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Harmonic Measure, Reduced Extremal Length and Quasicircles

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Doctor of Philosophy

Mathematics

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An abstract of

A dissertation submitted to the Faculty of the

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Abstract

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It is well known that there is a close connection between the analytic behavior of the sewing homeomorphism induced by a Jordan curve and the geometry of this Jordan domain. For example, the sewing homeomorphism is quasisymmetric if and only if the Jordan domain is a quasidisk. This dissertation is devoted to the further study of this type of connection. Several equivalent conditions are established for sewing homeomorphism to be bi-Lipschitz or bi-Hölder. In particular, we explore these conditions by using conformal invariants such as harmonic measure, extremal distance and reduced extremal distance. Furthermore, some parallel conditions for a quasicircle are obtained.

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Chapter 1

Introduction

1.1 Extremal length

In this section, we introduce a geometric method named the method of extremal length. This method has a profound influence on the theory of conformal mapping as well as the more general theory of quasiconformal mapping. The roots can be tracked back to the length-area comparisons, to the strip method and to even earlier works. In 1940's, extremal length was introduced as the measure of curve families which is invariant under a conformal mapping by Ahlfors. It is such a powerful tool for estimating conformal invariants, like harmonic measure, in terms of more geometric quantities. Actually most conformal invariants can be linked to extremal properties. More details about extremal length can be found in *[1]*, *[13]*, *[15]*, *[20]*, *[21]* and *[24]*.

Definition 1.1.1. *[2, p51] Let Ω be a domain in the complex plane. Suppose Γ is a curve family in Ω. Let P be the set of all non-negative Borel measurable functions on Ω. For each curve $\gamma \in \Gamma$, it has ρ length*

$$L(\gamma, \rho) = \int_\gamma \rho |dz|, \quad \rho \in P,$$

which may be infinite, and the domain Ω *has a* ρ *area*

$$A(\Omega, \rho) = \iint\limits_{\Omega} \rho^2 dxdy.$$

Then the extremal length of Γ *in* Ω *is defined as*

$$\lambda_\Omega(\Gamma) = \sup_\rho \frac{\inf_\gamma L(\gamma, \rho)^2}{A(\Omega, \rho)}, \tag{1.1.1}$$

where ρ *is subject to the condition* $0 < A(\Omega, \rho) < \infty$.

From this definiton, we can see that the extremal length is invariant under a conformal map f, since it is clear that $\Gamma(\gamma, \rho) = \Gamma(f(\gamma), \rho^*)$ and $A(\Omega, \rho) = A(f(\Omega), \rho^*)$ where $\rho = \rho^*(f(z))|f'(z)|$. Furthermore, $\lambda_\Omega(\Gamma)$ depends only on Γ and not on Ω. Therefore we simplify the notation to $\lambda(\Gamma)$.

Lemma 1.1.2. *(The Composition Laws) [2, p55] Let* Ω_1 *and* Ω_2 *be disjoint sets. Let* Γ_1, Γ_2 *consist of arcs in* Ω_1 *and* Ω_2 *respectively, and let* Γ *be a third set of arcs.*

(1) If every $\gamma \in \Gamma$ *contains a* $\gamma_1 \in \Gamma_1$ *and a* $\gamma_2 \in \Gamma_2$*, then*

$$\lambda(\Gamma) \geq \lambda(\Gamma_1) + \lambda(\Gamma_2).$$

(2) If every $\gamma_1 \in \Gamma_1$ *and every* $\gamma_2 \in \Gamma_2$ *contains a* $\gamma \in \Gamma$*, then*

$$\frac{1}{\lambda(\Gamma)} \geq \frac{1}{\lambda(\Gamma_1)} + \frac{1}{\lambda(\Gamma_2)}.$$

The composition laws are best illustrated by the following two examples.

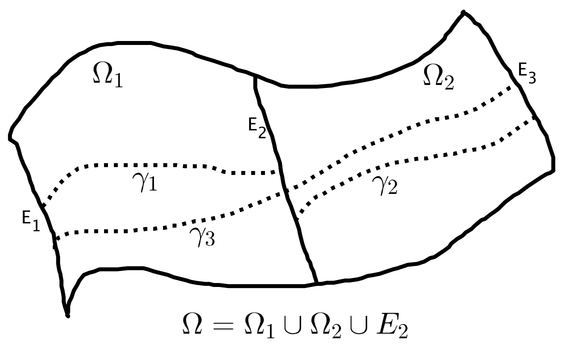

Figure 1.1: Every $\gamma \in \Gamma$ contains a $\gamma_1 \in \Gamma_1$ and a $\gamma_2 \in \Gamma_2$

Example 1.1.3. *In Figure 1.1, $\Omega = \Omega_1 \cup \Omega_2 \cup E_2$. Every arc in Ω from E_1 to E_3 contains an arc in Ω_1 from E_1 to E_2, and one in Ω from E_2 to E_3. Therefore part one of the composition laws implies*

$$\lambda(\Gamma) \geq \lambda(\Gamma_1) + \lambda(\Gamma_2).$$

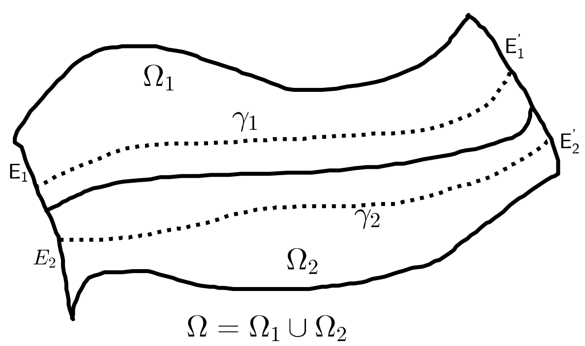

Figure 1.2: Every $\gamma_1 \in \Gamma_1$ and every $\gamma_2 \in \Gamma_2$ contains a $\gamma \in \Gamma$

Example 1.1.4. *In Figure 1.2, $\Omega = \Omega_1 \cup \Omega_2$. Every arc in Ω_1 from E_1 to E_1' and every arc in Ω_2 from E_2 to E_2' not only contains but actually is an arc in Ω from*

$E_1 \cup E_2$ to $E_1^{'} \cup E_2^{'}$. Therefore part two of the composition laws yields

$$\frac{1}{\lambda(\Gamma)} \geq \frac{1}{\lambda(\Gamma_1)} + \frac{1}{\lambda(\Gamma_2)}.$$

1.2 Modulus

In (1.1.1), the ratio is unchanged if the metric ρ is multiplied by a positive constant, because of the homogeneity. By normalizing the metric ρ, one can introduce the following definition.

Definition 1.2.1. *[26] Suppose Γ is a curve family in the plane. ρ is a non-negative Borel measurable function such that*

$$\int_{\gamma} \rho |dz| \geq 1 \tag{1.2.1}$$

for every locally rectifiable curve γ in Γ. Then the modulus of Γ is defined by

$$mod(\Gamma) = \inf_{\rho} \int \rho^2$$

where the infimum is taken over all ρ that satisfies (1.2.1).

From this definition, it can be easily deduced that modulus only depends on Γ and modulus is just the reciprocal of extremal length. It is a matter of taste whether one prefers to use extremal length or the modulus. Since extremal length is conformally invariant, the modulus is also conformally invariant.

Theorem 1.2.2. *[3, p8] Let Γ be a family of curves in a domain $D \subset \overline{\mathbb{C}}$ and $w = f(z)$ be a conformal map of D onto $f(D) \subset \overline{\mathbb{C}}$, then*

$$mod(\Gamma) = mod(f(\Gamma)).$$

We refer the reader to *[2]*, *[3]*, *[15]*, *[20]* and *[24]* for more details on extremal length and modulus.

1.3 Extremal distance

Extremal distance is the most useful example of extremal length and will be frequently used in this thesis.

Definition 1.3.1. *[13, p130] Let Ω be a domain in the plane, E and F be two disjoint subsets in the closure of Ω. The extremal distance between E and F relative to Ω is defined as*

$$d_\Omega(E,F) = \lambda(\Gamma),$$

where Γ is the family of connected arcs in Ω that join E and F.

Definition 1.3.2. *[13, p144] Let Ω be a Jordan domain, E be an arc on $\partial\Omega$ and $z_0 \in \Omega$. The extremal distance from z_0 to E is defined as*

$$\lambda(z_0, E, \Omega) = \sup_{\sigma} d_{\Omega\setminus\sigma}(\sigma, E),$$

where σ is a Jordan arc in Ω that joins z_0 and $\partial\Omega \setminus E$. The supremum is taken all over such Jordan arcs.

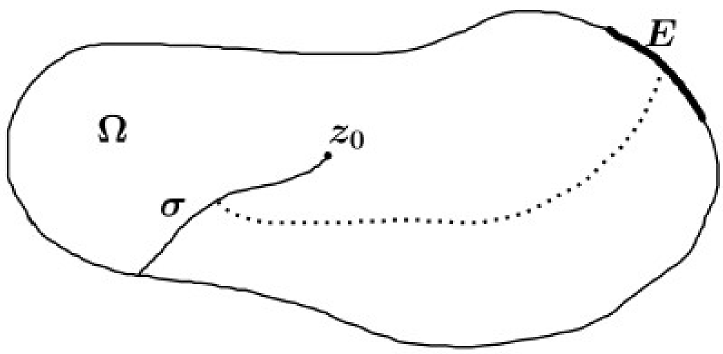

Figure 1.3: σ is the arc in Ω that joins z_0 and $\partial\Omega \setminus E$

One important property of extremal distance is the so called **symmetric principle** (see *[1]* and *[13]*). Here we introduce this principle in the upper half plane, actually it also works in the unit circle, because there exists a conformal map of $\overline{\mathbb{C}}$ that maps upper half plane to the unit disk.

Lemma 1.3.3. *[13, p137] Let Ω be a domain that is contained in the upper half plane, E and F be two disjoint subarcs of $\partial\Omega$, and let Ω^*, E^*, F^* be the reflections of Ω, E and F over the real axis, respectively. Set $\tilde{\Omega} = \Omega \cup \Omega^*$, $\tilde{E} = E \cup E^*$ and $\tilde{F} = F \cup F^*$, then*

$$d_{\tilde{\Omega}}(\tilde{E}, \tilde{F}) = \frac{1}{2} d_{\Omega}(E, F).$$

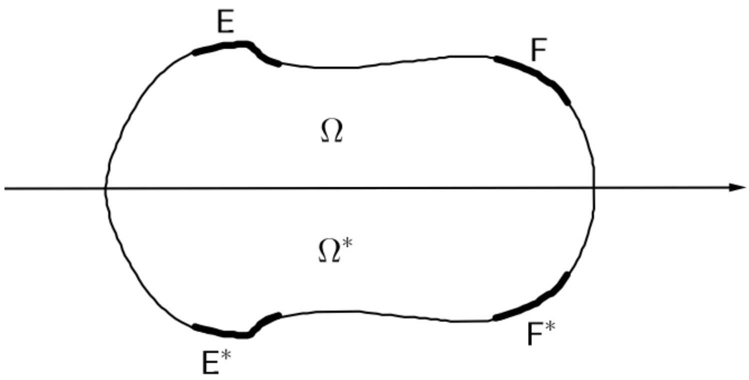

Figure 1.4: Ω^* is the reflection of Ω in the real axis

If Ω is the unit disk and E is an subarc on the boundary, then the extremal distance from 0 to E can be estimated by the capacity of this subarc.

Lemma 1.3.4. *[23, p212] Let E be a Borel set on $\partial\mathbb{D}$ and let $\Gamma_E(r)$ $(0 < r < 1)$ denote the family of all curves in $\{r < |z| < 1\}$ that connect E with $\{|z| = r\}$, then*

$$\frac{\sqrt{r}}{1+r} capE \leq e^{\frac{-\pi}{mod \Gamma_E(r)}} \leq \frac{\sqrt{r}}{1-r} capE,$$

for $0 < r \leq \frac{1}{3}$ and thus

$$capE = \lim_{r \to o} \frac{1}{\sqrt{r}} e^{\frac{-\pi}{mod \Gamma_E(r)}}.$$

In this lemma, "*cap*" means logarithmic capacity and is definied by using the Robin's constant for E. For more details, see *[2]*, *[13]* and *[21]*.

1.4 Quasiconformal mapping

The concept of quasiconformal mappings were introduced by Grötzsch, but the importance of quasiconformal mappings in complex analysis was first realized by

Ahlfors and Teichmüller. There are three definitions for quasiconformal mappings: metric definition, geometric definition and analytic definition. The reader is referred to *[1]*, *[15]*, *[16]*, *[21]* for details. In this section we introduce the analytic definition. This definition is useful in estimating the Hölder continuity of quasiconformal mappings.

Definition 1.4.1. *[15] A map $f : I \to \mathbb{C}$ is absolutely continuous on I, if for any $\epsilon > 0$, there is a positive number $\delta > 0$, such that for any finite sequence of pairwise disjoint subintervals (x_k, y_k) of I satisfying*

$$\sum_{k=1}^{n} |x_k - y_k| < \delta,$$

it holds that

$$\sum_{k=1}^{n} |f(x_k) - f(y_k)| < \delta.$$

Definition 1.4.2. *(Class ACL) [15] We say a continuous function f is absolutely continuous on lines (or ACL) in a domain $\Omega \subset \overline{\mathbb{C}}$ if for any rectangle $R = \{x+iy : a < x < b, c < y < d\}$, $\overline{R} \subset \Omega$, it has the following properties:*

(1) $f(x + iy)$ is absolutely continuous in x for a.e. $y \in [c, d]$.

(2) $f(x + iy)$ is absolutely continuous in y for a.e. $x \in [a, b]$.

Definition 1.4.3. *[13, p241] Let Ω and Ω' be domains in the extended plane, let $f : \Omega \to \Omega'$ be a homeomorphism which preserves the orientation, and let $K \geq 1$. Then we say f is a K-quasiconformal mapping if:*

(1) f is ACL in Ω.

(2) The derivatives

$$f_z = \frac{f_x - if_y}{2} \quad \text{and} \quad f_{\bar{z}} = \frac{f_x + if_y}{2}$$

satisfy

$$|f_{\bar{z}}| \leq \frac{K-1}{K+1} |f_z|$$

almost everywhere in Ω.

1.5 Quasicircle

Quasicircles were originally introduced independently by Pfuger and Tienari. In *[21]* and older articles, it was referred to as quasiconformal curve. Quasicircles play an important role in the theory of quasiconformal mappings and complex dynamical systems.

Definition 1.5.1. *[15] A domain Ω is a K-quasidisk if it is the image of an open disk or half plane under a K-quasiconformal self mapping of $\overline{\mathbb{C}}$. The boundary $\partial\Omega$ is called a quasicircle.*

By this definition, we see that quasicircle is the image of a unit circle under a quasiconformal mapping of the extended complex plane. The following result is called Ahlfors' two point inequality. It gives us a geometrically intuitive way to determine if a Jordan curve is a quasicircle or not.

Lemma 1.5.2. *[23, p94] A Jordan curve J is a quasicircle, if and only if there exists a constant $M \geq 1$, such that*

$$diam J(a,b) \leq M|a-b|$$

for all $a, b \in J$, where $J(a,b)$ is the smaller arc of J between a and b.

We note that a quasicircle J can be non-rectifiable. If J is a piecewise smooth quasicircle, then it has no cusps (of angle 0 or 2π).

1.6 Riemann mapping theorem and Schwarz—Christoffel formula

The Riemann mapping therorem is one of the most important results of complex analysis. It was first stated by Bernhard Riemann under the assumption that the boundary is piecewise smooth in 1851 in his PhD thesis. See *[7]* and *[8]* for more details.

Theorem 1.6.1. *[8, p160] (Riemann Mapping Theorem) Let G be a simply connected region which is not the whole plane and let $a \in G$. Then there is a unique analytic function $f : G \to \mathbb{C}$ having the following properties:*
(1) $f(a) = 0$ and $f'(a) > 0$;
(2) f is one-one;
(3) $f(G) = \{z : |z| < 1\}$.

