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The Reduced Unitary Whitehead Groups over Function Fields of $p$-adic Curves

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## Abstract

Reduced Unitary Whitehead Groups over Function Fields of $p$-adic Curves
By Zitong Pei

The study of the Whitehead groups of semi-simple simply connected groups is classical with an abundance of new open questions concerning the triviality of these groups. The Kneser-Tits conjecture on the triviality of these groups was answered in the negative by Platanov for general fields. There is a relation between reduced Whitehead groups and $R$-equivalence classes in algebraic groups.

Let $G$ be an algebraic group over a field $F$. Let $RG(F)$ be the equivalence class of the identity element in $G(F)$. Then $RG(F)$ is a normal subgroup of $G(F)$ and the quotient $G(F)/RG(F)$ is called the group of $R$-equivalence classes of $G(F)$. It is well known that for the semi-simple simply connected isotropic group $G$ over $F$, the Whitehead group $W(G, F)$ is isomorphic to the group of $R$-equivalence classes.

Suppose that $D_0$ is a central division $F_0$-algebra for a field $F_0$. If the group $G(F_0)$ of rational points is given by $SL_n(D_0)$ for an integer $n > 1$, then $W(G, F_0)$ is the reduced Whitehead group of $D_0$. Let $F$ be a quadratic field extension of $F_0$ and $D$ be a central division $F$-algebra. Suppose that $D$ has an involution of second kind $\tau$ such that $F^\tau = F_0$. If the hermitian form $h_\tau$ of $\tau$ is isotropic and the group $G(F_0)$ is given by $SU(D, h_\tau)$, then $W(G, F_0)$ is isomorphic to the reduced unitary Whitehead group of $D$.

Let $F_0$ be the function field of a $p$-adic curve. Let $F/F_0$ be a quadratic field extension. Let $A$ be a central simple algebra over $F$. Assume that the period of $A$ is 2 and $A$ has a unitary $F/F_0$ involution. We provide a proof for the triviality of the reduced unitary Whitehead group of $A$.

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# Chapter 1

# Introduction

## 1.1 An Introduction

The study of Whitehead groups of semi-simple, simply connected groups constitutes a classical field, yet it remains replete with numerous open questions concerning the triviality of these groups.

Let $F_0$ be a field. Typically, we assume that the characteristic of $F_0$ is not equal to 2 unless otherwise specified. We define $X$ as an $F_0$-variety when $X$ is a geometrically integral and separable scheme of finite type over $F_0$. $X$ is said to be $F_0$-rational if the function field of $X$ is purely transcendental over $F_0$. $X$ is said to be $F_0$-stably rational if $X \times_{F_0} \mathbb{A}^n_{F_0}$ is $F_0$-rational. Let $G$ be a smooth connected linear algebraic group over $F_0$. $G$ is said to be $F_0$-(stably) rational if its underlying variety if $F_0$-(stably) rational. Let $G(F_0)$ denote the group of $F_0$-rational points of $G$. Two points $\alpha, \beta \in G(F_0)$ are defined to be $R$-equivalent if there is a rational map $f : \mathbb{A}^1_{F_0} \dashrightarrow G$ such that $f(0) = \alpha$ and $f(1) = \beta$. The definition of $R$-equivalence was introduced by Manin [19] when studying cubic hypersurfaces.

For the smooth connected linear algebraic group $G$, $R$-equivalence is actually an equivalence relation; let $RG(F_0)$ denote the equivalent class of the identity $e \in$

$G(F_o)$. Then $RG(F_0)$ is a normal subgroup of $G(F_0)$; there is a bijection of sets $G(F_0)/RG(F_0) \longleftrightarrow G(F_0)/\sim$ (cf. [2], [28]). Therefore, the quotient set $G(F_0)/\sim$ has a group structure. The group of $R$-equivalence classes, $G(F_0)/RG(F_0)$, is very useful while studying the rationality problem for linear algebraic groups. The rationality problems for linear algebraic groups have a long history. We are interested in the case when $G$ is a semi-simple simply connected isotropic algebraic group. Let $G^+(F_0)$ denote the normal subgroup of $G(F_0)$ generated by the conjugates of $F_0$-points of the unipotent radical of a proper $F_0$ parabolic subgroup of $G$. The factor group $G(F_0)/G^+(F_0)$, denoted by $Wh(G, F_0)$, is called the Whitehead group for $G$ over $F_0$. In this case, $Wh(G, F_o) \cong G(F_o)/RG(F_o)$ ([28]). There is a conjecture on the Whitehead group for $G$:

**Conjecture** (Kneser-Tits)**.** *Is it true that $Wh(G, F_o)$ is trivial?*

We consider two special cases:

- Case I

  Let $A_0$ be a central simple $F_0$-algebra. Let $[A_0^*, A_0^*]$ denote the commutator subgroup of $A_0$. Let $SL_1(A_0) = \{a_0 \in A_0 | Nrd_{A_0/F_0}(a_0) = 1\}$. Let $SK_1(A_0) = SL_1(A_0)/[A_0^*, A_0^*]$. We call it the reduced Whitehead group of $A_0$. If $G(F_0) = SL_1(A_0)$, then $Wh(G, F_0) \cong G(F_0)/RG(F_0) \cong SK_1(A_0)$ ([28]).

- Case II

  Let $F/F_0$ be a quadratic field extension. Let $A$ be a central simple $F$-algebra. Assume that $A$ has a unitary $F/F_0$-involution $\tau$. Let $\Sigma'_\tau(A) = \{a \in A^* | Nrd_{A/F}(a) \in F_0\}$; let $\Sigma_\tau(A)$ denote the subgroup of $A^*$ generated by the set $\{a \in A^* | \tau(a) = a\}$. Assume that the hermitian form $h_\tau$ is non-singular and isotropic. Let $SK_1U(A, \tau) = \Sigma'_\tau(A)/\Sigma_\tau(A)$. We call it the reduced unitary Whitehead group of $A$. If $G(F_0) = SU(A, h_\tau)$, then $Wh(G, F_0) \cong G(F_0)/RG(F_0) \cong SK_1U(A, \tau)$([28]).

Platonov firstly proved that $SK_1(A_0)$ can be non-trivial; then Platonov & Yanchevskii proved that $SK_1U(A)$ can be non-trivial. See reference [4]. We will subsequently explore a result by Nivedita Bhaskhar ([3]), adding depth to our analysis.

Let $p$ be a prime; let $K$ be a $p$-adic field. Let $\mathcal{C}$ be a curve over $K$, i.e. a smooth, projective, and geometrically integral $K$-scheme of finite type with dimension 1; let $F_0$ be the function field of $\mathcal{C}$. Let $F/F_0$ be a quadratic field extension; let $A$ be a central simple $F$-algebra. Assume that the period of $A$ is a prime $\ell$; assume that $A$ has a unitary $F/F_0$-involution $\tau_A$. We start from a combined result from Nivedita and Yanchevskii.

**Theorem 1.1.1** (cf. [3], [32]). *If $\ell > 2, p \neq \ell$, and $F$ contains a primitive $\ell^{2th}$ root of unity, then $SK_1U(A, \tau_A) = 1$.*

Given the theorem, one might ask about the case when the period $\ell = 2$. We state the aim of the thesis, which includes the case $\ell = 2$.

**Theorem 1.1.2.** *If $\ell = 2$ and $p \neq \ell$, then $SK_1U(A, \tau_A) = 1$.*

## 1.2 Organization

Chapter 2 provides a detailed description of algebras with unitary involution, which serves as the foundational basis for our study.

Chapter 3 introduces the concepts of reduced Whitehead groups and reduced unitary Whitehead groups. Additionally, it includes a comprehensive discussion on $R$-equivalence.

Chapter 4 presents our primary methodological approach: patching. This technique, derived from algebraic geometry, is particularly advantageous in our case, especially when the base field is the function field of a $p$-adic curve.

Chapter 5 demonstrates a positive resolution to the main theorem utilizing the patching method.

# Chapter 2

# Algebras with Involutions

The focal point of this chapter is the exploration of basic objects known as central simple central algebras. While these entities can be analyzed within the broader framework of simple algebras, our discussion primarily revolves around the distinctive properties and principles governing central simple algebras. For those interested in delving deeper into related concepts, we suggest consulting authoritative texts such as [7] and [18], which offer comprehensive insights into general theories concerning simple algebras, separable algebras, and Azumaya algebras.

We assume that the characteristics of all the fields are not equal to 2.

## 2.1 Central Simple Algebras: An Introduction

Let $F$ be a field, and let $A$ be an $F$-algebra. We say that $A$ is a *central simple $F$-algebra* if the following three conditions are satisfied:

(1) $A$ has no nontrivial two-sided ideal;

(2) The central of $A$ is $F$;

(3) The dimension of $A$ as an $F$-vector space is finite.

For example, the ring of matrices $M_n(F)$ over a field $F$ is a central simple $F$-algebra.

In fact, every central simple algebra can be regarded as a matrix algebra over a division ring $D$. We are prepared to present the structure theorem of central simple algebras.

**Theorem 2.1.1** (Wedderburn, Theorem 3.2.6, [7])**.** *Let $A$ be a central simple algebra over a field $F$. Then $A$ is isomorphic to a matrix algebra $M_n(D)$ over a finite dimensional central division $F$-algebra $D$. The central division algebra $D$ and the number $n$ are uniquely determined by the isomorphism.*

There is a special case of the Theorem 2.1.1. Assume that $F$ is an algebraic closed field. Then $A \cong M_n(D)$ for a finite dimensional central division $F$-algebra $D$. Assume that there is an element $d \in D \setminus F$. Then there is a sub-algebra of $D$ generated by $F$ and $d$, which is denoted by $F[d]$. Since $D$ is a division ring and $dim_F(D)$ is finite, $F[d]$ is an integral domain and $dim_F(F[d])$ is finite. Therefore, $F[d]$ is a finite field extension of $F$. Since we assume that $F$ is algebraic closed, $F = F[d] = D$ and $A \cong M_n(F)$.

Let $A$ and $B$ be two central simple $F$-algebras. We say that $A$ and $B$ are *similar* if $A \cong M_{n_1}(D)$ and $B \cong M_{n_2}(D)$. We use the notation $Br(F)$ to denote the set of similarity classes of central simple $F$-algebras, and use $[A] \in Br(F)$ to represent the similarity class of $A$. For example, $[F] = [M_s(F)]$; $[A] = [A \otimes_F M_n(F)]$.

In fact, the set $Br(F)$ has a group structure by the following theorem.

**Theorem 2.1.2** (cf. Chapter 29, [18] )**.** *Let $A$, $B$, $A'$, and $B'$ be central simple $F$-algebras.*

*(1) $A \otimes_F B$ is a central simple $F$-algebra;*

*(2) Let $A^{op}$ denote the opposite algebra of $A$, then $A^{op}$ is a central simple $F$-algebra and $[A \otimes_F A^{op}] = [F] \in Br(F)$;*

*(3) If $[A] = [A']$ and $[B] = [B']$, then $[A \otimes_F A'] = [B \otimes_F B'] \in Br(F)$.*

Combined with the check of the associativity and commutativity, we obtain the following corollary.

**Corollary 2.1.3.** *Define an operation "+" on* $Br(F)$*:* $[A] + [B] = [A \otimes_F B]$*. Then* $(Br(F), +)$ *is a commutative group.*

By Theorem 2.1.2, the identity element of $Br(F)$, denoted by 0, is $[F]$; the inverse element of $[A]$, denoted by $-[A]$, is $[A^{op}]$. We call $(Br(F), +)$ the *Brauer group of the field* $F$, and $[A] \in Br(F)$ the *Brauer class of* $A$.

In analogy to the property observed in vector spaces, wherein any base change results in another vector space, one may inquire whether such a principle extends to central simple algebras. We shall now proceed to elucidate the notion of base change for central simple algebras.

**Proposition 2.1.4.** *Let* $A$ *be an* $F$*-algebra, and let* $E$ *be a field extension of* $F$*. Then* $A \otimes_F E$ *is a central simple* $E$*-algebra if and only if* $A$ *is a central simple* $F$*-algebra.*

*Proof.* cf. F12 & F15, Chapter 29, [18]; □

Therefore, there is a well-defined map induced by Proposition 2.1.4

$$Res_{E/F} : Br(F) \to Br(E),$$

which is called *restriction map with respect to the field extension* $E/F$. The kernel of the restriction map is denoted by $Br(E/F)$. By definition, $Br(E/F) = \{[A] \in Br(F) | [A \otimes_F E] = 0\}$.

In general, we say that $A$ is *split* over a field $E$ or $E$ is a *splitting field* of $A$ if $[A \otimes_F E] = 0 \in Br(E)$.

Let $A$ be a central simple algebra over a field $F$. If we fix an algebraic closure $F^{al}$ of $F$, then $A \otimes_F F^{al} \cong M_n(F^{al}) \in Br(F^{al}) = \{0\}$ by Proposition 2.1.4 and the discussion after Theorem 2.1.1. Thus $F^{al}$ is a splitting field of $A$ and $dim_F(A)$ is a square number. Since $A \cong M_n(D)$ by Theorem 2.1.1, $\sqrt{dim_F(A)} = n \cdot \sqrt{dim_F(D)}$.

In general, we use the notation $deg_F(A)$ to denote the square root $\sqrt{dim_F(A)}$ for a central simple $F$-algebra $A$, which is called the *degree* of $A$. Since the Brauer class $[A]$ is uniquely determined by the isomorphism $A \cong M_n(D)$, we call $deg_F(D)$ the *index of* $[A]$ and denote it by $ind_F(A)$.

Among the intriguing topics concerning splitting fields is the quest for a splitting field within sub-algebras of a central simple $F$-algebra $A$. This pursuit relies heavily on a renowned theorem known as the 'Centralizer Theorem.' For further elaboration, interested readers are directed to Theorem 14, Chapter 29 in [18]. At present, we shall confine our discussion to the outcomes pertaining to central division algebras.

**Theorem 2.1.5** (cf. Theorem 17, Chapter 29, [18])**.** *Let $D$ be a finite dimensional central division $F$-algebra. Then $D$ has a splitting field $E$ such that $E$ is a sub-algebra of $D$ and $[E : F] = deg_F(D)$.*

If $[A] = [D] \in Br(F)$, then $A$ has a splitting field $E$ such that $[E : F] = ind_F(A)$ by Theorem 2.1.5.

If our scope extends beyond sub-algebras of a central simple $F$-algebra $A$, we arrive at the following result.

**Proposition 2.1.6.** *$A$ has a splitting field $L$ which is a finite Galois field extension of $F$. Such $L$ cannot always be a sub-algebra of $A$.*

*Proof.* cf. Corollary 2.2.12, [8] □

It implies from Proposition 2.1.6 that

$$Br(F) = \bigcup_{L/F} Br(L/F)$$

where $L/F$ range over all finite Galois extensions of $F$ contained in an algebraic closure $F^{al}$.

Another famous result is the Skolem-Noether theorem of central simple algebras. Let $A$ be an algebra over a field $F$. We say that a $F$-automorphism $f$ of $A$ is *inner* if there is an invertible element $x$ of $A$ such that $f(a) = x^{-1} \cdot a \cdot x$ for each $a \in A$.

**Theorem 2.1.7** (Skolem-Noether)**.** *Let $A$ be a central simple $F$-algebra. Let $B_1$ and $B_2$ be simple sub-algebras of $A$. Then every $F$-isomorphism between $B_1$ and $B_2$ is from an inner automorphism of $A$.*

*Proof.* cf. Theorem 4.5.12, Chapter 5, [7]. □

By Theorem 2.1.7, we obtain that every $F$-automorphism of $A$ is inner. There is also a general result on Azumaya algebras. Let $R$ be a local ring and $\mathcal{A}$ an Azumaya algebra over $R$. Then every $R$-automorphism of $\mathcal{A}$ is inner. Refer to Section 8 of Chapter 7 in [7].

Fix an algebraic closure $F^{al}$ of $F$. We know that $A \otimes_F F^{al} \cong M_n(F^{al})$. Then there is an $F$-homomorphism $i : A \to M_n(F^{al})$ which is an injection. For each $a \in A$, $i(a)$ is a matrix over $F^{al}$. We denote the characteristic polynomial of the matrix $i(a)$ by

$$Red_a(X) = X^n + c_1 X^{n-1} + c_2 X^{n-2} + \cdots + c_n.$$

**Proposition 2.1.8.**

*(1) $Red_a(X)$ is a polynomial in the ring $F[X]$ and is independent of the choice of the injection $i$;*

*(2) The determinant of the matrix $i(a)$ is $(-1)^n c_n$;*

*(3) The trace of the matrix $i(a)$ is $-c_1$;*

*Proof.* cf. F23, [18]. □

The polynomial $Red_a(X) \in F[X]$ is called the *reduced characteristic polynomial* of $a$. We call $(-1)^n c_n$ the *reduced norm of* $a$ and denote it by $Nrd(a)$; we call $-c_1$ the reduced trace of $a$ and denote it by $Trd(a)$.

Therefore, we obtain a map $Nrd_F : A \to F$ which is called the *reduced norm map of* $A$, and $Trd_F : A \to F$ which is called the *reduced trace map of* $A$. It is not difficult to verify that the map $Nrd_F$ is multiplicative and $Trd_F$ is a $F$-linear map.

For $a \in A$, we also have a linear endomorphism $m_a : A \to A$ defined by $m_a(b) = ab$. We denote the characteristic polynomial of $m_a$ , the norm of $m_a$, and the trace of $m_a$ by $m_a ch(X)$, $N(a)$, and $Tr(a)$ respectively.

**Proposition 2.1.9.** *$m_a ch(X) = (Red_a(X))^n$, $N(a) = (Nrd(a))^n$, and $Tr(a) = n(Trd(a))$.*

*Proof.* cf. Proposition 2.6.3, [8]. □

To direct readers to additional results regarding the reduced norm map $Nrd_F$ in number theory or class field theory, please consult Section 2.6 of [8].

Now we briefly introduce using group cohomology to describe $Br(F)$, which is for the definition of cyclic algebras later. Since $Br(F) = \bigcup_{L/F} Br(L/F)$ for a fixed algebraic closure $F^{al}$, we can start from $Br(L/F)$ with $L/F$ a finite Galois field extension.

Let $G$ denote the Galois group of a finite Galois field extension of $L/F$. Then $G$ acts on the group $L^\times$, which satisfies $g(xy) = g(x)g(y)$ for $g \in G$, $x \in L^\times$, and $y \in L^\times$.

**Definition 2.1.10.** A map $\sigma : G \times G \to L^\times$ is called a *2-cocycle* of $G$ in $L^\times$ if $\sigma$ satisfies

$$\sigma(g_1, g_2g_3) \cdot \sigma(g_2, g_3) = \sigma(g_1g_2, g_3) \cdot g_3\big(\sigma(g_1, g_2)\big)$$

for $g_1, g_2, g_3 \in G$.

If $\sigma$ is a 2-cocycle which is defined by

$$\sigma(g_1, g_2) = h(g_2) \cdot g_2\big(h(g_1)\big) \cdot h(g_1g_2)^{-1}$$

where $h : G \to L^\times$ is a map satisfying $h(Id_G) = 1$, we call such $\sigma$ a *2-coboundary.* Let $Z^2(G, L^\times)$ denote the set of 2-cocyles of $G$ in $L^\times$, and $B^2(G, L^\times)$ the set of 2-coboundaries in $Z^2(G, L^\times)$.

**Proposition 2.1.11.**

*(1) $Z^2(G, L^\times)$ is an abelian group with identity element the trivial map, and $B^2(G, L^\times)$ is a subgroup of $Z^2(G, L^\times)$; denote the factor group $Z^2(G, L^\times)/B^2(G, L^\times)$ by $H^2(G, L^\times)$;*

*(2) There is an isomorphism of groups $\phi : H^2(G, L^\times) \cong Br(L/F)$;*

*(3) If $G =< \xi >$ is a finite cyclic group of degree $n$, then $H^2(G, L^\times) \cong Br(L/F) \cong L^\times/N_{L/F}(F^\times)$ and each element $[A] \in Br(L/F)$ is the Brauer class of a central simple central $F$-algebra given by the form*

$$\bigoplus_{j=0}^{n-1} y^j L.$$

*Moreover, $y^n = t$ for some $t \in F$ and $xy = y \cdot \xi(x)$ for each $x \in L$.*

*Proof.* For the proof, refer to Chapter 30 of [18]. □

In the view Proposition 2.1.6, $Br(F) = \bigcup_{L/F} Br(L/F)$ where $L/F$ range over all finite Galois extensions of under an algebraic closure. Then we can using (2) of Proposition 2.1.11 to imply the following theorem.

**Theorem 2.1.12** (Theorem 3, Chapter 30, [18])**.** *Let $A$ be a central simple algebra over a field $F$. Then*

*(1) $Br(F)$ is a torsion group and the order of $[A] \in Br(F)$ divides $ind_F(A)$;*

*(2) The order of $[A]$ and $ind_F(A)$ have same prime factors.*

We call the order of $[A]$ in $Br(F)$ the *exponent* of $A$ and denote it by $exp(A)$. By Theorem 2.1.12, $exp(A)$ divides $ind_F(A)$. An application of Theorem 2.1.12 is the following decomposition theorem.

**Theorem 2.1.13** (Primary Decomposition, cf. Proposition 2.8.13, [8]). *Let $A$ be a central simple $F$-algebra. If the prime decomposition of the degree of $A$ is $deg_F(A) = p_1^{s_1} p_2^{s_2} \cdots p_t^{s_t}$, then $A$ has a unique decomposition up to an isomorphism that*

$$A \cong_F A_1 \otimes_F A_2 \otimes_F \cdots \otimes_F A_t$$

*where each $A_i$ is a central simple $F$-algebra of degree $p_i^{s_i}$ for all $i = 1, 2, ..., t$. Furthermore, $A$ is a division algebra if and only if each $A_i$ is a division algebra for all $i = 1, 2, ..., t$.*

In the case (3) of Proposition 2.1.11, we call such a central simple $F$-algebra a *cyclic algebra* and denote it by $(L, \xi, t)$. For example, a cyclic algebra is a quaternion algebra if $L/F$ is a degree 2 extension and $ch(F) \neq 2$. Finally, we provide an important property of a cyclic algebra.

