Distribution Agreement

On K_t-saturated Graphs

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Abstract

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Let G be a graph on n vertices. Let H be a graph. Any H-free graph G is called H-saturated if the addition of any edge $e \notin E(G)$ results in H as a subgraph of G. The minimum size of an H-saturated graph on n vertices is called a saturation number, denoted by $sat(n, H)$. The edge spectrum for the family of graphs with property P is the set of all sizes of graphs with property P.

In this dissertation, we show the edge spectrum of K_4-saturated graphs. We also classify all K_4-saturated graphs of connectivity 2 and 3. Furthermore, we show that, for $n \geq 5t - 7$, there is an (n, m) K_t-saturated graph G if and only if G is complete $(t-1)$-partite graph or $(t-1)(n - \frac{t}{2}) - 2 \leq m \leq \lfloor \frac{(t-2)n^2 - 2n + (t-2)}{2(t-1)} \rfloor + 1$.

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To my daughter Riya

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Chapter 1

Introduction

Extremal numbers first appeared in a 1907 paper [8] of Mantel dealing with triangles. This was generalized to all complete graphs in a 1941 paper [9] of Paul Turán. In that paper, he determined the maximum number of edges in a K_t-free graph on r vertices. Furthermore, he found all K_t-free graphs with the maximum number of edges. Since then extremal numbers have been studied intensively by numerous mathematicians and this study created a new branch of graph theory, called Turán-type extremal graph theory. General extremal graph theory is the study of relations between various graph invariants, such as order, size, connectivity, minimum degree, maximum degree, chromatic number and diameter, as well as the values of these invariants which guarantee certain graph properties.

1.1 Definitions

A *graph* is a finite set of elements called vertices and a (possibly empty) set of 2-element subsets of vertices called edges. The *vertex set* of G is denoted by $V(G)$, while the *edge set* of G is denoted by $E(G)$. The cardinality of the vertex set is called the *order* of the graph. The cardinality of the edge set is called the *size* of the graph. An $(n, m) - graph$ has order n and size m. Two vertices are called *adjacent vertices* if they are two end vertices of an edge. Two edges are called *adjacent edges* if they share a commom end vertex. A vertex and an edge are *incident* if the vertex is one end of the edge. The *degree of a vertex* v, denoted by $d(v)$, in a graph G is the number of edges incident with v. By our definition, all graphs in this dissertation are simple graphs, i.e., finite graphs without loops (an edge joining a vertex to itself) or parallel edges (multiple edges sharing the same two end vertices).

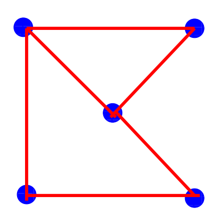

Figure 1.1: G

A graph G is called a *complete* graph if every two of its vertices are adjacent. Hence, for any complete graph G of order n, denoted by K_n, for all vertices $v \in V(G)$, $d(v) = n-1$. A *path* graph P_n is a sequence of alternating vertices and edges $v_1e_1...e_{n-1}v_n$, where no vertices are repeated and consecutive elements are incident, so that $|V(P_n)| = |E(P_n)| + 1 = n$. A graph G whose vertex set $V(G)$ can be partitioned into two distinct subsets X and Y such that each edge of G has one endpoint in X and one endpoint in Y is called a *bipartite* graph, denoted by $G(X \cup Y, E)$. A bipartite graph $G(X \cup Y, E)$ is called a *complete bipartite* graph, denoted by $K_{k,n-k}$, where $|X| = k$ and $|Y| = n-k$, if for all vertices $x \in X$ and $y \in Y$, the edge $xy \in E(G)$. A graph G is *k-partite*, $k \geq 1$, if it is possible to partition $V(G)$ into k subsets $V_1, V_2, ..., V_k$ (called *partite sets*) such that every element of $E(G)$ joins a vertex of V_i to a vertex of V_j, $i \neq j$. A *complete k-partite* graph G is a k-partite graph with partite sets $V_1, V_2, ..., V_k$ having the added property that if $u \in V_i$ and $v \in V_j$, $i \neq j$, then $uv \in E(G)$. The *join* $G = G_1 + G_2$ has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1) \text{ and } v \in V(G_2)\}$. Two vertices that are not adjacent in a graph G are said to be *independent*. A set S of vertices is independent if every two vertices of S are independent. All other terms not defined here can be found in [2].

1.2 Extremal Numbers

A graph G is called an *H-saturated graph* if G does not contain H as a subgraph but the addition of any edge $e \notin E(G)$ produces H as a subgraph. The maximum number of edges in a H-saturated graph of order n is called the *extremal number* and is denoted by $ex(n,H)$. $ex(n,\mathcal{F})$ is defined to be the maximum number of edges in a H-saturated graph of order n for any graph H in the family of graphs $\mathcal{F}$. $EX(n,H)$ is defined to be the set of H-saturated graphs of order n of size $ex(n,H)$. Extremal numbers have three nice properties. Let $\mathcal{F}$ be a family of graphs. Then $ex(n,\mathcal{F})$ satisfies:

1. $ex(n,\mathcal{F}) \leq ex(n+1,\mathcal{F})$;

2. If $\mathcal{F}_1 \subset \mathcal{F}$ then $ex(n,\mathcal{F}_1) \geq ex(n,\mathcal{F})$;

3. If H is a subgraph of G, then $ex(n,H) \leq ex(n,G)$.

In 1966, Erdős, Stone, and Simonovits showed the following relationship between the chromatic number of a graph and the extremal number of a graph, known as the Erdős-Stone-Simonovits Theorem.

Theorem 1.1 *For any simple graph G*

$$\lim_{n\to\infty} \frac{ex(n,G)}{n^2} = \frac{1}{2}\left(\frac{\chi(G)-2}{\chi(G)-1}\right)$$

In 1941 [9], Turán found the extremal number for complete graphs.

Theorem 1.2 *There is a unique extremal graph on n vertices that is K_{p+1}-free, namely $T_{n,p}$.*

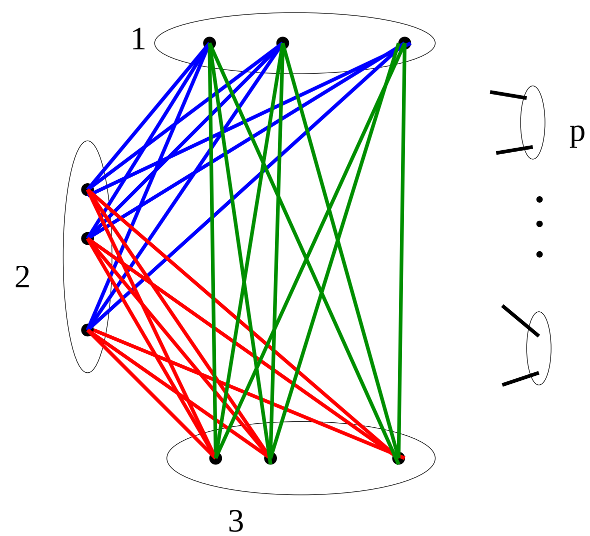

Figure 1.2: $T_{n,p}$

The graph $T_{n,p}$ is called the Turán graph and is a complete almost balanced p-partite graph on n vertices. For example, the Turán graph which is a K_4-free graph on 10 vertices has 33 edges, namely a $K_{4,3,3}$ graph. Note that the graph $K_{1,1,8}$ on 10 vertices and 17 edges is also a K_4-free graph. Hence, there exists other saturated graph that are not extremal graphs.

1.3 Saturation Numbers

The *saturation number* of a graph is the minimum number of edges in an H-saturated graph of order n and it is denoted by $sat(n, H)$. $SAT(n, H)$ is defined to be the set of H-saturated graphs of order n of size $sat(n, H)$. A fundamental open problem for saturation numbers is to find a result that is an analogue of Erdős-Stone-Simonovits Theorem. In 1964 [5], Erdős, Hajnal and Moon determined the saturation number for K_t:

Theorem 1.3 *For $t \geq 3$, $sat(n, K_t) = (t-2)(n-1) - \binom{t-2}{2}$.*

An example of a graph that attains this bound is $K_{t-2} + \overline{K}_{n-t+2}$, where $+$ denotes the join operation and $\overline{K}_{n-t+2}$ denotes $n - t + 2$ independent vertices, and is the uniqe minimum sized saturated graph for K_t. In fact, in [5] it is shown that $SAT(n, K_t)$ only contains $K_{t-2} + \overline{K}_{n-t+2}$. In particular, $sat(n, K_3) = n-1$. The *edge spectrum* for the family of graphs with property P is the set of all sizes of graphs with property P. In 1995, Barefoot et. al [1] proved the following result about the edge spectrum of K_3-saturated graphs.

Theorem 1.4 *Let $n \geq 5$ and m be nonnegative integers. There is an (n, m) K_3-saturated graph if and only if $2n - 5 \leq m \leq \lfloor (n-1)^2/4 \rfloor + 1$ or $m = k(n-k)$ for some positive integer k.*

Until now, this was the only result about the edge spectrum of saturated graphs. The goal of this dissertation is to expand upon Theorem 1.4 and determine the edge spectrum for any complete graph K_t, $t \geq 3$. In 1986, Kászonyi and Tuza [7] found the saturation number for paths, stars and matchings. A *star* S_{t+1} on $t+1$ vertices is simply the graph $K_{1,t}$. A set of pairwise independent edges of G is called a *matching* in G. The graph tK_2 is simply a union of t disjoint copies of the graph K_2.

Theorem 1.5

1. *For* $n \geq 3$, $sat(n, P_3) = \lfloor \frac{n}{2} \rfloor$.

2. *For* $n \geq 4$,

$$sat(n, P_4) = \begin{cases} n/2 & n \text{ even} \\ (n+3)/2 & n \text{ odd.} \end{cases}$$

3. *For* $n \geq 5$, $sat(n, P_5) = \lceil \frac{5n-4}{6} \rceil$.

4. *Let*

$$a_k = \begin{cases} 3 * 2^{t-1} - 2 & \text{if } k = 2t \\ 4 * 2^{t-1} - 2 & \text{if } k = 2t+1. \end{cases}$$

If $n \geq a_k$ *and* $k \geq 6$, *then*

$$sat(n, P_k) = n - \lfloor \tfrac{n}{a_k} \rfloor.$$

Theorem 1.6

$$sat(n, S_{k+1}) = \begin{cases} \begin{pmatrix} k \\ 2 \end{pmatrix} + \begin{pmatrix} n-k \\ 2 \end{pmatrix} & \text{if } k+1 \le n \le k + \frac{k}{2} \\ \lceil \frac{k-1}{2} n - \frac{k^2}{8} \rceil & \text{if } k + \frac{k}{2} \le n. \end{cases}$$

Theorem 1.7 *$sat(n, tK_2) = 3t - 3$ for $n \ge 3t - 3$.*

Furthermore, Kászonyi and Tuza also gave a general bound on the saturation number for every graph in [7].

