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Modular Linear Differential Equations and Deligne's Exceptional Series

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An abstract of
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Abstract

Modular Linear Differential Equations and Deligne's Exceptional Series
By Robert Dicks

In 1988, Mathur, Mukhi, and Sen studied rational conformal field theories in terms of differential equations satisfied by their characters. These differential equations are modular invariant, and the solutions they obtain for order 2 equations have relationships with certain Lie algebras. In fact, the Lie algebras in the Deligne Exceptional series appear, whose study is motivated by uniformities which appear in their representation theory. This thesis studies the Deligne Exceptional Series from these two perspectives, and gives a sequence of finite groups which has analogies with the Deligne series.

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Contents

List of Figures

1 Introduction

The point of this thesis is to describe and motivate the study of the *Deligne exceptional series of Lie groups and Lie algebras*, both from the perspective of representation theory of Lie groups and from the perspective of rational conformal field theories. They arise from this series of (adjoint) complex Lie groups:

$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8. \tag{1.1}$$

The beginning of this story dates back to the classification of simple complex Lie algebras given by Cartan and Killing in the 1800s. The classification theorem for simple complex Lie algebras states that any simple complex Lie algebra must fit into one of the following classes:

- (A_n)
- (B_n)
- (C_n)
- (D_n)
- $\{\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8\}$

The Lie algebras in the last group are called *exceptional Lie algebras*. Though this theorem gives exactly which simple complex Lie algebras there are, there is one way in which it is unsatisfactory. The exceptional Lie algebras are only so named because they do not fit into any of the first four families, which are called the *classical* families. The work of Deligne and other authors (see [16] for instance) begins with noticing some peculiar uniformities in the representation theory of groups in his series. To give an example: One such property is that there is a uniformizing parameter λ that allows for interpolation between the dimensions of their adjoint representations via this formula:

$$\frac{-2(\lambda+5)(\lambda-6)}{\lambda(\lambda-1)}.$$

To understand the origin of this parameter, suppose you fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, where $\mathfrak{g}$ is one of the Lie algebras of these groups. The Killing form on $\mathfrak{h}$ induces a bilinear form on $\mathfrak{h}^*$, its dual. The parameter is then $-6a$, where a is the square of the lengths of the highest roots for these Lie algebras as $\mathfrak{g}$ varies.

Some striking properties about the formula above is that it is a quotient of products of rational linear factors and that it is invariant under sending λ to $\lambda^* := 1 - \lambda$. For a set of virtual representations of certain extensions of these groups that includes the adjoint representation, Deligne gives more dimension formulas; they are all quotients of products of rational linear factors, and they are similarly invariant. These virtual representations are either irreducible, 0, or the negative of an irreducible, and they allow for uniform decomposition formulas of certain symmetric and exterior powers of the adjoint representations of these groups. The values of the Casimirs of these representations can also be written in terms of λ, and together with the decompositon formulas allowed for the computations of the dimension formulas.

On the physical side, the physicists Mathur, Mukhi, and Sen sought to study rational conformal field theories via modular linear differential equations, with a view towards their classification. Conformal field theory studies quantum field theories invariant under conformal transformations, those transformations which are angle-preserving. These rational conformal field theories have partition functions which are a sums of holomorphic and anti-holomorphic functions. The holomorphic parts of the partition function are called the *characters* of the theory. Moreover, these characters may be regarded as the solutions of a differential equation of order n, and this differential equation must be modular-invariant ([3]), which is why modular functions and modular forms play a role in this story. The definitions of modular function and modular form used in [3] are given in [5], and are reproduced in Section 4. The modular linear differential equations are defined with respect to this differential operator on weight r modular functions given in [1]:

$$\mathcal{D}^{(r)} := (\frac{\partial}{\partial\tau} - \frac{1}{6}i\pi r E_2(\tau))f(\tau), \tag{1.2}$$

where $E_2(\tau)$ is the Eisenstein series of weight 2. The reason the differential operator is defined in this way comes down to how derivatives of modular functions of weight r are not modular. However, adjusting by the multiple of $E_2(\tau)$ ensures that modularity is preserved, with weight $r+2$. The general order n modular linear differential equation is:

$$\mathcal{D}^n f + \sum_{k=0}^{n-1} \phi_k(\tau)\mathcal{D}^k f = 0,$$

where the weight r of f has been suppressed, and $\mathcal{D}^n$ is the operator (1.2) iterated n times and mapping f to a modular form of weight $r+2n$.

However, Mathur, Mukhi, and Sen are mainly concerned with the modular linear differential equations of order 2. In the course of solving modular linear differential equations of order 2, they obtain among their possible rational conformal field theories several Wess-Zumino-Witten models which have relations with the Deligne exceptional series of Lie algebras. This thesis will explain how this occurs.

In addition to these topics, this thesis also explores a series of finite groups which have analogies with the Deligne exceptional series of Lie groups. These groups are as follows:

$$L_2(7), 2.A_9, J_2, 2.Ru, 2.F_4(2), Fi22, HN, Th. \tag{1.3}$$

The last portion of the paper explores some properties satisfied by the representations of these finite groups, and ends with further questions that can be explored.

The structure of this paper is as follows. Sections 2 and 3 give the background on Lie groups and Lie algebras necessary to understand what the Deligne exceptional series is and what Deligne's observations are. Section 4 gives some facts on modular forms and modular functions, as well as the approach of Mathur, Mukhi, and Sen in classifying rational conformal field theories. Section 5 gives some background on the representation theory of finite groups necessary to understand the observations on the series of finite groups listed above, and ends with questions. Section 6 is the concluding section, which summarizes the contents of the thesis.

2 Lie groups/Lie algebras

2.1 Manifolds

To describe the exceptional series, some preliminaries are necessary. In particular, Lie groups and Lie algebras need to be defined. The notion of a Lie group relies on some manifold theory. The definitions given below come from [24]. We begin with the definition of a topological manifold:

Definition 2.1. Let M be a topological space. We say M is a *topological manifold* if it satisfies these properties:

- It is a Hausdorff space. That means, for distinct points $p, q \in M$, there are disjoint open sets U and V such that $p \in U$ and $q \in V$.
- It is second countable. That is, it admits a finite or countable basis for the topology.
- It is locally Euclidean of dimension n, for some positive integer n; each point of M has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$. This is the *dimension* of the manifold.

When studying manifolds, it is often necessary to consider the concept of a submanifold. These are defined in a natural way:

Definition 2.2. Let M be a topological manifold of dimension m, and let $N \subset M$. We say N is a (closed) *submanifold* of dimension n if it is a manifold under the induced topology. The number $m - n$ is called the *codimension* of the submanifold.

It is natural to endow topological manifolds with more structure, such as a differential structure. This allows tools from analysis to enrich their study. This motivates the definition of a smooth or differentiable manifold. To describe these, it is necessary to understand what a chart on M is:

Definition 2.3. Let M be a topological manifold. A *chart* on M is a pair (U, ϕ) where U is an open subset of M and $\phi : U \to \hat{U}$ is a homeomorphism from U to an open subset $\hat{U} = \phi(U) \subset \mathbb{R}^n$.

The idea behind using charts is to cover the manifold and form an atlas that will satisfy certain properties and allow for differential structure to be placed on the manifold. An atlas is defined in this way:

Definition 2.4. Let M be a topological manifold. An atlas for M is a collection of charts whose domains cover M.

For smooth manifolds, the notion of a smooth atlas is pertinent. It relies on what are called transition maps:

Definition 2.5. Let M be a topological manifold of dimension n, and let (U, ϕ) and (V, Ψ) be charts. If $U \cap V \neq \emptyset$, then the composite map $\Psi \circ \phi^{-1} : \phi(U \cap V) \to \Psi(U \cap V)$ is called the transition map from ϕ to Ψ.

Definition 2.6. An atlas $\mathfrak{A}$ for a topological manifold M is called a smooth atlas if any two charts are *smoothly compatible.* That is, for two charts (U, ϕ) and (V, Ψ), either $U \cap V$ is empty or the transition map from ϕ to Ψ is a C^∞ map.

Now the definition of a smooth manifold can be given:

Definition 2.7. Let M be a topological manifold.If $\mathfrak{A}$ is a maximal smooth atlas, then the pair $(M, \mathfrak{A})$ is called a smooth manifold.

