality we can assume that $\theta \in L_{P,\eta}^{nr}$. We have

$$\begin{aligned}\lambda = N_{L_{P,\eta}/F_{P,\eta}}(\mu) &= [N_{L_{P,\eta}^{nr}/F_{P,\eta}}(\theta)]^e[N_{L_{P,\eta}/F_{P,\eta}}(\sqrt[e]{u\pi})]^i \\ &= [N_{L_{P,\eta}^{nr}/F_{P,\eta}}(\theta)]^e[N_{L_\eta/F_\eta}(\sqrt[e]{u\pi})]^i.\end{aligned}$$

Let $\overline{\theta}$ be the image of θ in the residue field $L(P,\eta)$. We have $\overline{N_{L_{P,\eta}^{nr}/F_{P,\eta}}(\theta)} = N_{L(P,\eta)/F(P,\eta)}(\overline{\theta}) \in F(P,\eta)$. Since $L(P,\eta)$ is completion of $L(\eta)$ and by Hensel's lemma, there exists $\overline{\phi} \in L(\eta)$ such that $\overline{\theta}\cdot\overline{\phi}^{-1}$ is a N^{th} power in $L(P,\eta)$. Let $\phi \in L_\eta^{nr}$ such that $\overline{\phi}$ is the image of ϕ in $L(\eta)$. Then $N_{L(\eta)/F(\eta)}(\overline{\phi}) = \overline{N_{L_\eta^{nr}/F_\eta}(\phi)}$. We get that $\overline{N_{L_{P,\eta}^{nr}/F_{P,\eta}}(\theta\phi^{-1})}$ is a N^{th} power in $F(P,\eta)$. Hence, by Hensel's lemma,

$$N_{L_{P,\eta}^{nr}/F_{P,\eta}}(\theta\phi^{-1}) = N_{L_{P,\eta}^{nr}/F_{P,\eta}}(\theta)[N_{L_\eta^{nr}/F_\eta}(\phi)]^{-1}$$

is a N^{th} power in $F(P,\eta)$. Now $\lambda' = N_{L_\eta^{nr}/F_\eta}(\phi)$ has the desired property. $\square$

Lemma 5.2.19. *Let K be a complete discretely valued field with residue field κ and F be the function field of a smooth, projective, geometrically integral curve over K. Assume that κ is algebraically closed. Assume that $L_i/F, 1 \leq i \leq m$, are cyclic extensions with $[L_i : F] = p$, where p is a prime number. Assume that $p \neq char(\kappa)$. Let $T_{L/F}$ denote the multinorm torus associated to the extensions L_i/F. Let η be the generic point of one of the components of $X \setminus \mathcal{P}$ and let (P,η) be a branch. Assume that for at least one i, $L_{i,P,\eta} = L_i \otimes_F F_{P,\eta}$ is unramified over $F_{P,\eta}$. Then the natural map*

$$T_{L/F}(F_\eta) \to T_{L/F}(F_{P,\eta})/R$$

is surjective.

Proof. We can assume that all $L_{i,P,\eta}$ are fields. Otherwise, if $L_{i,P,\eta}$ is not a field for some i, then $L_{i,P,\eta}$ is a product of two copies of $F_{P,\eta}$. Thus, in this case, we get that $T_{L/F}(F_{P,\eta})/R = \{1\}$ by Lemma 2.4.6. Hence the conclusion of the lemma holds.

Let $(\mu_1, \ldots, \mu_{m-1}, \mu_m) \in T_{L/F}(F_{P,\eta})$. Let $\lambda_i = N_{L_{i,P,\eta}/F_{P,\eta}}(\mu_i)$ for all i, $1 \leq i \leq m$. Then $\prod_{i=1}^{m} \lambda_i = 1$. Without loss of generality, we can assume that the extension $L_{m,P,\eta}/F_{P,\eta}$ is unramified. Let $\pi \in F_\eta$ be a parameter. Then π is also a parameter in $F_{P,\eta}$ and $L_{m,P,\eta}$. By Lemma 5.2.18, for i, $1 \leq i \leq m-1$, we can find $\mu'_i \in L_{i,\eta}$ with such that $\lambda_i[N_{L_{i,\eta}/F_\eta}(\mu'_i)]^{-1} = \alpha_i^p$ for some $\alpha_i \in F_{P,\eta}^\times$. For $1 \leq i \leq m-1$, let $\lambda'_i = N_{L_{i,\eta}/F_\eta}(\mu'_i)$. Since $\prod_{i=1}^{m-1} \lambda_i = (\lambda_m)^{-1}$ and $\lambda_m = N_{L_{m,P,\eta}/F_{P,\eta}}(\mu_m)$, p divides $\text{val}_\pi(\prod_{i=1}^{m-1} \lambda_i)$ where val_π denotes the valuation on $F_{P,\eta}$ with parameter π. Since, for $1 \leq i \leq m-1$, $\lambda_i {\lambda'}_i^{-1} = \alpha_i^p$ with $\alpha_i \in F_{P,\eta}^\times$, p also divides $\text{val}_\pi(\prod_{i=1}^{m-1} {\lambda'}_i^{-1})$.