Theorem 1.8 *For every graph F, there exists a constant c such that $sat(n, F) < cn$.*

Recently, G. Chen, R. J. Faudree, and R. Gould [3] found the saturation number for a book graph and a generalized book graph. A *book* B_p is a union of p triangles sharing one edge. A generalized book $B_{b,p}$ is the union of p copies of K_{b+1} sharing a common K_b.

Theorem 1.9 *Let n and p be two positive integers such that $n \ge p^3 + p$. Then,*

$$sat(n, B_p) = \tfrac{1}{2}\left((p+1)(n-1) - \lceil \tfrac{p}{2} \rceil \lfloor \tfrac{p}{2} \rfloor + \theta(n,p)\right),$$

$$\textit{where } \theta(n,p) = \begin{cases} 1 & \textit{if } p \equiv n - p/2 \equiv 0 (\bmod 2) \\ 0 & \textit{otherwise.} \end{cases}$$

Now define $f(n, b, p) = (p + 2b - 3)(n - b + 1) - \lceil \frac{p}{2} \rceil \lfloor \frac{p}{2} \rfloor + (b - 1)(b - 2) + \theta(n, b, p)$, where

$$\theta(n,b,p) = \begin{cases} 1 & \text{if } p \equiv n - p/2 - b \equiv 0 (\bmod 2) \\ 0 & n \text{ otherwise.} \end{cases}$$

Theorem 1.10 *Let $n, b \geq 3$ and p be three positive integers such that $n \geq 4(p + 2b)^b$. Then*

$$sat(n, B_{b,p}) = \frac{1}{2} f(n, b, p).$$

Finding saturation number turns out to be hard in general. Saturation numbers do not hold nice monotonicity properties, similar to extremal numbers listed in section 1.2, as shown in the following examples.

1. $sat(n, H)$ is not monotone for the order of the graph H, n.

   Note that $sat(n, K_3) = n - 1$ and $sat(n + 1, K_3) = n$. Take a star on n and $n + 1$ vertices, respectively. Hence, $sat(n, K_3) \leq sat(n + 1, K_3)$.

   Furthermore, $sat(2k, P_4) = k$. Take k pairs of vertices and join each pair by an edge. Then, we obtain $sat(2k - 1, P_4) = k + 1$, by taking

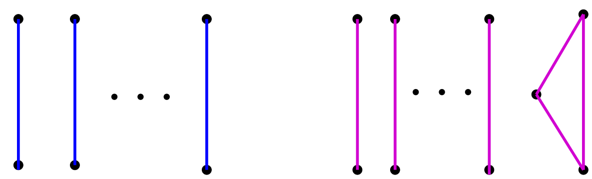

Figure 1.3: $G \in SAT(n, P_4)$ with $n = 2k$ and $n = 2k - 1$

$k-2$ pairs of vertices joined by an edge and the remaining three vertices making a K_3. Hence, with $n = 2k - 1$, $sat(n, P_4) > sat(n + 1, P_4)$.

2. $sat(n, \mathcal{F})$ in not monotone for family $\mathcal{F}$.

   Note that $sat(n, K_4) = 2n - 3$ and $sat(n, \{K_3, K_4\}) = n - 1$. Hence, $sat(n, K_4) > sat(n, \{K_3, K_4\})$.

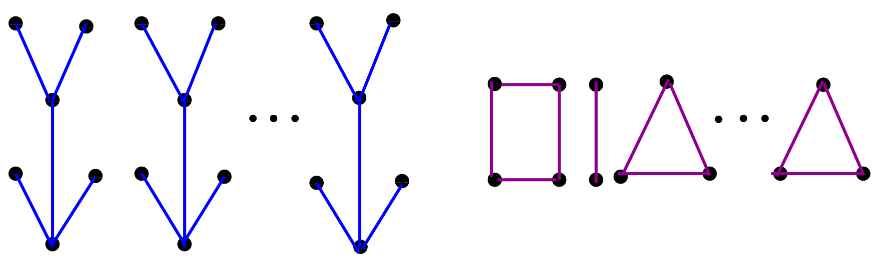

Figure 1.4: $G \in SAT(n, P_5)$ (Left graph, in blue) and $G \in SAT(n, \{P_5, S_4\})$ (Right graph, in magenta)

Let $n = 6k$ for some positive integer k. Then, $sat(n, P_5) = n-(\frac{n-2}{6}+1)$ (Figure 1.4) while $sat(n, \{P_5, S_4\}) = n - 1$ (Figure 1.5). So for large n, we have $sat(n, P_5) < sat(n, \{P_5, S_4\})$.

3. $sat(n, H)$ is not monotone for graph H.

   Note that $sat(n, K_3) = n - 1$ and $sat(n, K_4) = 2n - 3$. Hence, $sat(n, K_3) \leq sat(n, K_4)$.

   Furthermore, consider K_4 and the supergraph H obtained by attaching an edge to K_4. We know that $sat(n, K_4) = 2n - 3$. But for H we have, $sat(n, H) \leq \frac{3}{2}n$. So, $sat(n, K_4) \geq sat(n, H)$.

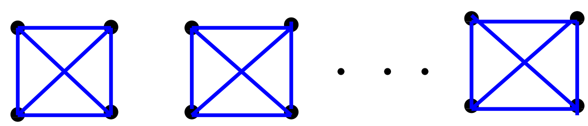

Figure 1.5: Shows $sat(n, H) \leq \frac{3n}{2}$

Most graph properties are monotone. And as we have just seen, being saturated is not a monotone property for graphs. Hence, it is interesting and important to study the edge spectrum of a H-saturated graph for any graph H. Complete graphs play a fundamental role in many aspects of graph theory, for example Ramsey Theory. Hence, it is important to study the edge spectrum of K_t-saturated graphs.

Chapter 2

K_4-Saturated Graphs

As we have seen in the inroduction, saturation numbers are not a monotone property. So it would be nice to know the sizes of all H-saturated graph for any graph H. In this chapter, we find the sizes of all K_4-saturated graphs. We know that any K_4-saturated graph has at least $2n-3$ edges and at most $\lfloor n^2/3 \rfloor$ edges and these bounds are sharp. The emphasis of this chapter is to prove that there is an (n, m) K_4-saturated graph G if and only if G is a complete tripartite graph or $3n-8 \leq m \leq \frac{n^2-n+4}{3}$. We show the lower bound of the interval using the minimum degree of the graph. We apply a result by Hanson and Toft [6] to show the upper bound of our result. We also found the exact structure of K_4-saturated graphs of connectivity 2 and 3. We also proved that K_4-saturated graphs have diameter 2 and are also 2-connected.

2.1 Results on K_4-saturated Graphs

The distance between two vertices u and v, denoted by $d(u, v)$, is the length of the shortest path connecting them. The diameter, denoted by $diam(G)$, of a graph G is the maximal distance between any two vertices on the graph. We have the following relationship between a K_4-saturated graph G and its diameter.

Proposition 2.1 *Let G be a K_4-saturated graph. Then $diam(G) = 2$.*

Proof. Let G be a K_4-saturated graph. Suppose $diam(G) \neq 2$, that is, suppose there exists vertices $u, v \in V(G)$ such that $d(u, v) = 3$. Say u, x, y, v is a $u - v$ distance 3 path. Then the addition of the edge uv must produce a K_4. So there must exist vertices a, b such that the induced subgraph $\langle u, a, b, v\rangle \cong K_4$. But now we have $d(u, v) = 2$, as u, a, v is such a path, a contradiction. $\square$

A vertex u is said to be *connected* to a vertex v in a graph G if there exists a $u - v$ path in G. A graph G is *connected* if every two of its vertices are connected. A graph that is not connected is *disconnected*. A *vertex-cut* in a graph G is a set U of vertices of G such that $G - U$ is disconnected. The *vertex connectivity* or simply the *connectivity* $\kappa(G)$ of a graph G is the minimum cardinality of a vertex-cut of G if G is not complete, and $\kappa(G) = n - 1$ if $G = K_n$ for some positive integer n. A graph G is said to be *k-connected*, $k \geq 1$, if $\kappa(G) \geq k$.

Proposition 2.2 *Let G be a K_4-saturated graph. Then G is 2-connected.*

Proof. Let G be a K_4-saturated graph. Suppose G is not 2-connected. Let u be the cut vertex and let A and B be components of $G - u$ with $x \in V(A)$ and $y \in V(B)$ (see Figure 2.1).

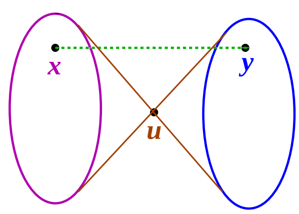

Figure 2.1: G

Now the addition of the edge xy creates a K_4. So there exists a vertex $w \neq u$ such that $wx \in E(G)$ and $wy \in E(G)$. Then x and y are connected in $G - u$, a contradiction. $\square$

Next we show that there is only one K_4-saturated graph with minimum degree two.

Theorem 2.3 *Let G be a K_4-saturated graph. Then either $G \cong K_{1,1,n-2}$ or $\delta(G) \geq 3$.*

Proof. Let G be a K_4-saturated graph. Let $v \in V(G)$ such that $N(v) = \{x, y\}$. Then for all the vertices $z \in V(G) - \{x, y, v\}$, the addition of the edge vz must produce a K_4 with vertex set $\{v, x, y, z\}$. Thus, $xy, xz, yz \in E(G)$. Since z was chosen arbitrarily, in fact, all vertices in $V(G) - \{x, y, v\}$ are adjacent to both x and y. So $K_{1,1,n-2} \subseteq G$. But $K_{1,1,n-2}$ is K_4-saturated, so $G \cong K_{1,1,n-2}$. □

Now, we know that we can obtain a K_4-saturated graph from a K_3-saturated graph joined with an independent set of vertices. We show that no other sizes of K_4-satuarted graphs exists in the following three lemmas. Before we prove these lemmas, we make the following two observations about K_4-saturated graphs G:

1. The neighborhood of every vertex contains an edge.
2. For all vertices $u, v \in V(G)$, $uv \notin E(G)$ if and only if there exists an edge in the common neighborhood of u and v.