In studying smooth manifolds, as with any structure, it is important to understand maps between them that preserve the smooth structure. The definition of a smooth map of manifolds is as follows:

Definition 2.8. Let M and N be smooth manifolds. We say $\rho : M \to N$ is a smooth map if for every point $p \in M$, there exist charts (U, ϕ) of p and (V, Ψ) of $\rho(p)$ such that $\rho(U) \subseteq V$ and the map $\Psi \circ \rho \circ \phi^{-1} : \phi(U) \to \mathbb{R}^n$ is a C^∞ map.

A natural way of understanding smooth manifolds is via their tangent spaces at certain points. For this paper, the tangent space to the identity of Lie groups will play an important role, as will the differential of a map of manifolds. The definition of the tangent space at a point relies on the notion of a tangent vector at a point:

Definition 2.9. Let M be a smooth manifold with $p \in M$, and let $F_p(M)$ be the family of real-valued functions on M that are differentiable at p. A function $r : F_p(M) \to \mathbb{R}$ is called a tangent vector at p if:

- It is linear, which means $r(af + bg) = ar(f) + br(g)$ for $f, g \in F_p(M)$ and $a, b \in \mathbb{R}$
- It is a derivation, which means $r(fg) = f(p)r(g) + g(p)r(f)$ for $f, g \in F_p(M)$.

There are operations on tangent vectors that make the tangent vectors at a point into a vector space:

Definition 2.10. Let M be a smooth manifold, and let $p \in M$. If t_1 and t_2 are tangent vectors at p, define the sum $(t_1 + t_2)(p) = t_1(p) + t_2(p)$ and the scalar multiplication by $(ct_1)(p) = ct_1(p)$. With these operations, the tangent vectors form a vector space called the tangent space of M at p.

2.2 Lie groups

With these preliminaries in mind, we can begin describing what Lie groups are. We begin by recalling the definitions of a Lie group and a Lie subgroup that appear in [1]:

Definition 2.11. We say a group G is a *real Lie group* if it is a C^∞ manifold such that the composition $* : G \times G \to G$ and the inversion $i : G \to G$ maps are smooth. If G and H are real Lie groups, we say $\rho : G \to H$ is a *map of real Lie groups* if it is simultaneously a smooth map and a group homomorphism.

Definition 2.12. Let G be a real Lie group. A subgroup $H \subset G$ is a *real Lie subgroup* if it is also a closed submanifold.

Remark. Let G be a real Lie group and let $H \subset G$ be a normal, real Lie subgroup. The quotient group G/H has the structure of a Lie group. The standard homomorphism theorems also hold for Lie groups. For instance, if $\rho : G \to H$ is a map of Lie groups, then it induces an isomorphism between $G/ker(\rho)$ and $\rho(G)$.

Examples of real Lie groups include $\mathbb{R}^n$ under addition and $\mathbb{R}^*$ under multiplication. An important class of examples of real Lie groups is given by the $n \times n$ invertible matrices, $GL(n, \mathbb{R})$, along with its subgroups. Examples include $SL(n, \mathbb{R})$, the $n \times n$ matrices over $\mathbb{R}$ having determinant 1, $SO(n, \mathbb{R})$, the $n \times n$ matrices over $\mathbb{R}$ having determinant 1 and whose inverse is given by its transpose.

The following definition of a representation of a real Lie group is natural, when the $n \times n$ invertible matrices are identified with the automorphisms of an n-dimensional real vector space V and written as $\mathrm{GL}(V)$ (i.e. in a coordinate-free way):

Definition 2.13. A *representation* of a real Lie group G on a finite dimensional, real vector space V is a map of real Lie groups $\rho : G \to GL(V)$. If the map is understood, then V is called a representation of G. A *subrepresentation* of V is a subspace $W \subset V$ if it is invariant under the action of G. We say V is *irreducible* if it has no nontrivial subrepresentations.

Remark. Let G be a Lie group and let V and W be representations of G. The direct sum and tensor product of V and W are also representations of G; the action is given by: $g(v, w) = (gv, gw)$ and $g(v \otimes w) = gv \otimes gw$, respectively, for $g \in G$.

Remark. For what follows, it is also necessary to consider *virtual representations* of Lie groups. If G is a Lie group, consider the free abelian group generated by its representations $\{\sum a_i V_i\}$, and quotient by the subgroup generated by elements of the form $V_i + V_{ii} - (V_i \oplus V_{ii})$. The elements of this group are the virtual representations of G.

The notion of a complex Lie group arises from replacing "C^∞ manifold" with "complex manifold" and "smooth" with "holomorphic" in the definition above. The definition of a representation of a complex Lie group and of a complex Lie subgroup is similarly adjusted. As in the real case, many important examples of complex Lie groups come from $GL(n, \mathbb{C})$ and its subgroups. In particular, all of the real Lie groups mentioned previously have complex analogues. These examples, together with the group $Sp(2n, \mathbb{C})$ of $2n \times 2n$ symplectic matrices over $\mathbb{C}$, will play a role in what follows.

Lie groups have algebraic and geometric structure; thus, understanding Lie groups necessarily involves exploiting the relationship between the two structures. One manifestation of this is the study of Lie groups via their associated Lie algebras. These are simpler objects that can be thought of as linear approximations of the groups. In spite of this, they are an effective tool for studying representations of Lie groups. An abstract Lie algebra is defined as follows in [10]:

Definition 2.14. A vector space $\mathfrak{g}$ over a field F (which will usually be taken to be $\mathbb{R}$ or $\mathbb{C}$ for what follows) is called a *Lie algebra* if it is equipped with a bilinear map $[\ ,\ ]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying:

- (skew-symmetry) $[x, x] = 0\ \ \forall x \in \mathfrak{g}$.
- (Jacobi identity) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

This bilinear map is called the *Lie bracket*. If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, then a *homomorphism* of Lie algebras is a map $\rho : \mathfrak{g} \to \mathfrak{h}$ preserving Lie brackets: $\rho([x, y]) = [\rho(x), \rho(y)]$.

Remark. One can endow any vector space with the structure of a Lie algebra if all brackets are defined to be 0.

Remark. Any associative algebra can be given the structure of a Lie algebra. The Lie bracket is given by the commutator of elements.

For a given Lie group G, it turns out that the tangent space to the identity, T_eG, can be given the structure of a Lie algebra, and this is the Lie algebra of G. First, note that the conjugacy action of G on itself gives a natural map $\Psi : G \to Aut(G)$. Second, differentiating this map gives an important representation of G on T_eG:

Definition 2.15. Let G be a Lie group. The *adjoint representation* of G, denoted by Ad: $G \to \mathrm{Aut}(T_eG)$, is the map obtained from differentiating the conjugate action of G on itself. The adjoint group of G is its image under the adjoint representation. It can be identified with $G/Z(G)$.

The tangent space at the identity of $\mathrm{Aut}(T_eG)$ can be identified with $\mathrm{End}(T_eG)$. Differentiating the adjoint representation of a Lie group G gives a map from T_eG to $\mathrm{End}(T_eG)$; This map is often denoted by ad: $T_eG \to \mathrm{End}(T_eG)$, such as in [10]. Finally, this map allows one to define a bilinear map on T_eG. More precisely, define [,]: $T_eG \times T_eG \to T_eG$ by $[X, Y] := \mathrm{ad}(X)(Y)$. This bilinear map will be skew-symmetric and have the Jacobi identity hold.

The tangent space to the identity of the Lie group $\mathrm{GL}(n, \mathbb{R})$ can be identified with the set of all $n \times n$ matrices over $\mathbb{R}$.(See [10]) It is denoted by $\mathfrak{gl}_n(\mathbb{R})$ when considered as a Lie algebra. The Lie bracket is given by the commutator of the matrices. If V is a real vector space, then $\mathrm{End}(V)$ can be given a Lie algebra structure, where the Lie bracket is the commutator of endomorphisms. When $\mathbb{R}$ is replaced by $\mathbb{C}$, these become examples of complex Lie algebras.

Just as Lie groups have representations, Lie algebras also have representations:

Definition 2.16. Let $\mathfrak{g}$ be a Lie algebra. A *representation* of $\mathfrak{g}$ on a finite-dimensional vector space V is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathrm{End}(V)$.