Let $F(\eta)$ and $F(m,\eta)$ be the residue field of F_η and $L_{m,\eta}$ respectively. Since κ is algebraically closed, $F(\eta)$ is a C_1 field([GS06, Thm 6.2.8, p-143]). Thus the norm map $N_{L(m,\eta)/F(\eta)}$ is surjective. Hence the norm map $N_{L_{m,\eta}/F_\eta}$ is surjective on units. Since p divides $\text{val}_\pi(\prod_{i=1}^{m-1} {\lambda'}_i^{-1})$, we can find a $\mu'_m \in L_{m,\eta}^\times$ with $N_{L_{m,\eta}/F_\eta}(\mu'_m) = \prod_{i=1}^{m-1} \lambda'^{-1}$. Then $(\mu'_1, \ldots, \mu'_m) \in T_{L/F}(F_\eta)$ and $N_{L_{i,P,\eta}/F_{P,\eta}}(\mu_i {\mu'}_i^{-1}) \in F_{P,\eta}^{\times p}$ for $1 \leq i \leq m$. Also, since the extensions $L_{i,P,\eta}/F_{P,\eta}$ are cyclic, $T_{L/F}(F_{P,\eta})/R = \{1\}$ by Lemma 2.4.2. Hence, the result follows by Lemma 5.2.14.

□

Lemma 5.2.20. *Let K be a complete discretely valued field with residue field κ and F be the function field of a smooth, projective, geometrically integral curve over K. Assume that κ is algebraically closed. Assume that $L_i/F, 1 \leq i \leq m$, are quadratic cyclic extensions. Assume that char$(\kappa) \neq 2$. Let $T_{L/F}$ denote the multinorm torus associated to the extensions L_i/F. Let η be the generic point of one of the components of $X \setminus \mathcal{P}$ and let (P, η) be a branch.*

Then the natural map

$$T_{L/F}(F_\eta) \to T_{L/F}(F_{P,\eta})/R$$

is surjective.

Proof. We can assume that all $L_{i,P,\eta}$ are fields. Otherwise, if $L_{i,P,\eta}$ is not a field for some i, then $L_{i,P,\eta}$ is a product of two copies of $F_{P,\eta}$. Thus, in this case, we get that $T_{L/F}(F_{P,\eta})/R = \{1\}$ by Lemma 2.4.6. Hence the conclusion of the lemma holds.

Let $\pi \in F_\eta$ be a parameter. Then π is also a parameter in $F_{P,\eta}$. Let δ be a parameter in $\kappa(\eta)$, the residue field of F_η. Then δ is also a parameter in the residue field of $F_{P,\eta}$.

Case A Let us assume that at least one of the extensions $L_{i,P,\eta}/F_{P,\eta}$ is unramified. Then we have the result by Lemma 5.2.19.

Case B Now we assume that all the extensions $L_{i,P,\eta}/F_{P,\eta}$ are ramified. There are only two non-isomorphic quadratic cyclic extensions of $F_{P,\eta}$ that are ramified, given by $F_{P,\eta}(\sqrt{\pi})$ and $F_{P,\eta}(\sqrt{\delta\pi})$. In this case, by Proposition 2.4.5, it is enough to consider the case when $m = 2$ and $L_{1,P,\eta}/F_{P,\eta}$ and $L_{2,P,\eta}/F_{P,\eta}$ are both ramified quadratic cyclic extensions. In this case, by Lemma 5.2.4, we have $T_{L/F}(F_{P,\eta})/R = \{1\}$. □

Lemma 5.2.21. *Let K be a complete discretely valued field with residue field κ and F be the function field of a smooth, projective, geometrically integral curve over K. Assume that $L_i/, 1 \leq i \leq m$, are finite Galois extensions of degree n_i, where n_i is a natural number coprime to char(κ). Let $T_{L/F}$ denote the multinorm torus associated to the extensions L_i/F. Let $\mathscr{X}$ be a regular proper model of F such that the union of $ram_{\mathscr{X}}(L_i/F)$ and the reduced special fibre X is a union of regular curves with normal crossings. Assume that the natural maps*

$$T_{L_i/F}(F_P) \to T_{L_i/F}(F_{P,U})/R$$

are surjective for $1 \leq i \leq m$. Let (P, U) be a branch.

Then the natural map

$$T_{L/F}(F_P) \to T_{L/F}(F_{P,U})/R$$

is surjective.