Lemma 2.4 *If G is a K_4-saturated graph of order n and $\delta(G) = 3$, then $|E(G)| \geq 3n - 8$.*

Proof. Let $v \in V(G)$ such that $N(v) = \{x, y, z\}$. Let $W = V(G) - \{v, x, y, z\}$. Note that the induced graph $\langle x, y, z \rangle$ is not isomorphic to K_3.

If $\langle x, y, z \rangle$ contains precisely one edge, say xy, then by observation 2, every vertex $w \in W$ must also be adjacent to x, y and W is therefore independent. But this would force $N(z)$ to be independent, contradicting observation 1.

If $\langle x, y, z \rangle$ contains precisely two edges, say xy, yz, then by observation 2, W can be partitioned into three sets W_{xy}, W_{yz}, W_{xyz}, where $W_{xy} = \{w \in W | N(w) \cap \{x, y, z\} = \{x, y\}\}$ with W_{yz} and W_{xyz} defined similarly (note that $W_{xz} = \emptyset$). Each of these are independent sets of vertices. Furthermore, if $w \in W_{xyz}$, then it has no adjacencies to W_{xy} or W_{yz}. Let $w_1 \in W_{xy}$ and $w_2 \in W_{yz}$. Then $N(w_1) \cap N(w_2) = \{y\}$. Thus, by observation 2, they must be adjacent. Hence, W_{xy} and W_{yz} must induce a complete bipartite graph. Let $|W_{xyz}| = h, |W_{yz}| = l, |W_{xy}| = n - h - l - 4$. Then,

$$
\begin{aligned}
|E(G)| &= 5 + 2l + 2(n - h - l - 4) + l(n - h - l - 4) + 3h \\
&= (2 + l)n + (1 - l)h - l^2 - 4l - 3.
\end{aligned}
$$

Let $f(l) = (2 + l)n + (1 - l)h - l^2 - 4l - 3$. Note if $l = 0$, then we are done by our previous argument. Now $f(1) = 3n - 8$. Furthermore, f is maximized at $l = \frac{n-h-4}{2}$ and is increasing between $l = 1$ and $l = \frac{n-h-4}{2}$. Note this is all of the relevant interval for l, since we can assume, without loss of generality, $|W_{yz}| \leq |W_{xy}|$. □

Lemma 2.5 *If G is a K_4-saturated graph of order n and $\delta(G) = 4$, then $|E(G)| \geq 3n - 8$.*

Proof. Let $v \in V(G)$ such that $d(v) = 4$ and $|V(G)| = n$. Let $N(v) = A = \{a_1, a_2, a_3, a_4\}$. Let $W = V(G) - A - \{v\}$. Then there must exist an edge $a_i a_j$ for some $i, j, 1 \leq i, j \leq 4$ as for any vertex $w \in W$, addition of the edge vw must create a K_4.

If $\langle A \rangle$ contains precisely one edge, say $a_1 a_2$, then every vertex $w \in W$ must also be adjacent to a_1, a_2 by observation 2 above and W is therefore independent. But this would force $N(a_3)$ and $N(a_4)$ to be independent, contradicting observation 1. So $\langle A \rangle$ must contain at least two edges. So now assume that there are precisely two edges among the vertices of A. Then we have two possibilities as shown in the Figure 2.2.

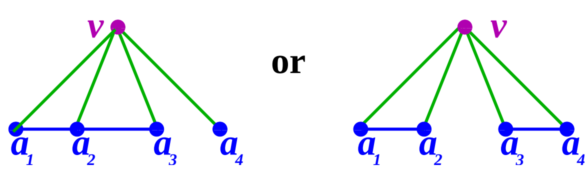

Figure 2.2: G

Pick the vertices a_i and a_j in different components of $\langle N(v) \rangle$. The addition of the edge $a_i a_j$must produce a K_4 with an edge in W, say $w_1 w_2$. But each vertex w_i, for $i = 1, 2$, must be adjacent to an edge in A, or else the addition of the edge vw_i would not produce a K_4. Hence, w_1 and w_2 are adjacent to at least three vertices of A.

The count below comes from counting the degree sum in the following parts: (1) the degree sum in $v \cup A$, (2) the degree sum from A to W, (3) the degree sum of vertices in W. In the last instance we use the assumption that $\delta(G) = 4$.

$$\begin{aligned} \sum_{x \in V(G)} d(x) &\geq (4 + 4 + 4) + (2(n-5) + 2) + (4(n-5)) \\ &= 6n - 16. \end{aligned}$$

Observe that if there are three edges among the vertices of A, we also obtain the same count. Hence, $|E(G)| \geq 3n - 8$. $\square$

Lemma 2.6 *If G is a K_4-saturated graph and $\delta(G) = 5$, then $|E(G)| \geq 3n - 8$.*

Proof. Note that $|E(G)| \geq \frac{5n}{2}$ and $\frac{5n}{2} \geq 3n - 8$ for $n \leq 16$.

Let $v \in V(G)$ with $d(v) = 5$ and $|V(G)| = n \geq 17$. Let $N(v) = A = \{a_1, a_2, a_3, a_4, a_5\}$. Let $W = V(G) - A - \{v\}$. We know there must exist an edge $a_i a_j$ for some i, j. In fact, as in Lemma 2.5, there must exist at least two edges. Furthermore, for any vertex $w \in W$, wa_i and wa_j must exist for some $i, j, 1 \leq i, j \leq 5$. Also, two of the vertices in W must be adjacent to at least 3 vertices in A, as shown in Lemma 2.5. Since $\delta(G) = 5$ and counting as we did in the previous lemma, we have

$$\begin{aligned} \sum_{x \in V(G)} d(x) &\geq (5+5+4) + (2(n-6)+2) + (5(n-6)) \\ &= 7n - 26 \end{aligned}$$

So, $|E(G)| \geq \frac{7n-26}{2} \geq 3n - 8$ for $n \geq 10$. □

Theorem 2.7 *Every 2-connected K_4-saturated graph of order n with $\delta(G) \geq 3$ has at least $3n - 8$ edges.*

Proof. From Lemmas 2.4 - 2.6, the result holds for graphs G with $3 \leq \delta(G) \leq 5$. For graphs G with $\delta(G) \geq 6$, $|E(G)| \geq \frac{6n}{2} = 3n$. □

Recall that the *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. In the following two theorems, we classify all K_4-saturated graphs G with $\kappa(G) = 2$ or $\kappa(G) = 3$.

Theorem 2.8 *If G is a K_4-saturated graph of order n with $\kappa(G) = 2$, then $G \cong K_{1,1,n-2}$.*

Proof. Let G be a K_4-saturated with $\kappa(G) = 2$. Let K be a minimal cut set of G and let C_1, C_2 be distinct components of $G - K$. Let $x_i \in V(C_i)$ for $i = 1, 2$. Then the addition of the edge x_1x_2 produces a K_4. So if $u, v \in K$, then $\langle x_1, x_2, u, v\rangle \cong K_4$. As x_1 (x_2) was an arbitrary vertex of C_1 (C_2), each vertex in C_1 (C_2) is also adjacent to u and v. So $G - K$ is an independent set and the result is shown. $\square$

Now we will classify all K_4-saturated graphs with $\kappa(G) = 3$. But first, we define the graph $W(a, b, c, d, e)$ to be a wheel on 5 sets totalling n vertices such that $a + b + c + d + e = n - 1$ and each of the 5 sets of the wheel are independent sets of sizes a, b, c, d, e, respectively, and two consecutive independent sets on the wheel form a complete bipartite subgraph.

Theorem 2.9 *If G is a K_4-saturated graph of order n with $\kappa(G) = 3$, then $G \cong K_{1,2,n-3}$ or $G \cong W(1, t, 1, r, s)$.*

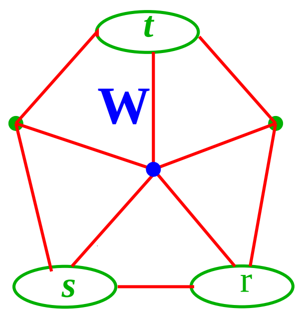

Figure 2.3: $W(1, t, 1, r, s)$

Proof. Let K be a minimal cut set of G, say $\{v_1, v_2, v_3\}$. Let C_1, C_2 be distinct components of $G - K$. Let $x_i \in V(C_i)$ for $i = 1, 2$. Then addition of the edge x_1x_2 produces a K_4. Without loss of generality, suppose $\langle x_1, x_2, v_1, v_2 \rangle \cong K_4$. If all the components of $G - K$ are trivial, then all are adjacent to all three vertices of K by the connectivity assumption. Now one of (but not both) v_1 or v_2 is adjacent to v_3 (say v_2) or else inserting the edge v_2v_3 would not produce a K_4. But now G is $K_{1,2,n-3}$.

We now assume C_1 is nontrivial. Let x_3 be a neighbor of x_1 in C_1. Inserting the edge x_2x_3, we know (without loss of generality) $\langle x_2, x_3, v_2, v_3 \rangle \cong K_4$, since x_3 cannot be adjacent to both v_1 and v_2. If there exists a vertex $x_4 \in V(C_2)$ such that $x_2x_4 \in E(G)$, then by a similar argument $x_4v_1, x_4v_2, x_4v_3 \in E(G)$. Hence, $\langle x_2, x_4, v_1, v_2 \rangle \cong K_4$. Hence, $V(C_2) = \{x_2\}$. In fact $G - \{K \cup C_1\}$ is an independent set. For every $w \in V(C_1)$, $N(w) \cap K = \{v_1, v_2\}$ or $N(w) \cap K =$

$\{v_2, v_3\}$ since w must be adjacent to an edge in K but cannot be in a K_4. Now partition C_1 into two classes A and B, where vertices of A are adjacent to v_1 and v_2, while vertices of B are adjacent to v_2 and v_3. Clearly, A is an independent set. Similarly, B is an independent set.