**Proposition 2.1.14** (cf. Chapter 30, [18]). *Let $L/F$ be a finite cyclic extension of degree $n$, and let $Gal(L/F) =< \xi >$. Then any cyclic $F$-algebra $(L, \xi, t)$ for some $t \in F$ has degree $n$ and contains $L$ as a maximal subfield.*

We will heavily use many principles of cyclic algebras in Chapter 5, which are derived from Galois cohomology and class field theory. Readers can find information from [5], [18], [21].

In general, the behavior of Brauer group $Br(F)$ will be very different when changing the base field $F$. For example, $Br(\mathbb{C}) = \{0\}$, $Br(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$, $Br(\mathbb{F}_q) = \{0\}$, and $Br(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}$. One can also use the language of central simple algebras to describe local class field theory. For reference, it can be found in [30].

We present several results regarding central simple algebras over complete discrete valued fields.

In the rest of this section, we assume that $F$ is a field with a discrete valuation $\omega$. Let $O_F$ denote the valuation ring of $F$; let $\overline{F}$ denote the residue field. Suppose that the characteristic of $\overline{F}$ is not equal to 2.

Let $D$ be a finite dimensional central division $F$-algebra. It is know that the valuation $\omega$ can be extended uniquely to a discrete valuation $v$ on $D$, i.e. a map $v : D \to \mathbb{Z} \cup \infty$ satisfying

(1) $v(d_1 \cdot d_2) = v(d_1) + v(d_2)$;

(2) $v(d_1 + d_2) \geq \min(v(d_1), v(d_2))$;

(3) $v(d_3) = \infty \Leftrightarrow d_3 = 0$.

Let $O_D$ denote the valuation ring of $D$, i.e. $O_D = \{d \in D | v(d) \geq 0\}$; let $I_D = \{d \in D^* | v(d) > 0\}$. In fact, $I_D$ is the unique two-sided maximal ideal of $O_D$. Let $\overline{D} = O_D/I_D$, which is called the *residue division ring*.

**Definition 2.1.15.**

(1) $D$ is called *unramified* if $[D : F] = [\overline{D} : \overline{F}]$ and $Z(\overline{D}) = \overline{Z(D)} = \overline{F}$.

(2 ) $D$ is called *nicely semi-ramified* if $D$ has a maximal subfield $E$ which is unramified over $F$ and a totally ramified maximal subfield $L$ over $F$ satisfying $\bar{v} : H \cong v(L^*)/v(F^*)$ for some subgroup $H$ of $L^*/F^*$.

(3) $D$ is called *unramified split* if $D \otimes_F F^{nr}$ is split, where $F^{nr}$ is the maximal unramified extension of $F$.

Here we have a decomposition of an unramified split $D$.

**Theorem 2.1.16** (Lemma 5.14, [14])**.** *Assume that $D$ is unramified split. Then there are finite dimensional central $F$-algebras $D_1$ and $D_2$ such that*

$$[D] = [D_1] + [D_2] \in Br(F),$$

*where $D_1$ is unramified over $F$ and $D_2$ is nicely semi-ramified over $F$.*

## 2.2 Unitary Involutions: Existence Criterion and Properties

In this section, we introduce involutions on central simple algebras. Let $A$ be a central simple algebra over a field $F$. The the transpose on the matrix algebra $M_n(F)$ is a classical example of an involution.

For the existence criterion, we will start from the algebra direction without using group cohomology. Therefore, readers can have a smooth transition from last section.

**Definition 2.2.1.** An *involution* on $A$ is a map $\tau : A \to A$ satisfying conditions:

(a) $\tau(a+b) = \tau(a) + \tau(b)$ for $a, b \in A$.

(b) $\tau(ab) = \tau(b)\tau(a)$ for $a, b \in A$.

(c) $\tau(\tau(a)) = a$ for $a \in A$.

Further,

(d) $\tau$ is called an *involution of the first kind* if $\tau(x) = x$ for every $x \in F$. Otherwise, $\tau$ is called an *involution of the second kind* or a *unitary involution.*

Let the couple $(A, \sigma)$ denote a central simple $F$-algebra $A$ with an involution $A$. For each morphism of $F$-algebras $f : (A, \sigma) \to (A', \sigma')$ , we always assume that $f \circ \sigma = \sigma' \circ f$.

If $\tau$ is a unitary involution on $A$, we use $F_0$ to denote the subfield of $F$ consisting of the $\tau$-stable elements. Therefore, $\tau(x) = x$ for $x \in F_0$ and $F/F_0$ is a degree 2 field extension. In this situation, we also say that $\tau$ is an *$F/F_0$-involution.*

For the existence of the involutions of the first kind, we have the following criterion.

**Theorem 2.2.2** (Albert, cf. Theorem 3.1, [16])**.** *Let $A$ be a central simple $F$-algebra. There is an involution of the first kind on $A$ if and only if $exp(A) \leq 2$.*

We now mainly focus on unitary involutions. In order to discuss the existence, we begin from the construction of the *corestriction map.*

Let $L/E$ be a field extension of degree 2, and let $G = Gal(L/E) =< \theta >$ be the Galois group. Let $B$ be an $L$-algebra. We introduce a new symbol $\theta(b)$ for each $b \in B$, and define

$$B_\theta = \{\theta(b) | b \in B\}.$$

The set $B_\theta$ is an $L$-algebra if we define operations:

(1) $\theta(b_1) + \theta(b_2) = \theta(b_1 + b_2)$,

(2) $\theta(b_1)\theta(b_2) = \theta(b_1 b_2)$,

(3) $\ell \cdot \theta(b_1) = \theta\big(\theta(\ell) \cdot b_1\big)$,

for $b_1$, $b_2 \in B$ and $\ell \in L$. The $L$-algebra $B_\theta$ is called the *conjugate algebra* of $B$ *with respect to* $L/E$.

Let $V = B_\theta \otimes_L B$. We define a map $\beta : V \to V$ by $\beta\big(\theta(b_1) \otimes b_2\big) = \theta(b_2) \otimes b_1$.

**Proposition 2.2.3** (cf. 3B, [16]).

*(1) $\beta$ is a $\theta$-semilinear map of vector spaces over $L$, which has properties $\beta(v_1 + v_2) = \beta(v_1) + \beta(v_2)$ and $\beta(\ell \cdot v) = \theta(\ell) \cdot \beta(v)$ for all $v_1, v_2, v \in V$ and all $\ell \in L$;*

*(2) $\beta$ is an automorphism of $E$-algebras.*

In (1) of Proposition 2.2.3, we call the map $\beta : V \to V$ the *switch map of* $V = B_\theta \otimes_L B$.

Now, we view $V = B_\theta \otimes_L B$ as an $E$-algebra and $B$ still an $L$-algebra. Then denote $cor_{L/E}(B) = \{v \in V | \beta(v) = v\}$, which is an $E$-sub-algebra of $V = B_\theta \otimes_L B$ by (2) of Proposition 2.2.3.

If $B$ is only a finite dimensional $L$-vector space, we can similarly define its *conjugate vector space* $B_\theta$ and thus the $L$-vector space $V = B_\theta \otimes_L B$. Proposition 2.2.3 will also be valid for such setting. Therefore, we have the *switch map* $\beta : V \to V$, and the $E$-sub-vector space $cor_{L/E}(B)$ of $V$.

**Proposition 2.2.4** (cf. Proposition 3.13, [16]). *Let $B'$ be an $L$-algebra. Let $W$ be a finite dimensional $L$-vector space. Then*

*(1)* $cor_{L/E}(B) \otimes_E L \cong_L V = B_\theta \otimes_L B$; $cor_{L/E}(B_\theta) \cong_E cor_{L/E}(B)$;

*(2)* $cor_{L/E}(B \otimes_L B') = cor_{L/E}(B) \otimes_E cor_{L/E}(B')$;

*(3)* $cor_{L/E}\big(End_L(W)\big) \cong_E End_E\big(cor_{L/E}(W)\big)$.

If $B$ is a central simple $L$-algebra, then $B_\theta \otimes_L B$ is also a central simple $L$-algebra by Theorem 2.1.2. By (1) of Proposition 2.2.4 and Proposition 2.1.4, $cor_{L/E}(B)$ is a central simple $E$-algebra. If $B'$ is a central simple $L$-algebra with $[B] = [B'] \in Br(L)$, then $[cor_{L/E}(B)] = [cor_{L/E}(B')] \in Br(E)$ by (2) and (3) of Proposition 2.2.4. Therefore, there is a map from $Br(L)$ to $Br(E)$ defined as follows.

**Definition 2.2.5** (corestriction map)**.** We define a map

$$Cor_{L/E} : Br(L) \to Br(E)$$

by

$$Cor_{L/E}([B]) = [cor_{L/E}(B)].$$

The map $Cor_{L/E}$ is called the *corestriction map* of $Br(L)$ *with respect to the quadratic extension* $L/E$.

In the last section, we already have a restriction map $Res_{L/E} : Br(E) \to Br(L)$ induced by the base change $L/E$. Therefore, it is reasonable to study the composition of two maps.

**Proposition 2.2.6** (Proposition 3.13 (5), [16])**.** *Let $\tilde{B}$ be a central simple $E$-algebra. Then*

$$Cor_{L/E} \cdot Res_{L/E}([\tilde{B}]) = [\tilde{B}] + [\tilde{B}] \in Br(E).$$

*Proof.* cf. Proposition 3. 13 (5) in [16]. It is an algebraic version proof without using group cohomology. □

Now we step into unitary involutions. Let $A$ be a central simple $F$-algebra. Assume that $F_0$ is a subfield of $F$ and $[F : F_0] = 2$. We provide a criterion of the existence of an $F/F_0$-involution on $A$.

**Theorem 2.2.7** (Albert-Riehm-Scharlau). *$A$ has an $F/F_0$-involution $\tau$ if and only if $Cor_{F/F_0}([A]) = 0 \in Br(F_0)$.*

*Proof.* cf. 3.B. [16]. □

Notice that in the case of Theorem 2.2.7, the Galois group $G = Gal(F/F_0) =< \tau|_F >$ . Although we globally assume that $ch(F_0) \neq 2$, Theorem 2.2.7 is valid for any separable quadratic extension $F/F_0$.

It is necessary to explain the construction of the corestriction map from purely algebraic aspect because we will frequently calculate the image of $Cor_{F/F_0}$ in Chapter 5. Abstract results from Galois cohomology are not enough in this thesis.

By Theorem 2.1.1, $A \cong_F M_n(D)$ for some finite dimensional division $F$-algebra. By Theorem 2.2.7, $A$ has an $F/F_0$-involution if and only if the division algebra $D$ has an $F/F_0$-involution. Then we can investigate unitary involutions on finite dimensional division algebras, which will provide a reason why one also uses "unitary" to describe the involutions of the second kind.

Let $D$ be a finite dimensional division $F$-algebra. Let $M$ be a finitely generated right $D$-module. Assume that $D$ has an $F/F_0$-involution $\tau$.

**Definition 2.2.8.** We say that a bi-additive map $h : M \times M \to D$ is a *hermitian form on $M$ with respect to $\tau$* if $h$ satisfies:

(1) $h(ad_1, bd_2) = \tau(d_1)h(b,a)d_2$ for all $a, b \in M$ and $d_1, d_2 \in D$;

(2) $h(a, b) = \tau\big(h(b, a)\big)$ for all $a, b \in M$.

The hermitian form $h$ is called *regular or non-singular* if the only element $a \in M$ such that $h(a, b) = 0$ for all $b \in M$ is $a = 0$. An injective map of right $D$-modules $u : M \to M'$ is called an *isometry* if $h'$ is a hermitian form on $M'$ and $h(a, b) = h'\big(u(a), u(b)\big)$ for all $a, b \in M$. All the bijective isometries of $M$ form its *unitary group* $\mathbf{U}(M, h)$.

We now assume that $h$ is regular. Let $M^*$ be the dual of the right $D$-module $M$. Although $M^*$ can be a left $D$-module naturally, we hope to define a right $D$-module structure on $M^*$. For each element $\alpha \in M^*$ and $d \in D$, we define $(\alpha \cdot d)(a) = \tau(d)\big(\alpha(a)\big)$ for each $a \in M$. Then one can verify that $M$ is a right $D$-module in this setting. Then we always consider $M^*$ as a right $D$-module.

For each map $f \in End_D(M)$, we define a map $T_f : M^* \to M^*$ by $T_f(\alpha)(a) = \alpha \circ f(a)$ for all $a \in M$. In fact, $T_f \in End_D(M^*)$, and we call it the *transpose of* $f$.

Consider a map $\hat{h} : M \to M^*$ given by $\hat{h}(a) : b \mapsto h(a, b)$ for all $a, b \in M$. Since we assume that $h$ is regular and $M$ is finite dimensional over $D$, we can very that $\hat{h}$ is an isomorphism of right $D$-modules.

**Theorem 2.2.9.** *Define a map* $\tau_h : End_D(M) \to End_D(M)$ *by* $\tau_h(f) = \hat{h}^{-1} \circ T_f \circ \hat{h}$. *Then:*

*(1)* $\tau_h$ *is a unitary involution on* $End_D(M)$*;*

*(2)* $\tau_h(a) = \tau(a)$ *for all* $a \in F$*;*

*(2) There is a one-to-one correspondence between regular hermitian forms on* $M$ *up to a multiplication of an element in* $F_0^\times$ *and the unitary involutions on* $End_D(M)$ *whose restrictions on* $F$ *agree with* $\tau|_F$*.*

The involution $\tau_h$ in Theorem 2.2.9 is called the *adjoint involution* with respect to $h$.

We assume that $A$ is a central simple $F$-algebra on which there are two unitary involutions $\tau_1$ and $\tau_2$. We say that $\tau_2$ is $\tau_1$-*centered* if $\tau_2|_F = \tau_1|_F$. Obviously, if $\tau_1$ is an $F/F_0$-involution then the same to $\tau_2$. For such $\tau_1$ and $\tau_2$, we have the following results.

**Lemma 2.2.10.** $\tau_2$ *is* $\tau_1$*-centered if and only if* $\tau_2 = Int(a) \circ \tau_1$ *for some* $a \in A^\times$. *Moreover,* $\tau_1(a) = a$.

*Proof.* One direction is obvious.

For another direction, we assume that $\tau_1|_F = \tau_2|_F$. By Theorem 2.1.7 (Skolem-Noether), there is an element $b \in A^\times$ such that $\tau_2 = Int(b) \circ \tau_1$. Then we have $\tau_1 \circ Int(b) = Int(\tau_1(b^{-1})) \circ \tau_1$, and then $\tau_2^2 = Int(b \cdot \tau_1(b^{-1})) = id_A$. Therefore, $b = c \cdot \tau_1(b)$ for some $c \in F$.

Since $\tau_1(c) \in F$, $b = c \cdot \tau_1\big(c \cdot \tau_1(b)\big) = c \cdot \tau_1(c) \cdot b$. Since $Gal(F/F_0) =< \tau_1|_F >$, $N_{F/F_0}(c) = 1$. By Hilbert 90, there is an element $d \in F^\times$ such that $c = \tau_1(d) \cdot d^{-1}$.

Let $a = bd$. Then $\tau_1(a) = \tau_1(d)\tau_1(b) = cd\tau_1(b) = cdc^{-1}b = db = bd = a$. Meanwhile, for any element $x \in A$, we have $a\tau_1(x)a^{-1} = bd\tau_1(x)d^{-1}b^{-1} = b\tau_1(x)b^{-1}$.

□

Another useful property of unitary involutions is from reduced characteristic polynomials. As before, $A$ is a central simple $F$-algebra.

**Proposition 2.2.11.** *Let $a \in A$, and let $Red_a(X)$ denote the reduced characteristic polynomial of $a$. Assume that $A$ has an $F/F_0$-involution $\tau$. Then $\tau|_F\Big(Red_a(X)\Big) = Red_{\tau(a)}(X) \in F[X]$.*

*Proof.* cf. Corollary 2.16 in [16]. □

From Proposition 2.2.11, we directly obtain that the reduced norm $Nrd_F\big(\tau(a)\big) = \tau\big(Nrd_F(a)\big) \in F$ and the reduced trace $Trd_F\big(\tau(a)\big) = \tau\big(Trd_F(a)\big) \in F$.

If $F_0^{sep}$ is a separable closure of $F_0$, then $F \otimes_{F_0} F_0^{sep} \cong F_0^{sep} \times F_0^{sep}$ and $A \otimes_{F_0} F_0^{sep}$ becomes a semi-simple $F_0^{sep}$-algebra. Therefore, we can extend the discussion of unitary involutions to the case when $A = A_1 \times A_2$, where $A_1$ and $A_2$ are central simple $F_0$-algebras. Similar definitions and properties can be found from 2.B, of [16].

Let $Q$ be a quaternion algebra over a field $F$.

Then we can write $Q$ as in the form of a cyclic algebra $(F(\sqrt{\alpha}), \xi, \beta)$ by the Section 2.1, where $F(\sqrt{\alpha})$ is a Galois field extension of $F$ with $Gal(F(\sqrt{\alpha})/F) =< \xi >$. Moreover, $Q$ has an $F$-basis $\{1, i, j, k\}$ satisfying $i^2 = \alpha$, $j^2 = \beta$, and $ij = -ji = k$.

Then each element $a \in Q$ has a form that $a = a_0 + a_1 i + a_2 j + a_3 k$ where $a_0, a_1, a_2, a_3 \in F$.

We consider the conjugation $\sigma : Q \to Q$ defined by $\sigma(a) = a_0 - a_1 i - a_2 j - a_3 k$. By a direct verification, we can prove that $\sigma$ is an involution of the first kind on $Q$. We call it the *canonical involution* on $Q$.

Now we consider unitary involutions on the quaternion $F$-algebra $Q$. Let $F_0$ be a subfield of $F$ with $[F : F_0] = 2$, and let $Gal(F/F_0) =< \eta >$.

**Theorem 2.2.12** (Albert, cf. Proposition 2.22, [16])**.** *Assume that the quaternion algebra $Q$ has an $F/F_0$-involution $\tau$ and $\tau|_F = \eta$. Then there is a unique quaternion $F_0$-sub-algebra of $Q$, denoted by $Q_0$, such that*

*(1) $Q \cong_F Q_0 \otimes_{F_0} F$;*

*(2) $\tau = \sigma_0 \otimes \eta$, where $\sigma_0$ is the canonical involution on $Q_0$.*

# Chapter 3

# Whitehead Groups

In this Chapter, we will discuss the reduced (unitary) Whitehead group of a central simple algebra. Then summarize connections among central simple algebras, linear algebraic groups, and rational problems on algebraic groups.

## 3.1 Reduced Whitehead Group: $SK_1$

Let $A$ be central simple algebra over a field $F$. By Proposition 2.1.8 of last chapter, we have the reduced norm map $Nrd_F : A \to F$. Let $SL_1(A) = \{a \in A | Nrd_F(a) = 1\}$.

We focus on the multiplicative group $A^\times$ of $A$. $SL_1(A)$ is a subgroup of $A^\times$. Let $[A^\times, A^\times]$ denote the commutator subgroup of $A^\times$. Then $[A^\times, A^\times]$ is a normal subgroup of $SL_1(A)$.

**Definition 3.1.1.** The factor group $SL_1(A)/[A^\times, A^\times]$ is called the *reduced Whitehead group* of the central simple $F$-algebra $A$, which is denoted by $SK_1(A)$.

There was a problem of Tannaka-Artin on the triviality of $SK_1(A)$ around the year 1943. Nakayama and Matsushima proved that $SK_1(A)$ is trivial if the base field $F$ is a local field. Later, Wang, in [29], proved the triviality of $SK_1(A)$ when $F$ is a global field. It also can be deduced from [29] that $SK_1(A)$ is trivial if $ind_F(A)$ is square free.

However, in [25], Platonov proved that $SK_1(A)$ can be non-trivial by constructing a specific example.

For the case that cohomological dimension $cd(F) \leq 3$ and $deg_F(A) = 4$, Rost proved that $SK_1(A)$ is trivial. Then Suslin made a conjecture: Is it true that $SK_1(A)$ is trivial for $cd(F) \leq 3$? The conjecture is still an open question. A special case of Suslin's conjecture was proved by Nivedita Bhaskhar.

**Theorem 3.1.2** (Nivedita Bhaskhar, [3])**.** *Let $F$ be the function field of a $p$-adic curve, and let $A$ be a central simple $F$-algebra. Assume that the exponent $exp(A)$ is a prime $\ell$. If $\ell \neq p$ and $F$ contains a primitive $\ell^2$th root of unity, then $SK_1(A)$ is trivial.*

By Theorem 2.1.1, $A \cong M_r(D)$ for a finite dimensional division $F$-algebra. $SK_1(A)$ only depends on the Brauer class $[D] \in Br(F)$.

**Proposition 3.1.3.** *There is an isomorphism of groups $SK_1(A) \cong SK_1(D)$.*

*Proof.* cf. §22, §23 in [6]. □

Moreover, we know that $A$ has a primary decomposition by Theorem 2.1.13. Assume that $deg_F(A) = p_1^{s_1} p_2^{s_2} \cdots p_t^{s_t}$ is the decomposition of primes. Then we can write $A \cong_F A_1^{s_1} \otimes A_2^{s_2} \otimes \cdots \otimes A_t^{s_t}$, where each $A_i$ is a finite dimensional central division $F$-algebra of degree $p_i^{s_i}$ for all $i = 1, 2, ..., t$. The next proposition provides a decomposition of $SK_1(A)$.

**Proposition 3.1.4.** *There is an isomorphism of groups*

$$SK_1(A) \cong SK_1(A_1) \times SK_1(A_2) \times \cdots \times SK_1(A_t).$$

*Proof.* cf. Lemma 5, Lemma 6, §23, [6].

□

The theory of $SK_1(A)$ is called the reduced $K$-theory. One can also define $SK_1(A)$ from the general algebraic $K$-theory.

## 3.2 Reduced Unitary Whitehead Group: $SK_1U$

Let $A$ be a central division algebra over a field $F$. Now, we assume that $A$ has an $F/F_0$-involution $\tau$. We continue to focus on the group $A^\times$.