Proof. Let $\mu_i \in L_{i,P,U}$ for $1 \leq i \leq m$ with $N_{L_{i,P,U}/F_{P,U}}(\mu_i) = \lambda_i$ such that $\prod_{i=1}^{m} \lambda_i = 1$. Let $n := \text{lcm}(n_i \mid 1 \leq i \leq m)$. By Lemma 4.2.7, we can write $\lambda_i = \lambda'_i \cdot (\alpha_i)^n$ for some $\alpha_i \in F_{P,U}$, where $\lambda'_i = u_i \pi_1^{s_i} \pi_2^{t_i}$ for some $u_i \in F_P$ and integers s_i, t_i. By Theorem 4.2.9, we can choose $\mu'_i \in L_{i,P}$ with $N_{L_{i,P}/F_P}(\mu'_i) = \lambda'_i$ for $1 \leq i \leq m-1$. Similarly, we choose $\mu'_m \in L_{m,P}$ with $N_{L_{m,P}/F_P}(\mu'_m) = (\prod_{i=1}^{m-1} \lambda'_i)^{-1}$. Then $(\mu'_1, \mu'_2, \ldots, \mu'_m) \in T_{L/F}(F_P)$. Now the result follows from Lemma 5.2.14.

□

Corollary 5.2.22. *Let K be a complete discretely valued field with residue field κ and F be the function field of a smooth, projective, geometrically integral curve over K. Assume that κ is algebraically closed. Assume that $L_i/F, 1 \leq i \leq m$, are finite Galois extensions of degree n_i, where n_i is a natural number coprime to char(κ). Let $T_{L/F}$ denote the multinorm torus associated to the extensions L_i/F. Let $\mathscr{X}$ be a regular proper model of F such that the union of ram$_{\mathscr{X}}(L_i/F)$ and the reduced special fibre X is a union of regular curves with normal crossings. Let (P, U) be a branch.*

Then the natural map

$$T_{L/F}(F_P) \to T_{L/F}(F_{P,U})/R$$

is surjective.

Proof. Since κ is algebraically closed by Lemma 4.4.1, we have: $T_{L_i/F}(F_{P,U})/R = \langle \rho_{n_i} \rangle$ for all $i; 1 \leq i \leq m$, where ρ_{n_i} is a primitive n_i^{th} root of unity in F. Thus the natural maps

$$T_{L_i/F}(F) \to T_{L_i/F}(F_{P,U})/R$$

are surjective for $1 \leq i \leq m$ since $\rho_{n_i} \in T_{L_i/F}(F)$. Thus, the result follows from (5.2.21).

□

Theorem 5.2.23. *Let K be a complete discretely valued field with residue field κ and F be the function field of a smooth, projective, geometrically integral curve over K. Assume that κ is algebraically closed. Let $\mathscr{X}$ be a regular proper model of F such that the union of ram$_{\mathscr{X}}(L_i/F)$ and the reduced special fibre X is a union of regular curves with normal crossings. Let $L_i/F, 1 \leq i \leq m$, be quadratic cyclic extensions. Assume that* *char*$(\kappa) \neq 2$. *Let* $T_{L/F}$ *denote the multinorm torus associated to the extensions* L_i/F. *If the graph associated to* F *is a tree then* $Ш_X(F, T_{L/F}) = 0$.

Proof. The result follows from Proposition 5.2.13, Lemma 5.2.16, Lemma 5.2.20, and Corollary 5.2.22.

□

Theorem 5.2.24. *Let* K *be a complete discretely valued field with residue field* κ *algebraically closed. Let* F *be the function field of a smooth, projective, geometrically integral curve over* K. *Let* $L_i/F, 1 \leq i \leq m$, *be quadratic cyclic extensions. Assume that char*$(\kappa) \neq 2$. *Let* $L = \prod_{i=1}^{m} L_i$. *If the graph associated to* F *is a tree then* $Ш(F, T_{L/F}) = 0$.

Proof. Let $\mathscr{X}$ be a regular proper model of F. By [Lip75, p-193], there exists a sequence of blow-ups $\mathscr{X}_0 \to \mathscr{X}$ centered at closed points of $\mathscr{X}$ such that the union of $ram_{\mathscr{X}}(L/F)$ and the reduced special fibre X_0 of $\mathscr{X}_0$ is a union of regular curves with normal crossings. Now by Theorem 5.1.3, it is enough to show that for any blowup $\mathscr{Y}$ of $\mathscr{X}_0$, $Ш_{\mathscr{Y}}(F, T_{L/F}) = 0$. Thus the result follows from Theorem 5.2.23.

□

Chapter 6

Examples of failure of local-global principle for norm one tori and multinorm tori over semi-global fields

In this chapter, we give examples of failure of local-global principle for norm one tori (see Corollary 6.0.11) and multinorm tori (see Corollary 6.0.12) over semi-global fields. We again use field patching and R-equivalence.

Let K be a complete discretely valued field with residue field algebraically closed. Colliot-Thélène, Parimala and Suresh ([CTPS16, Section 3.1. & Proposition 5.9.]) constructed a function field of a smooth, projective, geometrically integral curve over K and a Galois extension L/F with Galois group $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ such that the local-global principle fails for the norm one torus $T_{L/F}$ associated to L/F. They use higher reciprocity laws to detect nontrivial elements in $Ш(F, T_{L/F})$. In this section, we produce examples of Galois extensions L/F with Galois group $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$ and using patching techniques, we show that $Ш(F, T_{L/F})$ is nontrivial.