We claim $\langle A \cup B \rangle$ is a complete bipartite graph. Let $a \in A, b \in B$ such that $ab \notin E(G)$. Then addition of the edge ab must produce a K_4. Since A and B are independent sets, the edge they have in common has to be in K, a contradiction. Hence, $\langle A \cup B \rangle$ is complete bipartite. Thus, if $|A| = s$ and $|B| = r$, using v_2 as the center of the wheel, we have $G \cong W(1, t, 1, r, s)$, where $3 + t + r + s = n$. $\square$

2.2 Hanson and Toft Result

In this section, we give a construction due to Hanson and Toft [6] that produces the maximum size of a K_t-saturated graph G on $n \geq t + 2$ vertices with $\chi(G) \geq t$. The graphs in the family $\mathcal{T}_n^t$ are on n vertices consisting of a complete $(t-1)$-partite graph on $n-2$ vertices, with classes of independent points $C_1, C_2, ..., C_{t-1}$, together with two adjacent vertices x and y and where each vertex of C_1 is joined to precisely one of x or y, x and y are each adjacent to at least one vertex of C_1, no vertex of C_2 is adjacent to either x or y and all vertices of C_i, $i > 2$ are adjacent to both x and y. For $t \geq 3$, define $T'_{t-1,n}$ to be graphs in $\mathcal{T}^t_n$ for which $|C_1|+1, |C_2|+2, |C_3|, ..., |C_{t-1}|$ are equal or as equal as possible. For $n \geq 3t-4$ we can describe $T'_{t-1,n}$ as follows: let $n+1 = k(t-1)+r$, $0 \leq r < t-1$ and let G denote a member of $\mathcal{T}^t_{n_0}$ on $n_0 = n-r$ vertices and $e_0 = e(T_{t-1,n-r}) - (k-2)$ edges where the classes C_i satisfy $|C_1| = k-1, |C_2| = k-2$ and $|C_i| = k$, $i > 2$ (G is unique up to adjacencies of x and y to class C_1). Define $T'_{t-1,n}$ to be a graph G with one vertex added to precisely r of the classes $C_1, ..., C_{t-1}$. Note that the graphs $T'_{t-1,n}$ are maximal, with respect to the number of edges, in the family $\mathcal{T}^t_n$. Then Hanson and Toft [6] showed the following result.

Theorem 2.10 *Let G be a K_t-saturated graph on $n \geq t+2 \geq 5$ vertices with $\chi(G) \geq t$, then G is a $T'_{t-1,n}$ graph.*

Theorem 2.11 *If G is a K_4-saturated graph of order n and G is not complete tripartite, then $|E(G)| \leq \frac{n^2-n+4}{3}$.*

Proof. Let G be a K_4-saturated graph of order n. Suppose G is not a complete tripartite graph. Since G is not tripartite, $\chi(G) \geq 4 = t$. Hence, by Theorem 2.10, $|E(G)| \leq |E(T'_{3,n})|$. For $n+1 = 3k+r$, a straight forward computation shows, $|E(T'_{3,n})| \leq \frac{n^2-n+4}{3}$. In fact, when $r = 0$, $|E(T'_{3,n})| = \frac{n^2-n+4}{3}$. Hence, $|E(G)| \leq \frac{n^2-n+4}{3}$. □

2.3 The Edge Spectrum of K_4-Saturated Graphs

We now show a K_4 analogue of Theorem 1.4

Theorem 2.12 *Let $n \geq 5$ and m be nonnegative integers. There is an (n, m) K_4-saturated graph G if and only if $3n-8 \leq m \leq \frac{n^2-n+4}{3}$ or $m = rs+st+rt$ for some positive integers r, s, t where $n = r + s + t$.*

Proof. Let $n \geq 5$ and m be nonnegative integers. Let G be an (n, m) K_4-saturated graph. If G is a tripartite graph, then G must be a complete tripartite graph, otherwise an edge may be added without creating a K_4. Now if $G \cong K_{1,1,n-2}$, then $m = 2n - 3$ and clearly $r = s = 1$ while $t = n - 2$, otherwise $m = rs + st + rt$ for some positive integers r, s, t such that $n = r + s + t$.

Now let G be a nontripartite graph. Then from Theorem 2.7, Theorem 2.9, and Theorem 2.11, we have that $3n - 8 \leq m \leq \frac{n^2-n+4}{3}$.

It is sufficient to construct an (n,m) K_4-saturated graph for each value of m. If $m = 2n-3$, then $G \cong K_{1,1,n-2}$. If $m = rs+st+rt$ for some positive integers r,s,t where $n = r+s+t$, then $G \cong K_{r,s,t}$ with m edges.

Now let $3n-8 \leq m \leq \frac{n^2-n+4}{3}$. From Theorem 3.2 (from Chapter 1, Introduction), there exist a K_3-saturated graph H of order $n-q$ with $2(n-q)-5 \leq |E(H)| \leq \lfloor\frac{(n-q-1)^2}{4}\rfloor+1$, for $1 \leq q \leq n-5$. Take $G \cong H+\overline{K_q}$. Then G is clearly a K_4-saturated graph. Now $2(n-q)-5+(n-q)q \leq |E(G)| \leq \lfloor\frac{(n-q-1)^2}{4}\rfloor+1+(n-q)q$. For $q=1$, $3n-8 \leq |E(G)| \leq \lfloor\frac{n^2+4}{4}\rfloor$; for $q=2$, $4n-13 \leq |E(G)| \leq \lfloor\frac{n^2+2n-3}{4}\rfloor$; . . . ; for $q = n-5$, $5n-20 \leq |E(G)| \leq 10n-20$. It can be easily seen that these intervals overlap. Furthermore, we obtain the lower bound when $q=1$, $3n-8$. Now let $f(q) = \lfloor\frac{(n-q-1)^2}{4}\rfloor+1+(n-q)q$. Then $f(q)$ is maximum when $q = \frac{n+1}{3}$ and $f(\frac{n+1}{3}) = \frac{n^2-n+4}{3}$. □

In the above Theorem, the lower and upper bounds are achieved. For example, for the following graphs G_1 (Figure 4) and G_2 (Figure 5), lower and upper bounds, respectively, are achieved.

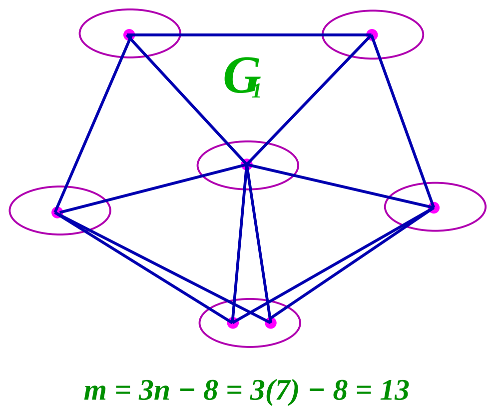

Figure 2.4: G_1

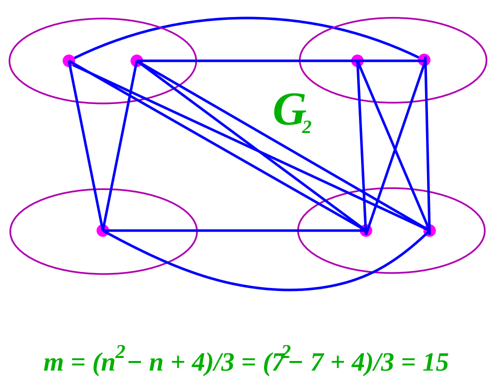

Figure 2.5: G_2

Chapter 3

K_t-Saturated Graphs

It is known that any K_t-saturated graph has at least $(t-2)(n-1) - \binom{t-2}{2}$ edges and at most $\lfloor \frac{(t-2)n^2}{2(t-1)} \rfloor$ edges and these bounds are sharp. The emphasis of this chapter will be on determining the sizes of K_t-saturated graphs of order n, that is the edge spectrum of K_t-saturated graphs. We prove that, for $n \geq 5t-7$, there is an (n, m) K_t-saturated graph G if and only if G is complete $(t-1)$-partite graph or $(t-1)(n-\frac{t}{2})-2 \leq m \leq \lfloor \frac{(t-2)n^2-2n+(t-2)}{2(t-1)} \rfloor + 1$. We show the lower bound of the interval for small minimum degree and chromatic number of a graph first. Then we apply a result by Duffus and Hanson [4] for $\delta(G) \geq t+2$ and $\chi(G) \geq t+1$. We again apply the result by Hanson and Toft [6] to show the upper bound of our result.

3.1 Lower Bound for K_t-saturated Graphs

In trying to establish a lower bound on the size of a K_t-saturated graph that is not isomorphic to the Erdős, Hajnal, and Moon graph $K_{t-2} + \overline{K_{n-t+2}}$, the chromatic number of the graph in question will be very important. Also, convexity considerations will play a role in minimizing the size of the graph. We develop a series of results based upon the chromatic number and subsequent minimum degree of the graph. First, we make the following observation.

Proposition 3.1 *Let G be a K_t-saturated graph. Then G is $(t-2)$-connected.*

Proof. Let G be a K_t-saturated graph. Suppose G is not $(t-2)$-connected. Let $U = \{u_1, u_2, ..., u_{t-3}\}$ be the cut set and let A and B be components of $G - U$ with $x \in V(A)$ and $y \in V(B)$. Now the addition of the edge xy creates a K_t. A contradiction as $\{x, y\} \cup U$ forms at most a K_{t-1}. $\square$

Theorem 3.2 *If G is a complete $(t-1)$-partite K_t-saturated graph and G is not isomorphic to $K_{t-2} + \overline{K_{n-t+2}}$, then $|E(G)| > (t-1)(n - \frac{t}{2}) - 2$.*

Proof. Let G be a complete $(t-1)$-partite K_t-saturated graph of order $n \geq t+1$. Let $C_1, C_2, C_3, ..., C_{t-1}$ be the $(t-1)$ partite sets of G. Suppose G is not isomorphic to $K_{t-2} + \overline{K_{n-t+2}}$. Now, by the Erdős, Hajnal, Moon

Theorem, G will have the minimum number of edges if $(t-2)$ sets, say $1, 2, ..., t-2$, each contain one vertex and the $(t-1)$-st set contains the remaining vertices. Since G is not isomorphic to $K_{t-2} + \overline{K_{n-t+2}}$, at least one of the $(t-2)$ sets contains more than one vertex. In fact, G will have the minimum number of edges if exactly one of the $(t-2)$ sets contains exactly two vertices. Without loss of generality, say set 1 contains two vertices. Now, the number of edges in G is as follows:

$$\begin{aligned}
|E(G)| &= \frac{1}{2}((t-3)(n-1) + 2(n-2) + (n-t+1)(t-1)) \\
&= \frac{1}{2}(2nt - 2n + t - t^2 - 2) \\
&= nt - n - \frac{t(t-1)}{2} - 1 \\
&= (t-1)(n - \frac{t}{2}) - 1 \\
&> (t-1)(n - \frac{t}{2}) - 2.
\end{aligned}$$

□

Note that, any $(t-1)$-partite graph that is K_t-saturated must be a complete $(t-1)$-partite graph. Hence, the case $\chi(G) = t-1$ is handled. Next, make the following two useful observations about K_t-saturated graphs.