The set $\{a \in A^\times | Nrd_F(a) \in F_0\}$ is actually a subgroup of $A^\times$, which is denoted by $\Sigma'_\tau(A)$. Another set $\{a \in A^\times | \tau(a) = a\}$ may not be a subgroup of $A^\times$, but we can have a subgroup of $A^\times$ generated by the set and denote it by $\Sigma_\tau(A)$.

**Lemma 3.2.1.** *$\Sigma_\tau(A)$ is a normal subgroup of $\Sigma'_\tau(A)$.*

*Proof.* Let $a \in \Sigma_\tau(A)$, and let $b \in \Sigma'_\tau(A)$. Then $a = y_1 y_2 \cdots y_m$, where $\tau(y_i) = y_i$ for all $i = 1, 2, ..., m$. Since $Nrd_F(a) = Nrd_F(y_1 \cdots y_m) \in F$ and $Nrd_F(a) = \tau\big(Nrd_F(a)\big)$, $a \in \Sigma'_\tau(A)$.

Since $Nrd_F\big(\tau(aba^{-1})\big) = Nrd_F(b) \in F_0$, $\Sigma_\tau(A)$ is a normal subgroup of $\Sigma'_\tau(A)$.

□

**Definition 3.2.2.** The factor group $\Sigma'_\tau(A)/\Sigma_\tau(A)$ is called the *reduced unitary Whitehead group* of $A$ with respect to $\tau$, which is denoted by $SK_1U(A, \tau)$.

For $SK_1U(A, \tau)$, one can also have the question: Is it true that $SK_1U(A, \tau)$ is trivial? There are examples of trivial $SK_1U(A, \tau)$. V.P.Platonov and V.I.Yanchevskii proved that $SK_1U(A, \tau) = 1$ for global fields around the year 1973; V.I.Yanchevskii proved its triviality when $F$ is perfect and $cd(F) \leq 2$. In [32], V.I.Yanchevskii discussed the case when $F$ is an Henselian discretely valued field.

$SK_1U(A, \tau)$ actually depends on the class of $\tau$-centered involutions.

**Lemma 3.2.3.** *Assume that $A$ has two unitary involutions $\tau_1$ and $\tau_2$. If $\tau_2$ is $\tau_1$-centered, then*

*(1)* $\Sigma'_{\tau_1}(A) = \Sigma'_{\tau_2}(A)$;

*(2)* $\Sigma_{\tau_1}(A) = \Sigma_{\tau_2}(A)$.

*Therefore,* $SK_1U(A, \tau_1) = SK_1U(A, \tau_2)$.

*Proof.* (1) Since $\tau_1|_F = \tau_2|_F$, $\tau_2$ is also an $F/F_0$-involution. Then $\Sigma'_{\tau_1}(A) = \Sigma'_{\tau_2}(A)$.

(2) Let $x \in A^\times$ such that $\tau_1(x) = x$. Then $\tau_1(x) = a\tau_2(x)a^{-1} = x$ and $\tau_2(xa) = \tau_2(a)\tau_2(x) = a\tau_2(x) = ax$ by Lemma 2.2.10. Thus $x = (xa)a^{-1} \in \Sigma_{\tau_2}(A)$. □

$SK_1U(A, \tau)$ also depends on the Brauer class $[A] \in Br(F)$.

**Theorem 3.2.4.** *Let $D$ be a finite dimensional central division $F$-algebra such that $[D] = [A] \in Br(F)$. Then there is an $F/F_0$-involution $\tau_D$ on $D$ such that $SK_1U(D, \tau_D) \cong SK_1U(A, \tau)$.*

*Proof.* cf. Lemma 2 & Lemma3, [15] . □

Let $A = A_1 \otimes A_2 \otimes \cdots \otimes A_t$ be the primary decomposition of $A$ by Theorem 2.1.13. Then we have an isomorphism.

**Theorem 3.2.5.** *There exist an unitary involution $\tau_i$ on each primary component $A_i$ for each $i = 1, 2, ..., t$ such that*

$$SK_1U(A, \tau) \cong SK_1U(A_1, \tau_1) \times \cdots \times SK_1U(A_t, \tau_t).$$

Since both $SK_1(A)$ and $SK_1U(A, \tau)$ depend on the Brauer class, we may mainly focus on a finite dimensional division $F$-algebra $D$ satisfying $[D] = [A] \in Br(F)$ from now. We always assume that $D$ has an $F/F_0$-involution $\tau$ when mention $SK_1U(D, \tau)$.

There are some maps between $SK_1U(D, \tau)$ and $SK_1(D)$, which will provide ideas on calculating them.

For each $x \in \Sigma'_\tau(D)$, we have $x = \tau(x) \cdot a_x$ for some $a_x \in SL_1(D)$ since $Nrd_F(x) =$

$Nrd_F(\tau(x)) \in F_o$. Then we define a map

$$\phi : SK_1U(D,\tau) \to SK_1(D) = SL_1(D)/[D^\times, D^\times]$$

by $\phi(\overline{x}) = \overline{a_x}$. We claim that $\phi$ is a homomorphism.

**Lemma 3.2.6.** *The map $\phi$ defined above is a homomorphism of groups, and the exponent of the kernel of $\phi$ divides 2.*

*Proof.* If $\overline{x} = \overline{y} \in SK_1U(D,\tau)$, then $x = z_1 z_2 \cdots z_m y$ and $z_1, ..., z_m$ are $\tau$-invariant elements in $D^\times$. Since $x = \tau(x)a_x$ and $y = \tau(y)a_y$ for some $a_x, a_y \in SL_1(D)$, $a_x a_y^{-1} = \tau(x^{-1})xy^{-1}\tau(y) = \tau(z_1 \cdots z_m y)^{-1}(z_1 z_2 \cdots z_m)\tau(y)$. By induction, it can be verified that $a_x a_y^{-1} \in [D^\times, D^\times]$. Then $\phi(\overline{x}) = \phi(\overline{y}) \in SK_1(D)$ if $\overline{x} = \overline{y} \in SK_1U(D,\tau)$.

Let $\overline{x_1}, \overline{x_2} \in SK_1U(D,\tau)$. Then $\phi(\overline{x_1}) \cdot \phi(\overline{x_2}) = \overline{a_{x_1}} \cdot \overline{a_{x_2}}$. Assume that $\phi(\overline{x_1} \cdot \overline{x_2}) = \overline{a_3}$, then $x_1 x_2 = \tau(x_1 x_2)a_3$.

By calculation, $\tau(x_1)a_1\tau(x_2) = \tau(x_2)\tau(x_1)a_3 a_2^{-1}$ and then $\big(\tau(x_2^{-1})a_1\tau(x_2)a_1^{-1}\big) = \big(\tau(x_2^{-1})\tau(x_1^{-1})\tau(x_2)\tau(x_1)\big)a_3(a_1 a_2)^{-1}$. Therefore, $a_3(a_1 a_2)^{-1} \in [D^\times, D^\times]$ and then $\phi$ is a homomorphism.

Suppose that $\phi(\overline{x}) = 0$. Then $x^2 = x \cdot \tau(x) \cdot a_x \in \Sigma_\tau(D)$. Since $[D^\times, D^\times] \subset \Sigma_\tau(D)$ (cf. Theorem 2, §2, Chapter 4, Part II [4] ) and $x\tau(x) \in \Sigma_\tau(D)$, it follows that $x^2 \in \Sigma_\tau(D)$ and the exponent of $ker(\phi)$ divides 2.

□

**Proposition 3.2.7.**

*(1) Let $n$ be the index of $D$. Then $a^n \in \Sigma_\tau(A)$ for each $a \in \Sigma'_\tau(A)$;*

*(2) If $n$ is odd and $SK_1(D) = 1$, then $SK_1U(D,\tau) = 1$ .*

*Proof.* For (1), refer to Corollary 2.5, [32].

(2) Suppose that $n$ is odd and $SK_1(D)$ is trivial. Then by Lemma 3.2.6, the group $SK_1U(D,\tau)$ is 2-torsion. By (1), $SK_1U(D,\tau)$ is $n$-torsion. Since $n$ is odd, $SK_1U(D,\tau)$ is trivial. □

On the other hand, we can also construct a map from $SK_1(D)$ to $SK_1U(D,\tau)$.

**Corollary 3.2.8.** *There is an exact sequence of groups:*

$$SK_1(D) \xrightarrow{f_1} SK_1U(D,\tau) \xrightarrow{f_2} \frac{F_0^* \cap Nrd_{D/F}(D)}{< Nrd_{D/F}(a) | a \in \Sigma(D) >} \to 1,$$

*where $f_1$ is induced by $SL_1(D) \to SK_1U(D,\tau)$ and $f_2$ is induced by $Nrd_{D/F}$ : $D^\times \to F^\times$.*

*Proof.* For any element $\overline{x} \in SK_1(D)$, let $f_1(\overline{x}) = \overline{x} \in SK_1U(D,\tau)$. Since $[D^\times, D^\times] \subset \Sigma(D)$, $f_1$ is well defined.

For any element $\overline{y} \in SK_1U(D,\tau)$, let $f_2(\overline{y}) = \overline{Nrd_{D/F}(y)}$. Then $f_2$ is surjective.

Since the definition of $SK_1(D) = SL_1(D)/[D^\times, D^\times]$, $im(f_1) \subset ker(f_2)$.

Suppose that $\overline{y} \in ker(f_2)$. Then $y = a_y \cdot y'$ for some $a_y \in SL_1(D)$ and some $y' \in \Sigma(D)$. Therefore, $\overline{y} = \overline{a_y} = f_1(\overline{a_y}) \in SK_1U(D,\tau)$ and $ker(f_2) \subset im(f_1)$.

□

## 3.3 $R$-Equivalence and Whitehead Groups of Algebraic Groups

The $R$-equivalence is defined by Manin ([19]), which is an equivalence relation on the rational points of an algebraic variety.

Let $F$ be a field. Let $X \to Spec(F)$ be a variety, which is separable, geometrically integral, and finite type. Assume that $a, b \in X(F)$. If there is an $F$-rational map $\phi : \mathbb{A}^1_F \dashrightarrow X$ such that 0 maps to $a$ and 1 maps to $b$, we say $a, b$ are *directly R-equivalence.*

**Definition 3.3.1.** The equivalence relation generated by the directly $R$-equivalence on the set $X(F)$ is called $R$-equivalence on $X(F)$.

If a variety $G/F$ is a connected linear algebraic group over $F$, then strictly $R$-equivalence is same as $R$-equivalence. This can be proved by a right translation of $G$. In particular, we denote the equivalence class of the identity element in $G(F)$ as $RG(F)$. We can verify that $RG(F)$ is a subgroup of $G(F)$. Since a conjugation map on $G$ is a rational map, $RG(F)$ is moreover a normal subgroup of $G(F)$.

**Definition 3.3.2.** The group $G(F)/RG(F)$ is called the *group of $R$-equivalence classes* of $G(F)$.

The group of R-equivalence classes, $G(F)/RG(F)$, is very useful while studying the rationality problem for algebraic groups,i.e. the problem to determine whether the variety of an algebraic group is rational or stably rational.

For $G$, a smooth connected linear algebraic group defined over $F$, we say that $G$ is *rational* if its function field is purely transcendental over $F$. We say that $G$ is $F$-*stably rational* if $G \times_F \mathbb{A}^n_F$ is rational for some $n \in \mathbb{N}$. If $G$ is $F$-stably rational, then $G(F)/RG(F) = 1$. Thus, if one can establish non-triviality of the group $G(E)/RG(E)$ just for one field extension $E/F$, the group $G$ is not $F$-stably rational.

Let $F_0$ be a field. Let $G$ be a semi-simple, simply connected, isotropic, and simple algebraic group over $F_0$. Let $G(F_0)$ be the $F_0$-rational points subgroup of $G$, and $G(F_0)^+$ be the normal subgroup of $G(F_0)$ generated by the $F_0$-rational points of the unipotent radicals of parabolic $F_0$-subgroups of G. We call $G(F_0)/G(F_0)^+$ the *Whitehead group* for $G$ over $F_0$, denoted by $W(G, F_0)$.

**Theorem 3.3.3** (Voskresenskii, [28])**.**

*(1) Suppose that $D_0$ is a central division $F_0$-algebra. If $G(F_0) = SL_n(D_0)$ for some $n > 1$, then*

$$W(G, F_0) \cong SK_1(D_0).$$

*Let $F$ be a quadratic field extension of $F_0$ and $D$ be a central division $F$-algebra. (2)Suppose that $D$ has an involution of second kind $\tau$ such that $F^\tau = F_0$. If the* *hermitian form $h_\tau$ is isotropic and $G(F_0) = SU(D, h_\tau)$, then*

$$W(G, F_0) \cong SK_1U(D, \tau).$$

The *Kneser-Tits conjecture* predicted the triviality of the group $W(G, F_0)$. Due to the former results on $SK_1(A)$ and $SK_1U(A, \tau)$, the Kneser-Tits conjecture is not valid in general. We refer to [9] for more details and a summary of $W(G, F_0)$.

**Theorem 3.3.4** (Theorem 7.6, [24])**.** *Let $F$ be a non-Archimedean locally compact field. Then the Kneser-Tits conjecture holds for any simple simply connected $F$-isotropic group $G$, i.e. $W(G, F) = \{1\}$.*

Over number fields, the conjecture is proven to be true (cf. [9]). Further the conjecture is also proven to hold for fields of cohomological dimension at most 2 (cf. [9]).

# Chapter 4

# Patching

Patching techniques were developed by Harbater, Hartmann and Krashen (cf. [10], [11], [12], [13]) to study torsors under linear algebraic groups. One of the arithmetic applications of patching is certain forms of 'local-global principles'. Firstly, I introduce the case on vector spaces over given fields, which is the base for this topic.

## 4.1 Patching for Vector Spaces

Let $\mathcal{F} = \{F_i\}_{i\in I}$ be a finite inverse system of fields and inclusions, whose inverse limit (in the category of rings) is a field $F$. For $i$, $j \in I$, we write $F_i \hookrightarrow F_j$ if $i \succ j$. We now define the *category of vector spaces patching problems for* $\mathcal{F}$, named PP($\mathcal{F}$), as followings:

$$Object : \Big(\mathcal{V} = \{V_i\}_{i\in I} \quad ; \quad \nu_{i,j} : V_i \otimes_{F_i} F_j \cong_{F_j} V_j, i \succ j\Big),$$

and

$$Morphism : \Big(f : \{V_i\}_{i\in I} \to \{V_i'\}_{i\in I}\Big) := \Big\{\phi_i : V_i \to_{F_i} V_i' \Big| \phi_j \circ \nu_{i,j} = \nu'_{i,j} \circ (\phi_i \otimes_{F_i} F_j), i \succ j\Big\}_{i,j\in I},$$

where each $V_i$ is a finite dimensional $F_i$-vector space for each $i \in I$.

Then there is a functor

$$\beta : \mathrm{Vect}(F) \to \mathrm{PP}(\mathcal{F})$$

$$V \mapsto \mathcal{V} = \{V \otimes_F F_i\}_{i \in I}$$

from the category of finite dimensional $F$-vector spaces to the category of vector space patching problems for $\mathcal{F}$.

**Definition 4.1.1.** We say that a vector space $V$ over $F$ is a *solution* to a patching problem $\mathcal{V}$ if $\beta(V)$ is isomorphic to $\mathcal{V}$ in the category $\mathrm{PP}(\mathcal{F})$.

In my current research, it asks for more condition on the inverse system.

**Definition 4.1.2** (cf. Definition 2.1, [13])**.** A *factorization inverse system* over a field $F$ is a finite inverse system of fields whose inverse limit is $F$, and whose index set $I$ has the following property: There is a partition $I = I_v \cup I_e$ into a disjoint union such that for each index $k \in I_e$, there are exactly two elements $i, j \in I_v$ for which $i, j \succ k$, and there are no other relation in $I$.

For example, let $F_1$, $F_2$, and $F = F_1 \cap F_2$ be sub-fields of a given field $F_0$. Then $\mathcal{V} = \{F_i\}_{i \in I}$ with $I = \{0, 1, 2\}$ is a factorization inverse system with $\varprojlim F_i = F$.

Now we can describe the idea of patching for vector spaces. Let $\mathcal{F} = \{F_i\}_{i \in I}$ be a finite inverse system. The aim is to study the case where the functor $\beta : \mathrm{Vect}(F) \to \mathrm{PP}(\mathcal{F})$ is an equivalence of categories, which can have the following motivation.

If $\mathcal{F} = \{F_i\}_{i \in I = I_e \cup I_v}$ is a factorization system, there is an ordered triple $(l, r, k)_{l \in I_v, r \in I_v, l, r \succ k \in I_e}$ for each $k \in I_e$. In fact, the factorization system is a finite multi-graph with an orientation for each $(l, r, k)$(cf. section 2.1, [12]). Assume that a vector space $V$ over $F$ is a solution of a patching problem $\mathcal{V} = \{V_i\}_{i \in I}$. Then

$$\beta(V) \cong \Big(\mathcal{V} = \{V_i\}_{i \in I} \quad ; \quad \nu_{i,j} : V_i \otimes_{F_i} F_j \cong_{F_j} V_j, i \succ j\Big)$$

$$= \Big(\mathcal{V} = \{V_i\}_{i\in I} \quad ; \quad \mu_k : V_l \otimes_{F_l} F_k \cong_{F_k} V_r \otimes_{F_r} F_k, (l,r,k)_{l\in I_v, r\in I_v, l,r\succ k\in I_e}\Big),$$

where $\mu_k = \nu_{r,k}^{-1} \circ \nu_{l,k}$.

Then, we can recover the structure of $V/F$ through the factorization system whose vector spaces satisfy the isomorphism condition over some common field extensions if $\beta$ is an equivalence of categories. In fact, the idea is similar to the definition of 'sheaf'.

Before stating a general proposition on the equivalence of $\beta$, we need the following definition.

**Definition 4.1.3** (cf. [13], Section 2)**.** Let $\mathcal{F} = \{F_i\}_{i\in I=I_v\cup I_e}$ be a factorization inverse system with inverse limit a field $F$. Let $G$ be a linear algebraic group over $F$. We say that *simultaneous factorization holds for $G$ over $\mathcal{F}$* if for any collection of elements $A_k \in G(F_k)$, for $k \in I_e$, there exist elements $A_i \in G(F_i)$ for all $i \in I_v$ such that $A_k = A_r^{-1}A_l \in G(F_k)$ for each such triple $(l,r,k)_{l\in I_v, r\in I_v, l,r\succ k\in I_e}$.

Now, we have the following result:

**Proposition 4.1.4** (Proposition 2.2, [13])**.** *Let $\mathcal{F}$ be a factorization inverse system over a field $F$. Then the functor $\beta : \mathrm{Vect}(F) \to \mathrm{PP}(\mathcal{F})$ is an equivalence if and only if simultaneous factorization holds for $\mathrm{GL_n}$ over $\mathcal{F}$ for every $n \geq 1$.*

## 4.2 Local-Global Principles over Arithmetic Curves

We will see that patching can be applied to linear algebraic groups defined over a function field of an arithmetic curve.

#### Resolution of Singularities

We state the general theory on resolution of singularities here, which can help to get an ideal model of a $p$-adic curve $X$ and simplify the structure of a division algebra later(cf. [26]).

Let $S$ be a Dedekind scheme. We call an integral, projective, flat $S$-scheme $\pi : \mathscr{X} \to S$ of dimension 2 a *fibered surface over* $S$. We have the following 'embedded resolution':

**Theorem 4.2.1** (Theorem 9.2.26, [17]). *Assume that $\mathscr{X} \to S$ is a regular fibered surface. Let $D$ be an effective Cartier divisor on $\mathscr{X}$. Suppose that the scheme $D$ is excellent. Then there exists a projective bi-rational morphism $f : \mathscr{X}' \to \mathscr{X}$ with $\mathscr{X}'$ regular, such that $f^*(D)$ is a divisor with normal crossings.*

Notice that any fibered surface $\mathscr{X}$ is excellent if $S$ is excellent. For example, $S = Spec(R)$ for some complete discrete valuation ring $R$. We call a regular fibered surface $\mathscr{X} \to S$ over a Dedekind scheme $S$ of dimension 1 an *arithmetic surface*. Then we have the following corollary:

**Corollary 4.2.2** (Corollary 9.2.30, [17]). *Let $\mathscr{X} \to S$ be an arithmetic surface that has only a finite number of singular fibers. Then there exists a projective bi-rational morphism $\mathscr{X}' \to \mathscr{X}$ such that $\mathscr{X}' \to S$ is an arithmetic surface with normal crossings.*

Next, recall the definition of models of algebraic curves:

**Definition 4.2.3** (cf. Chapter 10, [17]). Let $S$ be a Dedekind scheme of dimension 1, with function field $K$. Let $X$ be a normal, connected, projective curve over $K$. We call a normal fibered surface $\mathscr{X} \to S$ together with an isomorphism $f : \mathscr{X} \times_S Spec(K) \cong X$ a *model of* $X$ *over* $S$.

For example, $Proj\ \mathbb{Z}[x, y, z]/(x^q + y^q + z^q)$ is a model of the projective curve over $\mathbb{Q}$ defined by the same equation for some square free integer $q \geq 1$.

Since the above theorem is valid for regular fibered surfaces, we may ask if there is always some regular model of the given curve. In fact, we have a positive answer when $S$ is affine.

**Theorem 4.2.4** (Proposition 10.1.8, [17]). *Suppose that $S = Spec(R)$ is an affine Dedekind scheme of dimension 1, with function field $K$. Let $X$ be a smooth projective curve of geometric genus $g \geq 1$ over $K$. Then $X$ admits a regular model $\mathscr{X} \to S$ with normal crossings.*

**Branches**

Let $R$ be a complete discrete valuation ring with a uniformizer $t$. Let $K$ denote the fraction field of $R$ and $k$ denote the residue field of $R$.

Let $F$ be the function field of a smooth projective geometrically integral curve $X \to Spec(K)$. We assume that $\mathscr{X} \to Spec(R)$ is a regular model.

Let $X_o$ denote the reduced closed fiber of $\mathscr{X}$. For each point $P \in X_o$, let $R_P = \mathscr{O}_{\mathscr{X},P}$. We denote the completion of $R_P$ as $\widehat{R}_P = \lim\limits_{\leftarrow} \frac{\mathscr{O}_{\mathscr{X},P}}{(\mathfrak{m}_{\mathscr{X},P})^n}$, and demote the fraction field $F_P = Frac(\widehat{R}_P)$.