Remark. Just as we can use two representations of a Lie group to build larger representations via direct sums and tensor products, we can do the same for Lie algebra representations. The tensor product of two Lie algebra representations, however, needs defined in such a way that a Lie group acting on a tensor product of its representations induces a representation of its Lie algebra on the tensor product. It is defined in this way for $X \in \mathfrak{g}$ to serve this purpose:

$$X(v \otimes w) = Xv \otimes w + v \otimes Xw.$$

The adjoint representation for a Lie algebra associated to a Lie group G is the map ad: $T_eG \to \mathrm{End}(T_eG)$. There is also a definition of adjoint representation for abstract Lie algebras:

Definition 2.17. Let $\mathfrak{g}$ be a Lie algebra. The *adjoint representation* of $\mathfrak{g}$ is the map $\rho : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ defined by $\rho(X)(Y) := [X, Y]$.

There are some basic definitions for Lie algebras that allow for deeper understanding of their adjoint representations. The first relevant notion is that of a subalgebra of a Lie algebra. Studying subalgebras of Lie algebras sheds light on Lie subgroups of Lie groups. These definitions [10]:

Definition 2.18. Let $\mathfrak{g}$ be a Lie algebra. A subset $\mathfrak{h} \subset \mathfrak{g}$ is called a subalgebra of $\mathfrak{g}$ if it is a subspace of $\mathfrak{g}$ that is closed under brackets. That is, for $X, Y \in \mathfrak{h}$, $[X, Y] \in \mathfrak{h}$.

An important way of constructing larger Lie algebras is via the direct sum. This is defined in the following way in [10]:

Definition 2.19. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras over a field F. The direct sum $\mathfrak{g} \oplus \mathfrak{h}$ is the vector space $\{(g, h) \, | g \in \mathfrak{g}, h \in \mathfrak{h}\}$ with the Lie bracket being defined componentwise.

An important notion for groups is that of a normal subgroup. It allows for the left cosets to have a group structure. The corresponding notion for rings is that of an ideal, endowing cosets with a ring structure. The notion of an ideal for a Lie algebra plays a similar role:

Definition 2.20. Let $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. We say $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ if for all $Y \in \mathfrak{g}$, $[X, Y] \in \mathfrak{h}$.

Remark. Let $\mathfrak{g}$ and $\mathfrak{h}$, and let $\rho : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. The kernel $ker(\rho)$ consists of all elements $X \in \mathfrak{g}$ such that $\rho(X) = 0$. It is an ideal of $\mathfrak{g}$. This is analogous to how kernels of group homomorphisms are normal subgroups, as well as how kernels of ring homomorphisms are ideals.

Remark. For a subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the bracket on $\mathfrak{g}$ induces a well-defined bracket on the quotient space $\mathfrak{g}/\mathfrak{h}$ if and only if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. This is called the quotient Lie algebra of $\mathfrak{g}$ by $\mathfrak{h}$. There is an analogue for Lie algebras of the usual homomorphism theorems. For instance, if $\rho : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras, then $\mathfrak{g}/\ker(\rho) \simeq \rho(\mathfrak{g})$ as Lie algebras.

The kernel of the adjoint representation is distinguished:

Definition 2.21. Let $\mathfrak{g}$ be a Lie algebra. The center of $\mathfrak{g}$, denoted by $Z(\mathfrak{g})$, consists of all elements $X \in \mathfrak{g}$ such that the bracket $[X, Y]$ vanishes for all $Y \in \mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is said to be abelian if it coincides with its center.

The Lie groups occuring in the Deligne exceptional series are simple complex Lie groups, and there is a corresponding notion of simplicity for Lie algebras. These notions are defined as follows in [10]:

Definition 2.22. A complex Lie group G is called simple if it is a non-abelian, connected Lie group which has no nontrivial, connected, normal subgroups.

Definition 2.23. A Lie algebra $\mathfrak{g}$ is simple if its dimension is greater than one and it contains no nontrivial ideals. A *semisimple Lie algebra* is a direct sum of simple Lie algebras.

Remark. A Lie group is simple if and only if its Lie algebra is simple. Additionally, a simple complex Lie group need not be simple as an abstract group. (See [10])

Lie algebras and their representations shed light on Lie groups and their representations. This allows for the classification of simple Lie groups to be approached via the corresponding classification for simple complex Lie algebras. To understand this in greater detail, it should be noted that, for any Lie algebra $\mathfrak{g}$, there is a Lie group whose Lie algebra is $\mathfrak{g}$. However, there is no one-to-one correspondence between Lie groups and their Lie algebras; two Lie groups may have the same Lie algebra. Taking quotients and extensions by finite groups does not change the Lie algebra of a Lie group.

However, there are weaker statements that can be made. If one restricts to simply connected Lie groups, then given a Lie algebra, there is a unique Lie group attached to it. This is summarized in the following theorem, a proof of which appears in [10]:

Theorem 2.24. For any Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ or $\mathbb{C}$, there is a unique, simply connected Lie group having $\mathfrak{g}$ as its Lie algebra. Furthermore, all Lie groups having $\mathfrak{g}$ as their Lie algebra are quotients of this group by a discrete subgroup of its center.

As a consequence, there is a unique simply connected complex group having as its Lie algebra one of the simple complex Lie algebras.

2.3 Classification of Simple Complex Lie Algebras

Killing and Cartan were the first mathematicians to classify simple complex Lie algebras.([25]) This paper follows the approach by Eugene Dynkin in 1947. Given the complex Lie groups $SL(n,\mathbb{C})$, $SO(n,\mathbb{C})$, and $Sp(2n,\mathbb{C})$, denote their Lie algebras by $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{so}_n(\mathbb{C})$, and $\mathfrak{sp}_{2n}(\mathbb{C})$ respectively. The classification theorem, which is stated in [10], claims that these are the only simple complex Lie algebras except for finitely many exceptions:

Theorem 2.25 (Classification of simple complex Lie algebras)**.** Every simple complex Lie algebra fits into one of five categories:

- $A_n = \mathfrak{sl}_{n+1}(\mathbb{C})$
- $B_n = \mathfrak{so}_{2n+1}(\mathbb{C})$
- $C_n = \mathfrak{sp}_{2n}(\mathbb{C})$
- $D_n = \mathfrak{so}_{2n}(\mathbb{C})$
- $\{\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8\}$

The five exceptions in the last category are called the exceptional Lie algebras, and Lie groups which have one of the exceptions as Lie algebras are called exceptional Lie groups.

In order to study this theorem more closely, it is necessary to study the roots and weights of a Lie algebra. They rely on studying Cartan subalgebras of Lie algebras. These definitions come from [10]:

Definition 2.26. Let $\mathfrak{g}$ be a Lie algebra. The lower central series of $\mathfrak{g}$ is inductively defined as follows in [10]:

$\mathcal{D}_1(\mathfrak{g}) := [\mathfrak{g}, \mathfrak{g}]$

$\mathcal{D}_n(\mathfrak{g}) := [\mathfrak{g}, \mathcal{D}_{n-1}(\mathfrak{g})]$

We say $\mathfrak{g}$ is *nilpotent* if its lower central series terminates at some point.

Definition 2.27. Let $\mathfrak{g}$ be a complex Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a nilpotent subalgebra. We say $\mathfrak{h}$ is a *Cartan subalgebra* if it is *self normalizing*; that is, $N_{\mathfrak{g}}(\mathfrak{h}) = \{a \in \mathfrak{g} | [a, \mathfrak{h}] \in \mathfrak{h}\} = \mathfrak{h}$.

Remark. There are alternative definitions for what a Cartan subalgebra is. See [10]

Remark. Let G be a Lie group over $\mathbb{R}$ or $\mathbb{C}$ with Lie algebra $\mathfrak{g}$, and let $T \subset G$ be a torus, which is a subgroup that is maximal among abelian connected subgroups. Then, the Lie algebra of T is a Cartan subalgebra of $\mathfrak{g}$. (See [10])

Definition 2.28. Let $\mathfrak{g}$ be a complex Lie algebra with $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra and V a finite-dimensional representation of $\mathfrak{g}$. An *eigenvector* for $\mathfrak{h}$ is a vector $v \in V$ that is an eigenvector for every $H \in \mathfrak{h}$. An *eigenvalue* for $\mathfrak{h}$ will be an element $\alpha \in \mathfrak{h}^*$ such that for every $H \in \mathfrak{h}$, $H(v) = \alpha(H)v$. The non zero eigenvalues of $\mathfrak{h}$ acting on V are called the *weights* of the representation. The representation V decomposes in this way:

$$V = \bigoplus V_\alpha$$

The V_α are one-dimensional eigenspaces corresponding to the eigenvalues. The V_α for eigenvalues α are called *weight spaces*.