1. The neighborhood of every vertex contains a copy of K_{t-2}.

2. For every pair of vertices $u, v \in V(G)$, $uv \notin E(G)$ if and only if there exists a copy of K_{t-2} in their common neighborhood.

Theorem 3.3 *Let G be a K_t-saturated graph with $\chi(G) = t$, then $|E(G)| \geq (t-1)(n - \frac{t}{2}) - 2$.*

Proof. Let G be a K_t-saturated graph of order n with $\chi(G) = t$. Let $C_1, C_2, ..., C_t$ be t independent sets based on the t color classes. Suppose G has the minimum number of edges. Now G will have minimum number of edges if $(t-1)$ sets, say $C_1, C_2, ..., C_{t-1}$, will contain exactly one vertex each and set C_t will contain the remaining $n-t+1$ vertices. It is easy to see that if you increase the number of vertices in any of the sets $C_1, C_2, ..., C_{t-1}$, you increase the number of edges as G is K_t-saturated and $\chi(G) = t$. Some of these graphs will not satisfy the given conditions and can be ignored. Now by observation 1, G contains at least one copy of K_{t-2} and by observation 2, every pair of nonadjacent vertices must be adjacent to a copy of K_{t-2}. So without loss of generality, say sets $C_1, C_2, ..., C_{t-2}$ make up K_{t-2}. Since G is a K_t-saturated graph, every vertex of the set C_t must be adjacent to the vertices in sets $C_1, C_2, ..., C_{t-2}$. Hence, now either the vertex in set C_{t-1} is adjacent to all the vertices of sets $C_1, C_2, ..., C_{t-2}$ or all the vertices of the set C_t and $t-3$ sets of the remaining $t-2$ sets $C_1, C_2, ..., C_{t-2}$. In either case,

$\chi(G) = t-1$. Hence, G cannot have t partite sets with $t-1$ partite sets of the size 1 and one partite set of size $n-t+1$.

So now suppose G has exactly one vertex in $t-2$ sets, say $C_1, C_2, ..., C_{t-2}$, two vertices in one set, say C_{t-1}, and the remaining $n-t$ vertices in one set, say C_t. As stated earlier, by observation 1, G contains at least one copy of K_{t-2} and by observation 2, every pair of nonadjacent vertices must be adjacent to a copy of K_{t-2}. So without loss of generality, say sets $C_1, C_2, ..., C_{t-2}$ make up a K_{t-2}. Since G is a K_t-saturated graph, one of the following two options for the remaining edges must hold:

1. Every vertex of the set C_{t-1} is adjacent to every vertex of the set C_t and every vertex of the sets C_{t-1} and C_t is adjacent to exactly $t-3$ of the $t-2$ sets.

2. Every vertex of the sets C_{t-1} and C_t is adjacent to every vertex of the sets $C_1, C_2, ..., C_{t-2}$ and there are no edges between vertices of the sets C_{t-1} and C_t.

Again, in both cases, $\chi(G) = t-1$. Hence, G cannot have t such color classes.

Hence, assume G has exactly one vertex in $t-3$ sets, say $C_1, C_2, ..., C_{t-3}$, exactly two vertices in two sets, say C_{t-2}, C_{t-1}, and the remaining $n-t-1$ vertices in one set, say C_t. Let $V(C_i) = v_i$ for $1 \leq i \leq t-3$, $V(C_{t-2}) = \{x_1, x_2\}$, $V(C_{t-1}) = \{y_1, y_2\}$, and $V(C_t) = \{w_1, w_2, ..., w_{n-t-1}\}$. Now from an argument similar to the ones above, we have $v_iv_j \in E(G)$ for all $1 \leq i < j \leq t-3$, $w_ix_2, w_iy_2 \in E(G)$ and $w_iv_j \in E(G)$ for $1 \leq i \leq n-t-1$, $1 \leq j \leq t-3$. Also, $x_iv_j, y_iv_j \in E(G)$ for $i = 1, 2$ and $1 \leq j \leq t-3$, $x_1y_1, x_1y_2, x_2y_1 \in E(G)$. Hence, the number of edges in G is

$$\begin{aligned}
|E(G)| &= \frac{(t-3)(t-4)}{2} + 2(n-t-1) + (n-t-1)(t-3) + 2(t-3) + \\
&\quad 2(t-3) + 3 \\
&= n(t-1) + \frac{-t^2+t-4}{2} \\
&= n(t-1) - \frac{t(t-1)}{2} - 2 \\
&= (t-1)(n - \frac{t}{2}) - 2.
\end{aligned}$$

□

We next consider the following.

Lemma 3.4 *If G is a K_t-saturated graph with $\chi(G) \geq t+1$, then $\delta(G) \geq t$.*

Proof. Let G be a K_t-saturated graph of order n with $\chi(G) \geq t+1$. Then by observation 1, $\delta(G) \geq t-2$. Assume $\delta(G) = t-2$. Let $v \in V(G)$ such that $d(v) = t-2$. Let $N(v) = \{v_1, v_2, ..., v_{t-2}\}$ and $W = V(G) - \{v \cup N(v)\}$. Then by observation 1, $\langle N(v) \rangle \cong K_{t-2}$. Then by observation 2, every vertex $u \in W$, $uv_i \in E(G)$ for $1 \leq i \leq t-2$. Hence, we can color vertices $\{v_1, v_2, ..., v_{t-2}\}$ with colors $\{1, 2, ..., t-2\}$, respectively, and the remaining vertices v and $u \in W$ by color $t-1$. Now $\chi(G) = t-1$, a contradiction.

Assume $\delta(G) = t-1$. Let $v \in V(G)$ such that $d(v) = t-1$. Let $N(v) = \{v_1, v_2, ..., v_{t-1}\}$ and $W = V(G) - \{v \cup N(v)\}$. Then by observation 1, $\langle N(v) \rangle$ contains a copy of K_{t-2}. Then $\langle N(v) \rangle$ contains either one or two copies of K_{t-2}. Suppose $\langle N(v) \rangle$ contains exactly one copy of K_{t-2}, say $K_{t-2} \cong \langle v_1, v_2, ..., v_{t-2} \rangle$. Then by observation 2, every vertex $u \in W$, $uv_i \in E(G)$ for $1 \leq i \leq t-2$. Hence, W must be an independent set. Furthermore, we note that v_{t-1} must be adjacent to each vertex of W for otherwise, by observation 2, v_{t-1} is adjacent to $v_1, v_2, ..., v_{t-2}$, producing a K_t, a contradiction. Without loss of generality, $v_j v_{t-1} \in E(G)$ for $1 \leq j \leq t-4$ as $N(v)$ contains exactly one copy of K_{t-2}. Then the addition of the edge $v_{t-1}v_{t-3}$ should create a K_t. Since $N(v_{t-1}) \cap N(v_{t-3}) = W \cup \{v, v_1, v_2, ..., v_{t-4}\}$, the addition of the edge $v_{t-1}v_{t-3}$ creates only a K_{t-1}, contradiction.

So suppose $\langle N(v)\rangle$ contains two copies of K_{t-2}, say $K^1_{t-2} \cong \langle v_1, v_2, ..., v_{t-2}\rangle$ and $K^2_{t-2} \cong \langle v_2, v_3, ..., v_{t-1}\rangle$. Then by observation 2, every vertex $u \in V(G) - \{v \cup N(v)\}$, $uv_i \in E(G)$ for $1 \leq i \leq t-2$ or $uv_i \in E(G)$ for $2 \leq i \leq t-1$ or $uv_i \in E(G)$ for $1 \leq i \leq t-1$. Define

$$A = \{u \in V(G) - \{v \cup N(v)\} | uv_i \in E(G), i = 1, 2, ..., t-2\},$$

$$B = \{u \in V(G) - \{v \cup N(v)\} | uv_i \in E(G), i = 2, 3, ..., t-1\},$$

$$C = \{u \in V(G) - \{v \cup N(v)\} | uv_i \in E(G), i = 1, 2, ..., t-1\}.$$

Note that if there is an edge in either of the sets, A, B, C, then K_t already exists as the vertices in these sets are adjacent to at least one copy of K_{t-2}. Hence, the sets A, B, C are independent sets. Furthermore, there is no edge between vertices of the sets A and C, otherwise K_t already exists. Similarly, there is no edge between vertices of the sets B and C.

Claim: $\langle A \cup B\rangle$ is a complete bipartite graph.

Suppose not. Then there exists vertices $u \in A$ and $v \in B$ such that $uv \notin E(G)$. Now addition of the edge uv should create a K_t. Since $N(u) \cap N(v) = \{v_2, v_3, ..., v_{t-2}\}$, addition of the edge uv creates a K_{t-1}, a contradiction. Hence, $\langle A \cup B\rangle$ is a complete bipartite graph.

Now color the vertex v with color 1, vertex v_{t-1} with color 2, and vertices $v_1, v_2, ..., v_{t-2}$ with colors $2, 3, ..., t-1$, respectively. Also, color the sets A and C with color 1 and color the set B with color t. Hence, $\chi(G) = t$, a contradiction. Hence, $\delta(G) \geq t$. $\square$

Lemma 3.5 *If G is a K_t-saturated graph with $\delta(G) \geq t$, $\chi(G) \geq t+1$, and $v \in V(G)$ with $d(v) = \delta(G)$, then $N(v)$ contains at least three copies of K_{t-2}.*

Proof. Assume G is a K_t-saturated graph with $\delta(G) \geq t$ and $\chi(G) \geq t+1$. Let $v \in V(G)$ with $d(v) = \delta(G)$. Let $N(v) = \{v_1, v_2, ...v_{\delta(G)}\}$, $W = \{v\} \cup N(v)$ and $U = V(G) - W$.

Suppose $N(v)$ contains exactly one copy of K_{t-2}. Without loss of generality, let $K_{t-2} \cong \langle v_1, v_2, ..., v_{t-2} \rangle$. As G is a K_t-saturated graph and there is exactly one copy of K_{t-2}, v_{t-1} is adjacent to at most $t-4$ of the vertices $v_1, v_2, ..., v_{t-2}$, say $v_1, v_2, ..., v_{t-4}$. Also, for every $u \in U$, $uv_i \in E(G)$, for $i = 1, 2, ..., t-2$. Hence, U is an independent set. Now addition of the edge $v_{t-2}v_{t-1}$ must create a K_t. But this fails as only a K_{t-1} is formed containing $v_1, v_2, ..., v_{t-4}, v_{t-2}, v_{t-1}$ and either v or at most one of the vertices from U.