For each generic point $\eta \in X_o$ and each non-empty open subset $U$ of $\overline{\{\eta\}} \subset X_o$, we denote the set of regular functions on $U$ by $R_U = \{f \in F | f \in \mathscr{O}_{\mathscr{X},Q} \text{ for each } Q \in U\}$. Denote the $(t)$-adic completion of $R_U$ by $\widehat{R}_U$, and let $F_U = Frac(\widehat{R}_U)$. Then we have $F \subset F_U \subset F_\eta$ if $U \subset \overline{\{\eta\}} \subset X_o$.

If $P \in X_o$ is a closed point in $X_o$ and $P \in \overline{\{U\}} = \overline{\{\eta\}} \subset X_0$ for some non-empty open subset $U \subset \overline{\{\eta\}}$ , then we can find a height one prime ideal $\mathfrak{P}$ of $\widehat{R}_P$ containing $t$. We call such $\mathfrak{P}$ a *branch* on $U$ at $P$. Let $\widehat{R}_\mathfrak{P}$ denote the completion of the localization of $\widehat{R}_P$ at $\mathfrak{P}$; let $F_\mathfrak{P}$ denote the fraction field of $\widehat{R}_\mathfrak{P}$. Then we have $F \subset F_U, F_P \subset F_\mathfrak{P}$ in this case.

Now, let $\mathscr{P}$ be a non-empty finite set of closed points of $X_o$ that contains all the closed points in which distinct irreducible components of $X_o$ meet and at least one point on each component of $X_o$. Let $\mathscr{U}$ be the collection of irreducible components of $X_o \backslash \mathscr{P}$. Let $\mathscr{B}$ be the collection of branches of $X_o$ at points of $\mathscr{P}$. This yields a finite inverse system $\mathcal{F}$ of fields $F_P$, $F_U$, $F_\mathfrak{P}$ for $P \in \mathscr{P}$, $U \in \mathscr{U}$, and $\mathfrak{P} \in \mathscr{B}$. In fact, the inverse limit of $\mathcal{F}$ is the field $F$ (Proposition 3.3, [13]), and we have the following proposition:

**Proposition 4.2.5** (Corollary 3.4, [13]). *The finite inverse system $\mathcal{F}$ given above is a factorization inverse system with inverse limit $F$. For this system, the base change functor $\beta : \mathrm{Vect}(F) \to \mathrm{PP}(\mathcal{F})$ is an equivalence of categories.*

In [13], the above proposition can be proved if the set $\mathscr{P} = g^{-1}(\infty)$ for some finite morphism $g : \mathscr{X} \to \mathbb{P}^1_R$ . We can always find such morphism $g$ in our situation.

**Proposition 4.2.6** (Proposition 3.3, [13]). *Let $W$ be finite set of closed points of $\mathscr{X}$. Write $\infty \in \mathbb{P}^1_k \subset \mathbb{P}^1_R$. There is a finite morphism $g : \mathscr{X} \to \mathbb{P}^1_R$ such that $g^{-1}(\infty) = W$ if and only if $W$ meets each irreducible component of $\mathscr{X}$ non-trivially.*

### Factorization for Rational Connected Linear Algebraic Group

We keep the above notations. A connected linear algebraic group $G$ over $F$ is *rational* if it is a rational $F$-variety, i.e. $G$ is bi-rational to $\mathbb{P}^N_F$.

In the above proposition, we know that there exists a morphism $g : \mathscr{X} \to \mathbb{P}^1_R$ such that $\mathscr{P} = g^{-1}(\infty)$. Then let $\mathscr{V}$ be the collection of irreducible components $V$ of $g^{-1}(\mathbb{A}^1_k)$, and recall that $\mathscr{B}$ is the collection of all branches $\mathfrak{P}$ at the points of $\mathscr{P}$.

We have the following factorization theorem:

**Theorem 4.2.7** (Theorem 3.6, [11]). *Let $G$ be a rational connected linear algebraic group over $F$. Suppose that there is an element $x_{\mathfrak{P}} \in G(F_{\mathfrak{P}})$ for each branch $\mathfrak{P} \in \mathscr{B}$. Then there is an element $x_P \in G(F_P)$ for each $P \in \mathscr{P}$, and an element $x_V \in G(F_V)$ for each $V \in \mathscr{V}$, such that $x_{\mathfrak{P}} = x_P \cdot x_V$ for every branch $\mathfrak{P} \in \mathscr{B}$ at a point $P \in \mathscr{P}$ with $\mathfrak{P}$ lying on the closure of some $V \in \mathscr{V}$.*

In the above theorem, each product $x_P \cdot x_V$ is taken in $G(F_{\mathfrak{P}})$ with respect to the inclusion $F_P, F_V \to F_{\mathfrak{P}}$.

We keep the notations. Assume that a linear algebraic group $G$ acts on a variety $H$ over a field $E$. We say that $G$ *acts transitively on the points of* $H$ if every field extension $E'$ of $E$ the induced action of the group $G(E')$ on the set $H(E')$ is transitive. Here we have a local-global principle for homogeneous spaces:

**Theorem 4.2.8** (Theorem 3.7, [11]). *Let $G$ be a rational connected linear algebraic group over $F$ which acts transitively on the points of an $F$-variety $H$. Then, $H(F) \neq \emptyset$ if and only if $H(F_P) \neq \emptyset$ for each $P \in \mathscr{P}$ and $H(F_V) \neq \emptyset$ for each $V \in \mathscr{V}$.*

**Local-Global Principles on $X_o$**

We keep all the notations in **Branches**. In the classical case of local-global principles over a number field $E$, there is the obstruction

$$Ш_{\Omega}(E,G) = ker\Big(H^1(E,G) \to \prod_{v\in\Omega} H^1(E_v,G)\Big)$$

to the validity of this local-global principle, where $G \to Spec(E)$ is a linear algebraic group.

In the case $F$, we have the similar situation if considering prime divisors on a regular model $\mathscr{X} \to Spec(R)$. Actually, we know that such a regular model always exists in our case. Then we can write:

$$Ш(F,G) = ker\Big(H^1(F,G) \to \prod_{D\,\text{prime}} H^1(F_{v_D},G)\Big).$$

If consider all the points of $X_o = (\mathscr{X} \times_{Spec(R)} Spec(k))_{red}$, we can define the obstruction

$$Ш_{\mathscr{X},X_o}(F,G) = ker\Big(H^1(F,G) \to \prod_{P\in X_o} H^1(F_P,G)\Big).$$

Notice that $P$ can be a generic point of $X_o$ in the sense of **Branches**.

Now, we always assume that $\mathscr{X}$ is a regular model with closed fiber $X_o$. For each generic point $\eta$ of $X_o$, there is a relation between the fields $F_\eta$ and $F_U$ for each non-empty open $U \subset \overline{\{\eta\}}$ in $X_o$. By Section 3.2.1 of [12], there is a procedure which is a kind of henselization.

Let $R^h_\eta = \lim\limits_{\to} \widehat{R}_V$, where $V$ ranges over the non-empty open subsets of $X_o$ that do not meet any other irreducible component of $X_o$. Let $F^h_\eta = Frac(R^h_\eta)$. Since $R_V \hookrightarrow R_\eta$ for each $V \in \overline{\{\eta\}}$, $F^h_\eta \hookrightarrow F_\eta$.

**Lemma 4.2.9** (Lemma 3.2.1, [12]). *Let $\overline{\{\eta\}} = C_\eta \subset X_o$ be an irreducible component, and $U_\eta \subset C_\eta$ be a non-empty open subset meeting no other component. Then $R^h_\eta$ is a Henselian discrete valuation with respect to the $\eta$-adic valuation, having residue field $k(U_\eta) = k(C_\eta)$. The field $F^h_\eta$ is the filtered direct limit of the fields $F_V$, where $V$ ranges over the non-empty open subsets of $U_\eta$.*

Later, we will see that $F_\eta$ can be approximated by $F_{U_\eta}$ with the help of the above lemma and our next results. Currently, we have another approximation result on smooth commutative group schemes.

**Proposition 4.2.10** (Proposition 3.2.2, [12]). *Let $G$ be a smooth commutative group scheme over $F$. If $\alpha \in H^n(F_{U_\eta}, G)$ satisfies $\alpha \otimes F_\eta = 0$, then $\alpha \otimes F_V = 0$ for some Zariski open neighborhood $V$ of $\eta$ in $U$.*

By the above proposition, we can shrink a open subset $U_\eta$.

Finally, there is a local-global principle on $X_o$.

**Theorem 4.2.11** (Theorem 3.2.3, [12]). *Let $G$ be a commutative linear algebraic group over $F$, and $n \geq 1$. Assume that either*

*(1) $G = \mathbb{Z}/m\mathbb{Z}(r)$, where $m$ is an integer not divisible by $char(k)$, and where either $r = n-1$ or else $[F(\mu_m) : F]$ is prime to $m$; or*

*(2) $G = \mathbb{G}_m$, $char(k) = 0$, and $K$ contains a primitive $m$-th root of unity for all $m \geq 1$.*

*Then*

$$0 \to H^n(F, G) \to \prod_{P \in X_o} H^n(F_P, G).$$

# Chapter 5

# $SK_1U$ over Function Fields of $p$-adic Curves

In this chapter we prove the following theorem.

**Theorem 5.0.1.** *Let $K$ be a $p$-adic field with $p \geq 3$, i.e. a finite extension of $\mathbb{Q}_p$, $R$ be the valuation ring of $K$, and $k$ be the residue field. Let $X \to Spec(K)$ be a smooth projective curve and $F_0$ be the function field of $X$. Let $F$ be a quadratic field extension of $F_0$ and $F = F_0(\sqrt{d})$. Let $D$ be a central division algebra over $F$ of period 2 with an $F/F_0$-involution $\tau$. Then the reduced unitary Whitehead group $SK_1U(D,\tau)$ is trivial.*

## 5.1 The Plan of the proof

According to Corollary 3.2.8, there is an exact sequence of groups:

$$SK_1(D) \xrightarrow{f_1} SK_1U(D,\tau) \xrightarrow{f_2} \frac{F_0^* \cap Nrd_{D/F}(D)}{< Nrd_{D/F}(a) | a \in \Sigma(D) >} \to 1.$$

Since period of $D$ is 2, We know that $SK_1(D) = 1$ [29]. We prove that the third item $\frac{F_0^* \cap Nrd_{D/F}(D)}{<Nrd_{D/F}(a)|a\in\Sigma(D)>}$ is trivial.

We start from choosing a suitable model $\mathscr{X} \to Spec(R_0)$. By Theorem 10.1.8 of

[17], we can firstly assume that $\mathscr{X}$ is regular.

Let $\mathscr{X}^{(1)}$ be the set of codimenson one points of $\mathscr{X}$. For $P \in \mathscr{X}$, let $v_P$ denote the discrete valuation on $F$ given by $P$. For any element $y \in F$, we define the *support of $y$ in $\mathscr{X}$* as

$$supp_{\mathscr{X}}(y) = \{P \in \mathscr{X}^{(1)} | v_P(y) \neq 0\}.$$

Since $\mathscr{X}$ is proper over an affine scheme, $supp_{\mathscr{X}}(y)$ is a finite set.

Let $\tilde{\mathscr{X}}$ be the normal closure of $\mathscr{X}$ in $F$. Since $p \neq 2$, each point $P \in \tilde{\mathscr{X}}^{(1)}$ induces a residue map $\partial_{v_P} : H^2(F, \mu_2) \to H^1(\kappa(P), \mathbb{Z}/2\mathbb{Z})$. We define the *ramification locus of $D$ in $\mathscr{X}$*:

$$ram_{\mathscr{X}}(D) = \{P \in \mathscr{X}^{(1)} | \partial_{v_Q}([D]) \neq 0 \ for\ some\ Q \in \tilde{\mathscr{X}}\ lying\ over\ P\}.$$

Let $\lambda \in F_0^*$. We can choose $\mathscr{X} \to Spec(R)$ a regular model of $X$ with the reduced special fibre $X_0$ such that $ram_{\mathscr{X}}(D) \cup supp_{\mathscr{X}}(\lambda) \cup supp_{\mathscr{X}}(d) \cup X_0$ is a union of normal crossing regular curves (ref). Since period of $D$ is 2, by a theorem of

Suppose $\lambda \in F_0^* \cap Nrd_{D/F}(D)$. To show that $\lambda \in < Nrd_{D/F}(a) | a \in \Sigma(D) >$, we construct three quadratic extensions $L_i/F_0$ and $\mu_i \in L_i^*$ such that

i) $\prod_i N_{L_i/F_0}(\mu_i) = \lambda$

ii) $\text{ind}(D \otimes L_i) \leq 2$

iii) $\mu_i \in L_i^* \cap Nrd(D \otimes L_i)$.

Since $\text{ind}(D \otimes L_i) \leq 2$, $\mu_i \in < Nrd_{D \otimes L_i/F \otimes L_i}(a) | a \in \Sigma(D \otimes L_i) >$. Since $\prod_i N_{L_i/F_0}(\mu_i) = \lambda$, it follows that $\lambda \in < Nrd_{D/F}(a) | a \in \Sigma(D) >$.

To construct $L_i$ and $\mu_i \in L_i$, using methods of Parimala, Preeti and Suresh ([21]), first we construct such extensions $L_i$ and $\mu_i \in L_i$ locally over complete discretely valued fields and over fraction fields of two dimensional complete regular local rings of dimension 2 with some compatibility conditions. Then using the patching techniques of Harbater, Hartmann and Krashen. we get the required extensions $L_i$ and $\mu_i \in L_i$

over $F_0$.

## 5.2 Preliminaries

**Lemma 5.2.1.** *Let $R$ be a complete regular local ring with maximal ideal $(\pi, \delta)$, field of fractions $F$ and residue field $\kappa$. Suppose that char($\kappa$) not equal to 2. Let $F_\pi$ be the completion of $F$ at the discrete valuation given by $(\pi)$. Let $a = u\pi^\epsilon\delta^{\epsilon'} \in R$ with $u \in R$ a unit and $\epsilon, \epsilon' \in \mathbb{Z}$. Then for any element $\mu_\pi \in F_\pi(\sqrt{a})^*$, there exists $\mu \in F(\sqrt{a})^*$ such that $\mu = w\pi^r\delta^s\sqrt{a}^{s'}$ with $w \in R[\sqrt{a}]$ a unit, $r, s, s' \in \mathbb{Z}$ and $\mu_\pi\mu^{-1} \in F_\pi(\sqrt{a})$ is a unit at $\pi$ and maps to 1 modulo $\pi$.*

*Proof.* Note that if $\epsilon$ or $\epsilon'$ is even, then $R[\sqrt{a}]$ is a regular local ring with maximal ideal $(\pi_1, \delta_1)$ for some primes $\pi_1, \delta_1 \in \{\pi, \delta, \sqrt{a}\} \subset R[\sqrt{a}]$(cf. Lemma 3.1, [22]). In this case $F_\pi(\sqrt{a})$ is a complete discretely valued field with $\pi_1$ as a parameter and residue field a complete discretely valued field with valuation ring $R/(\pi)[\sqrt{\bar{a}}]$ and $\bar{\delta}_1$ as a parameter. Suppose both $\epsilon$ and $\epsilon'$ are odd. Then $F_\pi(\sqrt{a})$ is a complete discretely valued field with $\pi_1 = \sqrt{a}$ as a parameter and residue field is also a complete discretely valued field with valuation ring $R/(\pi)$ and $\bar{\delta}_1 = \bar{\delta}$ a parameter.

Let $\mu_\pi \in F_\pi(\sqrt{a})^*$. Then $\mu_\pi = \mu_1\pi_1^r$ for some $\mu_1 \in F_\pi(\sqrt{a})^*$ a unit at $\pi_1$ and $r \in \mathbb{Z}$. Let $\bar{\mu}_1$ be the image of $\mu$ in the residue field of $F_\pi(\sqrt{a})$. Then, we have $\bar{\mu}_1 = \bar{w}\bar{\delta}_1^s$ for some unit $w \in R[\sqrt{a}]$. Let $\mu = w\pi_1^r\delta_1^s$. Then $\mu_\pi\mu^{-1} \in F_\pi[\sqrt{a}]$ is a unit at $\pi$ and maps to 1 in $\kappa(\pi)$. □

**Lemma 5.2.2.** *Let $R$ be a complete regular local ring with maximal ideal $(\pi, \delta)$, field of fractions $F$ and residue field $\kappa$. Suppose that char($\kappa$) not equal to 2. Let $\lambda = u\pi^r\delta^s \in F^*$ with $u \in R^*$. Let $F_\pi$ be the completion of $F$ at the discrete valuation given by $(\pi)$. Let $n \geq 1$, $a_{i\pi} \in F_\pi^*$ and $\mu_{i\pi} \in L_{i\pi} = F_\pi[X]/(X^2 - a_{i\eta})$ for $1 \leq i \leq n$ with $\prod_i N_{L_{i\pi}/F_\pi}(\mu_{i\pi}) = \lambda$. Then there exist $a_i = u_i\pi^{\epsilon_i}\delta^{\epsilon_i'} \in R$ with $u_i \in R^*$, $\mu_i = w_i\pi^{r_i}\delta^{s_i}\sqrt{a_i}^{s_i'}$ for some $w_i \in R[X]/(X^2 - a_i)^*$ such that* *i)* $a_{i\pi}a_i^{-1} \in F_\pi^{*2}$ *for all* $i$

*i)* $\prod_i N_{F[X]/(X^2-a_i)/F}(\mu_i) = \lambda$

*ii) there is an isomorphism* $\phi_i : F_\pi[X]/(X^2 - a_{i\eta}) \to F_\pi[X]/(X^2 - a_i)$ *with* $\phi_i(\mu_{i\pi})\mu_i^{-1} \in F_\pi[X]/(X^2 - a_i)^{2^m}$ *for all* $m \geq 1$.

*Proof.* Applying (5.2.1) for $a_{i\pi}$ with $a = 1$, we get $a_i = u_i\pi^{\epsilon_i}\delta^{\epsilon_i'}$ with $u_i \in R^*$ such that $a_ia_{i\pi} \in F_\pi^{*2}$. Hence replacing $a_{i\pi}$ by $a_i$ we assume that $\mu_i \in F_\pi[X]/(X^2 - a_i)$.

Let $1 \leq i \leq n$. Suppose $a_i$ is a square in $F$. Then $F_\pi[X]/(X^2 - a_i) = F_\pi \times F_\pi$ and $\mu_{i\pi} = (\mu'_{i\pi}, \mu''_{i\pi})$. Let $\mu'_i, \mu''_i \in F$ be as in (5.2.1) corresponding to $\mu'_{i\pi}$ and $\mu''_{i\pi}$ and $\mu_i = (\mu'_i, \mu''_i) \in F[X]/(X^2 - a_i)$. Suppose $a_i$ is not a square. Let $\mu_i \in F$ be as in (5.2.1) corresponding to $\mu_{i\pi}$. Then $\lambda^{-1}\prod_i N_{F[X]/(X^2-a_i)/F}(\mu_i)$ is a unit in $R$ and maps to 1 in the residue field $\kappa$ of $R$. Since $\mathrm{char}(\kappa) \neq 2$, there exists $\theta \in R^*$ which maps to 1 in $\kappa$ and $\lambda^{-1}\prod_i N_{F[X]/(X^2-a_i)/F}(\mu_i) = \theta^2$. Replacing $\mu_1$ by $\mu_1\theta$, we have the required $\mu_i$. □

**Lemma 5.2.3.** *Let* $R$ *be a complete regular local ring with maximal ideal* $(\pi, \delta)$*, field of fractions* $F$ *and residue field* $\kappa$*. Suppose that* $\kappa$ *is a finite field with char*$(\kappa)$ *not equal to 2. Let* $D$ *be a quaternion algebra over* $F$ *which is unramified on* $R$ *except possibly at* $(\pi)$ *and* $(\delta)$*. Let* $a = u\pi^\epsilon\delta^\epsilon \in R$ *with* $u \in R^*$ *and* $\mu = w\pi^r\delta^s\sqrt{a}^{s'} \in F[X]/(X^2 - a)$ *for some* $w \in R[X]/(X^2 - a_i)^*$*. If* $\mu$ *is a reduced norm from* $D \otimes F_\pi[X]/(X^2 - a)$*, then* $\mu$ *is a reduced norm from* $D \otimes F[X]/(X^2 - a)$*.*

*Proof.* Suppose that $\epsilon$ or $\epsilon'$ is even. Then $R[X]/(X^2 - a)$ is regular and the result follows from (Lemma 6.5, [21]).

Suppose both $\epsilon$ and $\epsilon'$ are odd. Then $a = u\pi\delta a_1^2$ for some $a_1 \in F^*$. Without loss of generality, we assume that $a = u\pi\delta$. Since $\kappa$ is a finite field, we have $D = (v, \pi)$ or $(v, \delta)$ or $(v, \pi\delta)$ or $(v_1\pi, v_2\delta)$ for some units $v, v_1, v_2 \in R$ (cf. Lemma 3.6, [31]). Since $(v, \pi\delta) \otimes F[X]/(X^2 - a) \simeq (v, u) \otimes F[X]/(X^2 - a)$ is split and $(v_1\pi, v_2\delta) \otimes F[X]/(X^2 - a) \simeq (v_1v_2u, \delta) \otimes F[X]/(X^2 - a)$, without loss of generality we assume that $D = (v, \pi)$

or $(v, \delta)$ for some unit $v \in R$. Since $(v, \pi) \otimes F[X]/(X^2 - a) \simeq (v, \delta) \otimes F[X]/(X^2 - a)$, we assume that $D = (v, \delta)$.

Suppose $D \otimes F_\pi[X]/(X^2 - a)$ is split. Since $D = (v, \delta)$ is unramified and $F_\pi[X]/(X^2 - a)$ is ramified at $\pi$, $D \otimes F_\pi$ is split by contradiction. Hence, by (Corollary 5.6, [21]), $D$ is split and $\mu$ is a reduced norm $D$.