Definition 2.29. Let $\mathfrak{g}$ be a Lie algebra with $\mathfrak{h}$ a Cartan subalgebra. Let $\mathfrak{h}$ act on $\mathfrak{g}$ by the adjoint representation. Then the weights are called the *roots* of the Lie algebra, and the corresponding weight spaces are called *root spaces*. The lattice generated by the roots $\Lambda_R \subset \mathfrak{h}^*$ is called the *root lattice*.

For what follows, it is necessary to introduce a notion of ordering on the roots of a Lie algebra. This is accomplished by considering a real linear functional $l : \Lambda_R \to \mathbb{R}$ that is irrational with respect to this lattice. Denote by R the roots of $\mathfrak{g}$, and let $R^+ := \{\alpha \in \mathrm{R} | l(\alpha) > 0\}$ with R^- defined similarly. The sets R^+ and R^- are called positive roots and negative roots with respect to the ordering. They allow for discussion of highest weight vectors, which are defined as follows in [10]:

Definition 2.30. Let V be a representation of a Lie algebra $\mathfrak{g}$. A vector $v \in V$ is called a *highest weight vector* if it is an eigenvector for the action of $\mathfrak{h}$ and is annihilated by the positive root spaces of $\mathfrak{g}$. The weight of the highest weight vector for an irreducible representation is called the *highest weight* of the representation. The term *dominant weight* is also used.

The following result from [10] gives some facts on highest weight vectors for semisimple complex Lie algebras:

Theorem 2.31. Let $\mathfrak{g}$ be a semisimple complex Lie algebra. Then: every finite-dimensional representation of $\mathfrak{g}$ has a highest weight vector; moreover, it is unique up to scalars.

The classification theorem hinges on studying the geometry of the roots of simple Lie algebras. A special symmetric, bilinear form called the Killing form allows for a deeper understanding of this geometry, and it is useful for the study of semisimple Lie algebras in general. It is defined as follows in [10]:

Definition 2.32. Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $X, Y \in \mathfrak{g}$. The Killing form is given by:

$$B(X,Y) = Tr(ad(X) \circ ad(Y)).$$

For what follows, it is necessary to collect some facts on the Killing form. Firstly, we have this theorem from [10]:

Theorem 2.33. if $\mathfrak{g}$ is a semisimple Lie algebra with a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, then the Killing form restricted to $\mathfrak{h}$ determines an isomorphism between $\mathfrak{h}$ and its dual $\mathfrak{h}^*$. Thus, it induces a symmetric bilinear form on $\mathfrak{h}^*$, and this is also called the Killing form.

For a given semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, understanding how the Killing form on $\mathfrak{h}^*$ is related to the roots of the Lie algebra is important for what follows. With this in mind, we have this theorem from [10]:

Theorem 2.34. The roots of $\mathfrak{g}$ span a real subspace of $\mathfrak{h}^*$ on which the Killing form is positive definite.

Thus, the Killing form on this space makes it into a Euclidean space. With the Killing form, it becomes an *abstract root system*:

Definition 2.35. Let E be a Euclidean space with a positive definite symmetric bilinear form $(,)$. Let ϕ be a finite subset of E satisfying:

- ϕ spans E
- If $\alpha \in E$, then the only other integer multiple of α in E is $-\alpha$.
- For $\alpha, \beta \in \phi$, $\beta - (\beta, \alpha)\alpha$ is in E.
- For $\alpha, \beta \in \phi$, the real number

  $$2\frac{(\beta, \alpha)}{(\alpha, \alpha)}$$

  is an integer.

The proof of the classification theorem given by Dynkin relies on the Dynkin diagram associated to a root system. This is a graph constructed from a subset of the roots of a Lie algebra called the simple roots. Nodes of the diagram correspond to simple roots, and lines between the nodes are determined by the angle between the roots. The last condition for an abstract root system may be interpreted as the following condition: for $\alpha, \beta \in \phi$, the real number $2cos(\theta)\frac{||\beta||}{||\alpha||}$ is an integer, with θ being the angle between α and β. This gives restrictions on the angles between simple roots, which are defined as follows in [10]:

Definition 2.36. Let $\mathfrak{g}$ be a semisimple Lie algebra with a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let R be the roots of $\mathfrak{g}$ with a linear map $l : \Lambda_R \to \mathbb{R}$ that allows for a choice of positive roots R^+. We say a root is *simple* if it is not the sum of two other positive roots.

The last condition for an abstract root system puts severe restrictions on the angles between the roots. The possible angles are the following: $\pi/6,\ \pi/4,\ \pi/3,\ \pi/2,\ 2\pi/3,\ 3\pi/4,\ 5\pi/6$.

Now we describe how the Dynkin diagram of a root system is obtained:

Definition 2.37. Let $\mathfrak{g}$ be a semisimple Lie algebra with a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let R be the roots of $\mathfrak{g}$ with a linear map $l : \Lambda_R \to \mathbb{R}$ that allows for a choice of positive roots R^+. The *Dynkin diagram* of $\mathfrak{g}$ is a graph with a node for each simple root; the nodes are joined by a number of lines depending on the angle between them:

- $\pi/2$: no lines
- $2\pi/3$: one line
- $3\pi/4$: two lines
- $5\pi/6$: three lines

Additionally, if the simple roots are connected by more than one line, then there is an arrow pointing from the longer root to the shorter root; otherwise, there is no arrow pointing from one to the other.

For simple complex Lie algebras, the roots form a root system that is *irreducible*. These are root systems which are not reducible, and they are defined in [26]:

Definition 2.38. Let E be a Euclidean space with $R \subset E$ a root system. We say R is *reducible* if it can be partitioned as $\Psi \cup \Psi'$ with every root in Ψ being orthogonal to every root in Ψ'. Otherwise, it is called *irreducible*.

Dynkin's approach relies on classifying Dynkin diagrams attached to irreducible root systems. The following theorem, relying only on Euclidean geometry, gives this classification:

Theorem 2.39. The only Dynkin diagrams for irreducible root systems are those given in figure 1.

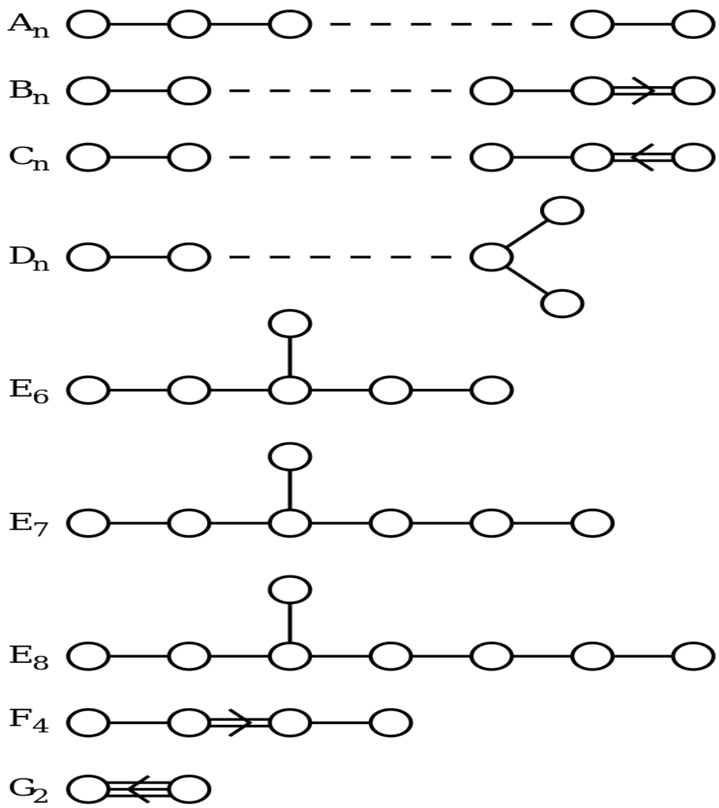
Remark. This figure comes from R.A. Nonenmacher. See [27].