Riemann mapping theorem tells us that there exists a conformal mapping that maps the unit disk onto any simply connected domain ohter than the whole plane. But it doesn't give an explicit formula. Two German mathematicians H. A. Schwarz and E. B. Christoffel discovered this conformal mapping when the domain is a polygon independently (See *[9]*).

Theorem 1.6.2. *[9, p10] Let P be the interior of a polygon Γ having vertices w_1, $w_2, \cdots, w_n$ and interior angles $\alpha_1\pi, \alpha_2\pi, \cdots, \alpha_n\pi$ in counterclockwise order. Let f be any conformal map from the upper half plane to P with $f(\infty) = w_n$. Then*

$$f(z) = A + C\int^{z}\prod_{k=1}^{n-1}(\xi - z_k)^{\alpha_k - 1}d\xi$$

for some complex constants A and C, where $w_k = f(z_k)$ for $k = 1, \cdots, n-1$.

1.7 Reduced extremal distance

The extremal distance between two arcs will tend to ∞ if one of them shrinks to a point. However, their difference may be a finite number. Therefore, one can define reduced extremal distance as follows. The reader is referred to *[2]*, *[20]* and *[24]* for details.

Definition 1.7.1. *[20, p241] let Ω be a domain, bounded or unbounded, in the extended complex plane, E be any set on $\partial\Omega$, and $z_0 \in \Omega$ be a finite point. Let $\{z : |z_0 - z| \le r\}$ be contained in Ω and $\Delta_r = \{z : |z_0 - z| = r\}$. The extremal distance between the Δ_r and E relative to Ω is denoted by $\lambda(E, \Delta_r, \Omega)$. Then the* *reduced extremal distance between* z_0 *and* E *is defined to be*

$$\delta(z_0, E, \Omega) = \lim_{r \to 0} [\lambda(E, \Delta_r, \Omega) - \lambda(\partial\Omega, \Delta_r, \Omega)], \qquad if \quad z_0 \neq \infty.$$

In case $z_0 = \infty \in \Omega$, $\lambda(E, \Delta_r, \Omega)$ *will mean the extremal distance between* $\{z : |z| = r\}$ *and* E *relative to* Ω.

$$\delta(\infty, E, \Omega) = \lim_{r \to \infty} [\lambda(E, \Delta_r, \Omega) - \lambda(\partial\Omega, \Delta_r, \Omega)], \qquad if \quad z_0 = \infty.$$

Chapter 2

Conformal invariants

It is well known that the extremal distance and reduced extremal distance are two conformal invariants. In this Chapter, we will discuss these two invariants on the unit circle and give a comparision.

2.1 Extremal domains for modulus

In this section, we introduce three extremal domains. The moduli of these extremal domains play an important role in the estimate of modulus or extremal distance and will be used frequently in this paper.

Let G be a doubly connected domain in the finite plane, C_1 and C_2 be the bounded and unbounded component of its complement.

2.1.1 Grötzsch extremal domain

If C_1 is the unit disk $\overline{\mathbb{D}}$, C_2 contains the point $R > 1$, then the maximal modulus of curve family that separates C_1 and C_2 is obtained in the following case:

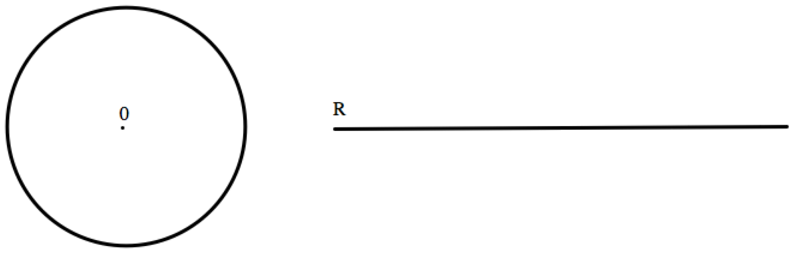

Figure 2.1: Grötzsch extremal domain

This domain is called Grötzsch extremal domain. The modulus is denoted by $\frac{1}{2\pi}\log\Phi(R)$.

2.1.2 Teichmüller extremal domain

If C_1 contains 0 and -1, C_2 contains a point with modulus P, then the maximal modulus of curve family that separates C_1 and C_2 is obtained in the following case:

Figure 2.2: Teichmüller extremal domain

This domain is called Teichmüller extremal domain. The modulus is denoted by $\frac{1}{2\pi}\log\Psi(P)$.

2.1.3 Mori extremal domain

If $diam(C_1\bigcap\mathbb{D})\geq\lambda$, C_2 contians the origin, then the maximal mudulus of curve family that separates C_1 and C_2 is obtained in the following case:

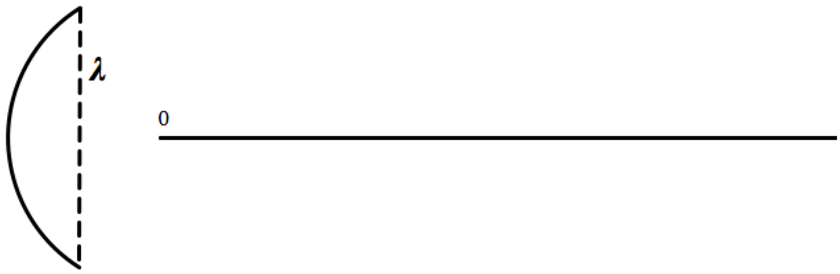

Figure 2.3: Mori extremal domain

This domain is called Mori extremal domain and the modulus is denoted by $\frac{1}{2\pi}\log X(\lambda)$.

There are some useful relations between the above moduli. For more details, see *[1]*, *[3]* and *[21]*.

$$\Phi(R)^2 = \Psi(R^2-1), \tag{2.1.1}$$

$$X(\lambda) = \Phi(\frac{\sqrt{4+2\lambda}+\sqrt{4-2\lambda}}{\lambda}), \tag{2.1.2}$$

$$16P \le \Psi(P) \le 16(P+1), \tag{2.1.3}$$

$$\lim_{P\to\infty}\frac{\log\Psi(P)}{\log P} = 1. \tag{2.1.4}$$

For simplification of notation, we introduce the functions $\mu(r)$ and $\Lambda(P)$ to denote the modulus of the Grötzsch extremal domain and Teichmüller extremal domain respectively:

$$\mu(r) = \frac{1}{2\pi}\log\Phi(\frac{1}{r}), \quad \Lambda(P) = \frac{1}{2\pi}\log\Psi(P). \tag{2.1.5}$$

This together with (2.1.1), yields that:

$$\mu(r) = \frac{1}{2}\Lambda(\frac{1}{r^2}-1), \tag{2.1.6}$$

for $0 < r < 1$. Furthermore, by applying the symmetry principle to the Teichmüller extremal domain, we can get $\Lambda(R) = \frac{1}{2}\Lambda(R)^+$. Here $\Lambda(R)^+$ is the modulus of curve familiy in the upper half plane that seperate C_1 and C_2. By the

Schwarz-Christoffel mapping theorem, the function

$$w = \int_0^z \frac{1}{\sqrt{z(z+1)(z-P)}} dz$$

maps the upper half plane to a rectangle. Choose different P, we get two different rectangles:

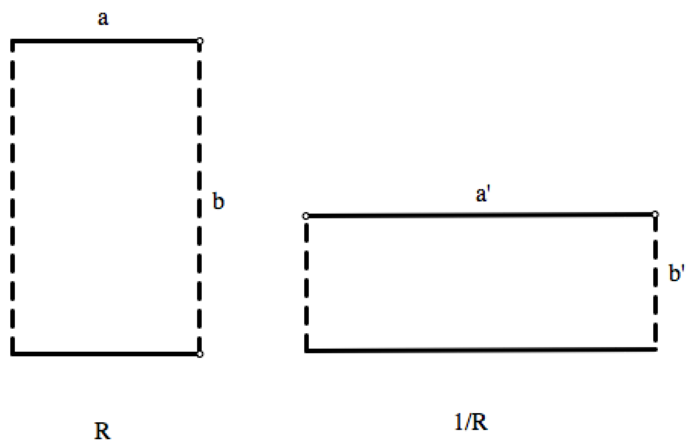

Figure 2.4: The images of upper half plane under different S-C maps

The modules of these rectangles are:

$$\Lambda(R) = \frac{1}{2}\Lambda(R)^+ = \frac{b}{2a} \quad and \quad \Lambda(\frac{1}{R}) = \frac{1}{2}\Lambda(\frac{1}{R})^+ = \frac{b'}{2a'},$$

where

$$a = \int_0^R \frac{dz}{\sqrt{z(z+1)(R-z)}}, \quad a' = \int_0^{\frac{1}{R}} \frac{dz}{\sqrt{z(z+1)(\frac{1}{R}-z)}},$$

$$b = \int_R^\infty \frac{dz}{\sqrt{z(z+1)(z-R)}}, \quad b' = \int_{\frac{1}{R}}^\infty \frac{dz}{\sqrt{z(z+1)(z-\frac{1}{R})}}.$$

After a simple calculation, one can see that

$$b' = \sqrt{R}a \quad and \quad a' = \sqrt{R}b.$$

Therefore, the product of $\Lambda(R)$ and $\Lambda(\frac{1}{R})$ is

$$\Lambda(R)\Lambda(\frac{1}{R}) = \frac{b}{2a}\frac{b'}{2a'} = \frac{b}{2a}\frac{\sqrt{R}a}{2\sqrt{R}b} = \frac{1}{4}, \tag{2.1.7}$$

for any $R > 0$.

2.2 Estimate of extremal distance in the unit disk

In this section, we give an estimate of extremal distance on the unit circle by using the modules of extremal domains.

Theorem 2.2.1. *Let $\mathbb{D}$ be the unit disk and E be an arc on the boundary of $\mathbb{D}$ with central angle α. Then*

$$\lambda(0, E, \mathbb{D}) = 2\mu(\sin\frac{\alpha}{4}) = \Lambda(\frac{\cos^2\frac{\alpha}{4}}{\sin^2\frac{\alpha}{4}}).$$

Furthermore, let $\tilde{E} = \partial\mathbb{D} \setminus E$ be the complement of E on $\partial\mathbb{D}$. Then

$$\lambda(0, E, \mathbb{D})\lambda(0, \tilde{E}, \mathbb{D}) = \frac{1}{4}.$$

Proof. Suppose first that $\alpha \in [0, \pi]$, by the symmetry principle, one can convert this domain to the Mori domain and deduce that

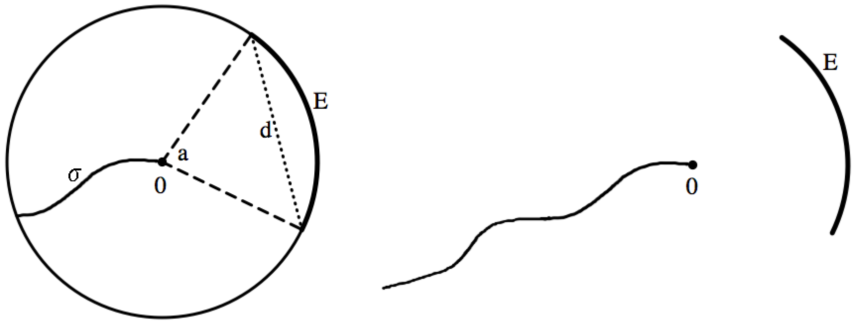

Figure 2.5: Application of symmetry principle

$$\lambda(0, E, \mathbb{D}) = \frac{1}{\pi} \log X(d),$$

where $d = 2\sin\frac{\alpha}{2}$ is the diameter of E.

Thus, by (2.1.2) and (2.1.5), a simple calculation yields that

$$\begin{aligned}
\lambda(0, E, \mathbb{D}) &= \frac{1}{\pi} \log \Phi(\frac{\sqrt{4+2d}+\sqrt{4-2d}}{d}) \\
&= 2\mu(\frac{d}{\sqrt{4+2d}+\sqrt{4-2d}}) \\
&= 2\mu(\frac{2\sin\frac{\alpha}{2}}{\sqrt{4+4\sin\frac{\alpha}{2}}+\sqrt{4-4\sin\frac{\alpha}{2}}}) \\
&= 2\mu(\frac{2\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{4}+\cos\frac{\alpha}{4}+\cos\frac{\alpha}{4}-\sin\frac{\alpha}{4}}) \\
&= 2\mu(\frac{2\sin\frac{\alpha}{2}}{2\cos\frac{\alpha}{4}}) \\
&= 2\mu(\sin\frac{\alpha}{4}).
\end{aligned}$$

By (2.1.6),

$$\mu(\sin\frac{\alpha}{4}) = \frac{1}{2}\Lambda(\frac{1}{\sin^2\frac{\alpha}{4}} - 1) = \frac{1}{2}\Lambda(\frac{\cos^2\frac{\alpha}{4}}{\sin^2\frac{\alpha}{4}}),$$

and thus

$$\lambda(0, E, \mathbb{D}) = 2\mu(\sin\frac{\alpha}{4}) = \Lambda(\frac{\cos^2\frac{\alpha}{4}}{\sin^2\frac{\alpha}{4}}).$$

Next, assume that $\alpha \in [\pi, 2\pi]$. In this case we may assume, after a rotation if necessary, that E is the arc on the unit circle joining $e^{-i\frac{\alpha}{2}}$ to $e^{i\frac{\alpha}{2}}$ counterclockwise. One can verify that the following transformation

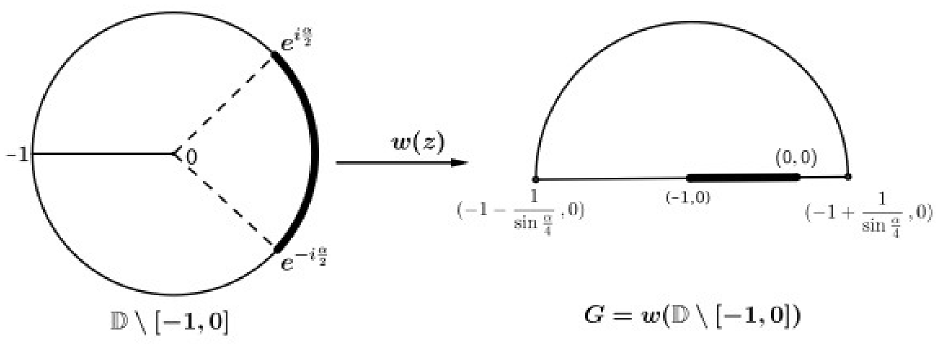

Figure 2.6: w maps $\mathbb{D}\backslash[-1, 0]$ to G

$$w(z) = \frac{\sqrt{z} - e^{i\frac{\alpha}{4}}}{\sqrt{z} + e^{i\frac{\alpha}{4}}} \cdot \frac{e^{i\frac{\alpha}{4}} + e^{-i\frac{\alpha}{4}}}{e^{i\frac{\alpha}{4}} - e^{-i\frac{\alpha}{4}}}$$

maps the slit disk $\mathbb{D}\backslash[-1,0]$ conformally onto the upper half disk denoted by $G = w(\mathbb{D}\backslash[-1,0])$, centered at $(-1,0)$ with radius $\frac{1}{\sin\frac{\alpha}{4}}$ and

$$w(e^{i\frac{\alpha}{2}}) = 0, \quad w(e^{-i\frac{\alpha}{2}}) = -1.$$

The diameter of G on the real line joins the two points

$$\left(-1 - \frac{1}{\sin\frac{\alpha}{4}}, 0\right) \quad \text{and} \quad \left(-1 + \frac{1}{\sin\frac{\alpha}{4}}, 0\right),$$

which are the images of points $z_1 = e^{\pi i}$ and $z_2 = e^{-\pi i}$ under the map of w. By the conformal invariance of extremal length, one can see that

$$\lambda(0, E, \mathbb{D}) = \lambda(E, [-1,0], \mathbb{D}) = \lambda([-1,0], \partial G \cap \mathbb{H}, G).$$

By associating the extremal length $\lambda([-1,0], \partial G \cap \mathbb{H}, G)$ with the modulus of an appropriate Grötzsch domain, tedious but routine calculation yields that

$$\lambda(0, E, \mathbb{D}) = 2\mu(\sin\frac{\alpha}{4}) = \Lambda(\frac{\cos^2\frac{\alpha}{4}}{\sin^2\frac{\alpha}{4}}).$$

Finally, let $\tilde{E} = \partial\mathbb{D} \setminus E$ be the complement of E on $\partial\mathbb{D}$. Then the central angle with respect to arc $\tilde{E}$ is $2\pi - \alpha$. And the extremal distance is

$$\lambda(0, \tilde{E}, \mathbb{D}) = \Lambda(\frac{\cos^2\frac{2\pi-\alpha}{4}}{\sin^2\frac{2\pi-\alpha}{4}}) = \Lambda(\frac{\sin^2\frac{\alpha}{4}}{\cos^2\frac{\alpha}{4}}).$$

Hense by (2.1.6), one can deduce that

$$\lambda(0, E, \mathbb{D})\lambda(0, \tilde{E}, \mathbb{D}) = \frac{1}{4}.$$

$\square$

2.3 Comparision of extremal distance and reduced extremal distance

In order to prove the equivalent conditions for the sewing homeomorphism of a Jordan domain to be bi-Lipschitz or bi-Hölder , we establish the following comparison result between extremal distance $\lambda(z_0, E, \Omega)$ and reduced extremal distance $\delta(z_0, E, \Omega)$, which also has its own interest. Without loss of generality, due to the conformal invariance, we may assume $\Omega = \mathbb{D}$, $z_0 = 0$.