Suppose $D \otimes F_\pi[X]/(X^2 - a)$ is non-split. Since $a = u\pi\delta$, we have $\mu = w\pi^r\delta^s\sqrt{u\pi\delta}^{s'} \in F[X]/(X^2 - a)$ for some $w \in R^*$. Since $\sqrt{u\pi\delta}$ is a parameter in $F_\pi[X]/(X^2 - a)$ and $D \otimes F_\pi[X]/(X^2 - a) \simeq (v, \delta) \otimes F_\pi[X]/(X^2 - a)$ is unramified at $\pi$, $\sqrt{u\pi\delta}$ is not a reduced norm from $D \otimes F_\pi[X]/(X^2 - a)$. Hence $s'$ is even and $w\pi^r\delta^s$ is a reduced norm from $D \otimes F_\pi[X]/(X^2 - a)$. Since $D \otimes F[X]/(X^2 - a) \simeq (v, \pi) \otimes F[X]/(X^2 - a) \simeq (v, \pi) \otimes F[X]/(X^2 - a)$, $-\pi$ and $-\delta$ are reduced norms from $D \otimes F[X]/(X^2 - a)$. Since $\kappa$ is a finite field and $u, v \in R^*$, $\pm u$ is a norm from the extension $F(\sqrt{v})$ and hence $\lambda$ is a reduced norm from $D \otimes F[X]/(X^2 - a)$. □

## 5.3 Weak Approximations over Global Fields

In this section we prove certain weak approximations over global fields.

**Proposition 5.3.1.** *Let $\kappa$ be a global field of characteristic not 2 and $d, w \in \kappa^*$. Let $S$ be a finite set of paces of $\kappa$. Let $u \in N_{\kappa(\sqrt{d})/\kappa}(\kappa(\sqrt{d})^*)N_{\kappa(\sqrt{w})/\kappa}(\kappa(\sqrt{w})^*$. For each $\nu \in S$, let $y_\nu \in \kappa_\nu(\sqrt{d})$ and $z_\nu \in \kappa_\nu(\sqrt{w})$ be such that $N_{\kappa_\nu(\sqrt{d})/\kappa_\nu}(y_\nu)N_{\kappa_\nu(\sqrt{w})/\kappa_\nu}(z_\nu)$ is close to $u$. Then there exist $y \in \kappa(\sqrt{d})$ and $z \in \kappa(\sqrt{w})$ be such that $y$ is close to $y_\nu$ and $z$ is close to $z_\nu$ for all $\nu \in S$ and $N_{\kappa(\sqrt{d})/\kappa}(y)N_{\kappa(\sqrt{w})/\kappa}(z) = u$.*

*Proof.* By the strong approximation theorem for global fields (cf. Section 15, Chapter II, [5]), there are elements $y \in \kappa(\sqrt{d})$ and $z \in \kappa(\sqrt{w})$ satisfying the required principles. □

**Lemma 5.3.2.** *Let $\kappa$ be a global field of characteristic not 2. Let $d_0, w_0 \in \kappa^*$ and $S_0$ a finite set of places of $\kappa$. For each place $\nu \in S_0$, suppose that there are elements $x_\nu \in$*

$\kappa_\nu^*$, $y_\nu \in \kappa_{1\nu} = \kappa_\nu(\sqrt{x_\nu})$ *such that* $y_\nu \in N_{\kappa_{1\nu}(\sqrt{w_0})/\kappa_{1\nu}}(\kappa_{1\nu}(\sqrt{w_0})^*)N_{\kappa_{1\nu}(\sqrt{d_0 w_0})/\kappa_{1\nu}}(\kappa_{1\nu}(\sqrt{d_0 w_0})^*)$. *Then there exist* $x \in \kappa$ *and* $y \in \kappa_1 = \kappa(\sqrt{x})$ *such that*

*i)* $x$ *is close to* $x_\nu$ *and* $y$ *is close to* $y_\nu$ *for all* $\nu \in S$

*ii)* $y \in N_{\kappa_1(\sqrt{w_0})/\kappa_1}(\kappa_1(\sqrt{w_0})^*)N_{\kappa_1(\sqrt{d_0 w_0})/\kappa_1}(\kappa_1(\sqrt{d_0 w_0})^*)$.

*Proof.* Let $x \in \kappa$ be close to $x_\nu$ for all $\nu \in S$ and $\kappa_1 = \kappa(\sqrt{x})$. Then $\kappa_1 \otimes \kappa_\nu = \kappa_{1\nu}$. Let $z_{1\nu} \in \kappa_{1\nu}(\sqrt{w_0})^*$ and $z_{2\nu} \in \kappa_{1\nu}(\sqrt{d_0 w_0})^*$ such that

$$y_\nu = N_{\kappa_{1\nu}(\sqrt{w_0})/\kappa_{1\nu}}(z_{1\nu})N_{\kappa_{1\nu}(\sqrt{d_0 w_0})/\kappa_{1\nu}}(z_{1\nu}).$$

Let $z_1 \in \kappa_1(\sqrt{w_0})^*$ and $z_2 \in \kappa_1(\sqrt{d_0 w_0})^*$ close to $z_{1\nu}$ and $z_{2\nu}$ respectively for all $\nu \in S$. Let

$$y = N_{\kappa_1(\sqrt{w_0})/\kappa_1}(z_1)N_{\kappa_1(\sqrt{d_0 w_0})/\kappa_1}(z_2).$$

Then $x$ and $y$ have the required properties. □

**Lemma 5.3.3.** *Let* $\kappa$ *be a global field of characteristic not 2. Let* $u_0, b_0, c_0, w_0, d_0 \in \kappa^*$ *and* $S_0$ *a finite set of places of* $\kappa$ *containing all the places where at least one of* $b_0$, $c_0$, $d_0$, $w_0$ *and* $u_0$ *is not a unit. Suppose that* $u_0 \in N_{\kappa(\sqrt{w_0},\sqrt{d_0})/\kappa(\sqrt{d_0})}(\kappa(\sqrt{w_0},\sqrt{d_0})^*)$. *For each place* $\nu \in S_0$, *suppose we have given* $x_\nu \in \kappa_\nu^*$, $y_{1\nu} \in \kappa_{1\nu} = \kappa_\nu(\sqrt{x_\nu})$, $y_{2\nu} \in \kappa_{2\nu} = \kappa_\nu(\sqrt{w_0})$ *and* $y_{3\nu} \in \kappa_3 = \kappa_\nu(\sqrt{d_0 w_0})$ *such that*

*i)* $\prod_i N_{\kappa_{i\nu}/\kappa_\nu}(y_{i\nu}) = u_0$

*ii)* $y_{1\nu} \in N_{\kappa_{1\nu}(\sqrt{w_0})/\kappa_{1\nu}}(\kappa_{1\nu}(\sqrt{w_0})^*)N_{\kappa_{1\nu}(\sqrt{d_0 w_0})/\kappa_{1\nu}}(\kappa_{1\nu}(\sqrt{d_0 w_0})^*)$.

*Then there exist* $x \in \kappa^*$ *and* $y_1 \in \kappa_1 = \kappa(\sqrt{x})$, $y_2 \in \kappa_2 = \kappa(\sqrt{w_0})$, $y_3 \in \kappa_3 = \kappa(\sqrt{d_0 w_0})$ *such that*

*i)* $x$ *is close to* $x_\nu$ *and* $y_i$ *is close to* $y_{i\nu}$ *for all* $\nu \in S$ *and* $i = 1, 2, 3$.

*ii)* $\prod_i N_{\kappa_i/\kappa}(y_i) = u_0$

*iii)* $y_1 \in N_{\kappa_1(\sqrt{w_0})/\kappa_1}(\kappa_1(\sqrt{w_0})^*)N_{\kappa_1(\sqrt{d_0 w_0})/\kappa_1}(\kappa_1(\sqrt{d_0 w_0})^*)$.

*Proof.* By (5.3.2), there exist $x \in \kappa$ and $y_1 \in \kappa_1 = \kappa(\sqrt{x})$ such that $x$ is close to

$x_\nu$, $y_1$ is close to $y_{1\nu}$ for all $\nu \in S$ and $y_1 = N_{\kappa_1(\sqrt{w_0})/\kappa_1}(z_1)N_{\kappa_1(\sqrt{d_0w_0})/\kappa_1}(z_2)$ for some $z_1 \in \kappa_1(\sqrt{w_0})^*$ and $z_2 \in \kappa_1(\sqrt{d_0w_0})^*$.

Let $u_1 = N_{\kappa_1/\kappa}(y_1)$, $u_{12} = N_{\kappa_1(\sqrt{w_0})/\kappa}(z_1)$ and $u_{13} = N_{\kappa_1(\sqrt{d_0w_0})/\kappa}(z_2)$. Then $u_1 = u_{12}u_{13}$ and $u_{12} \in N_{\kappa_2/\kappa}(\kappa_2^*)$ and $u_{13} \in N_{\kappa_3/\kappa}(\kappa_3^*)$.

Let $u_2 = u_0u_1^{-1}$. Since $y_1$ is close to $y_{1\nu}$ for all $\nu \in S$, $N_{\kappa_{2\nu}/\kappa_\nu}(y_{2\nu})N_{\kappa_{3\nu}/\kappa_\nu}(y_{3\nu})$ is close to $u_2$. Hence, by (5.3.1), there exist $y_2 \in \kappa_2$ and $y_3 \in \kappa_3$ which are close to $y_{2\nu}$ and $y_{3\nu}$ respectively for all $\nu \in S$ such that $N_{\kappa_2/\kappa}(y_2)N_{\kappa_3/\kappa}(y_3) = u_2$. Then $x_1$, $y_1$, $y_2$ and $y_3$ have the required properties. □

**Proposition 5.3.4.** *Let $\kappa$ be a global field of characteristic not 2. Let $u_0, b_0, c_0 \in \kappa^*$ and $S_0$ a finite set of places of $\kappa$ containing all the places where at least one of $u_0$, $b_0$, $c_0$ is not a unit or $(b_0, c_0)$ is nontrivial at $\nu$. Suppose for every $\nu \in S$, we have given $x_\nu, w_\nu \in \kappa_\nu$, $y_{1\nu} \in \kappa_{1\nu} = \kappa_\nu[X]/(X^2 - x_\nu)$ and $y_{2\nu} \in \kappa_{2\nu} = \kappa_\nu[X]/(X^2 - w_\nu)$ such that*

*i) $N_{\kappa_{1\nu}/\kappa_\nu}(y_{1\nu})N_{\kappa_{2\nu}/\kappa_\nu}(y_{2\nu}) = u_0$*

*ii) $(b_0, c_0)$ splits over $\kappa_{1\nu}$ and $\kappa_{2\nu}$.*

*Then there exist units $x, w \in \kappa^*$, $y_1 \in \kappa_1 = \kappa[X]/(X^2 - x)$ and $y_2 \in \kappa_2 = \kappa[X]/(X^2 - w)$ such that*

*i) $x$, $w$, $y_1$ and $y_2$ are close to $x_\nu$, $w_\nu$, $y_{1\nu}$ and $y_{2\nu}$ respectively for all $\nu \in S$*

*ii) $N_{\kappa_1/\kappa}(y_1)N_{\kappa_2/\kappa}(y_2) = u_0$,*

*iii) $(b_0, c_0)$ splits over $\kappa_1$ and $\kappa_2$.*

*Proof.* Let $x, w \in \kappa$ be close to $x_\nu$ and $w_\nu$ for all $\nu \in S$. Since $(b_0, c_0)$ splits over $\kappa_\nu$ for places $\nu \notin S$ and $(b_0, c_0)$ splits over $\kappa_{1\nu}$ and over $\kappa_{2\nu}$ for all $\nu \in S$, $(b_0, c_0)$ splits over $\kappa_1$ and $\kappa_2$ .

Let $y_1 \in \kappa_1$ close to $y_{1\nu}$ for all $\nu \in S$. Let $u_1 = N_{\kappa_1/\kappa}(y_1)$. Then $N_{\kappa_{2\nu}/\kappa_\nu}(y_{2\nu})$ is close to $u_1^{-1}u_0$ for all $\nu \in S$. Hence there exists $y'_{2\nu} \in \kappa_{2\nu}$ which is close to $y_{2\nu}$ such that $N_{\kappa_{2\nu}/\kappa_\nu}(y'_{2\nu}) = u_1^{-1}u_0$. Since $\kappa_2/\kappa$ is a quadratic extension, there exists $y_2 \in \kappa_2$ such that $N_{\kappa_2/\kappa}(y_2) = u_1^{-1}u_0$ and $y_2$ is close to $y'_{2\nu}$ for all $\nu \in S$.

Hence $x, w, y_1$ and $y_2$ have the required properties. □

## 5.4 Complete Discretely Valued Fields

Let $R_0$ be a complete discretely valued field with residue field $\kappa$ a positive characteristic global field of characteristic not equal to 2 and $F_0$ the field of fractions. Let $d \in R_0$ be a non-square and $F = F_0(\sqrt{d})$. Then the residue field of $F$, denoted by $\kappa_F$, is a global field which is isomorphic to $\kappa$ or $\kappa(\sqrt{\bar{d}})$. Let $R$ be the integral closure of $R_0$ in $F$. Let $D$ be a central division algebra over $F$ with a $F/F_0$-involution $\tau$.

Suppose that $\pi_F$ is a parameter of $F$ and per$(D) = 2$. Then there is an unramified cyclic extension $E/F$ with $Gal(E/F) =<\sigma>$ such that $[D] = [D'] + [(E, \sigma, \pi_F)] \in H^2(F, \mu_2)$ and $ind(D) = ind(D' \otimes_F E) \cdot [E : F]$ for some $[D'] \in H^2_{nr}(F, \mu_2) \cong H^2(\kappa_F, \mu_2)$ (cf. Lemma 4.2, [21]; Theorem 5.6, [14]). In particular, $(E, \sigma, \pi_F)$ is a cyclic division $F$-algebra of $ind\big((E, \sigma, \pi_F)\big) \leq 2$ (cf. Section 4, [21]; Corollary d, Chapter 15, [23]). Since $\kappa_F$ is a global field, $per(D') = ind(D') \leq 2$ (cf. 4.5, §4, Chapter 3, Part II, [4]). Then ind$(D) \leq 4$.

We know that $SK_1U(D, \tau)$ is trivial (cf. Corollary 4.16, Corollary 4.17, [32]). The aim of this section is to show that given $\lambda \in F_0^*$ which is a reduced norm from $D$, there exist $a_1, a_2, a_3 \in F_0^*$ and $\mu_i \in L_i = F_0[X]/(X^2 - a_i)$ which approximates some given elements such that $\prod_i N_{L_i/F_0}(\mu_i) = \lambda$ and $ind(D \otimes F_0 L_i) \leq 2$. This is required for our main result and also this gives an alternative proof of the fact that $SK_1U(A, \tau)$ is trivial.

**Lemma 5.4.1.** *Suppose that the valuation of $d$ is even, then $D = (b, c) \otimes (w, \pi)$ for some units $b, c, w \in R_0$ and a parameter $\pi$ of $F_0$.*

*Proof.* Assume that $[D] \neq 0 \in H^2(F, \mu_2)$. Since the valuation of $d \in F_0$ is even, the extension $F/F_0$ is unramified.

Let $\pi_0$ be a parameter of $F_0$. By the discussion above, we can find an unramified cyclic extension $E/F$ with $Gal(E/F) =<\sigma>$ such that $[D] = [D'] + [(E, \sigma, \pi_0)]$, $ind(D') \leq 2$, and $ind\big((E, \sigma, \pi_0)\big) \leq 2$.

Since $D$ has a unitary $F/F_0$-involution and $F/F_0$ is unramified, there are elements $b, c, x, y$ in $F_0$ such that $[D'] = [(b, c) \otimes_{F_0} F] \in H^2_{nr}(F, \mu_2) \cong H^2(\kappa_F, \mu_2)$ and $[(E, \sigma, \pi_0)] = [(x, y) \otimes_{F_0} F] \in H^2(F, \mu_2)$ (cf. Proposition 2.22, [16]).

Assume that $[D'] \neq 0 \in H^2(F, \mu_2)$. Since $[D'] \in H^2_{nr}(F, \mu_2)$, $b$ and $c$ are units in $R_0$ (cf. Section 4, [21]).

Assume that $[(E, \sigma, \pi_0)] \neq 0 \in H^2(F, \mu_2)$. Also notice that the unramified extension $E/F$ is a splitting field of the division $F$-algebra $(E, \sigma, \pi_0)$. Since $[(E, \sigma, \pi_0)] \notin H^2_{nr}(F, \mu_2)$, $[(x, y) \otimes_{F_0} F] = [(u_1\pi_0, u_2\pi_0) \otimes_{F_0} F]$ or $[(u_3, u_4\pi_0) \otimes_{F_0} F]$ for some units $u_1, u_2, u_3, u_4$ in $R_0$. The former case cannot happen since the non-split quaternion $F$-algebra $(u_1\pi_0, u_2\pi_0)$ is not split over any unramified extension of $F$. Therefore, we can let $w = u_3$ and $\pi = u_4\pi_0$.

□

**Lemma 5.4.2.** *Suppose that the valuation of $d$ is odd . Then $D = (b, c)$ for some units $b, c \in R_0$.*

*Proof.* Assume that $[D] \neq 0 \in H^2(F, \mu_2)$. Since the valuation of $d \in F_0$ is odd, there is a parameter $\pi_0$ of $F_0$ such that $F_0(\sqrt{\pi_0}) \cong F = F_0(\sqrt{d})$ and $\sqrt{\pi_0}$ is a parameter of $F$.

By (Lemma 4.2, [21]) and the discussion before Lemma 5.3.5, there is an unramified extension $E/F$ with $Gal(E/F) =<\sigma>$ such that $[D] = [D'] + [(E, \sigma, \sqrt{\pi_0})] \in H^2(F, \mu_2)$, $ind(D') \leq 2$, and $ind\big((E, \sigma, \sqrt{\pi_0})\big) \leq 2$.

Since $D$ has a unitary $F/F_0$-involution, there are elements $b, c, x, y$ in $F_0$ such that $[D'] = [(b, c) \otimes_{F_0} F] \in H^2_{nr}(F, \mu_2) \cong H^2(\kappa, \mu_2)$ and $[(E, \sigma, \sqrt{\pi_0})] = [(x, y) \otimes_{F_0} F] \in H^2(F, \mu_2)$ (cf. Proposition 2.22, [16]).

Assume that $[D'] \neq 0 \in H^2(F, \mu_2)$. Since $[D'] \in H^2_{nr}(F, \mu_2)$, $b$ and $c$ are units in $R_0$.

Assume that $[(E,\sigma,\sqrt{\pi_0})] \neq 0 \in H^2(F,\mu_2)$. Since $F \cong F_0(\sqrt{\pi_0})$ and $[(E,\sigma,\sqrt{\pi_0})] \notin H^2_{nr}(F,\mu_2)$, the quaternion $F$-algebra $(x,y)\otimes_{F_0} F$ has to be split over $F$. Then $[D] \cong_F [(b,c)] \in H^2(F,\mu_2)$ for units $b,c$ in $R_0$.

□

**Proposition 5.4.3.** *Suppose $d$ is a unit in $R_0$. Suppose $D=(b,c)\otimes(w,\pi)$ for some units $b,c,w \in R_0$, $\pi \in R_0$ a parameter. Let $u \in R_0$ be a unit. Let $S_0$ be a finite set of places of $\kappa$ containing all the places $\{\nu\}$ where at least one of $\bar{u}$, $\bar{b},\bar{c},\bar{w},\bar{d}$ is not a unit or $(\bar{b},\bar{c})$ is not split. For each place $\nu \in S_0$, suppose that we have given $x_\nu \in \kappa_\nu - \kappa_\nu^{*2}$, $y_{1\nu} \in \kappa_{1\nu} = \kappa[X]/(X^2-x_\nu)$, $y_{2\nu} \in \kappa_{2\nu} = \kappa_\nu[X]/(X^2-\bar{w})$ and $y_{3\nu} \in \kappa_{3\nu} = \kappa_\nu[X]/(X^2-\bar{w}\bar{d})$ such that*

*i) $\prod_i N_{\kappa_{i\nu}/\kappa_\nu}(y_{i\nu}) = \bar{u}$*

*ii) $y_{1\nu} \in N_{\kappa_{1\nu}(\sqrt{\bar{w}},\sqrt{\bar{d}})/\kappa_{1\nu}(\sqrt{\bar{d}})}(\kappa_{1\nu}(\sqrt{\bar{w}},\sqrt{\bar{d}})^*)$.*

*Then there exist units $a \in R_0$, $\mu_1 \in R_0[X]/(X^2-a)$, $\mu_2 \in R_0[X]/(X^2-w)$, $\mu_3 \in R_0[X]/(X^2-wd)$ such that*

*i) $\bar{a}$ is close to $x_\nu$, $\bar{\mu}_i$ is close to $y_{i\nu}$ for all $\nu \in S_0$ and $i=1,2,3$*

*ii) $\prod_i N_{L_i/F_0}(\mu_i) = u$, where $L_1 = F_0[X]/(X^2-a)$, $L_2 = F_0[X]/(X^2-w)$ and $L_3 = F_0[X]/(X^2-dw)$*

*iii) $(b,c)\otimes L_1$ is split*

*iv) $\mu_i$ is a reduced norm from $D\otimes_{F_0} L_i$ for $i=1,2,3$.*

*Proof.* Since $\kappa_{1\nu}(\sqrt{\bar{w}},\sqrt{\bar{d}})/\kappa_{1\nu}$ is a bi-quaternion extension, it can be verified that $N_{\kappa_{1\nu}(\sqrt{\bar{w}})/\kappa_{1\nu}}(\kappa_{1\nu}(\sqrt{\bar{w}})^*)\cdot N_{\kappa_{1\nu}(\sqrt{\bar{d}\bar{w}})/\kappa_{1\nu}}(\kappa_{1\nu}(\sqrt{\bar{d}\bar{w}})^*) = N_{\kappa_{1\nu}(\sqrt{\bar{w}},\sqrt{\bar{d}})/\kappa_{1\nu}(\sqrt{\bar{d}})}(\kappa_{1\nu}(\sqrt{\bar{w}},\sqrt{\bar{d}})^*)$. Then by (5.3.3), there exist $x \in \kappa$ and $y_1 \in \kappa_1 = \kappa(\sqrt{x})$, $y_2 \in \kappa_2 = \kappa(\sqrt{\bar{w}})$, $y_3 \in \kappa_3 = \kappa(\sqrt{\bar{d}\bar{w}})$ such that

i) $x$ is close to $x_\nu$ and $y_i$ is close to $y_{i\nu}$ for all $\nu \in S_0$ and $i=1,2,3$.

ii) $\prod_i N_{\kappa_i/\kappa}(y_i) = \bar{u}$

iii) $y_1 \in N_{\kappa_1(\sqrt{\bar{w}})/\kappa_1}(\kappa_1(\sqrt{\bar{w}})^*)N_{\kappa_1(\sqrt{\bar{d}\bar{w}})/\kappa_1}(\kappa_1(\sqrt{\bar{d}\bar{w}})^*)$.