The completion of the classification theorem for simple complex Lie algebras relies on this fact from [10]:

Theorem 2.40. There exists, for every Dynkin diagram of an irreducible root system, a simple complex Lie algebra whose roots have that Dynkin diagram. Furthermore, it is unique up to isomorphism.

The groundwork has been laid; the next chapter delves into what the exceptional series is, and why it is interesting to study.

Figure 1: The Classification of simple complex Lie Algebras



3 Deligne Exceptional Series

3.1 Background

In the paper [7], Deligne considers the adjoint complex Lie groups in this series:

$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8. \tag{3.1}$$

For the Dynkin diagrams of their Lie algebras, an automorphism is a permutation of the nodes leaving the diagram invariant. [22] The automorphism groups of the Dynkin diagrams for the groups in this series are trivial for every group except for A_2, D_4, and E_6; for A_2 and E_6, they are cyclic of order 2, and for D_4, it is S_3. Deligne extends the groups in (1) by taking semidirect products with these automorphism groups. This is the exceptional series of Lie groups, and the groups are interesting to study because of the uniform behavior that appears in their representation theory.

Let G be such a group, and let $\mathfrak{g}$ be its Lie algebra. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Recall that the Killing form on $\mathfrak{h}$ induces a bilinear form on $\mathfrak{h}^*$, also called the Killing form. Denote by Φ the Killing form on $\mathfrak{h}^*$. Deligne considers the Killing Form on $\mathfrak{h}^*$. For a Lie algebra $\mathfrak{g}$ for a group G, let α be the highest root. Deligne denotes $\Phi(\alpha, \alpha)$ by k, and he evaluates k for each group G in the series. The values are as follows: $1/2, 1/3, 1/4, 1/6, 1/9, 1/12, 1/18$, and $1/30$ as G varies through the groups in the series. The role of k is to define the parameters a and a^*. Deligne defines a to be either k or $1/6 - k$, and he defines a^* to be $-(1/6) - a$.

These parameters a and a^* also appear in Deligne's study of the Casimir invariants of representations of these groups. These are defined as follows in [10]:

Definition 3.1. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ of dimension n and let V be a representation of $\mathfrak{g}$. If $\{\beta_i\}$ is a basis for $\mathfrak{g}$, let $\{\beta_i^*\}$ be the dual basis with respect to the Killing form. That is, if $B(,) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, then $\{\beta_i^*\}$ satisfies $B(\beta_i, \beta_j^*) = \delta_{ij}$, where δ_{ij} is the Kronecker delta function. *The Casimir endomorphism of V with respect to the Killing form* is defined as the endomorphism

$$C(v) = \sum_{i=1}^{n} \beta_i \beta_i^*(v)$$

for $v \in V$.

Remark. One can also describe this in terms of the universal enveloping algebra of $\mathfrak{g}$, but this is unnecessary for what follows. (See [10])

If V is an irreducible representation, then Schur's lemma gives this consequence:

Theorem 3.2. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, and let V be an irreducible representation. Then the Casimir endomorphism acts as a scalar multiple of the identity. This scalar multiple is called the *Casimir invariant* of V. In particular, the Casimir invariant of the adjoint representation is 1.

3.2 Deligne's observations

Deligne's observations can now be described. Each group in the series has a corresponding family of virtual representations, which are denoted by X_i $(0 \leq i \leq 4)$, Y_i $(0 \leq i \leq 3)$, Y_i^* $(0 \leq i \leq 3)$, A, C, and C^*. which behave in the following ways:

1. For a given G, they are all irreducible, 0, or the negative of an irreducible representation. In particular, the trivial representation and the adjoint representation occur among this family.

2. There are uniform direct sum decomposition formulas for low degree symmetric and exterior powers of the adjoint representations of groups in the series. In the following examples from [7], Deligne writes them in this way:

   - $Sym^2\mathfrak{g} = \mathbb{1} + Y_2 + Y_2^*$
   - $\bigwedge^2 \mathfrak{g} = \mathfrak{g} + X_2$
   - $Sym^3\mathfrak{g} = \mathfrak{g} + X_2 + A + Y_3 + Y_3^*$
   - $\bigwedge^3 \mathfrak{g} = \mathbb{1} + X_2 + Y_2 + Y_2^* + X_3$

3. The dimension formulas above allow for decompositions of some of the X_i representations in terms of exterior and tensor powers of the adjoint representation. As an example, we have:

$$X_2 = \bigwedge^2 \mathfrak{g} - \mathfrak{g}^2 + \mathfrak{g}$$

4. The Casimir invariants for the virtual representations are given by these values:

   - $X_n : n$
   - $Y_n : n + (n^2 - n)a$
   - $Y_n^* : n + (n^2 - n)a^*$

- $A : 8/3$
- $C : 3 + 3a$
- $C^* : 3 + 3a^*$.

5. Let $\lambda := 6a$ and $\lambda^* := 6a^*$. Then the dimensions of the virtual representations are rational functions of λ, where the numerator and denominator factor as products of linear factors in λ. Deligne calls this a miracle. Example dimension formulas are given by:

   $dim\ X_1 = 2\frac{(\lambda+5)(\lambda-6)}{\lambda(\lambda+1)}$

   $dim\ X_2 = 5\frac{(\lambda+3)(\lambda+5)(\lambda-4)(\lambda-6)}{\lambda^2(\lambda-1)^2}$

   $dim\ X_3 = -10\frac{(\lambda+2)(\lambda+4)(\lambda+5)(\lambda-3)(\lambda-5)(\lambda-6)}{\lambda^3(\lambda-1)^3}$

   $dim\ Y_2 = -90\frac{(\lambda+5)(\lambda-4)}{\lambda^2(\lambda-1)(2\lambda-1)}$

   $dim\ A = -27\frac{(\lambda+3)(\lambda+4)(\lambda+5)(\lambda-4)(\lambda-5)(\lambda-6)}{\lambda^2(\lambda-1)^2)(3\lambda-1)(3\lambda-2)}$

6. The dimension formulas allow one to express the dimension of each symmetric and exterior power as a rational function in λ.

7. For the decomposition formulas in observation (2), one can express the trace of the Casimir endomorphism as a rational function of λ.

These observations motivated Deligne to conjecture the existence of a category satisfying certain properties that should encapsulate and explain these observations in some way. This conjecture is currently unproven, and there is evidence that it is not true given in [4]. Nevertheless, these observations are striking in that they give relations among the exceptional Lie groups and their Lie algebras. The classification theorem only gives that the exceptional Lie algebras are the exceptions to the four classical families; these observations suggest that their is more to be explored in having the exceptional algebras form their own family. More work on the exceptional series is given in [8].

4 Rational Conformal Field Theory

In the paper [3], Mathur, Mukhi, and Sen study differential equations in order to make a contribution to the classification of rational conformal field theories. Rational conformal field theories are two-dimensional conformal field theories with a finite number of primary fields. For a more detailed study of 2-dimensional conformal field theory, see [15].

These differential equations are satisfied by the characters associated to any given theory. These characters come from the partition functions of these theories. The partition functions are sums of holomorphic and anti-holomorphic functions, and the holomorphic functions appearing are referred to as the characters of any given theory. The characters are modular functions [9], and the differential equations they satisfy are modular-invariant.

4.1 Modular Forms

The following definition of modular functions appears in [5]. Other sources would call these meromorphic modular forms, such as [14]. They have nice transformation properties with respect to a certain class of 2×2 matrices over the integers:

Definition 4.1. Consider $SL_2(\mathbb{Z})$, the matrices over the integers of determinant 1. The group $\Gamma := SL_2(\mathbb{Z})/\pm I$ is called the modular group.

Remark. Some authors say $SL_2(\mathbb{Z})$ is the modular group. (For instance, see [17])

Remark. The modular group can also be called the projective special linear group $PSL_2(\mathbb{Z})$. The modular group can also be defined as the linear fractional transformations of the upper-half plane $\mathfrak{H} := \{\tau \in \mathbb{C} : Im(\tau) > 0\}$:

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{4.1}$$

such that $ad - bc = 1$ and $a, b, c, d \in \mathbb{Z}$. This gives an isomorphic group. ([5])

The definition of a modular function depends on the notion of a weakly modular function. These are defined as follows:

Definition 4.2. A function f on the upper half plane $\mathfrak{H} := \{\tau \in \mathbb{C} : Im(\tau) > 0\}$ is called a weakly modular function of weight 2k if it satisfies:

- It is meromorphic on $\mathfrak{H}$.
- The relation $f(\frac{a\tau+b}{c\tau+d}) = (c\tau + d)^{2k} f(\tau)$ holds for matrices in $SL_2(\mathbb{Z})$.