Theorem 2.3.1. *Denote the central angle with respect to an arc $E \subset \partial\mathbb{D}$ by $\alpha(E)$.*
(1) If $0 \le \alpha(E) \le \pi$, then

$$\delta(0, E, \mathbb{D}) + \frac{3}{2\pi}\ln 2 \le \lambda(0, E, \mathbb{D}) \le \delta(0, E, \mathbb{D}) + \frac{2}{\pi}\ln 2;$$

(2) If $\pi \le \alpha(E) < 2\pi$, then

$$\frac{1}{\frac{8}{\pi}\ln 2 - \frac{2}{\pi}\ln(1 - e^{-2\pi\delta(0,E,\mathbb{D})})} \le \lambda(0, E, \mathbb{D}) \le \frac{1}{\frac{6}{\pi}\ln 2 - \frac{2}{\pi}\ln(1 - e^{-2\pi\delta(0,E,\mathbb{D})})}.$$

Proof. For simplicity of notation, we write $\alpha(E)$ as α for a fixed arc $E \subset \partial\mathbb{D}$. First assume that $0 \le \alpha \le \pi$. By Theorem 2.2.1 we have

$$\lambda(0, E, \mathbb{D}) = 2\mu(\sin\frac{\alpha}{4}) = \Lambda\left(\frac{\cos^2\frac{\alpha}{4}}{\sin^2\frac{\alpha}{4}}\right).$$

It follows from estimate (2.1.3) and (2.1.5) on Teichmüller function that

$$\frac{1}{2\pi}\ln 16\frac{\cos^2\frac{\alpha}{4}}{\sin^2\frac{\alpha}{4}} \le \lambda(0, E, \mathbb{D}) \le \frac{1}{2\pi}\ln 16\left(\frac{\cos^2\frac{\alpha}{4}}{\sin^2\frac{\alpha}{4}} + 1\right).$$

Furthermore, since for $0 \le \alpha \le \pi$

$$\frac{1}{2\pi}\ln\cos^2\frac{\alpha}{4} \ge -\frac{1}{2\pi}\ln 2,$$

routine estimates yield that

$$-\frac{1}{\pi}\ln\sin\frac{\alpha}{4}+\frac{3}{2\pi}\ln 2\le\lambda(0,E,\mathbb{D})\le-\frac{1}{\pi}\ln\sin\frac{\alpha}{4}+\frac{2}{\pi}\ln 2.$$

On the other hand, by the definition of reduced extremal distance, it can be calculated as

$$\delta(0,E,\mathbb{D})=\lim_{r\to 0}[\lambda(\Delta_r,E,\mathbb{D})-\lambda(\Delta_r,\partial\mathbb{D},\mathbb{D})].$$

By Lemma 1.3.4,

$$\begin{aligned}\delta(0,E,\mathbb{D})&=\lim_{r\to 0}[\frac{1}{\pi}\ln\frac{1}{\sqrt{t}\sin\frac{\alpha}{4}}-\frac{1}{2\pi}\ln\frac{1}{r}]\\&=\frac{1}{\pi}\ln\frac{1}{\sin\frac{\alpha}{4}}\\&=-\frac{1}{\pi}\ln\sin\frac{\alpha}{4}.\end{aligned}\tag{2.3.1}$$

Thus we obtain the desired inequalities

$$\delta(0,E,\mathbb{D})+\frac{3}{2\pi}\ln 2\le\lambda(0,E,\mathbb{D})\le\delta(0,E,\mathbb{D})+\frac{2}{\pi}\ln 2,$$

when $0\le\alpha\le\pi$.

Next assume that $\pi\le\alpha<2\pi$. By Theorem 2.2.1 and relation (2.1.7), we have

$$\lambda(0,E,\mathbb{D})=\Lambda\left(\frac{\cos^2\frac{\alpha}{4}}{\sin^2\frac{\alpha}{4}}\right)=\frac{1}{4}\frac{1}{\Lambda\left(\frac{\sin^2\frac{\alpha}{4}}{\cos^2\frac{\alpha}{4}}\right)}.$$

Thus it follows from (2.1.3) and (2.1.5) that

$$\begin{aligned}\Lambda\left(\frac{\sin^2\frac{\alpha}{4}}{\cos^2\frac{\alpha}{4}}\right)&\le\frac{1}{2\pi}\ln 16(\frac{\sin^2\frac{\alpha}{4}}{\cos^2\frac{\alpha}{4}}+1)\\&\le\frac{2}{\pi}\ln 2-\frac{1}{\pi}\ln\cos\frac{\alpha}{4},\end{aligned}$$

and

$$\Lambda\left(\frac{\sin^2\frac{\alpha}{4}}{\cos^2\frac{\alpha}{4}}\right) \geq \frac{1}{2\pi}\ln 16(\frac{\sin^2\frac{\alpha}{4}}{\cos^2\frac{\alpha}{4}}) \geq \frac{3}{2\pi}\ln 2 - \frac{1}{\pi}\ln\cos\frac{\alpha}{4},$$

therefore

$$\frac{1}{\frac{8}{\pi}\log 2 - \frac{4}{\pi}\ln\cos\frac{\alpha}{4}} \leq \lambda(0,E,\mathbb{D}) \leq \frac{1}{\frac{6}{\pi}\ln 2 - \frac{4}{\pi}\ln\cos\frac{\alpha}{4}}. \quad (2.3.2)$$

Finally, taking into account that $\delta(0,E,\mathbb{D}) = -\frac{1}{\pi}\ln\sin\frac{\alpha}{4}$, we obtain that

$$\cos^2\frac{\alpha}{4} = 1 - e^{-2\pi\delta(0,E,\mathbb{D})}. \quad (2.3.3)$$

Applying (2.3.3) to (2.3.2), it follows that

$$\frac{1}{\frac{8}{\pi}\log 2 - \frac{2}{\pi}\ln(1-e^{-2\pi\delta(0,E,\mathbb{D})})} \leq \lambda(0,E,\mathbb{D}) \leq \frac{1}{\frac{6}{\pi}\ln 2 - \frac{2}{\pi}\ln(1-e^{-2\pi\delta(0,E,\mathbb{D})})},$$

when $\pi \leq \alpha < 2\pi$. □

We close this chapter by deriving two corollaries from Theorem 2.3.1. With the same notation as in Theorem 2.3.1, we have

Corollary 2.3.2.

(a) $\lim_{\alpha(E)\to 0}\frac{\lambda(0,E,\mathbb{D})}{\delta(0,E,\mathbb{D})} = 1,$

(b) $\lim_{\alpha(E)\to 2\pi}\frac{\lambda(0,E,\mathbb{D})}{\delta(0,E,\mathbb{D})} = +\infty.$

Proof. For the proof of (a), dividing the inequalities in Theorem 2.3.1 part (1) by $\delta(0,E,\mathbb{D})$, we obtain that

$$1 + \frac{\frac{3}{2\pi}\ln 2}{\delta(0,E,\mathbb{D})} \leq \frac{\lambda(0,E,\mathbb{D})}{\delta(0,E,\mathbb{D})} \leq 1 + \frac{\frac{2}{\pi}\ln 2}{\delta(0,E,\mathbb{D})}.$$

Furthermore, by (2.3.1),

$$\delta(0,E,\mathbb{D}) = -\frac{1}{\pi}\ln\sin\frac{\alpha}{4} \to \infty$$

as $\alpha(E) \to 0$. Thus, by the squeeze theorem in Calculus, it follows that

$$\lim_{\alpha(E)\to 0} \frac{\lambda(0,E,\mathbb{D})}{\delta(0,E,\mathbb{D})} = 1.$$

For the proof of (b), by Theorem 2.3.1, we have

$$\begin{aligned}\frac{1}{\frac{8}{\pi}\ln 2 - \frac{2}{\pi}\ln(1-e^{-2\pi\delta(0,E,\mathbb{D})})} &\le \lambda(0,E,\mathbb{D}) \\ &\le \frac{1}{\frac{6}{\pi}\ln 2 - \frac{2}{\pi}\ln(1-e^{-2\pi\delta(0,E,\mathbb{D})})}.\end{aligned}$$

Dividing each side of the above inequalities by $\delta(0,E,\mathbb{D})$, we obtain that

$$\frac{\lambda(0,E,\mathbb{D})}{\delta(0,E,\mathbb{D})} \ge \frac{1}{\delta(0,E,\mathbb{D})[\frac{8}{\pi}\log 2 - \frac{2}{\pi}\ln(1-e^{-2\pi\delta(0,E,\mathbb{D})})]}. \tag{2.3.4}$$

Now we need to show the right sides of (2.3.4) tend to ∞ as $\alpha(E) \to 2\pi$. Consider the function

$$f(x) = x[\frac{8}{\pi}\ln 2 - \frac{2}{\pi}\ln(1-e^{-2\pi x})].$$

As $x \to 0$,

$$\begin{aligned}\lim_{x\to 0} f(x) &= \lim_{x\to 0} x[\frac{8}{\pi}\ln 2 - \frac{2}{\pi}\ln(1-e^{-2\pi x})] \\ &= \lim_{x\to 0} \frac{-\frac{2}{\pi}\frac{1}{(1-e^{-2\pi x})}(2\pi e^{-2\pi x})}{-\frac{1}{x^2}} \\ &= \lim_{x\to 0} \frac{4x^2}{1-e^{-2\pi x}} \\ &= \lim_{x\to 0} \frac{8x}{2\pi e^{-2\pi x}} = 0.\end{aligned}$$

Now replace x by $\delta(0,E,\mathbb{D})$, we obtain the following limit

$$\lim_{\alpha(E)\to 2\pi} \frac{1}{\delta(0,E,\mathbb{D})[\frac{8}{\pi}\log 2 - \frac{2}{\pi}\ln(1-e^{-2\pi\delta(0,E,\mathbb{D})})]} = \infty.$$

Applying this limit to (2.3.4), we get

$$\lim_{\alpha(E)\to 2\pi} \frac{\lambda(0,E,\mathbb{D})}{\delta(0,E,\mathbb{D})} = +\infty.$$

□

Corollary 2.3.3. *For any arc $E \subset \partial\mathbb{D}$, $\lambda(0, E, \mathbb{D}) > \delta(0, E, \mathbb{D})$.*

Proof. If the central angle with respect to E is no more than π, by Theorem 2.3.1 part (1), we have the following inequality

$$\delta(0, E, \mathbb{D}) + \frac{3}{2\pi}\ln 2 \le \lambda(0, E, \mathbb{D}) \le \delta(0, E, \mathbb{D}) + \frac{2}{\pi}\ln 2.$$

It is easy to see that $\lambda(0, E, \mathbb{D}) > \delta(0, E, \mathbb{D})$.

When the central angle with respect to E is greater than π, by (2.3.1),

$$0 < \delta(0, E, \mathbb{D}) < \frac{1}{2\pi}\ln 2.$$

By Theorem 2.3.1, we get

$$\lambda(0, E, \mathbb{D}) \ge \frac{1}{\frac{8}{\pi}\ln 2 - \frac{2}{\pi}\ln(1 - e^{-2\pi\delta(0,E,\mathbb{D})})}.$$

Next, we show that

$$\frac{1}{\frac{8}{\pi}\ln 2 - \frac{2}{\pi}\ln(1 - e^{-2\pi\delta(0,E,\mathbb{D})})} > \delta(0, E, \mathbb{D}).$$

For any $x \in (0, \frac{1}{2\pi}\ln 2)$, consider the function

$$f(x) = \frac{1}{\frac{8}{\pi}\ln 2 - \frac{2}{\pi}\ln(1 - e^{-2\pi x})}.$$

By some calculations, we have

$$f'(x) = \frac{\pi^2 e^{-2\pi x}}{(1 - e^{-2\pi x})(4\ln 2 - \ln(1 - e^{-2\pi x}))^2}$$

and

$$f''(x) = \frac{2\pi^3 e^{-2\pi x}(2e^{-2\pi x} - (4\ln 2 - \ln(1 - e^{-2\pi x})))}{(1 - e^{-2\pi x})^2(4\ln 2 - \ln(1 - e^{-2\pi x}))^3}.$$

It is easy to see that for any $x \in (0, \frac{1}{2\pi}\ln 2)$, $f'(x) > 0$.

Now we consider the concavity of $f(x)$. Let

$$g(x) = 2e^{-2\pi x} - (4\ln 2 - \ln(1 - e^{-2\pi x})).$$

Then the derivative of $g(x)$ is

$$g'(x) = 2\pi e^{-2\pi x}\left(\frac{1}{1 - e^{-2\pi x}} - 2\right),$$

which is positive for any $x \in (0, \frac{1}{2\pi}\ln 2)$. Since $g(0) = -\infty$ and $g(x) < 0$, for any $x \in (0, \frac{1}{2\pi}\ln 2)$, it follows that $f''(x) < 0$. So $f(x)$ is concave down. By the proof of Corollary 2.3.2, we know

$$\lim_{x\to 0}\frac{f(x)}{x} = +\infty$$

and

$$f\left(\frac{1}{2\pi}\ln 2\right) = \frac{\pi}{10\ln 2} > \frac{1}{2\pi}\ln 2.$$

Taking into account of the concavity of $f(x)$, it follows that

$$\frac{1}{\frac{8}{\pi}\ln 2 - \frac{2}{\pi}\ln(1 - e^{-2\pi\delta(0,E,\mathbb{D})})} > \delta(0, E, \mathbb{D}),$$

which implies

$$\lambda(0, E, \mathbb{D}) > \delta(0, E, \mathbb{D}).$$

□

Chapter 3

Equivalent conditions for Bi-Lipschitz sewing homeomorphism

In this chapter we establish some equivalent conditions for the sewing homeomorphism h_Ω of a Jordan domain to be bi-Lipschitz by using harmonic measure, extremal distance and reduced extremal distance in Ω.

3.1 Harmonic measure

Before proceeding to the main result of this section, we recall the definition of harmonic measure and its relation to extremal distance. In general, harmonic measure in a Jordan domain is defined using the solution of a Dirichlet problem. The reader is referred to *[5]*, *[6]*,*[13]*, *[15]*, *[18]* for more details. Since it is conformally invariant, it can also be defined more intuitively as follows.

Definition 3.1.1. *Let E be an arc on the unit circle $\partial\mathbb{D}$. Then the harmonic measure of E with respect to the origin 0 in $\mathbb{D}$ is defined as*

$$\omega(0, E, \mathbb{D}) = \frac{\alpha(E)}{2\pi},$$

where $\alpha(E)$ is the central angle of E. Furthermore, for an arc E on the boundary of a Jordan domain Ω and a point $z_0 \in \Omega$, the harmonic measure $\omega(z_0, E, \Omega)$ is defined by means of a conformal map $f : \Omega \to \mathbb{D}$ with $f(z_0) = 0$ and the harmonic measure on $\mathbb{D}$ defined above.