Let $\nu$ be a place of $\kappa$. Suppose that $\nu \in S_0$. Then, by the choice, $[\kappa_\nu(\sqrt{x_\nu}), \kappa_\nu] = 2$. Since $\kappa_\nu$ is a local field, $(\bar{b}, \bar{c})$ is split over $\kappa_\nu(\sqrt{x_\nu}) = \kappa_\nu(\sqrt{x})$. Suppose that $\nu \notin S_0$. Then, by the choice of $S_0$, $(\bar{b}, \bar{c})$ is split over $\kappa_\nu$. Hence, by the theorem of Albert-Brauer-Hasse-Noether, $(\bar{b}, \bar{c})$ is split over $\kappa_1 = \kappa(\sqrt{x})$.

Let $a \in R_0$ be a lift of $x$, $\mu_1 \in R_0[X]/(X^2-a)$ a lift of $y_1$ and $\mu_2 \in R_0[X]/(X^2-w)$ a lift of $y_2$. Since $R_0$ is complete, there exists $\mu_3 \in R_0[X]/(X^2 - wd)$ which has a lift $y_3$ such that $\prod_i N_{L_i/F_0}(\mu_i) = u$.

Since $R_0$ is complete and $(\bar{b}, \bar{c})$ splits over $\kappa(\sqrt{x})$, $(b, c) \otimes L_1$ is split.

Since $y_1 \in N_{\kappa_1(\sqrt{\bar{w}}, \sqrt{\bar{d}})/\kappa_1(\sqrt{\bar{d}})}(\kappa_1(\sqrt{\bar{w}}, \sqrt{\bar{d}})^*)$, $R_0$ is complete and $\mu_1$ is a lift of $y_1$, $\mu_1 \in N_{L_1(\sqrt{w}, \sqrt{d})/L_1(\sqrt{d})}(L_1(\sqrt{w}, \sqrt{d})^*)$. Since $(b, c) \otimes L_1$ is split, $D \otimes L_1 = (w, \pi) \otimes L_1(\sqrt{d})$. Hence $\mu_1$ is a reduced norm from $D \otimes L_1$.

Let $i = 2, 3$. Since $\kappa$ is a global field, $y_i$ is a reduced norm from $(\bar{b}, \bar{c}) \otimes \kappa_i$. Since $R_0$ is complete, $\mu_i$ is a reduced norm from $(b, c) \otimes L_i(\sqrt{d}) = D \otimes L_i$.

Hence $a_1, \mu_1, \mu_2, \mu_3$ have the required properties. □

## 5.5 Two Dimensional Complete Fields

Let $R_0$ be a complete two dimensional regular local ring with $m = (\pi, \delta)$ the maximal ideal of $R_0$, $\kappa_0 = R/m$ and $F_0$ field of fractions of $R_0$. Suppose that $\kappa_0$ a finite field of char not equal to 2. Let $w \in R_0$ be a unit which is not a square in $R_0$ and $d = w$ or $\pi$. Let $F = F_0(\sqrt{d})$ and $R$ the integral closure of $R_0$ in $F$. Then $R$ is a regular local ring with maximal ideal $m_R = (\pi', \delta)$ with $\pi' = \pi$ if $d = w$ and $\pi' = \sqrt{\pi}$ if $d = \pi$ (cf. Lemma 3.1 & Lemma 3.2, [22]).

Let $D/F$ be a central division algebra of period 2 which is unramified on $R$ except possibly at $(\pi')$ and $(\delta)$. Since the residue field $\kappa_0$ is a finite field of $char(\kappa_0) \neq 2$, then the index of $D$ is 2 (cf. Proposition 3.5, [31]).

**Proposition 5.5.1.** *Suppose that $D$ admits a $F/F_0$-involution. Then there exists a* *quaternion division algebra $D_0/F_0$ such that $D \simeq D_0 \otimes F$ and*

*i) if $d = w$, then $D_0 = (\pi, \delta)$;*

*ii) if $d = \pi$, then $D_0 = (w, \delta)$.*

*Proof.* By (Proposition 2.22, [16]), there exists a quaternion division algebra $D_0/F_0$ such that $D \cong D_0 \otimes_{F_0} F$.

According to (Lemma 3.6 & Lemma 4.1, [31]) and considering the square classes of $F_0$, $D_0$ has the form $(\pi, \delta)$ or $(w, \delta)$. □

Let $\lambda = u\pi^r\delta^s \in F_0$ with $u \in R_0$ a unit and $r, s \in \mathbb{Z}$. Suppose that $\lambda$ is a reduced norm from $D$. In this section we construct quadratic extensions $L_1, L_2, L_3$ of $F_0$ and $\mu_i \in L_i$ with $\prod_i N_{L_i/F_0}(\mu_i) = \lambda$ and satisfying some other properties. These results are used in the proof of the main theorem. Let $\epsilon_1, \epsilon_2 \in \{0, 1\}$ such that $r = 2r_1 + \epsilon_1$ and $s = \epsilon_2 + 2s_1$ for some $r_1, s_1 \in \mathbb{Z}$. Then $\lambda = u\pi^{\epsilon_1}\delta^{\epsilon_2}(\pi^{r_1}\delta^{s_1})^2$.

We begin with the following.

**Proposition 5.5.2.** *Let $D_0$ be a quaternion division algebra over $F_0$ which is unramified on $R_0$ except possibly at $(\pi)$ and $(\delta)$. Suppose that $\lambda$ is a reduced norm from $D_0$. Then there exist $a_i \in F_0^*$ and $\mu_i \in L_i = F_0[X]/(X^2 - a_i)$ for $i = 1, 2$ such that*

*1) $a_1 = v\pi^{\epsilon_1}\delta^{\epsilon_2} \in F_0^* \setminus F_0^{*2}$ with $v \in R_0^*$*

*2) $a_2 \in R_0$ which is a unit at $(\pi)$, $(\delta)$, $\partial_\pi(D_0) = \bar{a}_2 \in \kappa(\pi)^*/\kappa(\pi)^{*2}$ and $\partial_\delta(D_0) = \bar{a}_2 \in \kappa(\delta)^*/\kappa(\delta)^{*2}$*

*3) $N_{L_1/F_0}(\mu_1)N_{L_2/F_0}(\mu_2) = \lambda$*

*4) $\mu_i$ is a reduced from from $D_0 \otimes L_i$ for $i = 1, 2$*

*5) $\mu_2 \in R_0[X]/(X^2 - a_2)$ a unit.*

*Proof.* By (Lemma 3.6 & Lemma 4.2, [31]), we have $D_0 = (v, \pi)$, $(v, \delta)$, $(v, \pi\delta)$ or $(v_1\pi, v_2\delta)$ for some units $v, v_1, v_2 \in R_0$. If $D_0 = (v, \pi)$, let $a_2 = v\delta^2 + \pi$. If $D_0 = (v, \delta)$, let $a_2 = v\pi^2 + \delta$. If $D_0 = (v, \pi\delta)$, let $a_2 = v$. If $D_0 = (v_1\pi, v_2\delta)$, let $a_2 = v_1\pi + v_2\delta$. Then $a_2$ satisfies the property *5)*. By checking the square classes, $D_0 \otimes L_2$ is trivial.

Suppose $\pm\lambda$ are not squares in $F_0$. Let $a_1 = -u\pi^{\epsilon_1}\delta^{\epsilon_2}$, $\mu_1 = \pi^{r_1}\delta^{s_1}\sqrt{a_1}$. Then, by (Lemma 6.2, [21]), $a_1, a_2, \mu_1$ and $\mu_2 = 1$ have the required properties.

Suppose that one of $\pm\lambda$ is a square in $F_0$. Suppose $\lambda$ is a square. Then $\epsilon_1 = \epsilon_2 = 0$, $u = u_1^2$ for some $u_1 \in R_0$ and $\lambda = u_1^2\pi^{2r_1}\delta^{2s_1}$. Suppose $D_0 = (v_1\pi, v_2\delta)$. Let $a_1 = w$, $\mu_1 = v_1^{r_1}v_2^{s_1}\pi^{r_1}\delta^{s_1}$ and $\mu_2 = u_1v_1^{-r_1}v_2^{-s_1}$. Then $a_1, a_2, \mu_1$ and $\mu_2$ have the required properties. Suppose $D_0 \neq (v_1\pi, v_2\delta)$. Let $a_1 = w$. Then $D_0 \otimes L_1$ is trivial. Hence $a_1, a_2, \mu_1 = u_1\pi^{r_1}\delta^{s_1}$ and $\mu_2 = 1$ have the required properties.

Suppose $\lambda$ is not a square in $F_0$. Then $-\lambda$ is a square in $F_0$. Then $\epsilon_1 = \epsilon_2 = 0$, $u = -u_1^2$ for some $u_1 \in R_0$, $-1$ is not a square in $F_0$ and $\lambda = -u_1^2\pi^{2r_1}\delta^{2s_1}$. In particular $-1$ is a reduced norm from $D_0$. Since $-1$ is not a square in $F_0$, it follows that $D_0 \neq (v_1\pi, v_2\delta)$. Let $a_1 = -1$. Since $\kappa_0$ is a finite field, there exists $\mu' \in F_0[X]/(X^2+1)$ such that $N_{L_1/F_0}(\mu') = -1$. Then $a_1 = -1$ $\mu_1 = \mu' u_1\pi^{r_1}\delta^{s_1}$ and $\mu_2 = 1$ have the required properties. □

**Lemma 5.5.3.** *Suppose that $d = w$. Let $a_2 = \pi + \delta$ and $a_3 = da_2$. There exist $a_1 = v\pi^{\epsilon_1}\delta^{\epsilon_2} \in F_0^* \setminus F_0^{*2}$ with $v \in R_0^*$ and $\mu_i \in L_i = F_0[X]/(x^2 - a_i)$ such that*

*1) $\prod_{i=1}^{3} N_{L_i/F_0}(\mu_i) = \lambda$*

*2) $\mu_i$ is a reduced from from $D \otimes_F FL_i$*

*3) $\mu_i \in L_i$ are units at $\pi$ and $\delta$ for $i = 1, 2$.*

*Proof.* Since $\kappa$ is a finite field, $R_0^*/R_0^{*2}$ has only one non trivial class and it is given by $d = w$. Since $F = F_0(\sqrt{d})$, every element in $R_0^*$ is a square in $R$. In particular $-1 \in R^{*2}$ and $\lambda = u_1^2\pi^r\delta^s$ for some $u_1 \in R$ with $u_1^2 = u$. Further $D_0 = (\pi, \delta)$.

Suppose that $\lambda$ is not a square in $F^*$. Then $\lambda = \pi^{\epsilon_1}\delta^{\epsilon_2}u_1^2\pi^{2r_1}\delta^{2s_1}$. Since $-1 \in F^{*2}$, both $\pm\lambda$ are not squares in $F^*$. Hence, by (Lemma 6.2, [21]), $a_1 = -u\pi^{\epsilon_1}\delta^{\epsilon_2}$, $\mu_1 = \pi^{r_1}\delta^{s_1}\sqrt{a_1}$, and $\mu_2 = \mu_3 = 1$ have the required properties.

Suppose that $\lambda$ is a square in $F^*$. Then $r = 2r_1$ and $s = 2s_1$. Suppose $\lambda$ is a square in $F_0$. Then $u_1 \in R_0^*$. Since $\pi$ and $\delta$ are reduced norms from $D_0$ and $D_0 \otimes F_0(\sqrt{a_2})$ is split, $a_1 = w$, $\mu_1 = \pi^{r_1}\delta^{s_1}$, $\mu_2 = u_1$ and $\mu_3 = 1$ have the required properties.

Suppose that $\lambda \notin F_0^{*2}$. Then $u = du_2^2$ for some unit $u_2 \in R_0$. Then $a_1 = w$, $\mu_1 = \pi^{r_1}\delta^{s_1}$, $\mu_2 = \sqrt{a_2}^{-1}$, $\mu_3 = u_2\sqrt{a_3}$ have the required properties. $\square$

**Lemma 5.5.4.** *Suppose that $d = \pi$. There exist $a_1 = v\pi^{\epsilon_1}\delta^{\epsilon_2} \in F_0^* \setminus F_0^{*2}$ with $v \in R_0^*$ and $\mu_1 \in L_1 = F_0[X]/(x^2 - a_1)$ and $\mu_2 \in L_2 = F_0[X]/(x^2 - w)$ such that*

*i) $N_{L_1/F_0}(\mu_1)N_{L_2/F_0}(\mu_2) = \lambda$*

*ii) $\mu_i$ is a reduced from from $D \otimes_F FL_i$*

*iii) $\mu_2 \in R_0[X]/(X^2 - w)^*$.*

*Proof.* Since $d = \pi$, we have $D_0 = (w, \delta)$ (5.5.1).

Suppose both $\pm\lambda$ are not squares in $F$. Then $a_1 = -u\pi^{\epsilon_1}\delta^{\epsilon_2}$, $\mu_1 = \pi^{r_1}\delta^{s_1}\sqrt{a_1}$ and $\mu_2 = 1$ have the required properties (Lemma 6.2, [21]).

Suppose that only one of $\pm\lambda$ is a square in $F$. Then $-1 \notin F_0^{*2}$ and $\lambda = \pm 1 \in F_0^*/F_0^{*2}$ or $\lambda = \pm\pi \in F_0^*/F_0^{*2}$. Further $D_0 = (-1, \delta)$.

Suppose $\lambda = \pm 1 \in F_0^*/F_0^{*2}$. Then $\lambda = \pm u_1^2\pi^{2r_1}\delta^{2s_1}$ for some $u_1 \in R_0^*$. Let $a_1 = -1$ Since $\kappa$ is a finite field, there exists $\mu_1' \in L_1 = F_0(\sqrt{a_1})$ with $N_{L_1/F_0}(\mu_1) = \pm 1$. Then $a_1$, $\mu_1 = u_1\pi^{r_1}\delta^{s_1}\mu_1'$, and $\mu_2 = 1$ have the required properties.

Suppose $\lambda = \pm\pi \in F_0^*/F_0^{*2}$. Then $\epsilon_1 = 1$, $\epsilon_2 = 0$ and $\lambda = \epsilon\pi u_1^2\pi^{2r_1}\delta^{2s_1}$ for some $u_1 \in R_0^*$ and $\epsilon = \pm 1$. Then $a_1 = -\pi$, $\mu_1 = u_1\pi^{r_1}\delta^{s_1}\sqrt{-\pi}$ , and $\mu_2 \in L_2 = F_0(\sqrt{-1})$ with $N_{L_2/F_0}(\mu_2) = \epsilon$ have the required properties.

Suppose both $\pm\lambda$ are squares in $F$. Then $-1 \in F^{*2}$. Since $d = \pi$, $-1 \in F_0^{*2}$.

Suppose $\lambda$ is a square in $F_0$. Then $a_1 = w$, $\mu_1 = \sqrt{\lambda}$ and $\mu_2 = 1$ have the required properties.

Suppose that $\lambda$ is not a square in $F_0^*$. Then $d\lambda \in F_0^{*2}$ and hence $\lambda = \pi u_1^2\pi^{2r_1}\delta^{2s_1}$ for some $u_1 \in R_0^*$. Then $a_1 = w\pi$, $\mu_1 = u_1\pi^{r_1}\delta^{s_1}\sqrt{w\pi}$ , and $\mu_2 \in L_2$ with $N_{L_2/F_0}(\mu_2) = -w^{-1}$ have the required properties. $\square$

**Lemma 5.5.5.** *Suppose $d$ is not a square and $D$ is ramified on $R$ at most at $\pi$. Then $D \otimes F_\pi$ is split.*

*Proof.* Suppose $d = w$ . Suppose $D$ is non split. Then, by (5.5.1), $D \simeq (\pi, \delta) \otimes F$. Then $D$ is ramified both at $(\pi)$ and $(\delta)$. This contradicts the assumption that $D$ is ramified at most at $\pi$. Hence $D$ is split.

Suppose $d = \pi$. Suppose $D$ is non split. Then, by (5.5.1), $D \simeq (w, \delta) \otimes F$ for unit $w \in R_0$ which is not a square. Since $F/F_0$ is ramified, $w \in R$ is not a square. In particular $D$ is ramified at $\delta$. This contradicts the assumption that $D$ is ramified at most at $\pi$. Hence $D$ is split. □

We end this section with the following.

**Proposition 5.5.6.** *Suppose that $D$ is ramified on $R$ at most at $\pi$. Let $n \geq 1$. Suppose there exist $a_{i\pi} \in F_{0\pi}$ and $\mu_{i\pi} \in L_{i\pi} = F_{0\pi}[X]/(X^2 - a_{i\pi})$ for $1 \leq i \leq n$ such that*

*i)* $\prod_i N_{L_{i\pi}/F_{0\pi}}(\mu_{i\pi}) = \lambda$

*ii)* $\mu_{i\pi}$ *is a reduced norm from* $D \otimes L_{i\pi}$ *for* $1 \leq i \leq n$.

*Then there exist $a_i \in F_0$ and $\mu_i \in L_i = F_0[X]/(X^2 - a_i)$ for $1 \leq i \leq n$ such that*

*i)* $\prod_i N_{L_i/F_0}(\mu_i) = \lambda$

*ii)* $\mu_i$ *is a reduced norm from* $D \otimes L_i$ *for* $1 \leq i \leq n$

*iii)* $a_{i\pi} a_i \in F^{*2}_{0,\pi}$ *for* $1 \leq i \leq n$

*iv) there is an isomorphism*

$$\phi_i : L_{i\pi} \simeq L_i \otimes F_\pi$$

*such that* $\phi_i(\mu_{i\pi})^{-1}\mu_i \in (L_i \otimes F_{0\pi})^{2^m}$ *for all* $m \geq 1$ *and* $1 \leq i \leq n$.

*Proof.* Apply (5.2.2) to $R_0$, $F_0$ and $a_{i\pi}$ with $a = 1$, and get $a_i \in F_0$ as in (5.2.2). Apply once again (5.2.2) to $R_0$, $F_0$, $a_i$ and $\mu_{i\pi}$, get $\mu_i \in L_i = F_0[X]/(X^2 - a_i)$ as in ( 5.2.2).

Suppose $d$ is not a square in $F_0$. Then, by (5.5.5), $D$ is split and hence $\mu_i$ are reduced norms from $D$.

Suppose $d$ is a square in $F_0$. Then $F = F_0 \times F_0$ and $D = D_0 \otimes F = D_0 \times D_0$. Since $\mu_{i\pi}$ are reduced norms from $D \otimes_F FL_i = D_0 \otimes L_i \times D_0 \otimes L_i$. Hence $\mu_{i\pi}$ are reduced norms from $D_0 \otimes L_i$ and by (5.2.3), $\mu_i$ are reduced norms from $D \otimes L_i$. $\square$

## 5.6 Choice at Nodal Points

Let $p \geq 3$ be a prime and $K$ be a $p$-adic field. Let $F_0$ be the function field of a curve over $K$ and $F = F_0(\sqrt{d})$ a quadratic field extension. Let $D$ be a central division algebra over $F$ with a $F/F_0$-involution. Let $\lambda \in F_0^* \cap Nrd(D)^*$.

Let $T$ be the valuation ring of $K$ and $k$ the residue field of $K$. Let $\mathscr{X}_0$ be regular proper model of $F_0$ over $T$ with the union of the ramification locus of $D$, support of $d$, support of $\lambda$ and the closed fibre $X_0$ of $\mathscr{X}_0$ is a union of regular curves with normal crossings. Further the integral closure $\mathscr{X}$ of $\mathscr{X}_0$ in $F$ is a is a regular proper model of $F$ (Proposition 8.3.8, [17]). Let $\mathscr{D}$ be the set of codimension one points of $\mathscr{X}_0$ consisting of support of $d$, support of $\lambda$, the closed fibre $X_0$ and the ramification locus of $D$ on $\mathscr{X}_0$. Let $P \in \mathscr{X}_0$ be a closed point. Then, by the choice of $\mathscr{X}_0$, there exist at most two codimension one points of $\mathscr{X}_0$ which are in $\mathscr{D}$ and passes through $P$. Further, since $\mathscr{X}$ is regular, there exists at most one codimension one point $\eta$ of $\mathscr{X}_0$ passing through $P$ such that $\nu_\eta(d)$ is odd.

Let $P \in \mathscr{X}_0$ be a closed point. Let $\hat{R}_{0P}$ be the completion of the local ring at $P$ on $\mathscr{X}_0$, $m_P$ the maximal ideal $\hat{R}_{0P}$, $F_{0P}$ the field of fractions of $\hat{R}_{0P}$ and $F_P = F_{0P} \otimes F$. Let $w_P \in \hat{R}_{0P}$ be a unit which is not a square in $\hat{R}_{0P}$. Since the residue field $\kappa(P)$ at $P$ is a finite field, any unit in $\hat{R}_{0P}$ is a square or $w_P$ times a square.

Let $\mathscr{P}_0$ be the finite set of closed points of $\mathscr{X}_0$ consisting of the points of intersection of two distinct codimension one points in $\mathscr{D}$.