***Multinorm tori over number fields and semi-global fields**:*

Let k be a number field and L_1, L_2 be two finite Galois extensions of k. Let $L = L_1 \times L_2$ and let $T_{L/k}$ be the multinorm torus associated to the extensions L_1 and L_2 over k. If L_1 and L_2 are linearly disjoint, then Demarche and Wei ([DW14, Theorem 1]) proved that the local-global principle holds for $T_{L/k}$. In this section, we also give an example to show that a similar result does not hold in general for function fields of curves *over a complete discretely valued field.*

Proposition 6.0.1. *Let A be a unique factorization domain and F be its fraction field. Let L/F be a finite Galois extension and B be the integral closure of A in L. Suppose that B is a unique factorization domain. Then every element in $T_{L/F}(F)$ can be written as $s\theta$ for some $s \in RT_{L/F}(L)$ and $\theta \in B$ a unit.*

Proof. Let $\lambda \in T_{L/F}(F)$. Then $\lambda \in L^\times$ and $N_{L/F}(\lambda) = 1$. Since L is the fraction field of B, $\lambda = \alpha\beta^{-1}$ for some $\alpha, \beta \in B$. Since $N_{L/F}(\lambda) = 1$, $N_{L/F}(\alpha) = N_{L/F}(\beta)$. Let $p \in B$ be a prime. Since A is a unique factorization domain, $pB \cap A = qA$ for some prime $q \in A$ and $N_{L/F}(p) = vq^r$ for some unit $v \in A$. Suppose that p divides α in B. Then $N_{L/F}(p)$ divides $N_{L/F}(\alpha)$ in A and hence q divides $N_{L/F}(\alpha)$. Since $N_{L/F}(\alpha) = N_{L/F}(\beta)$, there exists a prime $p' \in B$ such that p' divides β and $p'B \cap A = qA$. Since L/F is a Galois extension, there exists $\sigma \in \text{Gal}(L/F)$ such that $p = w\sigma(p')$ for some unit $w \in B$. Write $\alpha = p\alpha'$ and $\beta = p'\beta'$. Then $\lambda = \alpha\beta^{-1} = pp'^{-1}\alpha'\beta'^{-1} = \sigma p' p'^{-1} w\alpha'\beta'^{-1}$. Since B is a unique factorization domain, the proposition follows by induction on the number of prime factors of α in B. □

Proposition 6.0.2. *Let A be a complete regular local ring of dimension 2 with maximal ideal (π, δ), fraction field F and residue field κ. Let n be a positive integer which is coprime to char(κ). Let $L = F(\sqrt[n]{\pi}, \sqrt[n]{\delta})$. Suppose that F contains a primitive n^2- th root of unity ρ. Then*

$$T_{L/F}(F) = RT_{L/F}(F)\langle\rho\rangle.$$

Proof. Let B be the integral closure of A in L. Then B is a regular local ring of dimension 2 with fraction field L and residue field κ ([PS14, Corollary 3.3.]). Let $\lambda \in T_{L/F}(F)$. Then $\lambda \in L^\times$ with $N_{L/F}(\lambda) = 1$. Then by Proposition 6.0.1, there exists $s \in RT_{L/F}(F)$ and a unit $\theta \in B$ such that $\lambda = s\theta$. Since the residue fields of A and B are equal, there exists $\theta_1 \in A$ such that $\theta \equiv \theta_1$ modulo the maximal ideal of B. Since n is coprime to char(κ), by Hensel's lemma, we have $\theta = \theta_1\alpha^{n^2}$ for some unit $\alpha \in B$. Let $s_1 = N_{L/F}(\alpha)^{-1}\alpha^{n^2} \in L$. Then by Lemma 2.4.7, $s_1 \in RT_{L/F}(F)$. Let $a = \theta_1 N_{L/F}(\alpha) \in F$. Then $\theta = as_1$. Thus $\lambda = s\theta = sas_1 = ss_1a$. Since $N_{L/F}(\lambda) = 1 = N_{L/F}(s) = N_{L/F}(s_1)$, $1 = N_{L/F}(a) = a^{n^2}$. Thus $a \in \langle\rho\rangle$. Hence $\lambda = ss_1a \in RT_{L/F}(F)\langle\rho\rangle$. □

Lemma 6.0.3. *Let F be a complete discretely valued field with residue field κ and ring of integers R. Let n be a positive integer coprime to char(κ). Let $\pi \in R$ be a parameter and $u \in R$ a unit with $[F(\sqrt[n]{u})\colon F] = n$. Let $L = F(\sqrt[n]{u}, \sqrt[n]{\pi})$. Suppose that F contains a primitive n^2-th root of unity ρ. Then $\rho^t \in RT_{L/F}(F)$ if and only if n divides t.*

Proof. Let σ be the automorphism of L/F given by $\sigma(\sqrt[n]{\pi}) = \rho^n \sqrt[n]{\pi}$ and $\sigma(\sqrt[n]{u}) = \sqrt[n]{u}$ and τ be the automorphism L/F given by $\tau(\sqrt[n]{u}) = \rho^n \sqrt[n]{u}$ and $\tau(\sqrt[n]{\pi}) = \sqrt[n]{\pi}$. Then the Galois group of L/F is an abelian group of order n^2 generated by σ and τ and hence $RT_{L/F}(F)$ is generated by the set $\{\sigma(a)a^{-1}\tau(b)b^{-1} \mid a, b \in L^\times\}$. Since $\rho^n = \tau(\sqrt[n]{u})/\sqrt[n]{u} \in RT_{L/F}(F)$, $\rho^{nj} \in RT_{L/F}(F)$ for any integer j.