Since the extremal distance $\lambda(z_0, E, \Omega)$ is strictly decreasing with respect to E while the harmonic measure $\omega(z_0, E, \Omega)$ is strictly increasing, one can expect that there is some functional relation between these two conformal invariants. We quote the following result from *[13]*, which will be needed in this thesis.

Lemma 3.1.2. *[13, p145] Let Ω be a Jordan domain, E be a subarc of $\partial\Omega$ and $z_0 \in \Omega$. Then*

$$e^{-\pi\lambda(z_0,E,\Omega)} \leq \omega(z_0, E, \Omega) \leq \frac{8}{\pi} e^{-\pi\lambda(z_0,E,\Omega)}.$$

Moreover

$$\lim_{\lambda\to\infty} \omega(z_0, E, \Omega) e^{\pi\lambda(z_0,E,\Omega)} = \frac{8}{\pi},$$

and

$$\lim_{\lambda\to 0} \omega(z_0, E, \Omega) e^{\pi\lambda(z_0,E,\Omega)} = 1.$$

3.2 Equivalent conditions for h_Ω to be bi-Lipschitz homeomorphism

We are now ready to establish one of the main results in this paper, which gives several equivalent conditions for the sewing homeomorphism h_Ω to be a bi-Lipschitz map.

Theorem 3.2.1. *Suppose Ω is a Jordan domain with $z_0 \in \Omega$ and $z_0^* \in \Omega^* = \overline{\mathbb{C}} \backslash \overline{\Omega}$. Let f_1 and f_2 be Riemann mappings from Ω and Ω^* onto $\mathbb{D}$ and $\mathbb{D}^*$ with $f_1(z_0) = 0$ and $f_2(z_0^*) = \infty$, respectively. Let $h = f_2 \circ f_1^{-1} : \partial\mathbb{D} \to \partial\mathbb{D}$ be the sewing homeomorphism induced by Ω. Then the following conditions are equivalent:*

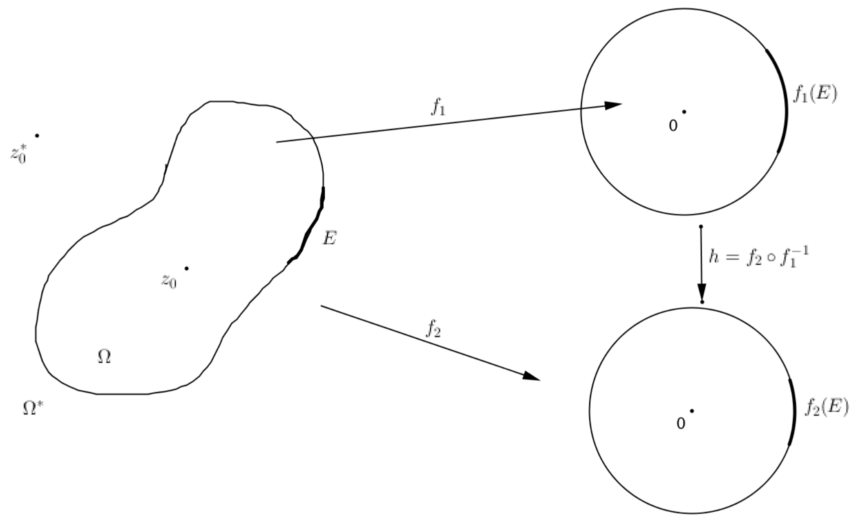

Figure 3.1: Sewing homeomorphism $h = f_2 \circ f_1^{-1}$

(1) h is a bi-Lipschitz homeomorphism, that is there exists a constant $L \geq 1$ such that for any $x, y \in \partial\mathbb{D}$,

$$\frac{1}{L}|y - x| \leq |h(y) - h(x)| \leq L|y - x|.$$

(2) There exists a constant $M \geq 1$, such that

$$\frac{1}{M} \leq \frac{\omega(z_0^*, E, \Omega^*)}{\omega(z_0, E, \Omega)} \leq M$$

for any subarc $E \subset \partial\Omega$.

(3) There exists a constant $N \geq 1$, such that

$$|\lambda(z_0^*, E, \Omega^*) - \lambda(z_0, E, \Omega)| \leq N$$

for any subarc $E \subset \partial\Omega$.

(4) There exists a constant $C \geq 1$, such that

$$|\delta(z_0^*, E, \Omega^*) - \delta(z_0, E, \Omega)| \leq C$$

for any subarc $E \subset \partial\Omega$.

3.3 Proof of Theorem 3.2.1

Roughly speaking, condition (2) says that the harmonic measures in Ω and Ω^* are comparable in the sense that their ratio is bounded above and below. Condition (3) (or (4)) reveals that the extremal distances (or reduced extremal distances) in Ω and Ω^* are comparable in the sense that their difference is bounded above and below. This section is devoted to the proof of Theorem 3.2.1, which requires some delicate analysis and estimates of various conformal invariants.

Proof. (1)$\Rightarrow$ (2): For any arc $E \subset \partial\Omega$, we consider two cases.

Case 1: $\omega(z_0, E, \Omega) \geq \frac{1}{\pi L}$. Since harmonic measure is less than or equal to 1,

$$\omega(z_0^*, E, \Omega^*) \leq 1 \leq \pi L\, \omega(z_0, E, \Omega).$$

This implies

$$\frac{\omega(z_0^*, E, \Omega^*)}{\omega(z_0, E, \Omega)} \leq \pi L.$$

Case 2: $\omega(z_0, E, \Omega) < \frac{1}{\pi L}$. In this case,

$$l(f_1(E)) = 2\pi\omega(z_0, E, \Omega) < \frac{2\pi}{\pi L} < \pi.$$

Claim: $l(f_2(E)) < \pi$.

If not, that is $l(f_2(E)) \geq \pi$, then there exists a subarc $E' \subset E$ such that $l(f_2(E')) = \pi$. Since

$$|f_1(E')| < 2\sin\frac{1}{L},$$

we have

$$\frac{|f_2(E')|}{|f_1(E')|} > \frac{2}{2\sin\frac{1}{L}} > L.$$

But by condition (1), we have

$$\frac{1}{L} \leq \frac{|f_2(E')|}{|f_1(E')|} \leq L,$$

which leads to a contradiction. This proves the above claim.

Thus in this case, we have both $l(f_1(E)) < \pi$ and $l(f_2(E)) < \pi$. Hence, using condition (1) and elementary inequalities on $\sin\theta$, one can derive that

$$\begin{aligned}\frac{\omega(z_0^*, E, \Omega^*)}{\omega(z_0, E, \Omega)} = \frac{l(f_2(E))}{l(f_1(E))} &= \frac{2\arcsin\frac{|f_2(E)|}{2}}{2\arcsin\frac{|f_1(E)|}{2}} \\ &\leq \frac{\frac{\pi}{2}|f_2(E)|}{|f_1(E)|} \leq \frac{\pi L}{2}.\end{aligned}$$

Let $M = \pi L$. Then in both case 1 and case 2, we have

$$\frac{\omega(z_0^*, E, \Omega^*)}{\omega(z_0, E, \Omega)} \leq M.$$

By symmetry, we also have

$$\frac{\omega(z_0^*, E, \Omega^*)}{\omega(z_0, E, \Omega)} \geq \frac{1}{M}.$$

Thus, the bi-Lipschitz condition (1) implies condition (2) with $M = \pi L$.

(2)⇒(3): By Lemma 3.1.2, we obtain the following inequalities:

$$e^{-\pi\lambda(z_0,E,\Omega)} \leq \omega(z_0, E, \Omega) \leq \frac{8}{\pi}e^{-\pi\lambda(z_0,E,\Omega)}$$

and

$$e^{-\pi\lambda(z_0^*,E,\Omega^*)} \leq \omega(z_0^*, E, \Omega^*) \leq \frac{8}{\pi}e^{-\pi\lambda(z_0^*,E,\Omega^*)}.$$

Combining these inequalities, we obtain

$$\begin{aligned}e^{-\pi\lambda(z_0^*,E,\Omega^*)} \leq \omega(z_0^*, E, \Omega^*) &\leq M\omega(z_0, E, \Omega) \\ &\leq \frac{8M}{\pi}e^{-\pi\lambda(z_0,E,\Omega)}.\end{aligned}$$

This implies that

$$\lambda(z_0^*, E, \Omega^*) - \lambda(z_0, E, \Omega) \geq \frac{\ln \frac{8M}{\pi}}{-\pi}.$$

By symmetry, we have

$$-N \leq \lambda(z_0^*, E, \Omega^*) - \lambda(z_0, E, \Omega) \leq N,$$

with $N = \frac{1}{\pi} \ln \frac{8M}{\pi}$.

(3)⇒(1): Suppose the central angle at 0 with respect to $f_1(E)$ in $\mathbb{D}$ is α, the central angle at 0 with respect to $f_2(E)$ in $\mathbb{D}$ is β. By symmetry, it is obvious that

$$\lambda(0, f_2(E), \mathbb{D}) = \lambda(\infty, f_2(E), \mathbb{D}^*).$$

By the representation of extremal distance in the unit disk,

$$\lambda(0, f_1(E), \mathbb{D}) = 2\mu\left(\sin\frac{\alpha}{4}\right) = \Lambda\left(\frac{\cos^2\frac{\alpha}{4}}{\sin^2\frac{\alpha}{4}}\right),$$

$$\lambda(0, f_2(E), \mathbb{D}) = 2\mu\left(\sin\frac{\beta}{4}\right) = \Lambda\left(\frac{\cos^2\frac{\beta}{4}}{\sin^2\frac{\beta}{4}}\right).$$

Taking into account of the continuity of sewing homeomorphism h, there exists a sufficiently small constant r_0 , such that when $|y - x| \leq r_0$, $|h(y) - h(x)|$ is also sufficiently small.

For any $x, y \in \partial\mathbb{D}$, let E be the pre-image under the mapping f_1 of the arc connecting x and y in $\partial\mathbb{D}$ such that $f_1(E)$ does not cover more than half of the unit circle. We consider two cases.

Case 1: $|y - x| \geq r_0$. Then

$$|h(y) - h(x)| \leq 2 \leq \frac{2}{r_0}|y - x|.$$

Case 2: $|y - x| < r_0$. Then

$$\begin{aligned}
\lambda(z_0^*, E, \Omega^*) - \lambda(z_0, E, \Omega) &= 2\mu(\sin\frac{\beta}{4}) - 2\mu(\sin\frac{\alpha}{4}) \\
&= \Lambda\left(\frac{\cos^2\frac{\beta}{4}}{\sin^2\frac{\beta}{4}}\right) - \Lambda\left(\frac{\cos^2\frac{\alpha}{4}}{\sin^2\frac{\alpha}{4}}\right) \\
&\leq \frac{1}{2\pi}\ln 16\left(\frac{\cos^2\frac{\beta}{4}}{\sin^2\frac{\beta}{4}} + 1\right) - \frac{1}{2\pi}\ln 16\left(\frac{\cos^2\frac{\alpha}{4}}{\sin^2\frac{\alpha}{4}}\right) \\
&= \frac{1}{\pi}\ln\frac{\sin\frac{\alpha}{4}}{\sin\frac{\beta}{4}} - \frac{1}{\pi}\ln\cos\frac{\alpha}{4}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\lambda(z_0^*, E, \Omega^*) - \lambda(z_0, E, \Omega) &= \Lambda\left(\frac{\cos^2\frac{\beta}{4}}{\sin^2\frac{\beta}{4}}\right) - \Lambda\left(\frac{\cos^2\frac{\alpha}{4}}{\sin^2\frac{\alpha}{4}}\right) \\
&\geq \frac{1}{2\pi}\ln 16\frac{\cos^2\frac{\beta}{4}}{\sin^2\frac{\beta}{4}} - \frac{1}{2\pi}\ln 16\left(\frac{\cos^2\frac{\alpha}{4}}{\sin^2\frac{\alpha}{4}} + 1\right) \\
&= \frac{1}{\pi}\ln\frac{\sin\frac{\alpha}{4}}{\sin\frac{\beta}{4}} + \frac{1}{\pi}\ln\cos\frac{\beta}{4}.
\end{aligned}$$

Combining the above two inequalities and taking into account of condition (3), one can deduce that there exists a positive constant M, such that

$$\frac{1}{M} \leq \frac{\sin\frac{\alpha}{4}}{\sin\frac{\beta}{4}} \leq M.$$

Since α and β are all sufficiently small, we have

$$\cos\frac{\alpha}{4} \geq \frac{1}{2}, \quad \cos\frac{\beta}{4} \geq \frac{1}{2}.$$

Hence it follows that

$$2\sin\frac{\beta}{4} \leq |h(y) - h(x)| = 2\sin\frac{\beta}{2} \leq 4\sin\frac{\beta}{4}$$

and

$$2\sin\frac{\alpha}{4} \leq |y - x| \leq 4\sin\frac{\alpha}{4}.$$

A simple calculation yields that

$$\frac{1}{2}\frac{\sin\frac{\beta}{4}}{\sin\frac{\alpha}{4}} \le \frac{|h(y)-h(x)|}{|y-x|} \le 2\frac{\sin\frac{\beta}{4}}{\sin\frac{\alpha}{4}}.$$

Therefore, we obtain that

$$\frac{1}{2M} \le \frac{|h(y)-h(x)|}{|y-x|} \le 2M,$$

which implies that h is a bi-Lipschitz homeomorphism.

(1)⇒(4): For any arc $E \subset \partial\Omega$, denote the two end points of E by $\widetilde{x}$ an $\widetilde{y}$, $x = f_1(\widetilde{x})$, $y = f_1(\widetilde{y})$. Suppose the central angle with respect to $f_1(E)$ is $\alpha(f_1(E))$. To show that h satisfies condition (4), there are three cases to be considered.

Case 1: $\alpha(f_1(E)) < 2\arcsin\frac{1}{L}$. In this case, it follows that

$$\begin{aligned} |f_2(\widetilde{x}) - f_2(\widetilde{y})| &= |f_2 \circ f_1^{-1}(x) - f_2 \circ f_1^{-1}(y)| \\ &= |h(x) - h(y)| \\ &\le L|x-y| = 2L\sin\frac{\alpha(f_1(E))}{2} \\ &< 2L\sin\left(\frac{1}{2}\cdot 2\arcsin\frac{1}{L}\right) = 2. \end{aligned}$$

Claim: $f_2(E)$ can not overlap half of the unit circle.

If $f_2(E)$ overlaps half or more of the unit circle, then there exists a subarc $E_0 \subset E$, such that $f_2(E_0)$ is exactly half of the unit circle. However, since

$$\alpha(f_1(E_0)) < \alpha(f_1(E)) < 2\arcsin\frac{1}{L},$$

the distance between two end points of $f_2(E_0)$ is less than 2, which contradicts the fact that $f_2(E_0)$ is half of the unit circle. Therefore $f_2(E)$ can not overlap half or more of the unit circle.