Now suppose $N(v)$ contains exactly two copies of K_{t-2}. Then without loss of generality, $K^{(1)}_{t-2} \cong \langle v_1, v_2, ..., v_{t-2} \rangle$ and $K^{(2)}_{t-2} \cong \langle v_2, v_3, ..., v_{t-1} \rangle$. As seen above, v_t is adjacent to at most $t-4$ vertices of $v_1, v_2, ..., v_{t-1}$, say

$v_2, v_3, ..., v_{t-3}$. Then for every $u \in U$, $uv_i \in E(G)$ for $i = 1, 2, ..., v_{t-2}$ or $i = 2, 3, ..., v_{t-1}$, or $i = 1, 2, ..., v_{t-1}$, or $i = 1, 2, ..., \delta(G)$. Let

$$A = \{u \in U | uv_i \in E(G) \text{ for } i = 1, 2, ..., t-2\},$$

$$B = \{u \in U | uv_i \in E(G) \text{ for } i = 2, 3, ..., t-1\},$$

$$C = \{u \in U | uv_i \in E(G) \text{ for } i = 1, 2, ..., t-1\},$$

$$D = \{u \in U | uv_i \in E(G) \text{ for } i = 1, 2, ..., \delta(G)\}.$$

It can be easily seen that the sets A, B, C, and D are independent sets. Furthermore, $A \cup B$ forms a complete bipartite graph, see Lemma 3.4. Also, there is no edge between two vertices from A and C, A and D, B and C, B and D, and C and D, otherwise K_t already exists. Now addition of the edge $v_1 v_t$ must create a K_t. But this fails as only a K_{t-1} is created containing the vertices $v_2, v_3, ..., v_{t-3}$ and either v or at most one vertex from $C \cup D$. Similarly, we obtain a contradiction if two copies of K_{t-2} intersect in less than $t-3$ vertices. Hence, $N(v)$ contains at least three copies of K_{t-2}. $\square$

Lemma 3.6 *If G is a K_t-saturated graph with $\delta(G) = t$, $\chi(G) = t+1$, and $v \in V(G)$ with $d(v) = \delta(G)$, then $N(v)$ contains exactly three copies of K_{t-2}.*

Proof. Assume G is a K_t-saturated graph with $\delta(G) = t$. Let $v \in V(G)$ with $d(v) = \delta(G) = t$. Let $N(v) = \{v_1, v_2, ...v_t\}$, $W = \{v\} \cup N(v)$ and $U = V(G) - W$. Then by Lemma 3.5, $N(v)$ contains at least three copies of K_{t-2}. Let

$$K_{t-2}^{(1)} \cong \langle v_1, v_2, ..., v_{t-2} \rangle,$$

$$K_{t-2}^{(2)} \cong \langle v_2, v_3, ..., v_{t-1} \rangle,$$

$$K_{t-2}^{(3)} \cong \langle v_2, v_3, ..., v_{t-2}, v_t \rangle.$$

Now by observation 2, every $u \in U$ is adjacent to at least one copy of K_{t-2}. So define the following sets as in Figure 3.1:

$$A = \{u \in U | uv_i \in E(G) \text{ for } i = 1, 2, ..., t-2\} \text{ (green)},$$

$$B = \{u \in U | uv_i \in E(G) \text{ for } i = 2, 3, ..., t-1\} \text{ (red)},$$

$$C = \{u \in U | uv_i \in E(G) \text{ for } i = 2, 3, ..., t-2, t\} \text{ (blue)},$$

$$D = \{u \in U | uv_i \in E(G) \text{ for } i = 1, 2, ..., t\} \text{ (black)}.$$

Then the sets A, B, C, and D are independent sets, otherwise a K_t already exists. Also, as seen earlier, $A \cup B$ and $A \cup C$ are complete bipartite graphs.

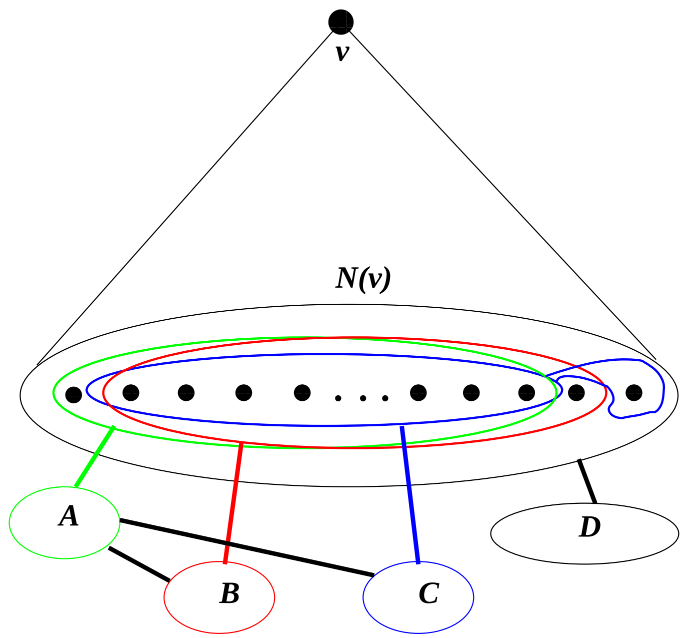

Figure 3.1: G

Furthermore, there is no edge between the vertices of the sets A and D, B and C, B and D, and C and D, otherwise a K_t already exists. Also, note that none of the sets A, B, C can be adjacent to more than one copy of K_{t-2}, otherwise a K_t already exists. Now it can be easily checked that G is a K_t-saturated graph.

Furthermore, it can be easily seen that $N(v)$ cannot contain more copies of K_{t-2} otherwise K_t already exists containing the vertex v and vertices from $N(v)$. Hence, $N(v)$ contains exactly three copies of K_{t-2}. $\square$

Lemma 3.7 *If G is a K_t-saturated graph with $\delta(G) = t$ and $\chi(G) \geq t + 1$, then $|E(G)| \geq (t-1)(n - \frac{t}{2}) - 2$.*

Proof. Suppose G is a K_t-saturated graph with $\delta(G) = t$ and $\chi(G) \geq t + 1$. Then by Lemma 3.6, $N(v)$ contains exactly three copies of K_{t-2} and we have the graph in Figure 1, as shown in Lemma 3.6. Note that we obtain minimum number of edges in the graph if $|A| = |B| = |C| = 1$, by convexity. Now, the minimum number of edges in the graph is

$$\begin{aligned}
|E(G)| &\geq t + \binom{t-3}{2} + 3(t-3) + 3(t-2) + 2 + t(n-t-4) \\
&= \frac{(t-3)(t-4)}{2} + 3t + nt - t^2 - 13 \\
&= n(t-1) - \frac{t(t-1)}{2} - 2 + n - t - 5 \\
&= (t-1)(n - \frac{t}{2}) - 2 + n - t - 5.
\end{aligned}$$

So $n - t - 5 \geq 0$ if $n \geq t + 5$. Hence, we have $|E(G)| \geq (t-1)(n - \frac{t}{2}) - 2$. □

Lemma 3.8 *If G is a K_t-saturated graph with $\delta(G) = t + 1$, $\chi(G) \geq t + 1$, and $v \in V(G)$ with $d(v) = \delta(G)$, then $N(v)$ contains at least five copies of K_{t-2}.*

Proof. Assume G is a K_t-saturated graph with $\delta(G) = t+1$ and $\chi(G) \geq t+1$. Let $v \in V(G)$ with $d(v) = \delta(G) = t+1$. Let $N(v) = \{v_1, v_2, ...v_{t+1}\}$, $W = \{v\} \cup N(v)$ and $U = V(G) - W$. Then by Lemma 3.6, $N(v)$ contains at least three copies of K_{t-2}.

Case 1: Supppose $N(v)$ contains exactly three copies of K_{t-2}.

Then we have one of the following three cases as in all other cases, G is not K_t-saturated graph:

(a) $K_{t-2}^{(1)} \cong \langle v_1, v_2, ..., v_{t-2}\rangle$, $K_{t-2}^{(2)} \cong \langle v_2, v_3, ..., v_{t-1}\rangle$, $K_{t-2}^{(3)} \cong \langle v_3, v_4, ..., v_t\rangle$, and $v_i v_{t+1} \in E(G)$ for $i = 3, 4, ..., t-2$.

(b) $K_{t-2}^{(1)} \cong \langle v_1, v_2, ..., v_{t-2}\rangle$, $K_{t-2}^{(2)} \cong \langle v_3, v_4, ..., v_t\rangle$, $K_{t-2}^{(3)} \cong \langle v_4, v_5, ..., v_{t+1}\rangle$, and $v_1 v_{t+1} \in E(G)$.

(c) $K_{t-2}^{(1)} \cong \langle v_1, v_2, ..., v_{t-2}\rangle$, $K_{t-2}^{(2)} \cong \langle v_2, v_3, ..., v_{t-1}\rangle$, $K_{t-2}^{(3)} \cong \langle v_2, v_3, ..., v_{t-2}, v_t\rangle$, and $v_i v_{t+1} \in E(G)$ for $i = 3, 4, ..., t-2$.