Remark. Note that $SL_2(\mathbb{Z})$ is generated by these two matrices:

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The preceding remark from [5] allows for the following proposition, which gives an equivalent condition for a function to be weakly modular:

Proposition 4.3. *Suppose f is a meromorphic function on the upper half plane $\mathfrak{H}$. Then f is weakly modular of weight $2k$ if and only if the relations:*

- $f(\tau + 1) = f(\tau)$
- $f(\frac{1}{\tau}) = \tau^{2k} f(\tau)$ *are satisfied.*

The fact that f is meromorphic at ∞ means it has a Laurent expansion in $q = e^{2\pi i\tau}$ in a disk around 0 of the form:

$$\sum_{n \geq n_0}^{\infty} a(n)q^n$$

for some integer n_0. Now the definition of a modular function can be given:

Definition 4.4. Let f be a weakly modular function. We say f is a modular function if it is meromorphic at ∞.

The derivative of a modular function is not modular. However, if one can adjust the derivative by a function with certain properties, one can again obtain a modular function. For this purpose, Mathur, Mukhi, and Sen employ the Eisenstein series of weight 2:

$$E_2(\tau) := 1 - \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

Where $\sigma_1(n)$ is the sum of divisors function. The weight 2 Eisenstein series is not a modular form, but it is a quasimodular form. It has the following transformation property:

$$E_2(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^2 E_2(\tau) + \frac{6c}{\pi i}(c\tau + d)$$

4.2 Modular Linear Differential Equations

For modular functions of weight r, Mathur, Mukhi and Sen use this differential operator:

$$\mathcal{D}^{(r)} := (\frac{\partial}{\partial \tau} - \frac{1}{6} i\pi r E_2(\tau)) f(\tau). \tag{4.2}$$

Additionally, if one suppresses the weight r and just writes $\mathcal{D}$ for the differential operator, denote by $\mathcal{D}^n$ the operator applied n times. The operator (4.2) maps modular functions of weight r to modular functions of weight $r + 2$. By previous work in [19, 20], it is shown that if a theory has n characters, then they can be regarded as n linearly independent solutions to an nth order differential equation. The general nth order differential equation they study is then given by:

$$\mathcal{D}^n f + \sum_{k=0}^{n-1} \phi_k(\tau) \mathcal{D}^k f = 0,$$

where the $\phi_k(\tau)$ are modular functions of weight $2(n - k)$. The following proposition gives a way of expressing the coefficients of this differential equation in terms its solutions. It appears in [19]:

Proposition 4.5. *Let $y_1, ..., y_n$ be linear independent meromorphic functions in a connected open subset $U \in \mathbb{C}$, and let V be their span over $\mathbb{C}$. Let E be a meromorphic vector field on U not identically zero. Then there exist unique meromorphic functions $k_1, ..k_n$ such that the space of solutions of*

$$E^n y + k_1 E^{n-1} y + ... + k_n y = 0$$

is V.

The proof given in [19] constructs these k_i explicitly as:

$$k_i = \frac{det \begin{bmatrix} y_1 & \dots & y_n \\ . & \dots & . \\ E^{i-2}y_1 & & \dots E^{i-2}y_n \\ E^i y_1 & & E^i y_n \\ \vdots & & \vdots \\ E^n y_1 & \dots & E^n y_n \end{bmatrix}}{det \begin{bmatrix} y_1 & \dots & y_n \\ Ey_1 & & \dots Ey_n \\ \vdots & & \vdots \\ E^{n-1}y_1 & \dots & E^{n-1}y_n \end{bmatrix}}.$$

When this result is applied to the differential equation they study, it yields the following expression of the coefficients in terms of the characters:

$$\phi_k(\tau) = (-1)^n \frac{W_k}{W}$$

where W_k and W are:

$$W_k = det \begin{bmatrix} f_1 & \dots & f_n \\ \cdot & \dots & \cdot \\ \mathcal{D}^{i-2} f_1 & & \dots \mathcal{D}^{i-2} f_n \\ \mathcal{D}^{i} f_1 & & \mathcal{D}^{i} f_n \\ \vdots & & \vdots \\ \mathcal{D}^{n} f_1 & \dots & \mathcal{D}^{n} f_n \end{bmatrix}$$

$$W = det \begin{bmatrix} f_1 & \dots & f_n \\ \mathcal{D} f_1 & & \dots \mathcal{D} f_n \\ \vdots & & \vdots \\ \mathcal{D}^{n-1} f_1 & \dots & \mathcal{D}^{n-1} f_n \end{bmatrix}.$$

4.3 Order 2 Equations

Mathur, Mukhi, and Sen are mainly concerned with the case $n = 2$; In particular, they consider what happens when the wronskian W has no zeroes in the upper-half plane. The functions ϕ_k then become modular forms in the sense of [5]:

Definition 4.6. Let f be a modular function. We say f is a *modular form* if it is holomorphic everywhere, including at ∞.

Remark. Other sources, such as [14], call this a holomorphic modular form, to distinguish between the meromorphic modular forms which served as the definition of weakly modular functions for [5].

Rational conformal field theories are intimately related with the representation theory of the Virasoro algebra. This is the unique central extension of the Witt algebra. The Witt algebra and central extensions of Lie algebras are defined as follows in [2]:

Definition 4.7. Let $\mathfrak{g}$ and Y be Lie algebras. An extension $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ by Y is a short exact sequence

$$0 \longrightarrow Y \longrightarrow \hat{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0.$$

We say this extension is *central* if the image of Y is contained in the center of $\hat{\mathfrak{g}}$ and *one-dimensional* if Y is.

Definition 4.8. Let $A := \mathbb{C}[t, t^{-1}]$ be the algebra of Laurent polynomials in one variable. The derivations on A can be given the structure of a Lie algebra, and this is called the *(complex) Witt algebra*.

Now the Virasoro algebra can be defined:

Definition 4.9. The *Virasoro algebra* is the unique, one-dimensional, central extension of the Witt algebra. It has generators $\{L_n | n \in \mathbb{Z}\}$ satisfying these relations:

$$[L_m, L_n] = (m-n)L_{n+m} + \frac{1}{12}(m^3 - m)\delta_{n+m,0}\mathbf{c}$$

The element $\mathbf{c}$ commutes with everything:

$$[\mathbf{c}, L_n] = 0 \ \ \forall n \in \mathbb{Z}$$

Attached to any conformal field theory is a vertex operator algebra, on which the Virasoro algebra acts; this is the Virasoro algebra of the theory. Vertex operator algebras arose from the notion of a vertex algebra introduced by Borcherds, whose definition comes from [12]:

Definition 4.10. A *vertex algebra* has the following data:

1. A $\mathbb{Z}_+$-graded vector space $V = \bigoplus_{m=0}^{\infty} V_n$ with $dim\ V_m < \infty$
2. A vector $|0\rangle \in V_0$ called the *vacuum vector*.
3. A linear operator $T : V \to V$ of degree one called the *translation operator*.

4. For each $a \in V$, a power series $a(z) = \sum_{m \in \mathbb{Z}} a_m z^{-n-m}$ if $a \in V_n$, for operators a_m mapping V_n to V_{n-m}, such that $a_n b = 0$ for n sufficiently large.

which satisfy these axioms:

1. $Y(|0\rangle, z) = Id_V$, the identity on V. Additionally, for $A \in V$, $Y(A, z)|0\rangle \in V[[z]]$ and $Y(A, z)|0\rangle|_{z=0} = A$.