Conversely, suppose $\rho^t \in RT_{L/F}(F)$ for some integer t. Without loss of generality, we may assume that $1 \leq t \leq n^2$. Then $\rho^t = a^{-1}\sigma(a)b^{-1}\tau(b)$ for some $a, b \in L$. Let $L' = F(\sqrt[n]{\pi})$. Since $\rho \in F$ and $N_{L/L'}(b^{-1}\tau(b)) = 1$, we have $\rho^{nt} = N_{L/L'}(a)^{-1}N_{L/L'}(\sigma(a))$. Let $c = N_{L/L'}(a) \in L'$. Since $\sigma(c) = \sigma(N_{L/L'}(a)) = N_{L/L'}(\sigma(a))$, we have $\sigma(c) = \rho^{nt}c$. Hence $\sigma(c^n) = (\sigma(c))^n = (\rho^{nt})^n c^n = c^n$. Since L'/F is a Galois extension with Galois group generated by σ, $c^n \in F$. Thus $c = \theta\sqrt[n]{\pi}^m$ for some integer m and $\theta \in F$. Since L/L' is an unramified extension of degree n and c is a norm from L/L', the valuation of c is divisible by n. Since $\theta \in F$ and $\sqrt[n]{\pi}$ is a parameter in L', $m = nr$ for some r. Hence $c \in F$ and $\rho^{nt} = c^{-1}\sigma(c) = 1$. Since ρ is a primitive n^2-th root of unity, n divides t. □

Notation 6.0.4. *Let A be a semi-local regular ring of dimension 2 with three maximal ideals m_1, m_2, m_3. Suppose that there exist three prime elements $\pi_1, \pi_2, \pi_3 \in A$ such that $m_1 = (\pi_2, \pi_3)$, $m_2 = (\pi_1, \pi_3)$ and $m_3 = (\pi_1, \pi_2)$. Suppose that $\pi_i \notin m_i$ for all i. Let $n \geq 2$ be an integer coprime to char(A/m_i) for all i. Let F be the fraction field of A. For $1 \leq i \leq 3$, let $\widehat{A}_{m_i}$ be the completion of A at m_i, F_{m_i} be the fraction field $\widehat{A}_{m_i}$ and F_{π_j} be the completion of F at the discrete valuation given by π_j. Let $1 \leq i \neq j, k \leq 3$. Since $m_i = (\pi_j, \pi_k)$, $\widehat{A}_{m_i}$ is a regular local ring with maximal ideal (π_j, π_k). In particular, π_j gives a discrete valuation on F_{m_i} which extends the discrete valuation on F given by π_j. Let F_{m_i,π_j} be the completion of F_{m_i} at the discrete valuation given by π_j. Then $F_{\pi_j} \subset F_{m_i,\pi_j}$. Let $L = F(\sqrt[n]{\pi_1\pi_2}, \sqrt[n]{\pi_2\pi_3})$. Suppose that F contains ρ, a primitive n^2-th root of unity.*

Corollary 6.0.5. *With notations as in Notation 6.0.4, we have*

$$T_{L/F}(F_{m_i}) = RT_{L/F}(F_{m_i})\langle\rho\rangle.$$

Proof. Since π_2 is a unit at m_2, we have $m_2A_{m_2} = (\pi_1\pi_2, \pi_3\pi_2)$. Hence, by Proposition 6.0.2, we have $T_{L/F}(F_{m_2}) = RT_{L/F}(F_{m_2})\langle\rho\rangle$. Since π_1 is a unit at m_1, $m_1A_{m_1} = (\pi_1\pi_2, \pi_1^{-1}\pi_3)$. Since $L = F(\sqrt[n]{\pi_1\pi_2}, \sqrt[n]{\pi_2\pi_3}) = F(\sqrt[n]{\pi_1\pi_2}, \sqrt[n]{\pi_1^{-1}\pi_3})$, by Proposition 6.0.2, we have $T_{L/F}(F_{m_1}) = RT_{L/F}(F_{m_1})\langle\rho\rangle$. Similarly, $T_{L/F}(F_{m_3}) = RT_{L/F}(F_{m_3})\langle\rho\rangle$. □