In each case, define the following sets as in Figure 3.2:

$$A = \{u \in U | uv_i \in E(G) \text{ for all } v_i \in V(K_{t-2}^{(1)})\} \text{ (green)},$$

$$B = \{u \in U | uv_i \in E(G) \text{ for all } v_i \in V(K_{t-2}^{(2)})\} \text{ (red)},$$

$$C = \{u \in U | uv_i \in E(G) \text{ for all } v_i \in V(K_{t-2}^{(3)})\} \text{ (blue)},$$

$$D = \{u \in U | uv_i \in E(G) \text{ for all } v_i \in N(v)\} \text{ (black)}.$$

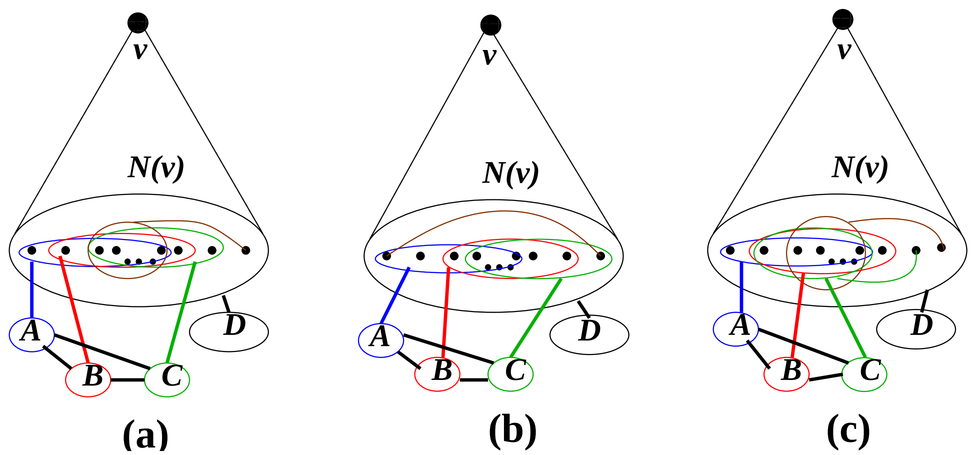

Figure 3.2: G

Furthermore, as seen earlier, $A \cup B$, $B \cup C$, and $A \cup C$ are complete bipartite graphs. Now the graphs in Figure 2(a) and 2(b) are not K_t-saturated graphs as the addition of the edge $v_t v_{t+1}$ does not create a K_t. Also, the graph in Figure 2(c) is not a K_t-saturated graph as the addition of the edge $v_1 v_{t+1}$ does not create a K_t.

Case 2: Suppose $N(v)$ contains exactly four copies of K_{t-2}.

Then, as seen above, we have one of the following three cases as in all other cases, G is not K_t-saturated graph:

(a) $K_{t-2}^{(1)} \cong \langle v_1, v_2, ..., v_{t-2} \rangle$, $K_{t-2}^{(2)} \cong \langle v_2, v_3, ..., v_{t-1} \rangle$, $K_{t-2}^{(3)} \cong \langle v_3, v_4, ..., v_t \rangle$, $K_{t-2}^{(4)} \cong \langle v_4, v_5, ..., v_{t+1} \rangle$, and $v_1 v_{t+1} \in E(G)$.

(b) $K_{t-2}^{(1)} \cong \langle v_1, v_2, ..., v_{t-2} \rangle$, $K_{t-2}^{(2)} \cong \langle v_2, v_3, ..., v_{t-1} \rangle$, $K_{t-2}^{(3)} \cong \langle v_3, v_4, ..., v_t \rangle$, and $K_{t-2}^{(4)} \cong \langle v_3, v_4, ..., v_{t-2}, v_t, v_{t+1} \rangle$.

(c) $K_{t-2}^{(1)} \cong \langle v_1, v_2, ..., v_{t-2} \rangle$, $K_{t-2}^{(2)} \cong \langle v_2, v_3, ..., v_{t-1} \rangle$, $K_{t-2}^{(3)} \cong \langle v_2, v_3, ..., v_{t-2}, v_t \rangle$, and $K_{t-2}^{(4)} \cong \langle v_2, v_3, ..., v_{t-2}, v_{t+1} \rangle$.

In each case, define the following sets as in Figure 3.3:

$$A = \{u \in U | uv_i \in E(G) \text{ for } v_i \in V(K_{t-2}^{(1)})\} \text{ (green)},$$

$$B = \{u \in U | uv_i \in E(G) \text{ for } v_i \in V(K_{t-2}^{(2)})\} \text{ (red)},$$

$$C = \{u \in U | uv_i \in E(G) \text{ for } v_i \in V(K_{t-2}^{(3)})\} \text{ (blue)},$$

$$D = \{u \in U | uv_i \in E(G) \text{ for } v_i \in V(K_{t-2}^{(4)})\} \text{ (magenta)},$$

$$E = \{u \in U | uv_i \in E(G) \text{ for } v_i \in N(v)\} \text{ (black)}.$$

Furthermore, as seen earlier, $A \cup B$, $B \cup C$, $C \cup D$, $A \cup C$, $A \cup D$, and $B \cup D$ are complete bipartite graphs. Now the graph in Figure $3(a)$ is not a K_t-saturated graph as the addition of the edge $v_1 v_t$ does not create a K_t. Also, the graph in Figure $3(b)$ is not a K_t-saturated graph as the addition of the edge $v_1 v_{t+1}$ does not create a K_t. Furthermore, $\chi(G) = t$ for the graph in Figure $3(c)$, a contradiction as $\chi(G) \geq t + 1$.

Hence, $N(v)$ must contain at least five copies of K_{t-2}. $\square$

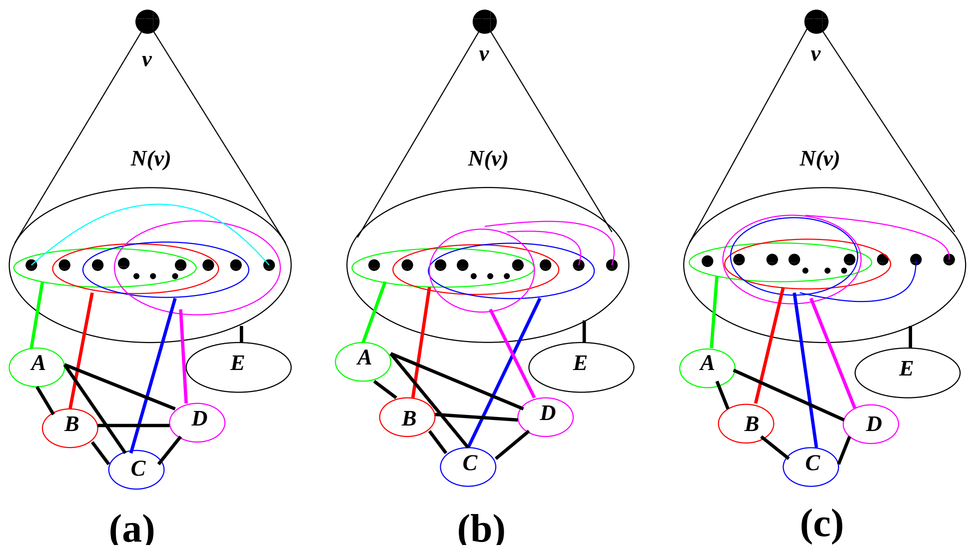

Figure 3.3: G

Lemma 3.9 *If G is a K_t-saturated graph with $\delta(G) = t+1$, $\chi(G) \geq t+1$, and $n \geq t+4$, then $|E(G)| \geq (t-1)(n - \frac{t}{2}) - 2$.*

Proof. Let G be a K_t-saturated graph with $\delta(G) = t+1$, $\chi(G) \geq t+1$, and $n \geq t+4$. Let $v \in V(G)$ such that $d(v) = \delta(G) = t+1$. Then by Lemma 3.8, $N(v)$ contains at least five copies of K_{t-2}. Suppose $N(v)$ contains exactly five copies of K_{t-2}. Let $K_{t-2}^{(1)} \cong \langle v_1, v_2, ..., v_{t-2}\rangle$, $K_{t-2}^{(2)} \cong \langle v_2, v_4, ..., v_{t-2}, v_t, v_{t+1}\rangle$, $K_{t-2}^{(3)} \cong \langle v_3, v_4, ..., v_{t-2}, v_{t-1}, v_t\rangle$, $K_{t-2}^{(4)} \cong \langle v_4, v_5, ..., v_{t+1}\rangle$, and $K_{t-2}^{(5)} \cong \langle v_1, v_4, ..., v_{t-2}, v_{t-1}, v_{t+1}\rangle$. Define the following sets as in Figure 3.4:

$$A = \{u \in U | uv_i \in E(G) \text{ for } v_i \in V(K_{t-2}^{(1)})\} \text{ (brown)},$$

$$B = \{u \in U | uv_i \in E(G) \text{ for } v_i \in V(K_{t-2}^{(2)})\} \text{ (red)},$$

$$C = \{u \in U | uv_i \in E(G) \text{ for } v_i \in V(K_{t-2}^{(3)})\} \text{ (green)},$$

$$D = \{u \in U | uv_i \in E(G) \text{ for } v_i \in V(K_{t-2}^{(4)})\} \text{ (magenta)},$$

$$E = \{u \in U | uv_i \in E(G) \text{ for } v_i \in V(K_{t-2}^{(5)})\} \text{ (blue)},$$

$$F = \{u \in U | uv_i \in E(G) \text{ for } v_i \in N(v)\} \text{ (black)}.$$

Then we obtain the graph as in Figure 3.4.

It can be easily checked that any other forms of five copies of K_{t-2} in $N(v)$ results in a graph that is not K_t-saturated. Also, if there is an edge between two vertices of the sets D and E, then G is not K_t-saturated graph as we already have a K_t. Furthermore, if one of the sets A, B, C is empty and the number of vertices in the remaining two sets is exactly one, then vertices in the sets D and E fails to meet minimum degree condition. If the either the set $D = \emptyset$ or $E = \emptyset$, but not both, then we do not have any problem. So if either one of sets A, B, C is empty or one of the sets D, E is empty, but not both, then we have more vertices among the remaining sets or more vertices in the set F. In either case, degree of the vertices in these sets is at least

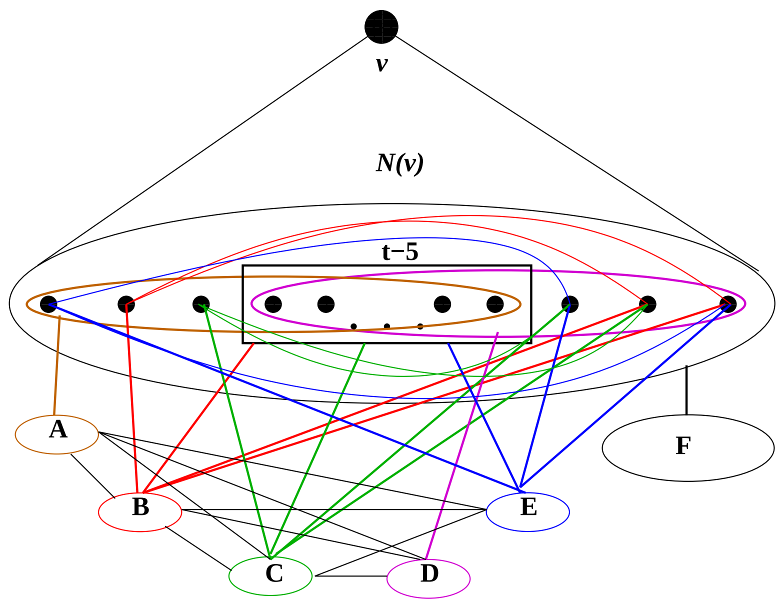

Figure 3.4: G

$t+1$. Furthermore, if the set $F = \emptyset$, then we still have minimum number of edges, as the vertices are moved to either of the remaining sets with having same or higher degree. Hence, we have the minimum number of edges if $|A| = |B| = |C| = |D| = |E| = 1$ and $|F| = n - t - 7$, by convexity. Hence, we have

$$
\begin{aligned}
|E(G)| &= t+1+\begin{pmatrix} t-5 \\ 2 \end{pmatrix} + 3+3+6(t-5)+6+(t+1)(n-t-7) \\
&\quad +5(t-2)+9 \\
&= \frac{(t-5)(t-6)}{2} + 4t+nt+n-t^2-25 \\
&= n(t-1) - \frac{t(t-1)}{2} - 2 - 2t + 2n - 8 \\
&= (t-1)(n-\frac{t}{2}) - 2 + 2n - 2t - 8 \\
&\geq (t-1)(n-\frac{t}{2}) - 2
\end{aligned}
$$

as $n \geq t+4$. Now it can be easily checked that as we increase number of copies of K_{t-2} in $N(v)$, then we increase number of edges. Hence, $|E(G)| \geq (t-1)(n-\frac{t}{2})-2$. $\square$