2. For every $A \in V$, $[T, Y(A, z)] = \frac{d}{dz} Y(A, z)$ and $T|0\rangle = 0$.

3. For every $A, B \in V$, there is a positive integer N such that $(z-w)^N [Y(A, z), Y(B, w)] = 0$

Adjusting this definition gives that of a vertex operator algebra [12]:

Definition 4.11. A vector space V over a field F is a *vertex operator algebra* if it is a vertex algebra with a $\mathbb{Z}$-grading $V = \bigoplus_{m=0}^{\infty} V_m$ and a vector $\omega \in V$ called the *conformal vector* satisfying:

1. $dim\ V_m < \infty$ for $m \in \mathbb{Z}$ and $dim\ V_m = 0$ for m sufficiently small.

2. for $m, n \in \mathbb{Z}$:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{n+m,0}\mathbf{c}$$

   where the $L(n)$ for $n \in \mathbb{Z}$ are linear operators on V given by $Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n) x^{-n-2}$ and $c \in \mathbb{C}$ is called the *central charge* of V.

3. $L(0)v = nv$ for $n \in \mathbb{Z}$ and $v \in V_n$

4. $Y(L(-1)v, x) = (d/dx) Y(v, x)$

Definition 4.12 (character of a vertex operator algebra)**.** If $V = \bigoplus_{n \in \mathbb{Z}} V_n$ is a vertex operator algebra with central charge c, then the *character* of V is defined as:

$$Ch(V) = \sum_{n \in \mathbb{Z}} dim\ (V_n) q^{n - \frac{c}{24}},$$

where $q = e^{2\pi i \tau}$.

Remark. Any vertex operator algebra has *modules*, and any module has a character. The above definition comes from considering the vertex operator algebra as a representation of itself, and defining the character accordingly. For the definition of the character of any module, see [17].

Vertex operator algebras were introduced by Frenkel, Lepowsky, Meurman to construct the Moonshine module $V^\natural$. A conformal field theory also comes with a Hilbert space on which the Virasoro algebra of the theory acts. It is a highest weight module, and so it decomposes into a direct sum of highest weight modules (submodules of highest weight modules are highest weight modules). These highest weight modules have highest weight vectors, and these are the *primary states* of the theory, the other vectors being *secondary states*.

Before studying the case of order 2 modular linear differential equations with no zeroes in W, Mathur, Mukhi, and Sen describe their general approach to classifying rational conformal field theories. They begin with an order n differential equation, and assume it has a power series solution in $q := e^{2\pi i \tau}$, and they describe how properties of characters of rational conformal field theories constrain the coefficients. In particular, for a given power series solution, the ratio of any coefficient to the leading coefficient should be rational. This counts the ratio of the number of secondary states at a given level to the number of states at the lowest level for a character. To define the level of a state, we need to define conformal dimensions for primary states. This definition appears in [15]:

Definition 4.13. *A primary state $|v\rangle$ with conformal dimension* Δ is a state satisfying

$$L_0|v\rangle = \Delta|v\rangle, L_{n>0}|v\rangle = 0$$

The *Verma module* $\mathcal{V}_\Delta$ is the representation of the Virasoro algebra with basis $\{\prod_{i=1}^{k} L_{-n_i}|v\rangle\}$, where the $\{n_i\}$are nonnegative integers satisfying $n_1 \leq n_2 \leq ... \leq n_k$, and k is some arbitrary nonnegative integer. The *level* of the state $\prod_{i=1}^{k} L_{-n_i}|v\rangle$ is $N := n_1 + n_2 + ... + n_k$.

However, more constraints are given by this fact: for the character of the vertex operator algebra of the theory, the number of states at the lowest level is always one, which means the ratios must be integral.

For equations of order 2 with no zeroes in W, the coefficients are modular forms of weights 0, 2, and 4. However, this fact from [11] restricts the possibilities for modular forms of these weights:

Theorem 4.14. Every modular form on the modular group is a $\mathbb{C}$-linear combination of the Eisenstein series E_4 and E_6.

Thus, we have as a corollary:

Theorem 4.15. There are no modular forms of weight 2, and, up to scaling, the only modular form of weight 4 is the Eisenstein series E_4.

Thus, one can write down the most general modular linear differential equation of order 2 as follows:

$$\frac{\partial^2}{\partial\tau^2} f - \frac{1}{3} i\pi E_2(\tau) \frac{\partial\tau}{\partial} f + \kappa\pi^2 E_4 f = 0$$

The above equation is based on these derivatives:

- $E_2' = \frac{E_2^2 - E_4}{12}$
- $E_4' = \frac{E_2 E_4 - E_6}{3}$
- $E_6' = \frac{E_2 E_6 - E_4^2}{2}$

They normalize the Eisenstein series so that their series expansion begins with 1, so for a solution of the form: $f = q^\alpha \sum_{n=0}^\infty f_n q^n$, they obtain the following equality:

$$4\alpha^2 - \frac{2}{3}\alpha - \kappa = 0$$

It follows from the quadratic formula that $\alpha = \frac{1}{12}(1 \pm \sqrt{36+\kappa})$. Let $y := +\sqrt{36+\kappa}$. Then $k_1 := \frac{f_1}{f_0} = \frac{10y^2+2y-12}{6-y}$, and this has to be an integer; moreover, one can use the expression for m_1 to solve for y: $y = \frac{1}{20}[-(k_1+2) \pm \sqrt{(k_1+2)^2 + 40(12+6k_1)}]$. A further constraint comes from the fact that $2(y-1)$ may be identified with the central charge of the theory. The characters, being modular functions, are determined by the growth in their coefficients, and the character given by setting α to be $\frac{1}{12}(1-y)$ corresponds to the character of the vertex operator algebra of the theory. Moreover, the central charge of the theory determines its polar term, which determines the growth of the coefficients, and as $\tau \to i\infty$, the character goes to $q^{\frac{1-y}{12}}$. This implies $2(y-1)$ is rational, meaning y is rational.

Thus, one can write $(k_1+2)^2 + 240(k_1+2)$ as r^2 for some integer r. This can be rewritten using the substitution $l = 120 + (k_1+2) - r$ in the following way: $k_1 + 2 = \frac{(120-l)^2}{2l} = 7200l - 120 - \frac{1}{2}l$. This implies l must be even and divide 7200. Moreover, one may assume r is positive since only r^2 appears in $(k_1+2)^2 + 240(k_1+2) = r^2$

These constraints give a finite number of possible values for l. Working backwards with these equations, one can calculate the values of m_1 and the central charge c of the candidate theory. Moreover, these equations allow one to calculate the dimension of the primary field of the other character, whose value of α must be $h - \frac{c}{24}$. Finally, using these possible dimensions and central charges, one can calculate the ratios $\frac{f_n}{f_0}$ for characters of the vertex operator algebra of the theory and see which ratios are integral up to a very high order.

Among the finite number of solutions for order 2 equations they obtain are a minimal model and several Wess-Zumino-Witten (WZW) models of level 1. The minimal models were introduced by Belavin, Polyakov, and Zamolodchikov in 1984 in [18]; they are the only rational conformal field theories with central charge less than 1. The WZW models are non-linear sigma models, which is to say they are scalar field theories with scalar fields defining maps from a Euclidean space to a manifold, with this manifold being a Lie group. If a real Lie group G is a target manifold for a WZW-model, the model is called the G WZW model. The level of the WZW model is the topological coupling factor, which helps to specify the maps from the Euclidean space to G. The candidate theories they find are given in this theorem:

Theorem 4.16. (Mathur-Mukhi-Sen 1988) The possibilities for two dimensional rational conformal field theories are:

1. $c = -22/5$ minimal model
2. $k = 1$ SU(2) WZW model
3. $k = 1$ SU(3) WZW model
4. $k = 1$ G_2 WZW model
5. $k = 1$ $SO(8)$ WZW model
6. $k = 1$ F_4 WZW model
7. $k = 1$ E_6 WZW model
8. $k = 1$ E_7 WZW model
9. $k = 1$ E_8 WZW model

The relationship between this classification and the Deligne exceptional series can now be understood. These are real Lie groups with real Lie algebras. For any real Lie algebra, there is the operation of complexification given in [23]:

Definition 4.17. Let $\mathfrak{g}$ be a real Lie algebra. The *complexification* of $\mathfrak{g}$ is the Lie algebra with vector space $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. If one views the elements of the complexification as pairs (u, v), with $u, v \in \mathfrak{g}$, then the bracket given by:

$$[(u_1, v_1), (u_2, v_2)] = ([u_1, u_2] - [v_1, v_2], [v_1, u_2] + [u_1, v_1])$$

The complexifications of the Lie algebras for the groups in the classification of Mathur, Mukhi, and Sen are exactly the simple complex Lie algebras occurring in the Deligne exceptional series.