By the relationship between bi-Lipschitz sewing homeomorphism and extremal distance, there exists a constant $N \geq 1$, such that

$$|\lambda(z_0^*, E, \Omega^*) - \lambda(z_0, E, \Omega)| \leq N.$$

Since $\alpha(f_1(E)) < 2\arcsin\frac{1}{L}$, $f_1(E)$ can not overlap half of the unit circle. Taking into account of the fact that $f_2(E)$ can not overlap half of the unit circle, by Theorem 2.3.1, we have

$$\begin{aligned}
&\delta(z_0^*, E, \Omega^*) - \delta(z_0, E, \Omega) \\
\leq\ & \lambda(z_0^*, E, \Omega^*) - \frac{3}{2\pi}\ln 2 - \left(\lambda(z_0, E, \Omega) - \frac{2}{\pi}\ln 2\right) \\
=\ & \lambda(z_0^*, E, \Omega^*) - \lambda(z_0, E, \omega) + \frac{1}{2\pi}\ln 2 \\
\leq\ & N + \frac{1}{2\pi}\ln 2
\end{aligned}$$

and

$$\begin{aligned}
&\delta(z_0^*, E, \Omega^*) - \delta(z_0, E, \Omega) \\
\geq\ & \lambda(z_0^*, E, \Omega^*) - \frac{2}{\pi}\ln 2 - \left(\lambda(z_0, E, \Omega) - \frac{3}{2\pi}\ln 2\right) \\
=\ & \lambda(z_0^*, E, \Omega^*) - \lambda(z_0, E, \omega) - \frac{1}{2\pi}\ln 2 \\
\geq\ & -N - \frac{1}{2\pi}\ln 2.
\end{aligned}$$

Case 2: $2\arcsin\frac{1}{L} \leq \alpha(f_1(E)) \leq 2\pi - 2\arcsin\frac{1}{L}$. In this case, it follows that

$$\begin{aligned}
|f_2(\widetilde{x}) - f_2(\widetilde{y})| &= |f_2 \circ f_1^{-1}(x) - f_2 \circ f_1^{-1}(y)| \\
&= |h(x) - h(y)| \\
&\geq \frac{1}{L}|x - y| = 2\frac{1}{L}\sin\frac{\alpha(f_1(E))}{2} \\
&\geq 2\frac{1}{L}\sin\left(\frac{1}{2}\cdot 2\arcsin\frac{1}{L}\right) \\
&= \frac{2}{L^2},
\end{aligned}$$

which implies that

$$2\arcsin\frac{1}{L^2} \leq \alpha(f_2(E)) \leq 2\pi - 2\arcsin\frac{1}{L^2}.$$

By the monotonicity of reduced extremal length, we have

$$\begin{aligned}
&\delta(z_0^*, E, \Omega^*) - \delta(z_0, E, \Omega) \\
&= \delta(\infty, f_2(E), \mathbb{D}^*) - \delta(0, f_1(E), \mathbb{D}) \\
&\leq -\frac{1}{\pi}\ln\sin\frac{2\arcsin\frac{1}{L^2}}{4} - \left(-\frac{1}{\pi}\ln\sin\frac{2\pi - 2\arcsin\frac{1}{L}}{4}\right) \\
&= \frac{1}{\pi}\ln\frac{\cos\frac{\arcsin\frac{1}{L}}{2}}{\sin\frac{\arcsin\frac{1}{L^2}}{2}}
\end{aligned}$$

and

$$\begin{aligned}
&\delta(z_0^*, E, \Omega^*) - \delta(z_0, E, \Omega) \\
&= \delta(\infty, f_2(E), \mathbb{D}^*) - \delta(0, f_1(E), \mathbb{D}) \\
&\geq -\frac{1}{\pi}\ln\sin\frac{2\pi - 2\arcsin\frac{1}{L^2}}{4} - \left(-\frac{1}{\pi}\ln\sin\frac{2\arcsin\frac{1}{L}}{4}\right) \\
&= \frac{1}{\pi}\ln\frac{\sin\frac{\arcsin\frac{1}{L}}{2}}{\cos\frac{\arcsin\frac{1}{L^2}}{2}}.
\end{aligned}$$

Case 3: $\alpha(f_1(E)) > 2\pi - 2\arcsin\frac{1}{L}$. In this case, denote the complement of $f_1(E)$ and $f_2(E)$ by $\widetilde{f_1(E)}$ and $\widetilde{f_2(E)}$, respectively. Same reason as the claim in case 1 induces that $\widetilde{f_2(E)}$ can not overlap half of the unit circle. That means

$$\alpha(f_2(E)) > \pi.$$

By the monotonicity of reduced extremal length, we get

$$\begin{aligned}
&\delta(z_0^*, E, \Omega^*) - \delta(z_0, E, \Omega) \\
&= \delta(\infty, f_2(E), \mathbb{D}^*) - \delta(0, f_1(E), \mathbb{D}) \\
&< -\frac{1}{\pi}\ln\sin\frac{\pi}{4} - 0 \\
&= \frac{1}{2\pi}\ln 2
\end{aligned}$$

and

$$\begin{aligned}
&\delta(z_0^*, E, \Omega^*) - \delta(z_0, E, \Omega) \\
&= \delta(\infty, f_2(E), \mathbb{D}^*) - \delta(0, f_1(E), \mathbb{D}) \\
&> 0 - \left(-\frac{1}{\pi}\ln\sin\frac{\pi}{4}\right) \\
&= -\frac{1}{2\pi}\ln 2.
\end{aligned}$$

Taking

$$C = \max\left\{N + \frac{1}{2\pi}\log 2, \quad \frac{1}{\pi}\left|\log\frac{\cos\frac{\arcsin\frac{1}{L}}{2}}{\sin\frac{\arcsin\frac{1}{L^2}}{2}}\right|\right\},$$

we have

$$|\delta(z_0^*, E, \Omega^*) - \delta(z_0, E, \Omega)| \leq C.$$

(4)⇒(1): For any $x, y \in \partial\mathbb{D}$, let E be the arc connecting x and y which does not overlap half of the unit circle. Since the sewing homeomorphism h is continuous, we can take r_0 sufficiently small, such that when $|x - y| < r_0$, $h(E)$ does not overlap half of the unit circle. Suppose the central angle with respect to E in $\mathbb{D}$ is α, the central angle with respect to $h(E)$ in $\mathbb{D}$ is β. We consider two cases.

Case 1: $|x - y| \geq r_0$. In this case, it follows that

$$|h(x) - h(y)| \leq 2 \leq \frac{2}{r_0}|x - y|.$$

Case 2: $|x - y| < r_0$. In this case, we have

$$\begin{aligned}
&|\delta(\infty, h(E), \mathbb{D}^*) - \delta(0, E, \mathbb{D})| \\
&= \left|-\frac{1}{\pi}\log\sin\frac{\beta}{4} - \left(-\frac{1}{\pi}\log\sin\frac{\alpha}{4}\right)\right| \\
&= \left|\frac{1}{\pi}\log\frac{\sin\frac{\beta}{4}}{\sin\frac{\alpha}{4}}\right|.
\end{aligned}$$

Taking into account of condition (4), one can deduce that there exists a constant C, such that

$$\frac{1}{C} \leq \frac{\sin\frac{\beta}{4}}{\sin\frac{\alpha}{4}} \leq C.$$

Since r_0 is sufficiently small, this implies that α and β are both sufficiently small. Then we have

$$\cos\frac{\alpha}{4} \geq \frac{1}{2}, \quad \cos\frac{\beta}{4} \geq \frac{1}{2}.$$

Therefore, we can deduce the following inequalities

$$2\sin\frac{\beta}{4} \leq |h(y) - h(x)| = 2\sin\frac{\beta}{2} \leq 4\sin\frac{\beta}{4}$$

and

$$2\sin\frac{\alpha}{4} \leq |x - y| = 2\sin\frac{\alpha}{2} \leq 4\sin\frac{\alpha}{4}.$$

By the above two inequalities, one can easily get

$$\frac{\sin\frac{\beta}{4}}{2\sin\frac{\alpha}{4}} \leq \frac{|h(y) - h(x)|}{|x - y|} \leq \frac{2\sin\frac{\beta}{4}}{\sin\frac{\alpha}{4}},$$

which implies

$$\frac{1}{2C} \leq \frac{1}{2}\frac{\sin\frac{\beta}{4}}{\sin\frac{\alpha}{4}} \leq \frac{|h(y) - h(x)|}{|y - x|} \leq 2\frac{\sin\frac{\beta}{4}}{\sin\frac{\alpha}{4}} \leq 2C.$$

Combining case 1 and case 2, we get that h is a bi-Lipschitz homeomorphism. $\square$

Chapter 4

Equivalent conditions for bi-Hölder sewing homeomorphism

To follow up on Theorem 3.2.1, one may ask the the following question. What can one say about the sewing homeomorphism h_Ω, if in the condition (3) or (4) of Theorem 3.2.1, the difference of the corresponding extremal distances or reduced extremal distance in Ω and in Ω^* is replaced by their ratio? In this chapter, we answer this and other questions by establishing equivalent conditions for h_Ω to be bi-Hölder.

4.1 Equivalent conditions for h_Ω to be bi-Hölder homeomorphism

Lemma 3.1.2 gives us the connection between extremal distance and harmonic measure. Theorem 2.3.1 tells us extremal distance and reduced extremal are comparable. Therefore the reduced extremal distance and harmonic measure should also be comparable. The following lemma shows us that we can estimate the harmonic measure by using the reduced extremal distance.

Lemma 4.1.1. *[13, p164] If Ω is a Jordan domain and if E is a finite union of* *closed arcs on $\partial\Omega$, then*

$$\omega(z_0, E, \Omega) \le e^{-\pi\delta(z_0,E,\Omega)}.$$

If E is a single arc, then

$$\omega(z_0, E, \Omega) = \frac{2}{\pi}\sin^{-1}(e^{-\pi\delta(z_0,E,\Omega)} \ge \frac{2}{\pi}e^{-\pi\delta(z_0,E,\Omega)}.$$

Theorem 4.1.2. *Suppose Ω is a Jordan domain with $z_0 \in \Omega$, and $z_0^* \in \Omega^* = \overline{\mathbb{C}}\backslash\overline{\Omega}$. Let f_1 and f_2 be Riemann mappings from Ω and Ω^* onto $\mathbb{D}$ and $\mathbb{D}^*$ with $f_1(z_0) = 0$ and $f_2(z_0^*) = \infty$, respectively. Let $h = f_2 \circ f_1^{-1}$ be the sewing homeomorphism induced by Ω. Then the following conditions are equivalent:*

(1) h is a bi-Hölder homeomorphism, that is, there exist constants $L \ge 1$, $0 < \beta \le 1$ such that

$$|h(y) - h(x)| \le L|y - x|^\beta \text{ and } |h^{-1}(y) - h^{-1}(x)| \le L|y - x|^\beta$$

for any $x, y \in \partial\mathbb{D}$.

(2) There exist constants $M \ge 1$ and $0 < \alpha \le 1$, such that

$$\omega(z_0^*, E, \Omega^*) \le M\omega^\alpha(z_0, E, \Omega) \text{ and } \omega(z_0, E, \Omega) \le M\omega^\alpha(z_0^*, E, \Omega^*)$$

for any subarc $E \subset \partial\Omega$.

(3) There exists a constant $N \ge 1$, such that

$$\frac{1}{N} \le \frac{\lambda(z_0^*, E, \Omega^*)}{\lambda(z_0, E, \Omega)} \le N$$

for any subarc $E \subset \partial\Omega$.

(4) There exists a constant $C \geq 1$, such that

$$\frac{1}{C} \leq \frac{\delta(z_0^*, E, \Omega^*)}{\delta(z_0, E, \Omega)} \leq C$$

for any subarc $E \subset \partial\Omega$ provided that $\delta(z_0, E, \Omega) \geq \frac{1}{2\pi} \log 2$.

4.2 Proof of Theorem 4.1.2

For the proof of Theorem 4.1.2, we deploy the same tools such as Theorem 2.2.1 and Lemma 3.1.2 as in the proof of Theorem 3.2.1. But the structure is some what different from the proof of Theorem 3.2.1.

Proof. (1) $\Rightarrow$ (2): Let L and β be as in condition (1). Fix any subarc $E \subset \partial\Omega$. We will divide into three cases:

Case 1: we first assume that both harmonic measures $w(z_0, E, \Omega)$ and $w(z_0^*, E, \Omega^*)$ are less than or equal to $\frac{1}{2}$:

$$w(z_0, E, \Omega) \leq \frac{1}{2}, \quad w(z_0^*, E, \Omega^*) \leq \frac{1}{2}.$$

In this case we have both

$$l(f_1(E)) \leq \pi \quad \textit{and} \quad l(f_2(E)) \leq \pi.$$

Thus condition (1) together with some elemetary estimates yields that

$$\begin{aligned} \frac{w(z_0^*, E, \Omega^*)}{w^\beta(z_0, E, \Omega)} = \frac{\frac{1}{2\pi} l(f_2(E))}{(\frac{1}{2\pi})^\beta l^\beta(f_1(E))} &\leq \frac{\frac{1}{2\pi}\frac{\pi}{2}|f_2(E)|}{(\frac{1}{2\pi})^\beta |f_1(E)|^\beta} \\ &\leq \frac{(2\pi)^\beta L}{4}, \end{aligned}$$

and, by symmetry, it follows that

$$\frac{w(z_0, E, \Omega)}{w^\beta(z_0^*, E, \Omega^*)} = \frac{\frac{1}{2\pi} l(f_1(E))}{(\frac{1}{2\pi})^\beta l^\beta(f_2(E))} \leq \frac{\frac{1}{2\pi}\frac{\pi}{2}|f_1(E)|}{(\frac{1}{2\pi})^\beta |f_2(E)|^\beta}$$
$$\leq \frac{(2\pi)^\beta L}{4}.$$

This verifies condition (2).

Case 2: we assume that one harmonic measure is less than or equal to $\frac{1}{2}$ and the other is greater than $\frac{1}{2}$:

$$w(z_0, E, \Omega) \leq \frac{1}{2}, \quad w(z_0^*, E, \Omega^*) > \frac{1}{2}.$$

In this case, one can choose a subarc $E_0 \subset E$ such that $w(z_0^*, E_0, \Omega^*) = \frac{1}{2}$. Then we have

$$w(z_0, E_0, \Omega) \leq \frac{1}{2}, \quad w(z_0^*, E_0, \Omega^*) = \frac{1}{2}.$$

Applying the above case to E_0, one can conclude that

$$\frac{w(z_0^*, E, \Omega^*)}{w^\beta(z_0, E, \Omega)} \leq \frac{2w(z_0^*, E_0, \Omega^*)}{w^\beta(z_0, E_0, \Omega)} = \frac{2\frac{1}{2\pi} l(f_2(E_0))}{[\frac{1}{2\pi} l(f_1(E_0))]^\beta}$$
$$\leq \frac{\frac{\pi}{2\pi}|f_2(E_0)|}{(\frac{1}{2\pi})^\beta |f_1(E_0)|^\beta}$$
$$\leq \frac{(2\pi)^\beta L}{2}.$$

Furthermore, it follows immediately that

$$\frac{\omega(z_0, E, \Omega)}{\omega^\beta(z_0^*, E, \Omega^*)} \leq \frac{1/2}{(1/2)^\beta} = \frac{2^\beta}{2}.$$

Thus condition (2) is verified in this case.

Case 3: we assume both harmonic measures $\omega(z_0, E, \Omega)$ and $\omega(z_0^*, E, \Omega^*)$ are greater than $\frac{1}{2}$:

$$w(z_0, E, \Omega) > \frac{1}{2}, \quad w(z_0^*, E, \Omega^*) > \frac{1}{2}.$$

In this case, it follows immediately that

$$\frac{\omega(z_0, E, \Omega)}{\omega^\beta(z_0^*, E, \Omega^*)} \le \frac{1}{(1/2)^\beta} = 2^\beta,$$

$$\frac{w(z_0^*, E, \Omega^*)}{w^\beta(z_0, E, \Omega)} \le \frac{1}{(1/2)^\beta} = 2^\beta.$$

Taking $M = \pi L$, $\alpha = \beta$, then the condition (2) is verified.

(2) $\Rightarrow$ (1): Fix $x, y \in \partial\mathbb{D}$. Let $E \subset \partial\Omega$ denote the image of the smaller component of $\partial\mathbb{D}\backslash\{x, y\}$ under f_1^{-1}. Then $f_1(E)$ is a subarc on $\partial\mathbb{D}$ joining x and y with central angle less than or equal to π. We note that

$$l(f_1(E)) = 2\pi w(z_0, E, \Omega) \le \pi,$$

and

$$l(f_2(E)) = 2\pi w(z_0^*, E, \Omega^*).$$

Thus the first inequality in condition (2) can be written as

$$\frac{l(f_2(E))}{2\pi} \le M\left(\frac{l(f_1(E))}{2\pi}\right)^\alpha.$$

Therefore, routine estimates yield that

$$\begin{aligned}|h(y) - h(x)| = |f_2(E)| \le l(f_2(E)) &\le \frac{2\pi M}{(2\pi)^\alpha} l^\alpha(f_1(E))\\ &\le \frac{2\pi M}{(2\pi)^\alpha}\left(\frac{\pi}{2}|f_1(E)|\right)^\alpha\\ &= \frac{2\pi M}{4^\alpha}|f_1(E)|^\alpha\\ &= \frac{2\pi M}{4^\alpha}|y - x|^\alpha.\end{aligned}$$

Thus we have

$$|h(y) - h(x)| \le \frac{2\pi M}{4^\alpha}|y - x|^\alpha.$$

By symmetry, applying the above argument to h^{-1}, we obtain that

$$|h^{-1}(y) - h^{-1}(x)| \leq \frac{2\pi M}{4^\alpha}|y-x|^\alpha.$$

This verifies condition (1) with $L = 2\pi M$ and $\beta = \alpha$.