Now let's define $E(n,t,\delta)$ and $\chi(n,t,\delta)$. $E(n,t,\delta)$ is defined to be the minimum number of edges that a graph with n vertices and minimum degree δ can have if the addition of any edge creates a K_t. Suppose G is a graph on n vertices with chromatic number $\chi(G) = t-1$ and minimum degree δ. Define $\chi(n,t,\delta)$ to be the minimum number of edges that G can have such that the addition of any edge to G increases the chromatic number. The following Theorem is shown by Duffus and Hanson in [4].

Theorem 3.10 *For integers $n, t,$ and δ where $2 \leq t \leq n$ and*

$t - 2 \leq \delta \leq \frac{t-2}{t-1}n,$

$$\frac{\delta+t-2}{2}(n-\delta-1)+\delta+\binom{t-2}{2} \leq E(n,t,\delta) \leq \chi(n,t,\delta).$$

Now applying the above theorem to a K_t-saturated graph G with $\chi(G) \geq t+1$, we obtain the following result.

Lemma 3.11 *If G is a K_t-saturated graph, $\delta(G) \geq t+2$, $\chi(G) \geq t+1$, and $n \geq 5t-7$, then $|E(G)| \geq (t-1)(n-\frac{t}{2})-2$.*

Proof. Let G be a K_t-saturated graph with $\delta(G) \geq t+2$, $\chi(G) \geq t+1$, and $n \geq 5t-7$. Applying the above Theorem, Theorem 3.10, with given parameters, we obtain the following:

$$\begin{aligned}
|E(G)| &\geq \frac{t+2+t-2}{2}(n-t-2-1)+t+2+\binom{t-2}{2} \\
&= (t-1)(n-\frac{t}{2})-2+n-5t+7 \\
&\geq (t-1)(n-\frac{t}{2})-2
\end{aligned}$$

as $n \geq 5t-7$. □

Theorem 3.12 *If G is a K_t-saturated graph, $\delta(G) \geq t$, $\chi(G) \geq t+1$, and $n \geq 5t-7$, then $|E(G)| \geq (t-1)(n-\frac{t}{2})-2$.*

Proof. Proof follows from Lemma 3.7, Lemma 3.9, and Lemma 3.11. □

Theorem 3.13 *If G is a K_t-saturated graph of order $n \geq 5t-7$ and G is not isomorphic to $K_{t-2}+\overline{K_{n-t+2}}$, then $E(G) \geq (t-1)(n-\frac{t}{2})-2$.*

Proof. Proof follows from Theorem 3.2, Theorem 3.3, and Theorem 3.12.

□

3.2 Upper Bound for K_t-saturated Graphs

From Theorem 2.10 by Hanson and Toft [6], we have the following result.

Theorem 3.14 *If G is not a complete $(t-1)$-partite K_t-saturated graph for $t \geq 5$, then $E(G) \leq \lfloor\frac{(t-2)n^2-2n+(t-2)}{2(t-1)}\rfloor+1$.*

Proof. Let G be a K_t-saturated graph of order n. Suppose G is not a complete $(t-1)$-partite graph. Since G is not $(t-1)$-partite, $\chi(G) \geq t$. Hence, by Theorem 2.10, $|E(G)| \leq |E(T'_{t-1,n})|$. Let $n+1=(t-1)s+r$, $0 \leq r < t-1$. We will consider two cases. One with $r=t-2$ and another with $0 \leq r \leq t-3$. For the first case, a straightforward computation shows, $|E(T'_{t-1,n})| = \frac{(t-2)n^2-2n(3-2t)-4t^2+16t-12}{2(t-1)} + 1 < F(n,t)$. Now for $0 \leq r \leq t-3$,

$$\begin{aligned}
|E(T'_{t-1,n})| &= \frac{(t-r-3)(t-r-4)}{2}s^2 + \frac{r(r-1)}{2}(s+1)^2 + \\
&\quad rs(s+1)(t-r-3) + (s-1)(s-2) + r(s-1)(s+1) \\
&\quad +(t-3-r)(s-1)s + r(s-2)(s+1) + (t-3-r)(s-2)s \\
&\quad +1 + s - 1 + 2r(s+1) + 2(t-3-r)s \\
&= \frac{(t-2)n^2 - 2n + t - 2 - tr + r + r^2}{2(t-1)} + 1.
\end{aligned}$$

Hence, it is easy to see that the number of edges is maximum when $r = 0$. So $|E(T'_{t-1,n})| = \frac{(t-2)n^2-2n+(t-2)}{2(t-1)} + 1$. □

3.3 Edge Spectrum of K_t-saturated Graphs

Theorem 3.15 *Let $n \geq 5t-7$ and m be nonnegative integers. There is an (n,m) K_t-saturated graph G if and only if G is a complete $(t-1)$-partite graph or $(t-1)(n-\frac{t}{2}) - 2 \leq m \leq \lfloor \frac{(t-2)n^2-2n+(t-2)}{2(t-1)} \rfloor + 1$.*

Proof. Let $n \geq 5t-7$ and m be nonnegative integers. Let G be an (n,m) K_t-saturated graph. If G is a $(t-1)$-partite graph, then G must be a complete $(t-1)$-partite graph, otherwise an edge may be added without creating a K_t.

Assume G is not a complete $(t-1)$-partite graph. Then from Theorem 3.13 and Theorem 3.14, $(t-1)(n-\frac{t}{2})-2 \leq m \leq \lfloor\frac{(t-2)n^2-2n+(t-2)}{2(t-1)}\rfloor+1$.

It is sufficient to construct an (n,m) K_t-saturated graph for each value of m. If $m=(t-2)(n-\frac{t-1}{2})$, then $G \cong K_{t-2}+\overline{K_{n-t+2}}$. If G is a complete $(t-1)$-partite graph, then G is a K_t-saturated graph.

Now to show the existence of a K_t-saturated graph for $(t-1)(n-\frac{t}{2})-2 \leq m \leq \lfloor\frac{(t-2)n^2-2n+(t-2)}{2(t-1)}\rfloor+1$, we use mathematical induction. We know the statement is true for $t=3,4$. Assume the statement is true for $t-1$. Let $(t-1)(n-\frac{t}{2})-2 \leq m \leq \lfloor\frac{(t-2)n^2-2n+(t-2)}{2(t-1)}\rfloor+1$. We want to construct a K_t-saturated graph for each value of m. Let $G \cong H+\overline{K_q}$, where H is a K_{t-1}-saturated graph of order $n-q$. Then by the induction hypothesis, $(t-2)(n-q-\frac{t-1}{2})-2 \leq |E(H)| \leq \lfloor\frac{(t-3)(n-q)^2-2(n-q)+(t-3)}{2(t-2)}\rfloor+1$. Hence, $(t-2)(n-q-\frac{t-1}{2})-2+(n-q)q \leq |E(G)| \leq \lfloor\frac{(t-3)(n-q)^2-2(n-q)+(t-3)}{2(t-2)}\rfloor+1+(n-q)q$. For $q=1$, $(t-1)(n-\frac{t}{2})-2 \leq |(E(G)| \leq \lfloor\frac{(t-3)(n-1)^2+2(t-3)(n-1)+(t-3)}{2(t-2)}\rfloor+1$, for $q=2$, $t(n-\frac{t}{2})-\frac{t}{2}-3 \leq |(E(G)| \leq \lfloor\frac{(t-3)(n-2)^2+2(2t-5)(n-2)+(t-3)}{2(t-2)}\rfloor+1$, and so on. Again, it can be easily seen that these intervals overlap. Furthermore, we obtain the lower bound when $q = 1$, $(t-1)(n-\frac{t}{2})-2$. Now let $f(q) = \lfloor\frac{(t-3)(n-q)^2-2(n-q)+(t-3)}{2(t-2)}\rfloor + 1 + (n-q)q$. Then $f(q)$ is maximum when $q = \frac{n+1}{t-1}$ and $f(\frac{n+1}{t-1}) = \lfloor\frac{(t-2)n^2-2n+(t-2)}{2(t-1)}\rfloor + 1$ $\square$

Chapter 4

Future Work

As mentioned earlier, finding saturation numbers is hard in general. A great deal of time and effort has been spend in finding saturation numbers for cycles. Still the only exact values known are for C_3, C_4, and C_5. There are also numerous other graphs whose saturation number is not known. Then the next question is finding the edge spectrum for those graphs. Here are some questions that can be considered next.

1. Edge spectrum for book and generalized book graphs.

2. Edge spectrum for paths.

3. Edge spectrum for a star.

4. Edge spectrum for complete bipartite graphs.

5. Edge spectrum for complete t-partite graphs.

6. Saturation numbers for cycles C_t, $t > 5$.

7. Edge spectrum for cycles.

Recently, D. West and P. Wenger (personal communcation) defined saturation unique graphs to be the ones that are H-saturated but the addition of any edge produces exactly one copy of H. Hence, the next natural question would be to find saturation unique graphs for complete graphs, bipartite graphs, paths, cycles, etc, if they exist. Furthermore, let $f(k)$ be the minimum number of copies of H in a H-saturated graph with k edges after we add any edge. Then finding this $f(k)$ for complete graphs, complete bipartite graphs, paths, cycles, etc., can be another future project.

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