Let $P \in \mathscr{P}_0$ and $\eta_1, \eta_2 \in \mathscr{D}$ such that $P \in \overline{\{\eta_1\}} \cap \overline{\{\eta_2\}}$. Then $m_P = (\pi_P, \delta_P)$ with $\eta_1$ and $\eta_2$ are given by primes $\pi_P$ and $\delta_P$ respectively at $P$, $d = d_1^2$ or $d = w_P d_1^2$ or

$d = w_P \pi_P d_1^2$ and $\lambda = u_P \pi_P^r \delta_P^s$ and $D$ is unramified at $P$ except possibly at $(\pi_P)$ and $(\delta_P)$, for some $u_P \in \hat{R}_{0P}$ units, $d_1 \in F_0^*$, $r = \nu_{\eta_1}(\lambda)$, $s = \nu_{\eta_2}(\lambda)$. Let $\epsilon_1, \epsilon_2 \in \{0,1\}$ and $r_1, s_1 \in \mathbb{Z}$ such that $r = 2r_1 + \epsilon_1$ and $s = 2s_1 + \epsilon_2$. Then $\lambda = u_P \pi_P^{\epsilon_1} \delta_P^{\epsilon_2} (\pi_P^{r_1} \delta_P^{s_1})^2$.

Suppose that period of $D$ is 2. Then $\mathrm{ind}(D) \leq 4$ (cf. [26]) and $\mathrm{ind}(D \otimes F_{0P}) \leq 2$ (cf. [20]). Then, there exists a central simple algebra $D_{0P}$ over $F_{0P}$ such that $D \otimes F_{0P} = D_{0P} \otimes F_P$ and $D_{0P}$ is unramified at $P$ except possibly at $\eta_1$ and $\eta_2$. Further if $D \otimes F_{0P}$ is a split algebra, we choose $D_{0P} = F_{0P}$ and if $D \otimes F_{0P}$ is not a split algebra, $D_{0P}$ be as in (5.5.1).

**Proposition 5.6.1.** *Suppose $\nu_{\eta_1}(d)$ and $\nu_{\eta_2}(d)$ are even. Then there exist $a_{iP}$, $\mu_{iP}$, $i = 1,2,3$ such that*

*1) $a_{1P} = v_P \pi_P^{\epsilon_1} \delta_P^{\epsilon_2} \in F_{0P} \setminus F_{0P}^{*2}$, $v_P$ a unit at $P$, $\mu_{1P} \in L_{1P} = F_{0P}[X]/(X^2 - a_{1P})$*

*2) $a_{2P} \in \hat{R}_{0P}$ a unit at $\eta_1$ and $\eta_2$ and $\partial_{\eta_i}(D_{0P}) = \bar{a}_{2P} \in \kappa(\eta_i)^*/\kappa(\eta_i)^{*2}$ for $i = 1,2$*

*3) $a_{3P} = d a_{2P}$,*

*4) $\mu_{iP} \in F_{0P}[X]/(X^2 - a_{iP})^*$ unit along $\pi$ and $\delta$ for $i = 2,3$*

*5) $\prod_i N_{L_{iP}/F_{0P}}(\mu_{iP}) = \lambda$, where $L_{iP} = F_0[X]/(X^2 - a_i)$ for $i = 1,2,3$*

*6) $\mu_{iP}$ is a reduced norm from $D \otimes L_{iP}$ for $i = 1,2,3$*

*Proof.* Suppose $D \otimes F_{0P}$ is a split algebra. Let $v_P$ a unit at $P$ such that $a_{1P} = v_P \pi_P^{\epsilon_1} \delta_P^{\epsilon_2} \in F_{0P} \setminus F_{0P}^{*2}$. Then $\mu_{1P} = \pi_P^{r_1} \delta_P^{s_1} \sqrt{a_{1P}}$, $a_{2P} = 1$, $\mu_{2P} = (-v_P^{-1} u_P, 1) = F_{0P} \times F_{0P} = F_{0P}[X]/(X^2 - 1)$ and $\mu_{3P} = 1$ have the required properties.

Suppose that $D \otimes F_{0P}$ is not a split algebra. Suppose $d$ is not a square in $F_{0P}$. Since $\nu_{\eta_1}(d)$ and $\nu_{\eta_2}(d)$ are even, $d = w_P d_1^2$ for some $d_1 \in F_{0P}^*$. Then, by (5.5.1), $D_{0P} = (\pi_P, \delta_P)$. Then $a_{1P}, a_{2P}, \mu_{iP}$ as in (5.5.3) have the required properties.

Suppose $d$ is a square in $F_{0P}$. Then $F \otimes F_{0P} = F_{0P} \times F_{0P}$ and $D \otimes F_{0P} = D_{0P} \times D_{0P}^{op}$ for some quaternion algebra $D_{0P}$ over $F_{0P}$. Further $D_{0P}$ is unramified at $P$ except possibly at $(\pi_P)$ and $(\delta_P)$. Let $a_{1P}$, $a_{2P}$ and $\mu_i \in L_{iP}$ be as in (5.5.2). Let $\mu_{3P} = 1$. Then $a_{1P}$, $a_{2P}$ and $\mu_{iP}$, $i = 1,2,3$ have the required properties. □

**Proposition 5.6.2.** *Suppose $\nu_{\eta_1}(d)$ is odd. Then there exist $a_{1P}$, $a_{2P}$, $\mu_{1P}$, $\mu_{2P}$ such that*

*1) $a_{1P} = v_P \pi_P^{\epsilon_1}\delta_P^{\epsilon_2} \in F_{0P} \setminus F_{0P}^{*2}$, $v_P$ a unit at $P$, $\mu_{1P} \in L_{1P} = F_{0P}[X]/(X^2 - a_{1P})$*

*2) $a_{2P} \in \hat{R}_{0P}$ a unit at $P$ and $\partial_{\eta_2}(D_{0P}) = \bar{a}_{2P} \in \kappa(\eta_2)^*/\kappa(\eta_2)^{*2}$*

*3) $\mu_{2P} \in \hat{R}_{0P}[X]/(X^2 - a_{2P})^*$*

*4) $N_{L_{1P}/F_{0P}}(\mu_{1P})N_{L_{2P}/F_{0P}}(\mu_{2P}) = \lambda$, where $L_{iP} = F_0[X]/(X^2 - a_i)$ for $i = 1, 2$*

*5) $\mu_{iP}$ is a reduced norm from $D \otimes L_{iP}$ for $i = 1, 2$*

*Proof.* Suppose $D \otimes F_{0P}$ is a split algebra. Let $v_P$ a unit at $P$ such that $a_{1P} = v_P\pi_P^{\epsilon_1}\delta_P^{\epsilon_2} \in F_{0P} \setminus F_{0P}^{*2}$. Then $\mu_{1P} = \pi_P^{r_1}\delta_P^{s_1}\sqrt{a_{1P}}$, $a_{2P} = 1$ and $\mu_{2P} = (-v_P^{-1}u_P, 1) = F_{0P} \times F_{0P} = F_{0P}[X]/(X^2 - 1)$ have the required properties.

Suppose $D \otimes F_{0P}$ is not a split algebra. Since $\nu_{\eta_1}(d)$ is even, by the choice of $\mathscr{X}_0$, $\nu_{\eta_2}(d)$ is even. Hence $d = v_P\pi_P d_1^2$ for some $v_1 \in \hat{R}_{0P}$ a unit and $d_1 \in F_0^*$. In particular $D_{0P} = (w_P, \delta_P)$. Let $a_{2P} = w_P$. Hence, by (5.5.4), there exist $a_{1P} = v_P\pi^{\epsilon_1}\delta^{\epsilon_2} \in F_{0P}^* \setminus F_{0P}^{*2}$ with $v_P \in \hat{R}_{0P}^*$ and $\mu_{1P} \in L_{1P} = F_{0P}[X]/(x^2 - a_{1P})$ and $\mu_{2P} \in L_{2P} = F_{0P}[X]/(x^2 - a_{2P})$ such that

i) $N_{L_{1P}/F_{0P}}(\mu_{1P})N_{L_{2P}/F_{0P}}(\mu_{2P}) = \lambda$

ii) $\mu_{iP}$ is a reduced from from $D \otimes L_{iP}$ for $i = 1, 2$

iii) $\mu_{2P} \in \hat{R}_{0P}[X]/(X^2 - a_{2P})^*$.

Then $a_{1P}$, $a_{2P}$, $\mu_{1P}$ and $\mu_{2P}$ have the required properties. □

## 5.7 Choices at Codimension One Points and Curve Points

Let $p \geq 3$ be a prime and $K$ be a $p$-adic field. Let $F_0$ be the function field of a curve over $K$ and $F = F_0(\sqrt{d})$ a quadratic field extension. Let $D$ be a central division algebra over $F$ with a $F/F_0$-involution. Suppose that period of $D$ is 2. Let $\lambda \in F_0^* \cap Nrd(D)^*$.

Let $\mathscr{X}_0$, $\mathscr{X}$, $\mathscr{D}$ and $\mathscr{P}_0$ be as in section 5.4. Let $\eta \in X_0$ be a codimension zero point. Let $\pi$ be a parameter at $\eta$. Then, by (5.4.1) and (5.4.2), we have $D \otimes F_{0\eta} = (b,c) \otimes (w,\pi)$ for some $b,c,w \in F_{0\eta}$. Let $D_{0\eta} = (b,c) \otimes (w,\pi)$. Write $\lambda = u\pi^r$ for some $u \in F_{0\eta}$ with $\nu_\eta(u) = 0$. Let $\epsilon \in \{0,1\}$ and $r_1 \in \mathbb{Z}$ such that $r = 2r_1 + \epsilon$. Then $\lambda = u\pi^\epsilon(\pi^{r_1})^2$. Let $\mathscr{P}_\eta = \mathscr{P}_0 \cap \overline{\{\eta\}}$.

**Proposition 5.7.1.** *Let $\eta \in X_0$ be a codimension zero point. Suppose $\nu_\eta(d)$ and $\nu_\eta(\lambda)$ are even. For each $P \in \mathscr{P}_\eta$, if $d$ is a unit at $P$ up to a square $F_{0P}$, let $a_{iP}$ and $\mu_{iP}$, $i = 1,2,3$ be as in (5.6.1) and if $d$ is not a unit at $P$ up to a square in $F_{0P}$, let $a_{iP}$ and $\mu_{iP}$, $i = 1,2$ be as in (5.6.2), $a_{3P} = da_{2P}$ and $\mu_{3P} = 1$. Then there exist $a_{1\eta}, a_{2\eta}, a_{3\eta} \in F_{0\eta}$ units at $\eta$ and $\mu_{i\eta} \in F_{0\eta}[X]/(X^2 - a_{i\eta})$ such that*

*i) $\prod_i N_{L_{i\eta}/F_{0\eta}}(\mu_{i\eta}) = \lambda$*

*ii) $\mu_{i\eta}$ is a reduced norm from $D \otimes L_{i\eta}$ for $i = 1,2,3$*

*iii) $ind(D \otimes F_{i\eta}) \leq 2^m$ for all $m \geq 1$, $P \in \mathscr{P}_\eta$, $i = 1,2,3$*

*iv) $a_{i\eta}a_{iP} \in F^{*2}_{0P,\eta}$ for $i = 1,2,3$*

*v) for $P \in \mathscr{P}_\eta$, there is an isomorphism*

$$\phi_{iP,\eta} : F_{0P,\eta}[X]/(X^2 - a_{i\eta}) \to F_{0P,\eta}[X]/(X^2 - a_{iP})$$

*such that*

$$\phi_{iP,\eta}(\mu_{i\eta})\mu_{iP}^{-1} \in (F_{0P,\eta}[X]/(X^2 - a_{iP}))^{2^m}$$

*for all $m \geq 1$ and $i = 1,2,3$.*

*Proof.* Since $\nu_\eta(d)$ is even, replacing $d$ by $d$ times a square in $F_{0\eta}$, we assume that $\nu_\eta(d) = 0$.

Let $\pi_\eta$ be a parameter at $\eta$ such that for every $P \in \mathscr{P}_\eta$, the maximal ideal at $P$ is given by $(\pi_\eta, \delta_P)$ for some prime $\delta_P$ because of normal crossings.

By (5.4.1) and (5.4.2), we have $D \otimes F_\eta = (b,c) \otimes (w, \pi_\eta)$ for some $b, c, w \in F_{0\eta}$ which are units at $\eta$. Let $D_{0\eta} = (b,c) \otimes (w, \pi_\eta)$, $u_0$, $b_0$, $c_0$, $d_0$ and $w_0$ be the images of $u, b, c, d, w$ in $\kappa(\eta)$. Since $\lambda = u\pi^r$ with $u \in F_{0\eta}$ a unit at $\eta$ and $\lambda$ is a reduced norm from $D \otimes F_\eta$, by the norm principle of bi-quadratic extensions, we have $u \in N_{F_{0\eta}(\sqrt{d},\sqrt{w})/F_{0\eta}(\sqrt{d})}((F_{0\eta}(\sqrt{d},\sqrt{w})^*)$. Hence $u_0 \in N_{\kappa(\eta)(\sqrt{d_0},\sqrt{w_0})/F_0(\sqrt{d_0}}(\kappa(\eta)(\sqrt{d_0},\sqrt{w_0})^*)$.

Since $\nu_\eta(\lambda) = 2r_1$, by the choice of $a_{1P}$ (5.6.1, 5.6.2), we have $a_{1P} = v_P\delta_P^{\epsilon_2}$ for some unit $v_P$ at $P$ and $\epsilon_2 \in \{0,1\}$. Further $a_{1P}$ is not square in $F_{0P}$. Since $F_{0P,\eta}$ is the completion of $F_{0P}$ at $\eta$, $a_{1P}$ is not a square in $F_{0P,\eta}$. Let $x_P = \bar{a}_{1P} = \bar{v}_P\bar{\delta}_P^{\epsilon_2}$.

Since $\mu_{iP} \in R_{0P}[X]/(X^2 - a_{iP})$ are units along $\eta$ for $i = 2, 3$ and $\prod_1^3 N_{L_{iP}/F_{0P}}(\mu_{iP}) = \lambda$, it follows that $\nu_\eta(N_{L_{1P}/F_{0P}}(\mu_{iP})) = \nu_\eta(\lambda)$. Since $L_{1P} \otimes F_{0P,\eta} = F_{0P,\eta}[X]/(X^2 - a_{1P})$ is unramified and $\nu_\eta(N_{L_{1P}/F_{0P}}(\mu_{1P})) = \nu_\eta(\lambda) = 2r_1$, we have $\mu_{1P} = y'_P\pi_\eta^{r_1}$ for some $y'_P \in L_{1P} \otimes F_{0P,\eta}$ unit in the valuation ring. Let $y_{1P} = \bar{y'}_P \in \kappa(\eta)_{1P} = \kappa(\eta)_P[X]/(X^2 - x_P)$.

For $i = 2, 3$, let $y_{iP}$ be the image of $\mu_{iP}$ in $\kappa(\eta)_P[X]/(X^2 - \bar{a}_{2P})$. By the choice (5.6.1, 5.6.2), we have $a_{2P} = \partial_\eta(D_{0\eta}) = \bar{w}$ and $a_{3P} = da_{2P}$. Then $y_{2P} = \bar{\mu}_{2P} \in \kappa(\eta)_{2P} = \kappa(\eta)_P[X]/(X^2 - \bar{a}_{2P}) = \kappa(\eta)_P[X]/(X^2 - \bar{w})$ and $y_{3P} = \bar{\mu}_{3P} \in \kappa(\eta)_{3P} = \kappa(\eta)_P[X]/(X^2 - \bar{a}_{3P}) = \kappa_\nu[X]/(X^2 - \bar{d}\bar{w})$.

Further we have

i) $\prod_i N_{\kappa(\eta)_{iP}/\kappa(\eta)_P}(y_{iP}) = \bar{u}$.

ii) $y_{1P} \in N_{\kappa(\eta)_P(\sqrt{\bar{d}},\sqrt{\bar{w}})/\kappa(\eta)_P}(\kappa(\eta)_P(\sqrt{\bar{d}},\sqrt{\bar{w}})^*)$.

Hence, by (5.4.3), there exists $a \in \hat{R}_{0\eta}^*$, $\mu_1 \in \hat{R}_{0\eta}[X]/(X^2 - a)$, $\mu_2 \in \hat{R}_{0\eta}[X]/(X^2 - w)$ and $\mu_3 \in \hat{R}_{0\eta}[X]/(X^2 - dw)$ such that

i) $\bar{a}$ is close to $x_P$ and $\bar{\mu}_i$ is close to $y_{iP}$ for all $P \in \mathscr{P}_0$ and $i = 1, 2, 3$

ii) $\prod_i N_{L_{i\eta}/F_{0\eta}}(\mu_i) = u$, where $L_{1\eta} = F_{0\eta}[X]/(X^2 - a)$, $L_{2\eta} = F_{0\eta}[X]/(X^2 - w)$ and $L_{3\eta} = F_{0\eta}[X]/(X^2 - dw)$

iii) $(b,c) \otimes L_{1\eta}$ is split

iv) $\mu_i$ is a reduced norm from $D \otimes_{F_{0\eta}} L_{i\eta}$ for $i = 1, 2, 3$.

Since $D \otimes L_{1\eta} = ((b,c) \otimes L_{1\eta}) \otimes (w, \pi_\eta) \otimes L_{1\eta}) = (w, \pi_\eta) \otimes L_{1\eta})$, $\pi_\eta$ is a reduced norm from $D \otimes L_{1\eta}$. Hence $a_{1\eta} = a$, $a_{2\eta} = w$, $a_{3\eta} = wd$, $\mu_{1\eta} = \mu_1 \pi_\eta^{r_1}$, $\mu_{2\eta} = \mu_2$ and $\mu_{3\eta} = \mu_3$ have the required properties. □

**Proposition 5.7.2.** *Let $\eta \in X_0$ be a codimension zero point. Suppose $\nu_\eta(d)$ is even and $\nu_\eta(\lambda)$ is odd. For each $P \in \mathscr{P}_\eta$, if $d$ is a unit at $P$ up to a square $F_{0P}$, let $a_{iP}$ and $\mu_{iP}$, $i = 1,2,3$ be as in (5.6.1) and if $d$ is not a unit at $P$ up to a square in $F_{0P}$, let $a_{iP}$ and $\mu_{iP}$, $i = 1,2$ be as in (5.6.2), $a_{3P} = da_{2P}$ and $\mu_{3P} = 1$. Then there exist $a_{1\eta}, a_{2\eta}, a_{3\eta} \in F_{0\eta}$ and $\mu_{i\eta} \in F_{0\eta}[X]/(X^2 - a_{i\eta})$ such that*

*i) $\prod_i N_{L_{i\eta}/F_{0\eta}}(\mu_{i\eta}) = \lambda$*

*ii) $\mu_{i\eta}$ is a reduced norm from $D \otimes L_{i\eta}$ for $i = 1,2,3$*

*iii) ind$(D \otimes F_{i\eta}) \leq 2$ for $i = 1,2,3$*

*iv) $a_{i\eta} a_{iP} \in F^{*2}_{0P,\eta}$ for $i = 1,2,3$*

*v) for $P \in \mathscr{P}_\eta$, there is an isomorphism*

$$\phi_{iP,\eta} : F_{0P,\eta}[X]/(X^2 - a_{i\eta}) \to F_{0P,\eta}[X]/(X^2 - a_{iP})$$

*such that*

$$\phi_{iP,\eta}(\mu_{i\eta})\mu_{iP}^{-1} \in (F_{0P,\eta}[X]/(X^2 - a_{iP}))^{2^m}$$

*for all $m \geq 1$ and $i = 1,2,3$.*

*Proof.* Since $\nu_\eta(\lambda)$ is odd, ind$(D)$ is at most 2 and $D \otimes F_{0\eta} = (w, \pi_\eta)$ for some parameter $\pi_\eta$ at $\eta$ and $w \in F_0^*$ a unit at $\eta$ (cf. Lemma 5.3.6; Corollary 5.6, [21] ).

Since $\nu(\lambda)$ is odd, $\pm\lambda$ is not a square in $F_{0P}$ for all $P \in \mathscr{P}_\eta$. Hence, by the choice of $a_{1P}$ and $\mu_{iP}$, we have $a_{1P}\lambda \in F^{*2}_{0P,\eta}$, $\mu_{1P}\sqrt{\lambda} \in F_{P\eta}(\sqrt{a_{1P}})^{*2}$. Further $wa_{2P} \in F^{*2}_{0P,\eta}$, $a_{3P} = a_{2P}d$, $\mu_{1P} = \mu_{2P} = 1$.

Let $a_{1\eta} = -\lambda$, $a_{2\eta} = w$, $a_{3\eta} = dw$, $\mu_{1\eta} = \sqrt{-\lambda}$ and $\mu_{2\eta} = \mu_{2\eta} = 1$. Since $\lambda$ is a reduced norm from $D$, we have $(\lambda) \cdot D = 0 \in H^3(F, \mu_2)$. Since $\nu(\lambda)$ is odd, if

$D \otimes F_\eta$ is not split, then by (Lemma 4.7, [21]), $\text{ind}(D \otimes F(\sqrt{a_{1\eta}}) < \text{ind}(D \otimes F_\eta)$. In particular $D \otimes F_\eta(\sqrt{a_{1\eta}})$ is split. Hence $a_{1\eta} = -\lambda$, $a_{2\eta} = w$, $a_{3\eta} = dw$, $\mu_{1\eta} = \sqrt{-a_{1\eta}}$ and $\mu_{2\eta} = \mu_{3\eta} = 1$ have the required properties. □

**Proposition 5.7.3.** *Let $\eta \in X_0$ be a codimension zero point. Suppose $\nu_\eta(d)$ is odd. For each $P \in \mathscr{P}_\eta$, let $a_{iP}$ and $\mu_{iP}$, $i = 1, 2$ be as in (5.6.2). Then there exist $a_{1\eta}, a_{2\eta} \in F_{0\eta}$ and $\mu_{i\eta} \in F_{0\eta}[X]/(X^2 - a_{i\eta})$ such that*

*1) $\prod_i N_{L_{i\eta}/F_{0\eta}}(\mu_{i\eta}) = \lambda$*

*2) $\mu_{i\eta}$ is a reduced norm from $D \otimes L_{i\eta}$ for $i = 1, 2$*

*3) ind$(D \otimes F_{i\eta}) \leq 2$ for $i = 1, 2$*

*4) $a_{i\eta} a_{iP} \in F^{*2}_{0P,\eta}$ for $i = 1, 2$*

*5) for $P \in \mathscr{P}_\eta$, there is an isomorphism*

$$\phi_{iP,\eta} : F_{0P,\eta}[X]/(X^2 - a_{i\eta}) \to F_{0P,\eta}[X]/(X^2 - a_{iP})$$

*such that*

$$\phi_{iP,\eta}(\mu_{i\eta})\mu_{iP}^{-1} \in (F_{0P,\eta}[X]/(X^2 - a_{iP}))^{2^m}$$

*for all $m \geq 1$ and $i = 1, 2$.*

*Proof.* By (5.4.1) and (5.4.2), we have $D \otimes F_\eta = (b, c)$ for some $b, c \in F_{0\eta}$ which are units at $\eta$. Let $D_{0\eta} = (b, c)$ and $u_0$, $b_0$, $c_0$ be the images of $u, b, c$ in $\kappa(\eta)$.