5 The Finite Group Series

5.1 Background

The Deligne Exceptional series of Lie groups have a parameter λ that allows one to interpolate between the dimensions of their adjoint representations. The adjoint representations of the last three groups have dimensions 78, 133, and 248 respectively. The last section of this thesis studies the representation theory of finite groups. The definition of representations of finite groups is similar to that of representations of Lie groups and appears in [10]:

Definition 5.1. Let G be a finite group. A *representation* of G on a finite-dimensional complex vector space V is a group homomorphism $\rho : G \to GL(V)$. When the map is understood, then V is called a *representation* of G. The dimension of V is called the *degree* of the representation.

Remark. The definitions of subrepresentations and irreducible representations for finite groups are exactly the same as the definitions given for Lie groups.

For what follows, it is necessary to consider faithful representations:

Definition 5.2. Let G be a finite group and $\rho : G \to GL(V)$ a representation of G on V. We say ρ is *faithful* if it is injective.

Just as there is a classification theorem for simple complex Lie algebras, there is a classification theorem for finite simple groups. For this classification, the sporadic groups are the analogues of the exceptional Lie algebras, and will be described in more detail later. There are three sporadic groups whose faithful representations of minimal degree have degrees 78, 133, and 248; they are the Fischer 22 group, the Harada-Norton group, and the Thompson group respectively, and they are denoted Fi22, HN, and Th.

This numerical coincidence motivates the study of the relationships between these sporadic groups, and brings up the question of whether or not other groups whose minimal degree faithful representations have degrees coinciding with the dimensions of the adjoint representations for the Deligne exceptional series have connections with these sporadic simple groups. The *finite groups series* consists of these sporadic groups, together with five other groups whose minimal degree faithful representations have degrees coinciding with the dimensions of the adjoint representations of the groups in Deligne's series. They are listed below:

$$L_2(7), 2.A_9, J_2, 2.Ru, 2.F_4(2), Fi22, HN, Th. \tag{5.1}$$

This series has analogies with the Deligne exceptional series. Just as the Deligne series is motivated by observations relating the exceptional Lie groups to other Lie groups, the observations motivating the study of this series came from sporadic simple groups. To be more precise, it is necessary to describe the classification theorem for finite simple groups:

Theorem 5.3. If G is a finite simple group, then it is one of:

- cyclic groups of prime order

- alternating groups A_n, with $n \geq 5$
- finite groups of Lie type
- The 26 exceptions to the above families.

The groups in the last category are called *sporadic simple groups*. Much like the theorem on classification for simple complex Lie algebras, this theorem only puts these groups together because they do not fit into the other families. This is why studying relationships among these groups is interesting. In order to describe the observations in detail, it is necessary to delve into what is called *character theory* of finite groups, in addition to complete reducibility of finite group representations. The approach here is that of chapter 2 in [10]. The complete reducibility of representations of finite groups is given below:

Theorem 5.4. Let G be a finite group with V a representation of G. Then V decomposes as a direct sum of irreducible representations in a unique way, up to isomorphism:

$$V = \bigoplus_{i=1}^{n} V_i$$

Characters of representations of finite groups are defined as follows:

Definition 5.5. Let G be a finite group with V a representation. the *character* of V is the complex valued function on G defined by:

$$\chi_V(g) = Tr(g|_V)$$

, the trace of g on V.

The utility of character theory comes from reducing problems about representations to computational problems. This manifests itself in the following result:

Proposition 5.6. *Let G be a finite group with representations V and W. The following character identities hold:*

- $\chi_{V\oplus W} = \chi_V + \chi_W$

- $\chi_{V\otimes W} = \chi_V \chi_W$

The result most useful for the observations below is as follows:

Theorem 5.7. Let G be a finite group. Then any representation of V is determined by its character up to isomorphism.

5.2 Observations

With this in mind, some observations that can be made about the groups in (5.1) are the following:

1. Each group has a collection of virtual representations whose dimensions coincide with the dimensions of some of the virtual representations for the groups in Deligne's series. The following calculations with characters from gap (see [21]) prove this. All such observations are given in what follows:

   - **for** $L_2(7)$

     $dim\ X_1 = \chi_2$

     $dim\ X_2 = 0$

     $dim\ X_3 = -\chi_2 + \chi_1$

     $dim\ Y_2 = \chi_4 - \chi_1$

     $dim\ Y_3 = \chi_5$

     $dim\ A = 0$

     $dim\ C = 0$

   - **for** $2.A_9$

     $dim\ X_1 = \chi_2$

     $dim\ X_2 = \chi_3 - \chi_1$

$dim\ X_3 = 0$

$dim\ Y_3 = \chi_{11} + \chi_2$

$dim\ C = \chi_{10} + \chi_3 + \chi_1$

- **for** J_2

$dim\ X_1 = \chi_2$

$dim\ X_2 = \chi_7 + \chi_2$

$dim\ X_3 = \chi_{12} + \chi_6 + \chi_1$

$dim\ Y_3 = \chi_{25} + \chi_7 - \chi_2$

$dim\ A = \chi_{23}$

$dim\ C = 2\chi_{25}$

- **for** $2.Ru$

$dim\ X_1 = \chi_{37}$

$dim\ X_2 = \chi_2 - \chi_{37}$

$dim\ Y_2 = \chi_{39} - 2\chi_{38} + 8\chi_1$

$dim\ C = \chi_{43} + 2\chi_{37} + 8\chi_1$

- **for** $2.F_4(2)$

$dim\ X_1 = \chi_{96}$

$dim\ Y_3 = \chi_{99}$

- **for** $Fi22$

$dim\ X_1 = \chi_2$

$dim\ X_2 = \chi_9 - \chi_8$

$dim\ X_3 = \chi_{24}$

$dim\ Y_2 = \chi_4 + \chi_5 - \chi_1$

$dim\ Y_3 = \chi_{12} + \chi_2$

- **for** HN

  $dim\ X_1 = \chi_2$

  $dim\ X_2 = \chi_6 - \chi_2$

  $dim\ X_3 = \chi_{20}$

- **for** Th

  $dim\ X_1 = \chi_2$

  $dim\ X_3 = \chi_{19}$

2. For most of the groups in (5.1), there are decomposition formulas for the tensor square of a minimal degree faithful representation of a group in the series that involve at most four summands. This representation will be treated as the "adjoint representation" of G. The following calculations depend on Proposition 5.4:

   - **for** $2.A_9$

     $\chi_2^2 = \chi_6 + \chi_5 + \chi_2 + \chi_1$

   - **for** $2.F_4(2)$

     $\chi_{96}^2 = \chi_5 + \chi_6$

   - **for** $Fi22$

     $\chi_2^2 = \chi_7 + \chi_6 + \chi_1$

   - **for** HN

     $\chi_2^2 = \chi_8 + \chi_7 + \chi_1$

   - **for** Th

     $\chi_2^2 = \chi_7 + \chi_6 + \chi_1$

As of yet, there are no explanations for why these relationships occur. Just as Deligne conjectures the existence of a category explaining his observations, one can ask whether or not there is a category with certain properties that help to see why these relationships hold.

To have a more precise conjecture, it would be helpful to have more observations of this kind.

Further questions regarding the groups in (5.1) deal with its connections with the Deligne series. The groups in Deligne's series have relationships with conformal field theory. The only case of the groups in (5.1) being used in conformal field theory is given by [6]. In [6], John Duncan constructs a vertex operator algebra on which $2.Ru$ acts.

Additionally, Deligne's series has been shown in [13] to be connected with algebraic geometry via the study of genus one curves and coregular representations of algebraic groups. Whether or not there are analogous applications to objects in algebraic geometry for this series has yet to be seen.

6 Conclusion

This thesis studied the Deligne exceptional series and how it appears in the classification of two dimensional rational conformal field theories. The work of Mathur, Mukhi, and Sen identified Wess-Zumino-Witten models which are related to the Deligne exceptional series of Lie algebras, but the study of this series is also motivated by representation theory via Deligne's observations. The finite group series given in the thesis have analogies with the Deligne exceptional series, and there are many topics for further research.

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