(3) ⇒ (2): By applying Lemma 3.1.2 repeatedly, one can deduce that

$$\begin{aligned}\omega(z_0, E, \Omega) \leq \frac{8}{\pi}e^{-\pi\lambda(z_0,E,\Omega)} &\leq \frac{8}{\pi}e^{-\frac{\pi}{N}\lambda(z_0^*,E,\Omega^*)}\\ &\leq \frac{8}{\pi}\omega^{\frac{1}{N}}(z_0^*, E, \Omega^*).\end{aligned}$$

By symmetry,

$$\omega(z_0^*, E, \Omega^*) \leq \frac{8}{\pi}\omega^{\frac{1}{N}}(z_0, E, \Omega).$$

Thus condition (2) holds with

$$M = \frac{8}{\pi}, \quad \alpha = \frac{1}{N}.$$

(2) ⇒ (3): The proof of this implication is more sophisticated than the others. Using Lemma 3.1.2 again, we obtain that

$$\begin{aligned}e^{-\pi\lambda(z_0^*,E,\Omega^*)} \leq \omega(z_0^*, E, \Omega^*) &\leq M\omega^\alpha(z_0, E, \Omega)\\ &\leq M(\frac{8}{\pi})^\alpha e^{-\pi\alpha\lambda(z_0,E,\Omega)}.\end{aligned}$$

It follows that

$$-\pi\lambda(z_0^*, E, \Omega^*) \leq \ln(M(\frac{8}{\pi})^\alpha) - \pi\alpha\lambda(z_0, E, \Omega).$$

Divide both sides by $\pi\alpha\lambda(z_0^*, E, \Omega^*)$,

$$-\frac{1}{\alpha} \leq \frac{\ln(M(\frac{8}{\pi})^\alpha)}{\pi\alpha}\frac{1}{\lambda(z_0^*, E, \Omega^*)} - \frac{\lambda(z_0, E, \Omega)}{\lambda(z_0^*, E, \Omega^*)}.$$

Rearrange the inequality, get

$$\frac{\lambda(z_0, E, \Omega)}{\lambda(z_0^*, E, \Omega^*)} \leq \frac{1}{\alpha} + \frac{\ln(M(\frac{8}{\pi})^\alpha)}{\pi\alpha} \frac{1}{\lambda(z_0^*, E, \Omega^*)}. \tag{4.2.1}$$

Choose subarc $E_0 \subset \partial\Omega$ such that

$$l(f_1(E_0)) = \frac{2\pi}{(2M)^{\frac{1}{\alpha}}},$$

and let

$$l_0 = l(f_1(E_0)), \quad \lambda_0 = \lambda(z_0, E_0, \Omega).$$

To verify condition (3), we divide the argument into two cases.

Case 1: we assume that $\lambda(z_0^*, E, \Omega^*) \geq \frac{1}{4\lambda_0}$. Then it follows immediately from (4.2.1) that

$$\frac{\lambda(z_0, E, \Omega)}{\lambda(z_0^*, E, \Omega^*)} \leq \frac{1}{\alpha} + \frac{4\lambda_0 \ln(M(\frac{8}{\pi})^\alpha)}{\pi\alpha}. \tag{4.2.2}$$

Case 2: we assume that $\lambda(z_0^*, E, \Omega^*) < \frac{1}{4\lambda_0}$.

By Theorem 2.2.1,

$$\lambda(z_0^*, \widetilde{E}, \Omega^*) = \frac{\frac{1}{4}}{\lambda(z_0^*, E, \Omega^*)} > \frac{\frac{1}{4}}{\frac{1}{4\lambda_0}} = \lambda_0,$$

where $\widetilde{E}$ is the complement of E on $\partial\Omega$.

Since the extremal distance is strictly decreasing and by the choice of λ_0, it follows that

$$l(f_2(\widetilde{E})) \leq \frac{2\pi}{(2M)^{\frac{1}{\alpha}}}.$$

Therefore

$$\begin{aligned} l(f_1(\widetilde{E})) = 2\pi\omega(z_0, \widetilde{E}, \Omega) &\leq 2\pi M \omega^\alpha(z_0^*, \widetilde{E}, \Omega^*) \\ &= 2\pi \cdot M \cdot \left(\frac{l(f_2(\widetilde{E}))}{2\pi}\right)^\alpha \\ &\leq 2\pi \cdot M \cdot \left(\frac{1}{(2M)^{\frac{1}{\alpha}}}\right)^\alpha = \pi. \end{aligned}$$

This together with Theorem 2.2.1 implies

$$\lambda(z_0, \widetilde{E}, \Omega) \geq \frac{1}{2}.$$

Hence, we deduce that

$$\begin{aligned}\frac{\lambda(z_0, E, \Omega)}{\lambda(z_0^*, E, \Omega^*)} &= \frac{\lambda(z_0^*, \widetilde{E}, \Omega^*)}{\lambda(z_0, \widetilde{E}, \Omega)} \\ &\leq \frac{1}{\alpha} + \frac{\ln(M(\frac{8}{\pi})^\alpha)}{\pi\alpha}\frac{1}{\lambda(z, \widetilde{E}, \Omega)} \\ &\leq \frac{1}{\alpha} + \frac{2\ln(M(\frac{8}{\pi})^\alpha)}{\pi\alpha}.\end{aligned}$$

By symmetry again, this together with (4.2.2) shows that condition (3) in Theorem 4.1.2 holds with the constant N determined by

$$N = \max\left\{\frac{1}{\alpha} + \frac{2\ln(M(\frac{8}{\pi})^\alpha)}{\pi\alpha}, \frac{1}{\alpha} + \frac{4\lambda_0\ln(M(\frac{8}{\pi})^\alpha)}{\pi\alpha}\right\}.$$

(4) $\Rightarrow$ (2): For the proof of the equivalence of condition (4) and condition (2), we use the relation between harmonic measure and reduced extremal distance in Lemma 4.1.1:

$$\frac{2}{\pi}e^{-\pi\delta(z_0,E,\Omega)} \leq \omega(z_0, E, \Omega) \leq e^{-\pi\delta(z_0,E,\Omega)}, \tag{4.2.3}$$

for any single subarc $E \subset \partial\Omega$.

Fix a subarc $E \subset \partial\Omega$ and denote the central angle of $f_1(E)$ by θ. To verify condition (2), we divide into two cases.

Case 1: we assume that $\theta \leq \pi$. By Lemma 1.3.4, we get

$$\begin{aligned}\delta(z_0, E, \Omega) &= \delta(0, f_1(E), \mathbb{D}) \\ &= \lim_{r\to 0}[\lambda(\Delta_r, f_1(E), \mathbb{D}) - \lambda(\Delta_r, \partial\mathbb{D}, \mathbb{D})] \\ &= \lim_{r\to 0}\{\frac{1}{\pi}\ln\frac{1}{\sqrt{r}\sin\frac{\theta}{4}} - \frac{1}{2\pi}\ln\frac{1}{r}\} \\ &= -\frac{1}{\pi}\ln\sin\frac{\theta}{4}\end{aligned}$$

Since $\theta \leq \pi$, it's easy to get

$$\delta(z_0, E, \Omega) \geq \frac{1}{2\pi} \log 2.$$

Thus it follows from (4.2.3) and condition (4) that

$$\begin{aligned} \omega(z_0^*, E, \Omega^*) \leq e^{-\pi\delta(z_0^*, E, \Omega^*)} &\leq \big(e^{-\pi\delta(z_0, E, \Omega)}\big)^{\frac{1}{C}} \\ &\leq \big(\frac{\pi}{2}\omega(z_0, E, \Omega)\big)^{\frac{1}{C}}. \end{aligned}$$

By symmetry, we have

$$\omega(z_0, E, \Omega) \leq \big(\frac{\pi}{2}\big)^{\frac{1}{C}} (\omega(z_0^*, E, \Omega^*))^{\frac{1}{C}}.$$

This verifies condition (2) with $M = (\pi/2)^{\frac{1}{C}}$ and $\alpha = 1/C$.

Case 2: we assume that $\theta > \pi$. Applying the above argument to the complement $\tilde{E}$ of E yields that

$$\omega(z_0^*, \tilde{E}, \Omega^*) \leq \big(\frac{\pi}{2}\big)^{\frac{1}{C}} (\omega(z_0, \tilde{E}, \Omega))^{\frac{1}{C}} \leq \big(\frac{\pi}{4}\big)^{\frac{1}{C}}.$$

Thus

$$\omega(z_0^*, E, \Omega^*) = 1 - \omega(z_0^*, \tilde{E}, \Omega^*) \geq 1 - \big(\frac{\pi}{4}\big)^{\frac{1}{C}}.$$

Hence, it follows that

$$\omega(z_0, E, \Omega) \leq 1 \leq M_1 \cdot (\omega(z_0^*, E, \Omega^*))^{\frac{1}{C}},$$

where

$$M_1 = \big[\frac{4^{1/C}}{4^{1/C} - \pi^{1/C}}\big]^{\frac{1}{C}}.$$

On the other hand, $\theta > \pi$ implies $\omega(z_0, E, \Omega) > \frac{1}{2}$, we obtain that

$$\omega(z_0^*, E, \Omega^*) \leq 1 \leq 2^{\frac{1}{C}} (\omega(z_0, E, \Omega))^{\frac{1}{C}}.$$

This completes the verification of condition (2) with constants

$$M = \max\{M_1,\ 2^{\frac{1}{C}}\},\quad \alpha = \frac{1}{C}.$$

(2) $\Rightarrow$ (4): To derive condition (4), we shall use both condition (2) and condition (3) since they are equivalent as shown above. Fix a subarc $E \subset \partial\Omega$ such that the central angle θ of $f_1(E)$ is less than or equal to π. Using (4.2.3) and condition (2), one can deduce that

$$\begin{aligned} e^{-\pi\delta(z_0^*,E,\Omega^*)} \leq \frac{\pi}{2}\omega(z_0^*, E, \Omega^*) &\leq \frac{\pi M}{2}\omega(z_0, E, \Omega)^{\alpha} \\ &\leq \frac{\pi M}{2}e^{-\pi\alpha\delta(z_0,E,\Omega)}. \end{aligned}$$

Taking the natural log of both sides

$$-\pi\delta(z_0^*, E, \Omega^*) \leq \ln(\frac{\pi M}{2}) - \pi\alpha\delta(z_0, E, \Omega).$$

Thus it follows that

$$\frac{\delta(z_0, E, \Omega)}{\delta(z_0^*, E, \Omega^*)} \leq \frac{1}{\alpha} + \frac{\ln(\frac{\pi M}{2})}{\pi\alpha} \cdot \frac{1}{\delta(z_0^*, E, \Omega^*)}. \tag{4.2.4}$$

Furthermore, since $\theta \leq \pi$, we have $\lambda(z_0, E, \Omega) \geq \frac{1}{2}$ by Theorem 2.2.1.
Thus condition (3) implies that

$$\lambda(z_0^*, E, \Omega^*) \geq \frac{\lambda(z_0, E, \Omega)}{N} \geq \frac{1}{2N}.$$

Since the reduced extremal distance is decreasing, which yields that

$$\delta(z_0^*, E, \Omega^*) \geq \delta_0 = -\frac{1}{\pi}\log\sin\frac{\theta_0}{4},$$

where the angle θ_0 is determined by

$$2\mu(\sin\frac{\theta_0}{4}) = \frac{1}{2N}.$$

Taking

$$C = \frac{1}{\alpha} + \frac{\ln(\frac{\pi M}{2})}{\pi\alpha} \cdot \frac{1}{\delta_0}.$$

This together with (4.2.4) yields that

$$\frac{\delta(z_0, E, \Omega)}{\delta(z_0^*, E, \Omega^*)} \leq \frac{1}{\alpha} + \frac{\ln(\frac{\pi M}{2})}{\pi\alpha} \cdot \frac{1}{\delta_0} = C.$$

Taking the reciprocal, get

$$\frac{\delta(z_0^*, E, \Omega^*)}{\delta(z_0, E, \Omega)} \geq \frac{1}{C},$$

which is the first inequality in condition (4).

To prove the second inequality in condition (4), we note that, by the symmetrical nature of (4.2.3) and condition (2), the same argument as in the proof of (4.2.4) yields that

$$\frac{\delta(z_0^*, E, \Omega^*)}{\delta(z_0, E, \Omega)} \leq \frac{1}{\alpha} + \frac{\ln(\frac{\pi M}{2})}{\pi\alpha} \cdot \frac{1}{\delta(z_0, E, \Omega)}. \tag{4.2.5}$$

Since $\theta \leq \pi$ implies that $\delta(z_0, E, \Omega) \geq \frac{1}{2\pi}\ln 2$, (4.2.5) shows that the second inequality in condition (4) holds with the constant

$$C = \frac{1}{\alpha} + \frac{2\ln(\frac{\pi M}{2})}{\alpha \ln 2}.$$

This completes the proof of Theorem 4.1.2.

Remark: It is worth noting that it is necessary to assume in condition (4) that the central angle θ of $f_1(E)$ is less than or equal to π. This is because when $\theta \to 2\pi$ condition (2) or (3) may not imply condition (4). □

Chapter 5

Harmonic measure property and quasicircle

In chapter 3 and 4, we established equivalent conditions for the sewing homeomorphism h_Ω of a Jordan domain to be bi-Lipschitz or bi-Hölder. Apparently, all bi-Lipschitz homeomorphisms are bi-Hölder and the converse is not true. However, there is an important class of homeomorphisms between these two, called *quasisymmetric* (or QS) homeomorphisms. It is well known that (see *[15]*, *[16]*, *[17]* *[19]*, *[22]* and *[27]*) the sewing homeomorphism h_Ω is quasisymmetric if and only if Ω is a quasidisk or, equivalently, $\partial\Omega$ is a quasicircle. In this chapter, we will establish some parallel equivalent conditions for Ω to be a quasidisk and give counterexamples to show that Theorem 3.2.1 condition (1) and Theorem 4.1.2 condition (1) are not equivalent to the condition that Ω is a quasicircle.

5.1 Preliminary

In this section, we will name two important properties. After that some definitions and lemmas about quasisymmetric and quasiconformal maps will be given. More details about quasiconformal maps can be found in *[1]*, *[3]*, *[16]* and *[21]*.

Definition 5.1.1. *A Jordan curve $J = \partial\Omega$ is said to have the harmonic measure property (or HMP) if there exist points $z_0 \in \Omega$, $z_0^* \in \Omega^*$, and a constant $M < \infty$,* *such that for any arc* $E \subset J$,

$$\frac{1}{M}\omega(z_0, E, \Omega) \leq \omega(z_0^*, E, \Omega^*) \leq M\omega(z_0, E, \Omega).$$

J is said to have the Hölder harmonic measure property (or Hölder HMP) if there exist points $z_0 \in \Omega$, $z_0^* \in \Omega^*$, and constants $0 < \alpha \leq 1$, $M < \infty$, such that for any arc $E \subset J$,

$$\frac{1}{M}\omega^{\frac{1}{\alpha}}(z_0, E, \Omega) \leq \omega(z_0^*, E, \Omega^*) \leq M\omega^{\alpha}(z_0, E, \Omega).$$

Definition 5.1.2. *[23, p109] A map h that maps $\partial\mathbb{D}$ into $\mathbb{C}$ is called quasisymmetric if it is injective and if there exists a strictly increasing continuous function $\lambda(x)$ $(0 \leq \lambda(x) < \infty)$ with $\lambda(0) = 0$ such that*

$$\left|\frac{h(z_1) - h(z_2)}{h(z_2) - h(z_3)}\right| \leq \lambda\left(\left|\frac{z_1 - z_2}{z_2 - z_3}\right|\right),$$

for all distinct z_1, z_2, $z_3 \in \partial\mathbb{D}$.