Corollary 6.0.6. *With notations as in Notation 6.0.4, we have*

$$T_{L/F}(F_{\pi_i}) = RT_{L/F}(F_{\pi_i})\langle\rho\rangle.$$

Proof. Let $\kappa(\nu_{\pi_i})$ be the residue field of F_{π_i}. The discrete valuation ν_{π_i} of F given by π_i has unique extension $\tilde{\nu}_{\pi_i}$ to L. Since F contains a primitive n^{th} root of unity, the residue field $\kappa(\tilde{\nu}_{\pi_i})$ of L at $\tilde{\nu}_{\pi_i}$ is a cyclic extension of $\kappa(\pi)$ of degree n. In particular, $T_{\kappa(\tilde{\nu}_{\pi_i})/\kappa(\nu_{\pi_i})}(\kappa(\nu_{\pi_i})) = RT_{\kappa(\tilde{\nu}_{\pi_i})/\kappa(\nu_{\pi_i})}(\kappa(\nu_{\pi_i}))$. Hence, by Lemma 4.1.2,

$$T_{L/F}(F_{\pi_i}) = RT_{L/F}(F_{\pi_i})\langle\rho\rangle.$$

□

Corollary 6.0.7. *Let F_{m_i,π_j} be as in Notation 6.0.4. Then $\rho^t \in RT_{L/F}(F_{m_i,\pi_j})$ if and only if n divides t.*

Proof. Since the residue field of F_{m_i,π_j} is a complete discretely valued field with the image of π_k $(k \neq i, j)$ as a parameter and the image of π_i as a unit, it is easy to see that $L \otimes_F F_{m_i,\pi_j} \simeq F_{m_i,\pi_j}(\sqrt[n]{v\pi_j}, \sqrt[n]{u})$ for some units u and v such that $[F_{m_i,\pi_j}(\sqrt[n]{u})\colon F_{m_i,\pi_j}] = n$. Thus, the corollary follows from Lemma 6.0.3. □

For each $1 \leq i \neq j \leq 3$, we have inclusions fields $F_{m_i} \to F_{m_i,\pi_j}$ and $F_{\pi_j} \to F_{m_i,\pi_j}$. Thus we have the induced homororphisms

$$\alpha_{ij}\colon T_{L/F}(F_{m_i})/R \to T_{L/F}(F_{m_i,\pi_j})/R$$

and

$$\beta_{ji}\colon T_{L/F}(F_{\pi_j})/R \to T_{L/F}(F_{m_i,\pi_j})/R.$$

Lemma 6.0.8. *The product map*

$$\phi\colon (\prod_{i=1}^{3} T_{L/F}(F_{m_i})/R) \times (\prod_{j=1}^{3} T_{L/F}(F_{\pi_j})/R) \to \prod_{1\leq i\neq j\leq 3} (T_{L/F}(F_{m_i,\pi_j})/R)$$

is not surjective.

Proof. Let $y_{12} = \rho \in T_{L/F}(F_{m_1,\pi_2})$ and $y_{ij} = 1 \in T_{L/F}(F_{m_i,\pi_j})$ for all $i \neq j$ and $(i,j) \neq (1,2)$. Then we show that $y = (y_{ij}) \in \prod_{1\leq i\neq j\leq 3} (T_{L/F}(F_{m_i,\pi_j})/R)$ is not in the image of ϕ.

Suppose y is in the image of ϕ. Then there exist $a_i \in T_{L/F}(F_{m_i})$ and $b_j \in T_{L/F}(F_{\pi_j})$ such that $\phi(a_1, a_2, a_3, b_1, b_2, b_3) = y$ modulo R-trivial elements. Then we have $\alpha_{12}(a_1)\beta_{21}(b_2) = y_{12} = \rho$ modulo R-trivial elements and $\alpha_{ij}(a_i)\beta_{ji}(b_j) \in RT_{L/F}(F_{m_i,\pi_j})$ for all $i \neq j$ and $(i,j) \neq (1,2)$. By Corollary 6.0.5 and Corollary 6.0.6, we have $a_i = c_i\rho^{s_i}$ for some $c_i \in RT_{L/F}(F_{m_i})$ and $b_j = d_j\rho^{t_j}$ for some $d_i \in RT_{L/F}(F_{\pi_j})$. Hence $a_i = \rho^{s_i}$ and $b_j = \rho^{t_j}$ modulo R-trivial elements. Since $\rho \in F$, $\alpha_{ij}(\rho) = \rho$ and $\beta_{ji}(\rho) = \rho$ for all $i \neq j$. We have $\rho = y_{12} = \alpha_{12}(a_1)\beta_{21}(b_2) = \rho^{s_1+t_2}$ modulo R-trivial elements. Hence, by Corollary 6.0.7, n divides $1 - s_1 - t_2$.