Lemma 5.1.3. *(Mori's Theorem) [1, p30] Let $f : \mathbb{D} \to \mathbb{D}$ be a K quasiconformal mapping, normalized by $f(0) = 0$. Then for any z_1, $z_2 \in \mathbb{D}$,*

$$|f(z_1) - f(z_2)| \leq 16|z_1 - z_2|^{\frac{1}{K}}.$$

and 16 cannot be replaced by a smaller constant.

Lemma 5.1.4. *[23, p110] If f and g map $\mathbb{D}$ and $\mathbb{D}^*$ conformally onto the inner and outer domains of a quasicircle, then $\phi = g^{-1} \circ f$ is a quasisymmetric map of $\partial\mathbb{D}$ onto $\partial\mathbb{D}$.*

Lemma 5.1.5. *[23, p112] Any sense-preserving homeomorphism ϕ of $\partial\mathbb{D}$ onto $\partial\mathbb{D}$ can be extend to a homeomorphism $\tilde{\phi}$ of $\overline{\mathbb{D}}$ onto $\overline{\mathbb{D}}$ that is real-analytic in $\mathbb{D}$ and has the following properties:*

(1) If σ, $\tau \in$ Möb($\mathbb{D}$) then the extension of $\sigma \circ \phi \circ \tau$ is given by $\sigma \circ \tilde{\phi} \circ \tau$.

(2) If ϕ is quasisymmetric then $\tilde{\phi}$ is quasiconformal in $\mathbb{D}$.

Lemma 5.1.6. *[18] Let J be a Jordan curve in the extended plane and let Ω, Ω^* be its complementary domains. The curve J is a quasicircle if and only if there exists two points $z_0 \in \Omega$, $z_0^* \in \Omega^*$ and a constant $M \geq 1$ such that for any two adjacent disjoint open subarcs E_1, E_2 of J such that*

$$\omega(z_0, E_1, \Omega) = \omega(z_0, E_2, \Omega)$$

we have

$$\frac{\omega(z_0^*, E_1, \Omega^*)}{\omega(z_0^*, E_2, \Omega^*)} \leq M.$$

5.2 Equivalent conditions for a quasicircle

In this section, we establish some parallel equivalent conditions for Ω to be a quasidisk.

Theorem 5.2.1. *Suppose Ω is a Jordan domain and let $\Omega^* = \overline{\mathbb{C}} \backslash \overline{\Omega}$. Then the following conditions are equivalent:*

(1) $J = \partial\Omega$ is a quasicircle;

(2) There exist $z_0 \in \Omega$, $z_0^* \in \Omega^*$ and a constant M such that

$$\frac{\omega(z_0^*, E_1, \Omega^*)}{\omega(z_0^*, E_2, \Omega^*)} \leq M$$

for any two adjacent disjoint arcs $E_1, E_2 \subset J$ with $\omega(z_0, E_1, \Omega) = \omega(z_0, E_2, \Omega)$;

(3) There exist $z_0 \in \Omega, z_0^* \in \Omega^*$ and a constant N such that

$$|\lambda(z_0^*, E_2, \Omega^*) - \lambda(z_0^*, E_1, \Omega^*)| \leq N$$

for any two adjacent disjoint arcs $E_1, E_2 \subset J$ with $\lambda(z_0, E_1, \Omega) = \lambda(z_0, E_2, \Omega)$;

(4) There exist $z_0 \in \Omega, z_0^* \in \Omega^*$ and a constant C such that

$$|\delta(z_0^*, E_2, \Omega^*) - \delta(z_0^*, E_1, \Omega^*)| \leq C$$

for any two adjacent disjoint arcs $E_1, E_2 \subset J$ with $\delta(z_0, E_1, \Omega) = \delta(z_0, E_2, \Omega)$.

5.3 Proof of Theorem 5.2.1

Proof. Lemma 5.1.6 establishes the equivalence of conditions (1) and (2). Next we prove the equivalence of conditions (2) and (3).

(2) $\Rightarrow$ (3): Assume that condition (2) holds with $z_0 \in \Omega$, $z_0^* \in \Omega^*$ and a constant M. By Lemma 3.1.2, it follows from condition (2) that

$$\begin{aligned} e^{-\pi\lambda(z_0^*, E_1, \Omega^*)} &\leq \omega(z_0^*, E_1, \Omega^*) \\ &\leq M\omega(z_0^*, E_2, \Omega^*) \\ &\leq M\frac{8}{\pi}e^{-\pi\lambda(z_0^*, E_2, \Omega^*)}. \end{aligned}$$

Taking the natural log on both sides leads to

$$-\pi\lambda(z_0^*, E_1, \Omega^*) \leq \ln(\frac{8M}{\pi}) - \pi\lambda(z_0^*, E_2, \Omega^*),$$

which yields that

$$\lambda(z_0^*, E_2, \Omega^*) - \lambda(z_0^*, E_1, \Omega^*) \leq \frac{1}{\pi}\ln(\frac{8M}{\pi}).$$

Thus, by symmetry, condition (3) holds with the same points $z_0 \in \Omega$, $z_0^* \in \Omega^*$ as in (2) and constant $N = \frac{1}{\pi}\ln\left(\frac{8M}{\pi}\right)$.

(3) $\Rightarrow$ (2): By condition (3), the inequality

$$|\lambda(z_0^*, E_2, \Omega^*) - \lambda(z_0^*, E_1, \Omega^*)| \leq N$$

holds with $z_0 \in \Omega,\ z_0^* \in \Omega^*$ and a constant N. After some algebraic manipulation, we get

$$-\pi\lambda(z_0^*, E_2, \Omega^*) \leq \pi N - \pi\lambda(z_0^*, E_1, \Omega^*).$$

Thus

$$e^{-\pi\lambda(z_0^*, E_2, \Omega^*)} \leq e^{\pi N} e^{-\pi\lambda(z_0^*, E_1, \Omega^*)}.$$

By Lemma 3.1.2, we get

$$\begin{aligned}\frac{\pi}{8}\omega(z_0^*, E_2, \Omega^*) &\leq e^{-\pi\lambda(z_0^*, E_2, \Omega^*)} \\ &\leq e^{\pi N} e^{-\pi\lambda(z_0^*, E_1, \Omega^*)} \\ &\leq e^{\pi N}\omega(z_0^*, E_1, \Omega^*).\end{aligned}$$

Therefore

$$\frac{\omega(z_0^*, E_2, \Omega^*)}{\omega(z_0^*, E_1, \Omega^*)} \leq \frac{8}{\pi} e^{\pi N},$$

this implies condition (2) with constant $M = \frac{8}{\pi} e^{\pi N}$.

Now we are going to establish the equivalence of conditions (2) and (4) by using Lemma 4.1.1.

(2) $\Rightarrow$ (4): Assume that condition (2) holds with $z_0 \in \Omega,\ z_0^* \in \Omega^*$ and a constant M. By Lemma 4.1.1, it follows from condition (2) that

$$\begin{aligned}\frac{2}{\pi} e^{-\pi\delta(z_0^*, E_1, \Omega^*)} &\leq \omega(z_0^*, E_1, \Omega^*) \\ &\leq M\omega(z_0^*, E_2, \omega^*) \\ &\leq M e^{-\pi\delta(z_0^*, E_2, \Omega^*)}.\end{aligned}$$

Taking the natural log on both sides yields that

$$-\pi\delta(z_0^*, E_1, \Omega^*) \leq \ln(\frac{\pi M}{2}) - \pi\delta(z_0^*, E_2, \Omega^*),$$

then, we get

$$\delta(z_0^*, E_2, \Omega^*) - \delta(z_0^*, E_1, \Omega^*) \leq \frac{1}{\pi}\ln(\frac{\pi M}{2}).$$

Thus, by symmetry, condition (4) holds with the same points $z_0 \in \Omega,\ z_0^* \in \Omega^*$ as in (2) and constant $C = \frac{1}{\pi}\ln(\frac{\pi M}{2})$.

(4) $\Rightarrow$ (2): By condition (4), we have

$$|\delta(z_0^*, E_2, \Omega^*) - \delta(z_0^*, E_1, \Omega^*)| \leq C,$$

for any two adjacent disjoint arcs $E_1, E_2 \subset J$ with $\delta(z_0, E_1, \Omega) = \delta(z_0, E_2, \Omega)$. After a simple calculation, we obtian that

$$-\pi\delta(z_0^*, E_1, \Omega^*) \leq C\pi - \pi\delta(z_0^*, E_2, \Omega^*).$$

Taking the natural log both sides yields that

$$e^{-\pi\delta(z_0^*, E_1, \Omega^*)} \leq e^{C\pi} e^{-\pi\delta(z_0^*, E_2, \Omega^*)}.$$

By Lemma 4.1.1,

$$\begin{aligned}\omega(z_0^*, E_1, \Omega^*) &\leq e^{-\pi\delta(z_0^*, E_1, \Omega^*)} \\ &\leq e^{C\pi} e^{-\pi\delta(z_0^*, E_2, \Omega^*)} \\ &\leq \frac{\pi}{2} e^{C\pi} \omega(z_0^*, E_2, \Omega^*).\end{aligned}$$

Therefore

$$\frac{\omega(z_0^*, E_1, \Omega^*)}{\omega(z_0^*, E_2, \Omega^*)} \leq \frac{\pi}{2} e^{C\pi},$$

which leads to condition (2) with constant $M = \frac{\pi}{2} e^{C\pi}$. □

5.4 HMP and quasicircle

Roughly speaking, a Jordan curve J has the HMP if the harmonic measures on both sides of the curve "looks alike" from both sides, it is tempting to use HMP to characterize quasicircles. In this direction, we have the following result.

Theorem 5.4.1. *If J has the HMP, then J is a quasicircle.*

Proof. Since J has the HMP, by the definition, there exist points $z_0 \in \Omega$, $z_0^* \in \Omega^*$, and a constant $M < \infty$, such that for any subarc $E \in J$,

$$\frac{1}{M} \le \frac{\omega(z_0^*, E, \Omega^*)}{\omega(z_0, E, \Omega)} \le M.$$

For any two adjacent disjoint arcs E_1, $E_2 \in J$, suppose they have the same harmonic measure from inside

$$\omega(z_0, E_1, \Omega) = \omega(z_0, E_2, \Omega). \tag{5.4.1}$$

By HMP, we get two inequalities:

$$\frac{1}{M} \le \frac{\omega(z_0^*, E_1, \Omega^*)}{\omega(z_0, E_1, \Omega)} \le M, \tag{5.4.2}$$

and

$$\frac{1}{M} \le \frac{\omega(z_0^*, E_2, \Omega^*)}{\omega(z_0, E_2, \Omega)} \le M. \tag{5.4.3}$$

Combine (5.4.1), (5.4.2) and (5.4.3), we obtain that

$$\begin{aligned}\frac{\omega(z_0^*, E_1, \Omega^*)}{\omega(z_0^*, E_2, \Omega^*)} &= \frac{\omega(z_0^*, E_1, \Omega^*)}{\omega(z_0, E_1, \Omega)} \frac{\omega(z_0, E_1, \Omega)}{\omega(z_0, E_2, \Omega)} \frac{\omega(z_0, E_2, \Omega)}{\omega(z_0^*, E_2, \Omega^*)} \\ &\le M^2.\end{aligned}$$

By Theorem 5.2.1, J is a quasicircle. □

Unfortunately, the converse of Theorem 5.4.1 is not true. On the one hand, there are some very complicated quasicircles that may have the harmonic measure property. Such examples can be found in *[4]* and *[25]*. On the other hand, there are some simple quasicircles that do not have this property. The following is such an example.

Example 5.4.2. *The boundary of a quarter plane does not have the HMP.*

Proof. Denote the quarter plane by Ω. For each pair of finite points z_1, z_2 on the boundary, we always have the inequality

$$\min_{j=1,2} dia(\gamma_j) \leq \sqrt{2}|z_1 - z_2|,$$

where *dia* denotes the Euclidean diameter and γ_1, γ_2 are the components of $\partial\Omega \setminus \{z_1, z_2\}$.

It is easy to see that the boundary of the quarter plane satisfies the Ahlfors' two point inequality. Therefore the boundary of the quarter plane is a quasicircle.

Now we show that it does not have the harmonic measure property. Suppose f maps the upper half plane onto the quarter plane and g maps the upper half plane onto the complement of the quarter plane. By the Schwarz-Christoffel formula,

$$f(z) = c_1\sqrt{z} \quad and \quad g(z) = c_2 z^{\frac{3}{2}},$$

where c_1, c_2 are two constants. Let $z_1 = f(i)$, $z_2 = g(i)$. Choose a small interval $[0, \delta]$. We are going to show that

$$\frac{\omega(z_1, [0, \delta], \Omega)}{\omega(z_2, [0, \delta], \Omega^*)} \to 0 \quad or \quad \infty$$

as $\delta \to 0$.

Since f and g are conformal mappings and conformal mapping doesn't change the harmonic measure, we obtian that

$$\frac{\omega(z_1, [0, \delta], \Omega)}{\omega(z_2, [0, \delta], \Omega^*)} = \frac{\omega(i, [0, f^{-1}(\delta)], \mathbb{H})}{\omega(i, [0, g^{-1}(\delta)], \mathbb{H})} = \frac{arctan(f^{-1}(\delta))}{arctan(g^{-1}(\delta))}.$$

By a simple calculation, we get

$$\lim_{\delta \to 0} \frac{arctan(f^{-1}(\delta))}{arctan(g^{-1}(\delta))} = \lim_{\delta \to 0} \frac{arctan(\frac{\delta^2}{c_1^2})}{arctan(\frac{\delta^{\frac{2}{3}}}{c_2^{\frac{2}{3}}})} = \lim_{\delta \to 0} c_3 \delta^{\frac{4}{3}} = 0,$$

where c_3 is a constant.

Therefore, we get

$$\frac{\omega(z_1, [0, \delta], \Omega)}{\omega(z_2, [0, \delta], \Omega^*)} \to 0, \quad as \quad \delta \to 0.$$

This implies that the boundary of the quarter plane does not have the harmonic measure property. □

Combining Lemma 5.1.3 and Theorem 4.1.2, we conclude this chapter by showing that quasicircles have the Hölder HMP.

Theorem 5.4.3. *If J is a quasicircle, then J has the Hölder HMP.*

Proof. If J is a quasicircle, by Lemma 5.1.4, the sewing homeomorphism h is quasisymmetric. By Lemma 5.1.5, h can be extended to a quasiconformal map of $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}}$. By Lemma 5.1.3, we know that it is Hölder continuous, therefore it is a bi-Hölder homeomorphism. By Theroem 4.1.2, h is a bi-Hölder homeomorphism implies that J has Hölder HMP. □

Chapter 6

Characterization of unit circle by using Robin Capacity

In connection with the theory of quasiconformal mappings, characterizations of quasicircles in particular, there are many ways to characterize the unit circle (See *[15]*). For example, a Jordan domain Ω is a quasicircle if and only if its QED constant $M(\Omega)$ is finite, and it is a disk if and only if $M(\Omega) = 2$. This constant can be used to reflect the geometry of a domain (See *[14]*, *[15]*, *[16]*, *[26]* for details). In this chapter, we will use a new index to characterize the unit circle. The idea comes from the observation that if a curve is very "close" to the unit circle, then the ratio of Robin capacity inside and outside the curve is close to 1.