Let $1 \leq i \neq j \leq 3$ with $(i,j) \neq (1,2)$. Then $1 = \alpha_{ij}(a_i)\beta_{ji}(b_j) = \rho^{s_i+t_j}$ modulo R-trivial elements. Hence $\rho^{s_i+t_j} \in RT_{L/F}(F_{m_i,\pi_j})$ and by Corollary 6.0.7, n divides s_i+t_j. Since n divides s_2+t_1 and s_3+t_1, n divides s_3-s_2. Since n divides s_1+t_3 and s_2+t_3, n divides s_1-s_2. Hence n divides s_1-s_3. Since n divides s_3+t_2, n divides s_1+t_2, which contradicts the fact that n divides $1-s_1-t_2$. □

Theorem 6.0.9. *Let K be a complete discretely valued field with residue field κ and ring of integers R. Let $\mathscr{X}$ be a regular integral surface proper over R and F be its fraction field. Let X denote the reduced special fibre of $\mathscr{X}$. Suppose that X is a union of regular curves with normal crossings. Suppose that there exist three three irreducible curves X_1, X_2 and X_3 regular on $\mathscr{X}$ such that $X_i \cap X_j$, $i \neq j$ has exactly one closed point. Let $n \geq 2$ be an integer coprime to char(κ). Suppose that K has a primitive n^2-th root of unity. Then there exists a Galois extension L/F of degree n^2 with Galois group isomorphic to $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$ such that the local-global principle fails for $T_{L/F}$.*

Proof. Let P_1, P_2 and P_3 be the points of $X_i \cap X_j$, $i \neq j$. Let A be the semi local ring at P_1, P_2 and P_3 on $\mathscr{X}$. Then A has three maximal ideals m_1, m_2 and m_3. Since $\mathscr{X}$ is regular and each X_i is regular on $\mathscr{X}$, there exist primes $\pi_1, \pi_2, \pi_3 \in A$ such that $m_i = (\pi_j, \pi_k)$ for all distinct i, j, k. Let $L = F(\sqrt[n]{\pi_1\pi_2}, \sqrt[n]{\pi_2\pi_3})$. Since K contains primitive n^{th} root of unity, L/F is a Galois extension with Galois group isomorphic to $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$. We claim that the local-global principle fails for $T_{L/F}$.

Let $\mathcal{P}$ be a finite set of closed points of X containing all the singular points of X. Let $X \setminus \mathcal{P} = \cup U_i$, with $U_i \subset X_i$ for $i = 1, 2, 3$. By [HHK15a, Corollary 3.6.], it is enough to show that the product map

$$\prod_{P \in \mathcal{P}} T_{L/F}(F_P) \times \prod_i T_{L/F}(F_{U_i}) \to \prod_{P, U_i} T_{L/F}(F_{P,U_i})$$

is not surjective. Since X_1, X_2, X_3 are the only curves in X passing through P_1, P_2 or P_3, it is enough to show that

$$\phi\colon \prod_{i=1}^{3} T_{L/F}(F_{P_i}) \times \prod_{j=1}^{3} T_{L/F}(F_{U_j}) \to \prod_{P_i, U_j} T_{L/F}(F_{P_i,U_j})$$

is not surjective. Since $F_{U_j} \subset F_{\pi_j}$ and $F_{P_i,U_j} = F_{P_i,\pi_j}$, ϕ factors as

$$\prod_{i=1}^{3} T_{L/F}(F_{P_i}) \times \prod_{j=1}^{3} T_{L/F}(F_{U_j}) \to \prod_{i=1}^{3} T_{L/F}(F_{P_i}) \times \prod_{j=1}^{3} T_{L/F}(F_{\pi_j}) \to \prod_{P_i, \pi_j} T_{L/F}(F_{P_i,\pi_j}).$$

Since, by Lemma 6.0.8,

$$\prod_{i=1}^{3} (T_{L/F}(F_{\pi_i})/R) \times \prod_{j=1}^{3} (T_{L/F}(F_{P_j})/R) \to \prod_{U_i, P_j} (T_{L/F}(F_{P_i,U_j})/R)$$

is not surjective, ϕ is not surjective. □

Remark 6.0.10. The above theorem for κ algebraically closed and $n = 2$ is proved by Colliot-Thélène, Parimala and Suresh ([CTPS16, Section 3.1. & Corollary 6.2.]).

Corollary 6.0.11. *Let K be a complete discretely valued field with residue field κ and ring of integers R. Let $t \in R$ be a parameter. Let $\mathscr{X} = Proj(R[x, y, z]/\langle xy(x + y - z) - tz^3 \rangle)$. Let X be the special fibre of $\mathscr{X}$. Then $X = Proj(\kappa[x, y, z]/\langle xy(x + y - z)\rangle)$ which is reduced. Then X has three irreducible components X_1, X_2, X_3 and X_i intersects X_j, $i \neq j$ at exactly one point. Let F be the function field of $\mathscr{X}$. Then $F \simeq K(x)[y]/\langle xy(x+y-1)-t\rangle$. Let $n \geq 2$ be coprime to char(κ). Suppose that K contains a primitive n^2-th root of unity. Let $L = F(\sqrt[n]{xy}, \sqrt[n]{y(x-1)})$. Then L/F is a Galois extension with Galois group isomorphic to $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$. By Theorem 6.0.9, the local-global principle for fails for $T_{L/F}$.*

Proof. Let $U = \operatorname{Spec}(R[x,y]/\langle xy(x+y-1)-t\rangle)$. Then U is an affine open subset of $\mathscr{X}$. Let $P_1 = (1,0)$, $P_2 = (0,1)$ and $P_3 = (0,0)$ be the three closed points of U. Let A be the semi local ring at P_1, P_2 and P_3 and let m_i be the maximal ideal of A corresponding to P_i. Then $m_1 = (x+y-1, y)$, $m_2 = (x, x+y-1)$ and $m_3 = (x,y)$. Hence, by Theorem 6.0.9, the local-global principle fails for $T_{L/F}$. $\square$

Corollary 6.0.12. *Let K be a complete discretely valued field with residue field κ and ring of integers R. Let $t \in R$ be a parameter. Let $\mathscr{X} = Proj(R[x,y,z]/\langle xy(x+y-z)(x-2z)-tz^4\rangle)$ and F be the function field of $\mathscr{X}$. Then $F \simeq K(x)[y]/\langle xy(x+y-1)(x-2)-t\rangle$. Let $\theta_1 = (x-2)/(x-2+xy(x+y-1))$ and $\theta_2 = (y-2)/(y-2+xy(x+y-1))$. Let $n \geq 2$ with $6n$ coprime to char(κ). Let $L_1 = F(\sqrt[n]{xy}, \sqrt[n]{y(x+y-1)})$ and $L_2 = F(\sqrt[n]{xy\theta_1}, \sqrt[n]{y(x+y-1)\theta_2})$. Then L_1 and L_2 are finite Galois extensions of F that are linearly disjoint. Let $L = L_1 \times L_2$. Then the local-global principle fails for the multinorm torus $T_{L/F}$.*

Proof. To show that the local-global principle fails for $T_{L/F}$, by [HHK09, Theorem 3.6.] and as in the proof of Theorem 6.0.9, it is enough to show that

$$\phi\colon \prod_{i=1}^{3}(T_{L/F}(F_{\pi_i})/R) \times \prod_{j=1}^{3}(T_{L/F}(F_{P_j})/R) \to \prod_{U_i,P_j}(T_{L/F}(F_{P_i,U_j})/R)$$

is not surjective.

Let $U = \operatorname{Spec}(R[x,y]/\langle xy(x+y-1)(x-2)-t\rangle)$. Then U is an affine open subset of $\mathscr{X}$. Let $P_1 = (1,0)$, $P_2 = (0,1)$, $P_3 = (0,0)$ and $Q = (2,2)$. Let A be the semi local ring at P_1, P_2, P_3 and Q. Let m_i be the maximal ideals of A corresponding to P_i and m be the maximal ideal corresponding to Q. Let $\pi_1 = x$, $\pi_2 = y$ and $\pi_3 = x+y-1$. Then $m_1 = (\pi_2, \pi_3)$, $m_2 = (\pi_1, \pi_3)$, $m_3 = (\pi_1, \pi_2)$. We also have $m = (x-2, y-2)$. Since $2 \neq \operatorname{char}(\kappa)$, $x-2$ and $y-2$ are units at m_i and $\theta_i = 1$ modulo m_j and π_j. Since n is coprime to char(κ), $\theta_i \in F_{P_i}^n$ and $\theta_i \in F_{\pi_j}^n$ for all i and j. Hence $L_1 \otimes_F F_{\pi_i} \simeq L_2 \otimes_F F_{\pi_i}$ and $L_1 \otimes_F F_{P_j} \simeq L_2 \otimes_F F_{P_j}$. By Proposition 2.4.3, we have $T_{L/F}(F_{\pi_i})/R \simeq T_{L_1/F}(F_{\pi_i})/\mathrm{R}$, $T_{L/F}(F_{P_j})/R \simeq T_{L_1/F}(F_{P_j})/R$ and $T_{L/F}(F_{U_i,P_j})/R \simeq T_{L_1/F}(F_{U_i,P_j})/R$. Since, by Lemma 6.0.8,

$$\prod_{i=1}^{3}(T_{L_1/F}(F_{\pi_i})/R) \times \prod_{j=1}^{3}(T_{L_1/F}(F_{P_j})/R) \to \prod_{U_i,P_j}(T_{L_1/F}(F_{P_i,U_j})/R)$$

is not surjective, ϕ is not surjective. Hence the local-global principle fails for $T_{L/F}$.

Since $\pi_1\pi_2 = xy = 4$ modulo m and $\pi_2\pi_3 = 6$ modulo m, we have $L_1 \otimes_F F_Q = F_Q(\sqrt[n]{4}, \sqrt[n]{6})$. Since $6n$ is coprime to $\mathrm{char}(\kappa)$, $L_1 \otimes_F F_Q/F_Q$ is unramified. Since $xy(x+y-1) = 12$ modulo m, $x-2+xy(x+y-1) = 12a^n$ for some $a \in F_Q$. Similarly, $y-2+xy(x+y+1) = 12b^n$ for some $b \in F_Q$. Hence $L_2 \otimes_F F_Q = F_Q(\sqrt[n]{(x-2)/3}, \sqrt[n]{(y-2)/2})$. Since the maximal ideal $m = (x-2, y-2)$ and $3, 2$ are units at m, $L_1 \otimes_F F_Q$ and $L_2 \otimes_F F_Q$ are linearly disjoint over F_Q. In particular, L_1 and L_2 are linearly disjoint over F. $\square$

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