6.1 Robin Function and Robin Capacity

Definition 6.1.1. *[10] Let Ω be a finitely connected domain in the extended complex plane $\bar{\mathbb{C}}$, containing the point at infinity and bounded by smooth Jordan curves. Let $A \subset \partial\Omega$ be an arbitrary closed subset and $B = \partial\Omega \setminus A$. For a fixed point $\xi \in \Omega$, the Robin function $R(z, \xi)$ is defined by the following requirements:*

(a) *$R(z, \xi)$ is harmonic in Ω and continuous in $\bar{\Omega}$ together with its first derivatives, except at $z = \xi$, where $R(z, \xi) + \log|z - \xi|$ is harmonic;*

(b) $R(z,\xi) = 0$ *for all* $z \in A$*;*

(c) $\frac{\partial R}{\partial n}(z,\xi) = 0$*, for all* $z \in B$*, where* $\frac{\partial}{\partial n}$ *denotes the inner normal derivative.*

For $\xi = \infty$*, the property (a) is modified to require* $R(z,\infty) - \log|z|$ *be harmonic in* Ω*.*

From this definition, one can easily see that Robin function is simply Green's function $g(z,\xi)$ on Ω if A is the whole boundary. For more details, see *[10]*, *[11]* and *[12]*.

Definition 6.1.2. *[10] The Robin capacity of* A *at* ξ *with respect to* Ω *is defined as*

$$\sigma(A) = e^{-\rho(A)},$$

where

$$\rho(A) = \lim_{z\to\xi}\{R(z,\xi) + \log|z-\xi|\}.$$

In particular, if $\xi = 0$*, then the Robin capacity of* A *at 0 is given by*

$$\sigma_0(A) = e^{-\rho_0(A)}, \quad \rho_0(A) = \lim_{z\to 0}(R(z,0) + \log|z|).$$

If $\xi = \infty$*, then the Robin capacity of* A *at* ∞ *is given by*

$$\sigma_\infty(A) = e^{-\rho_\infty(A)}, \quad \rho_\infty(A) = \lim_{z\to\infty}(R(z,\infty) - \log|z|).$$

Actually we can investigate Robin capacity from a more geometric viewpoint. One can use extremal distance to interpret Robin capacity. The following lemma builds the connection between Robin capacity and extremal distance. By this lemma, one can express the Robin capacity in terms of extremal distance.

Lemma 6.1.3. *[20, p243]: let* Ω *be a domain bounded by a finite number of closed Jordan curves, and* E *be a closed set consisting of a finite number of arcs on* $\partial\Omega$*,* $z_0 \in \Omega$*. Then*

$$\rho_{z_0}(E) = 2\pi \lim_{r\to 0}(\lambda(E,\Delta_r,\Omega) + \frac{1}{2\pi}\log r),$$

where $\lambda(E, \Delta_r, \Omega)$ *is the extremal distance between* E *and disk* $\Delta_r = \{z; |z - z_0| = r\}$.

6.2 Characterization of unit circle

In this section, we define an index $I(J)$, then use this index to characterize the unit circle.

Lemma 6.2.1. *[8, p130] (Schwarz's Lemma) Let* $D = \{z : |z| < 1\}$ *and suppose* f *is analytic on* D *with* $|f(z)| \leq 1$ *for* $z \in D$ *and* $f(0) = 0$. *Then*

$$|f'(0)| \leq 1 \quad and \quad |f(z)| \leq |z|$$

for all z *in the disk* D. *Moreover if* $|f'(0)| = 1$ *or if* $|f(z)| = |z|$ *for some* $z \neq 0$, *then there is a constant* c, $|c| = 1$, *such that* $f(w) = cw$ *for all* w *in* D.

In order to compare the Robin capacity on both sides of a Jordan curve, we introduce the following index $I(J)$.

Definition of $I(J)$: For a closed Jordan curve in the extended complex plane, without loss of generality, we assume that the curve J lies in the unit circle $\mathbb{D}$ and $J \bigcap \partial\mathbb{D} \neq \emptyset$. Then J separates the plane into two complementary domains: Ω and Ω^*. Suppose $0 \in \Omega$ and $\infty \in \Omega^*$. Define the index $I(J)$ as follows,

$$I(J) = \max[\sup_A(\frac{\sigma_0(A)}{\sigma_\infty(A)}), \sup_A(\frac{\sigma_\infty(A)}{\sigma_0(A)})],$$

where the supremum is taken over all non-degenerate subarcs A of J.

Theorem 6.2.2. *For a normalized Jordan curve* J *as in the above definition,* $I(J) = 1$ *if and only if* J *is the unit circle.*

Proof. If J is the unit circle, then by the Lemma 6.1.3,

$$\rho_0(A) = 2\pi \lim_{r \to 0}[\lambda(A, \Delta_r, \Omega) + \frac{1}{2\pi} \ln r] \qquad in \quad \Omega$$

and

$$\rho_\infty(A) = 2\pi \lim_{R\to\infty} [\lambda(A, \Delta_R, \Omega^*) - \frac{1}{2\pi} \ln R] \quad in \quad \Omega^*.$$

Using the map $f(z) = \frac{1}{z}$ and the conformal invarianve of extremal length, one easily obtains that

$$\rho_0(A) = \rho_\infty(A)$$

for any subarc A of J. Thus, it follows that

$$I(J) = 1.$$

In order to prove the other direction of the theorem by contradiction, we start with the assumption that J is a normalized Jordan curve that is not a unit circle. Then by Riemann mapping theorem, there exists a unique conformal map

$$\varphi(z) : \Omega \to \mathbb{D},$$

such that $\varphi(0) = 0$, $\varphi'(0) > 0$.

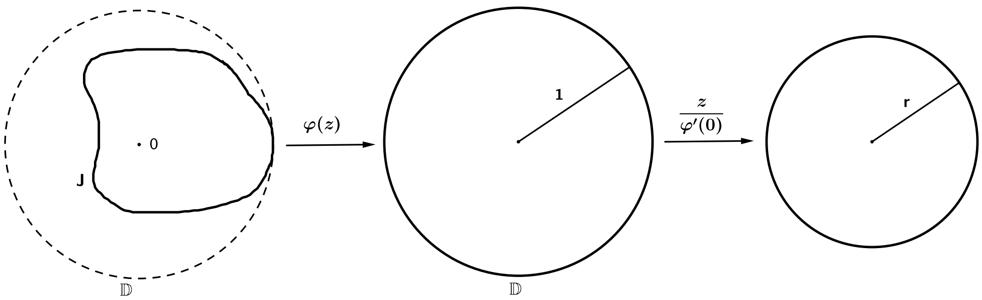

Figure 6.1: Map between J and the circle with radius r

Now define a map

$$g(z) = \frac{1}{\varphi'(0)}\varphi(z).$$

Then $g(0) = 0$ and $g'(0) = 1$. Furthermore, $g(z)$ is a conformal mapping of Ω and

$g(z)$ maps J to a circle with radius r. By a computation done in *[20]* (p238-p249), we obtain that

$$\rho_{g(0)}(g(A)) = \rho_0(A) + |\log|g'(0)|| = \rho_0(A).$$

By the definition of Robin capacity, we have

$$\sigma_0(A) = \sigma_0(g(A)).$$

Applying Schwartz Lemma to the map $\varphi^{-1} : \mathbb{D} \to \Omega \subseteq \mathbb{D}$, with $\varphi^{-1}(0) = 0$, we obtian that $|(\varphi^{-1})'(0))| \leq 1$, which yields that

$$|\varphi'(0)| = \frac{1}{|(\varphi^{-1})'(\varphi(0))|} \geq 1.$$

And hence

$$r = \frac{1}{|\varphi'(0)|} \leq 1.$$

Moreover, since J is a normalized Jordan curve other than the unit circle, by Schwartz Lemma, the strict inequality $|(\varphi^{-1})'(0))| < 1$ holds. Thus it follows that $r < 1$.

Now we need to calculate the Robin capacity of a subarc on the circle with radius $r < 1$. First, we recall some calculation on the unit disk $\mathbb{D}$. For any $A \subset \partial\mathbb{D}$, we have

$$\sigma_0(A) = e^{-\rho_0(A)},$$

where

$$\rho_0(A) = 2\pi \lim_{t\to 0}[\lambda(A, \Delta_t, \Omega) + \frac{1}{2\pi}\ln t].$$

By Lemma 1.3.4,

$$\lambda(A, \Delta_t, \Omega) = \frac{1}{\pi}\ln\frac{1}{\sqrt{t}\sin\frac{\alpha}{4}},$$

where α is the angle spanned by A. Thus, the Robin capacity of A is:

$$\begin{aligned}\sigma_0(A) &= e^{-\rho_0(A)} \\ &= e^{-2\pi \lim_{t\to 0}[\lambda(A,\Delta_t,\Omega)+\frac{1}{2\pi}\ln t]} \\ &= e^{-2\pi \lim_{t\to 0}[\frac{1}{\pi}\ln \frac{1}{\sqrt{t}\sin\frac{\alpha}{4}}+\frac{1}{2\pi}\ln t]} \\ &= e^{-[2\ln \frac{1}{\sin\frac{\alpha}{4}}]} \\ &= \sin^2\frac{\alpha}{4}.\end{aligned}$$

Next, consider the map $f(z) = rz$ which maps the unit disk to a disk with radius r. By the result of *[20]* (p238-p249), we get the following:

$$\rho_0(f(A)) = \rho_0(A) + \ln r.$$

Therefore the Robin capacity of $f(A)$ on the circle with radius r is given by

$$\begin{aligned}\sigma_0(f(A)) &= e^{-\rho_0(f(A))} = e^{-\rho_0(A)-\ln r} \\ &= \frac{1}{r}e^{-\rho_0(A)} = \frac{1}{r}\delta_0(A) \\ &= \frac{1}{r}\sin^2\frac{\alpha}{4}.\end{aligned}$$

If we take A to be the whole boundary J, then

$$\alpha = 2\pi \quad \textit{and} \quad \sigma_0(g(J)) = \frac{1}{r}.$$

We already know that $\sigma_0(J) = \sigma_0(g(J))$ and $r < 1$, therefore

$$\sigma_0(J) = \sigma_0(g(J)) = \frac{1}{r} > 1.$$

Finally, we need to estimate $\sigma_\infty(J)$. As we see in the following graph, each curve connecting J and Δ_R contains a subarc that connect $\partial\mathbb{D}$ and Δ_R.

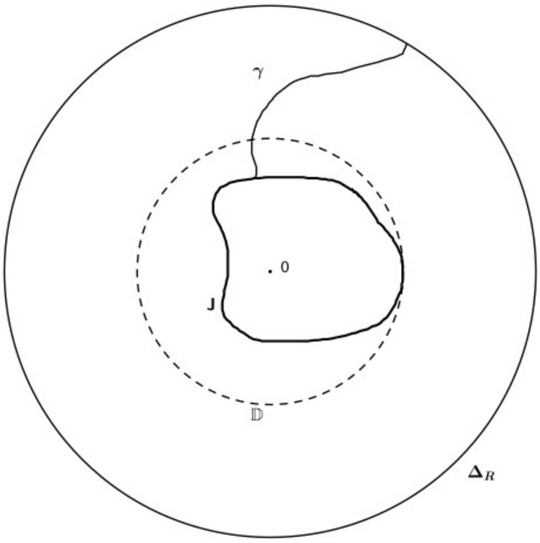

Figure 6.2: compare of $\lambda(J, \Delta_R, \Omega^*)$ and $\lambda(\partial\mathbb{D}, \Delta_R, \Omega^*)$

By Lemma 1.1.2 (the composition law), we get

$$\begin{aligned}
\rho_\infty(J) &= 2\pi \lim_{R\to\infty} (\lambda(J, \Delta_R, \Omega^*) - \frac{1}{2\pi}\ln R) \\
&\geq 2\pi \lim_{R\to\infty} (\lambda(\partial\mathbb{D}, \Delta_R, \Omega^*) - \frac{1}{2\pi}\ln R) \\
&= 2\pi \lim_{R\to\infty} (\frac{1}{2\pi}\ln R - \frac{1}{2\pi}\ln R) \\
&= 0.
\end{aligned}$$

Therefore

$$\sigma_\infty(J) = e^{-\rho_\infty(J)} \leq 1.$$

This together with the strict inequality $\sigma_0(J) = \frac{1}{r} > 1$ yields that

$$I(J) \geq \frac{\sigma_0(J)}{\sigma_\infty(J)} > 1,$$

which contradicts the assumption that $I(J) = 1$. This shows that if $I(J) = 1$, then J must be the unit circle.

□

Chapter 7

Future work

We close this thesis by proposing some future work motiveted by results obtained above.

As mentioned before, a Jordan curve is a quasicircle if and only if it "looks alike" from both sides of the curve. Based on this philosophy, many characterizations have been established for quasicircles (See *[15]*). One of these characterizations compares the harmonic measure of two adjacent subarcs, see Theorem 5.2.1. Another one is so called quasiextremal distance property (for more details, see *[14]*, *[16]* and *[26]*). It can be stated as follows.

A Jordan curve J in the complex plane is a quasicircle if and only if there exists a constant $M < \infty$ such that

$$\frac{1}{M} mod(A, B; \Omega) \leq mod(A, B; \Omega^*) \leq M mod(A, B; \Omega),$$

for any two disjoint continua A, $B \subset J$.

Note that both of these characterizations compare conformal invariants of two continua on the curve with respective to the two complementary domains Ω and Ω^*. Our goal is to characterize quasicircles by comparing some conformal invariants (such as the harmonic measure) of a single continuum on the curve with respect to the two complementary domains.

Actually, the relation between HMP (harmonic measure property - see Definition 5.1.1) and quasicircle is very complicated. On the one hand, there exist exotic and complex quasicircles that have the HMP. Such examples can be found in *[4]* and *[25]*. For instance, Bishop constructed a quasicircle which has HMP, but the Hausdorff dimension is between 1 and 2. On the other hand, some simple and "nice" quasicircles, such as the one in Example 5.4.2, do not have the HMP. The proof of Example 5.4.2 can be generalized to show that any Jordan curve with a "corner" does not have HMP. These together with other examples prompt us to make the following conjecture:

Conjecture 1: *For a locally rectifiable Jordan curve J, it has the HMP if and only if it is a symmetric quasicircle.*

A Jordan curve J is said to be a symmetric quasicircle if

$$\max_{w\in J(a,b)} \frac{|a-w|+|w-b|}{|a-b|} \to 1$$

as a, $b \in J$ and $|a-b| \to 0$, where $J(a,b)$ is the smaller arc of J between a and b (see *[23]*).

Note that locally rectifiable assumption is necessary since an arbitrary symmetric quasicircle may not have HMP. Here is such an example:

$$h(x) = \begin{cases} \frac{\ln 2+1}{\log^2 2}x & x \leq -\frac{1}{2}; \\ \frac{-t}{\ln(-t)} & -\frac{1}{2} < x < 0; \\ \frac{-t}{\ln t} & 0 \leq x < \frac{1}{2}; \\ \frac{\ln 2+1}{\ln^2 2}x & \frac{1}{2} \leq x. \end{cases}$$

$h(x)$ is the sewing homeomorphism induced by a Jordan domain Ω. Since

$$\lim_{t\to 0} \frac{h(x+t)-h(x)}{h(x)-h(x-t)}$$

converge uniformly to 1 for x near zero, this means the $\partial\Omega$ is symmetric quasicircle.

On the other hand, we can prove that

$$\lim_{x \to 0} \frac{h(x)}{x} = 0$$

which implies the curve does not have the HMP. Therefore we need to add an extra condition such as local rectifiability.

Another future project is to use the index $I(J)$ definied in previous chapter to characterize quasicircles. In chapter 6, we have proved that $I(J) = 1$ if and only if the curve is the unit circle. Recall that the index $I(J)$ is definied by the ratio of Robin capacities in the complementary domains and the Robin capacity can be expressed in terms of extremal distance. By the geometric definition of quasiconformal maps, we know that the extremal distance doesn't change too much under a quasiconformal map. This property prompts us to make the following conjecture:

Conjecture 2*: J is a quacicircle if and only if* $1 \le I(J) \le M$ *for some constant M.*

Without loss of generality, we assume J is a normalized Jordan curve. We can prove that if J is a quasicircle, then $I(J)$ is bounded by some constant. But the other direction is hard and we still have a lot of work to do.

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