Write $r = 2r_1 + \epsilon_1$ for some $r_1 \in \mathbb{Z}$ and $\epsilon_1 \in \{0, 1\}$. By the choice of $a_{1P}$, we have $a_{1P} = w_P^{\epsilon_P} \pi_\eta^{\epsilon_1}$ for some $w_P \in F_{0P,\eta}$ unit at $\eta$. Let $x \in \kappa(\eta)$ be close to $\bar{w}_P$ for all $P \in \mathscr{P}_\eta$. Let $a \in F_{0\eta}$ which maps to $x$ in $\kappa(\eta)$ and $a_{1\eta} = a\pi_\eta^{\epsilon_1}$. Then $a_{1\eta} a_{1P} \in F^{*2}_{0P,\eta}$ for all $P \in \mathscr{P}_\eta$. Let $L_{1\eta} = F_{0\eta}(\sqrt{a_{1\eta}})$ and $L_{1P,\eta} = F_{0P,\eta}(\sqrt{a_{1P}})$. Then $L_{1\eta} \otimes F_{0P,\eta} = L_{1P,\eta}$. Let $\kappa(\eta)_1$ be the residue field of $L_{1\eta}$. Then $\kappa(\eta)_{1P}$ is the residue field of $F_{0P,\eta}(\sqrt{a_{1P}})$.

Since $\mu_{2P} \in R_{0P}[X]/(X^2 - a_{2P})$ is a unit along $\eta$ and $N_{L_{1P}/F_{0P}}(\mu_{1P}) N_{L_{2P}/F_{0P}}(\mu_{2P}) = \lambda$, it follows that $\nu_\eta(N_{L_{1P}/F_{0P}}(\mu_{1P})) = \nu_\eta(\lambda) = 2r_1 + \epsilon_1$.

Suppose $\epsilon_1 = 0$. Then $L_{1\eta}/F_{0,\eta}$ is unramified and $\pi_\eta$ is a parameter in $F_{0P,\eta}(\sqrt{a_{1P}})$. Hence $\mu_{1P} = \theta_P \pi_\eta^{r_1}$ for some $\theta_P \in F_{0P,\eta}(\sqrt{a_{1P}})$ a unit at $\eta$. Suppose $\epsilon_1 = 1$. Then $L_{1\eta}/F_{0\eta}$ is ramified and $\sqrt{a_{1\eta}}$ is a parameter. Hence $\mu_{1P} = \theta_P \pi_\eta^{r_1}\sqrt{a_{1\eta}}$ for some $\theta_P \in F_{0P,\eta}(\sqrt{a_{1P}})$ a unit at $\eta$. In both cases, let $\theta \in \kappa(\eta)_1$ close to $\bar{\theta}_P$ for all $P \in \mathscr{P}_\eta$ and $\theta_1 \in L_{1\eta}$ which lifts $\theta$. If $\epsilon_1 = 0$, let $\mu_{1\eta} = \theta_1 \pi_\eta^{r_1}$ and if $\epsilon_1 = 1$, let $\mu_{1\eta} = \theta_1 \pi_\eta^{r_1}\sqrt{a_{1\eta}}$. Then $\mu_{1\eta}\mu_{1P} \in L_{1\eta,P}^{*2}$ and $\lambda N_{L_{1\eta}/F_{0\eta}}(\mu_{1\eta})^{-1}$ is a unit at $\eta$. Since $\nu(d)$ is odd, $F/F_0$ is ramified at $\eta$ and hence $\mu_{1\eta} = \mu'_{1\eta} g_\eta^2$ for some $\mu'_{1\eta} \in F \otimes L_{1\eta}$ a unit at $\eta$ and $g_\eta \in F \otimes L_{1\eta}$. Since $\kappa(\eta)_1$ is a global field, $\bar{\mu}'_{1\eta} \in \kappa(\eta)_1$ is a reduced norn from $(b_0, c_0) \otimes \kappa(\eta)_1$ (Albert-Brauer-Hasse-Noether). Hence $\mu_{1\eta}$ is a reduced norm from $D \otimes L_{1\eta}$.

Let $z_1 \in \kappa(\eta)$ be the image of $\lambda N_{L_{1\eta}/F_{0\eta}}(\mu_{1\eta})^{-1}$ and $y_{2P}$ be the image of $\mu_{2P}$ in $\kappa(\eta)_{2P} = \kappa(\eta)_P[X]/(X^2 - \bar{a}_{2P})$. By the choice of $\mu_{1\eta}$, it follows that $N_{\kappa(\eta)_{2P}/\kappa(\eta)_P}(y_{2P})$ is close to $z_1$. Hence, replacing $y_{2P}$ by some element which is close to $y_{2P}$, we assume that $N_{\kappa(\eta)_{2P}/\kappa(\eta)_P}(y_{2P}) = z_1$. In particular the quaternion algebra $(\bar{a}_{2P}, z_1)$ is split over $\kappa(\eta)_P$ for all $P \in \mathscr{P}_\eta$. Hence $\bar{a}_{2P}$ is a norm from the extension $\kappa(\eta)_P[X]/(X^2 - z_1)$. Let $\tilde{a}_{2P} \in \kappa(\eta)_P[X]/(X^2 - z_1)$ with norm equal to $\bar{a}_{2P}$. Let $\tilde{a}_2 \in \kappa(\eta)[X]/(X^2 - z_1)$ be close to $\tilde{a}_{2P}$ for all $P \in \mathscr{P}_\eta$ and $\bar{a}_2$ be the norm of $\tilde{a}_2$. Then $\bar{a}_2$ is close to $\bar{a}_{2P}$ for all $P \in \mathscr{P}_\eta$. Since the quaternion algebra $(\bar{a}_2, z_1)$ is split, $z_1$ is a norm from the extension $\kappa(\eta)_2 = \kappa(\eta)[X]/(X^2 - \bar{a}_2)$.

There exists $y_2 \in \kappa(\eta)_2$ which is close to $y_{2P}$ for all $P \in \mathscr{P}_\eta$ such that $N_{\kappa(\eta)_2/\kappa(\eta)}(y_2) = z_1$ since $\kappa(\eta)_2$ is a global field. Let $a_{2\eta} \in F_{0\eta}$ be a lift of $\bar{a}_2 \in \kappa(\eta)$ and $\mu_{2\eta} \in L_{2\eta} = F_{0\eta}[X]/(X^2 - a_{2\eta})$ be such that $N_{L_{2\eta}/F_{0\eta}}(\mu_{2\eta}) = \lambda N_{L_{1\eta}/F_{0\eta}}(\mu_{1\eta})^{-1}$. Since $\mu_{2\eta}$ is a unit at $\eta$ and $D$ is unramified at $\eta$, as above, $\mu_{2\eta}$ is a reduced norm from $D \otimes L_{2\eta}$.

Hence $a_{1\eta}$, $a_{2\eta}$, $\mu_{1\eta}$ and $\mu_{2\eta}$ have the required properties. □

**Proposition 5.7.4.** *Let $P \in \mathscr{X}_0$ be a closed point. Suppose that $P \notin \mathscr{P}_0$. Let $\eta \in D$ be the unique codimension one point with $P \in \overline{\{\eta\}}$. Let $a_{i\eta} \in F_{0\eta}$ and $\mu_{i\eta} \in L_{i\eta} = F_{0\eta}[X]/(X^2 - a_{i\eta})$ be as in (5.7.1, 5.7.2, 5.7.3). Then there exist $a_{iP} \in F_{0P}$*

*and* $\mu_{iP} \in L_{iP} = F_{0P}[X]/(X^2 - a_{iP})$ *such that*

*i)* $\prod_i N_{L_{iP}/F_{0P}}(\mu_{iP}) = \lambda$

*ii)* $\mu_{iP}$ *is a reduced norm from* $D \otimes L_{iP}$

*iii)* $a_{i\eta}a_{iP} \in F_{0P,\eta}^{*2}$

*iv) there is an isomorphism*

$$\phi_{iP,\eta} : F_{0P,\eta}[X]/(X^2 - a_{i\eta}) \to F_{0P,\eta}[X]/(X^2 - a_{iP})$$

*such that*

$$\phi_{iP,\eta}(\mu_{i\eta})\mu_{iP}^{-1} \in (F_{0P,\eta}[X]/(X^2 - a_{iP}))^{2^m}$$

*for all* $m \geq 1$.

*Proof.* Let $\pi_P$ be a prime defining $\eta$ at $P$. Since there is a unique codimension one point in $\mathscr{D}$, the support of $d$ at $P$ and the ramification locus of at $P$ is at most $\eta$. Hence, by (5.5.6), we have the required $a_{iP}$ and $\mu_{iP}$. □

## 5.8 Choice of $U$

Let $T$ be a complete discrete valuation ring with field of fractions $K$ and residue field $k$. Let $F_0$ be the function field of a curve over $K$ and $F = F_0(\sqrt{d})$ a quadratic étale extension. Let $D$ be a central division algebra over $F$ with period($D$) coprime to char($k$).

**Proposition 5.8.1.** *Let* $\mathscr{X}_0$ *be a normal proper model of* $F_0$ *over* $T$ *and* $X_0$ *the closed fibre of* $\mathscr{X}_0$. *Let* $\eta \in X_0$ *be a codimension zero point. Let* $\lambda \in F_0^* \cap Nrd(D)^*$, $m \geq 2$ *and* $M \geq 1$. *Suppose that for* $1 \leq i \leq m$, *there exist* $a_{i\eta} \in F_{0\eta}$, $\mu_{i\eta} \in L_{i\eta} = F_{0\eta}[X]/(X^2 - a_{i\eta})$ *such that*

*i)* $\prod_1^m N_{L_{i\eta}/F_{0\eta}}(\mu_{i\eta}) = \lambda$

*ii)* $\mu_{i\eta}$ *is a reduced norm from* $D \otimes L_{i\eta}$ *for all* $i$

*iii) ind*$(D \otimes L_{i\eta}) < M$ *for all* $i$.

*Then there exist a non-empty open proper subset* $U$ *of* $\overline{\{\eta\}}$ *and* $a_{iU} \in F_{0U}$, $\mu_{iU} \in L_{iU} = F_{0U}[X]/(X^2 - a_{iU})$ *such that*

*i)* $a_{iU}a_{i\eta} \in F_{0\eta}^{*2}$

*ii) there is an isomorphism*

$$\phi_{i,U_\eta} : F_{0,\eta}[X]/(X^2 - a_{iU}) \to F_{0,\eta}[X]/(X^2 - a_{i\eta})$$

*such that*

$$\phi_{i,\eta}(\mu_{iU})\mu_{i\eta}^{-1} \in (F_{0P,\eta}[X]/(X^2 - a_{i\eta}))^{*2^m}$$

*for all* $m \geq 1$ *and* $i = 1, 2, 3$.

*iii)* $\prod_1^m N_{L_{iU}/F_{0U}}(\mu_{iU}) = \lambda$ *for all* $i$

*ii)* $\mu_{iU}$ *is a reduced norm from* $D \otimes L_{iU}$ *for all* $i$

*iii) ind*$(D \otimes L_{iU}) < M$ *for all* $i$.

*Proof.* Since $F_{0\eta}$ is the completion of $F_0$ and char$(k) \neq 2$, there exists $a_i \in F_0^*$ such that $a_i a_{i\eta} \in F_{0\eta}^{*2}$. Thus, replacing $a_{i\eta}$ by $a_i$, we assume that $a_{i\eta} = a_i \in F_0^*$.

Since $L_{i\eta} = F_{0\eta}(\sqrt{a_i})$ is the completion of $L_i = F_0(\sqrt{a_i})$, there exists $\mu_i \in L_i$ close to $\mu_{i\eta}$ in $L_{i\eta}$. In particular $\theta_i = N_{L_i/F_0}(\mu_i)^{-1}N_{L_{i\eta}/F_{0\eta}}(\mu_{i\eta})$ is close to 1 in $F_{0\eta}$. Then $\theta = \prod_1^{m-1}\theta_i$ is close to 1 in $F_{0\eta}$. Let $\lambda_1 = \lambda(\prod_1^{m-1} N_{L_i/F_0}(\mu_i))^{-1} \in F_0$. Since $\prod_1^m N_{L_{i\eta}/F_{0\eta}}(\mu_{i\eta}) = \lambda$, we have

$$N_{L_m/F_0}(\mu_{m\eta}) = \lambda_1\theta^{-1}.$$

Since $\theta^{-1} \in F_{0\eta}$ is close to 1, $\theta^{-1} = N_{L_{m\eta}/F_{0\eta}}(\theta')$ for some $\theta' \in L_{m\eta}$ which is close to 1. In particular $\theta'$ is a reduced norm from $D \otimes L_{i\eta}$. Hence replacing $\mu_{m\eta}$ by $\mu_{m\eta}\theta'$, we assume that

$$N_{L_m/F_0}(\mu_{m\eta}) = \lambda_1.$$

Hence, by (Lemma 7.2, [21]), there exists a nonempty proper open subset $U_0$ of $\overline{\{\eta\}}$ and $\mu_{mU_0} \in L_m \otimes F_{0U_0}$ such that $\lambda_1 = N_{L_m/F_0}(\mu_{mU_0})$ and $\mu_{mU_0}$ is close to $\mu_{m\eta}$ in $L_m \otimes F_{0\eta}$.

Since $\text{ind}(D \otimes L_{i\eta}) < M$ for all $i$, there exist nonempty proper open subsets $U_i$ of $\overline{\{\eta\}}$ such that $\text{ind}(D \otimes L_{iU}) < M$ for all $i$.

Then $U = (\cap_i U_i) \cap U$, $a_{iU} = a_i$ for all $i$, $\mu_{iU} = \mu_i$ for $1 \leq i \leq m-1$ and $\mu_{mU} = \mu_{mU_0}$ have the required properties. □

## 5.9 The main theorem

**Theorem 5.9.1.** *Let $p \geq 3$ be a prime and $K$ be a $p$-adic field. Let $F_0$ be the function field of a curve over $K$ and $F = F_0(\sqrt{d})$ a quadratic field extension. Let $D$ be a central division algebra over $F$ with a $F/F_0$-involution. Suppose that period of $D$ is 2. Let $\lambda \in F_0^* \cap Nrd(D)^*$. Then there exist $a_i \in F_0^*$ and $\mu_i \in L_i = F_0[X]/(X^2 - a_i)$ for $i = 1, 2, 3$ such that*

*i)* $\prod_i N_{L_i/F_0}(\mu_i) = \lambda$

*ii)* $\mu_i$ *is a reduced norm from* $D \otimes L_i$ *for* $i = 1, 2, 3$

*iii)* *ind*$(D \otimes L_i) \leq 2$.

*Proof.* Let $T$ be the valuation ring of $K$ and $k$ the residue field of $K$. Let $\mathscr{X}_0$ be a regular proper model of $F_0$ over $T$ with the union of the ramification locus of $D$, support of $d$, support of $\lambda$ and the closed fibre $X_0$ of $\mathscr{X}_0$ is a union of regular curves with normal crossings. Further the integral closure $\mathscr{X}$ of $\mathscr{X}_0$ in $F$ is a is a regular proper model of $F$.

Let $\mathscr{D}$ be the set of codimension one points of $\mathscr{X}_0$ consisting of support of $d$, support of $\lambda$, the closed fibre $X_0$ and the ramification locus of $D$ on $\mathscr{X}_0$. Let $P \in \mathscr{X}_0$

be a closed point. Then, by the choice of $\mathscr{X}_0$, there exist at most two codimension one points of $\mathscr{X}_0$ which are in $\mathscr{D}$ and passes through $P$. Further, since $\mathscr{X}$ is regular, there exists at most one codimension one point $\eta$ of $\mathscr{X}_0$ passing through $P$ such that $\nu_\eta(d)$ is odd.

Let $\mathscr{P}_0$ be the finite set of closed points of $\mathscr{X}_0$ consisting of points of the intersection of the closures of any two distinct codimension one points in $\mathscr{D}$.

Let $P \in \mathscr{P}_0$ and $\eta_1, \eta_2 \in \mathscr{D}$ with $P \in \overline{\{\eta_1\}} \cap \overline{\{\eta_2\}}$. If $\nu_1(d)$ and $\nu_2(d)$ are even, then let $a_{iP}, \mu_{iP}$ for $i = 1, 2, 3$ be as in (5.6.1). If either $\nu_1(d)$ or $\nu_2(d)$ is odd, let $a_{iP}, \mu_{iP}$ for $i = 1, 2$ be as in (5.6.2) and $a_{3P} = da_{2P}$, $\mu_{3P} = 1$.

Let $\eta \in X_0$ be a codimension zero point. If $\nu(d)$ and $\nu(\lambda)$ are even, then let $a_{i\eta}, \mu_{i\eta}$ be as in (5.7.1) for $i = 1, 2, 3$. If $\nu(d)$ is even and $\nu(\lambda)$ is odd, then let $a_{i\eta}, \mu_{i\eta}$ be as in (5.7.2) for $i = 1, 2, 3$. If $\nu(d)$ is odd, then let $a_{i\eta}, \mu_{i\eta}$ be as in (5.7.3) for $i = 1, 2$ and $a_{3\eta} = a_{2\eta}d$, $\mu_{3\eta} = 1$.

Let $U_\eta$, $a_{iU_\eta}$ and $\mu_{iU_\eta}$ be as in (5.8.1). If necessary, replacing each $U_\eta$ by a open subset of $U_\eta$, we assume that $\mathscr{P}_0 \cap U_\eta = \emptyset$. Let $\mathscr{U} = \{U_\eta\}$.

Let $\mathscr{P} = X_0 \setminus \cup_\eta U_\eta$. Then $\mathscr{P}_0$ is a finit set of closed points of $\mathscr{P}_0 \subseteq \mathscr{P}$.

Let $P \in \mathscr{P} \setminus \mathscr{P}_0$. Then there is a unique codimension one point $\eta \in \mathscr{D}$. Let $a_{iP}$ and $\mu_{iP}$ for $i = 1, 2, 3$ be as in (5.7.4).

Let $P \in \mathscr{P}$ and $U \in \mathscr{U}$ with $P \in \overline{\{\eta\}}$. Then, by the choice of $a_{iP}$ and $a_{iU}$ we have $a_{iP} = \theta_{iP,\eta}^2 a_{iU}$ for some $\theta_{iP,U} \in F_{0U,P}^*$. Hence, by (Proposition 7.4, [21]), there exist $\theta_{iP} \in F_{0P}^*$ and $\theta_{iU} \in F_{0U}^*$ such that $\theta_{iP,\eta} = \theta_{iP}\theta_{iU}$. Thus $a_{iP}\theta_{iP}^{-2} = a_{iU}\theta_{iU}^2$ for all branches $(U, P)$. Hence there exist $a_i \in F_0^*$ such that $a_i = a_{iP} \in F_{0P}^*/F_{0P}^{*2}$ and $a_i = a_{iU} \in F_{0U}^*/F_{0U}^{*2}$ . Let $L_i = F_0[X]/(X^2 - a_i)$. Then, by (Theorem 5.1, [11]), $\mathrm{ind}(D \otimes L_i) \leq 2$ for all $i$.

Let $P \in X_0$ be a closed. Since $\kappa(P)$ is a finite field, there exists $t_P \geq 2$ such that $\kappa(P)$ has no $2^{t_P}$th primitive root of unity. Let $t > 2t_P$ for all $P \in \mathscr{P}$.

Let $P \in \mathscr{P}$. We have $\mu_{iP} \in F_{0P}[X]/(X^2 - a_i)$ and $\mu_{iU} \in F_{0U}[X]/(X^2 - a_i)$ such that $\mu_{iP}\mu_{iU}^{-1} \in (F_{0U,P}[X]/(X^2 - a_i))^{*2^m}$ for all $m \geq 1$. Hence $\mu_{iP} = \mu_{iU}\beta_{iU,P}^{2^{2t}}$ for some $\beta_{iU,P} \in L_i \otimes F_{0U,P}$. By (Proposition 7.4, [27]), there exist $\beta_{iP} \in L_i \otimes F_{0P}$ and $\beta_{iU} \in L_i \otimes F_{0U}$ such that $\beta_{iU,P} = \beta_{iU}\beta_{iP}$. In particular we have $\mu_{iP}\beta_{iP}^{-2^{2t}} = \mu_{iU}\beta_{iU}^{2^{2t}}$ for all branches $(U, P)$. Hence, by (Proposition 6.3, [10]), there exist $\mu_i \in L_i$ such that $\mu_i = \mu_{iP}\beta_{iP}^{-2^{2t}} = \mu_{iU}\beta_{iU}^{2^{2t}}$.

Let $\lambda_1 = \lambda N_{L_1/F_0}(\mu_1)^{-1}N_{L_2/F_0}(\mu_2)^{-1}$. For $\zeta \in \mathscr{P} \cup \mathscr{U}$, we have

$$\begin{aligned} \lambda_1 &= \lambda N_{L_1/F_0}(\mu_1)^{-1}N_{L_2/F_0}(\mu_2)^{-1} \\ &= N_{L_{1\zeta}/F_{0\zeta}}(\mu_{1\zeta})N_{L_{2\zeta}/F_{0\zeta}}(\mu_{2\zeta})N_{L_{3\zeta}/F_{0\zeta}}(\mu_{3\zeta})N_{L_1/F_0}(\mu_1)^{-1}N_{L_1/F_0}(\mu_2)^{-1} \\ &= N_{L_{1\zeta}/F_{0\zeta}}(\mu_{1\zeta}\mu_1^{-1})N_{L_{2\zeta}/F_{0\zeta}}(\mu_{2\zeta}\mu_2^{-1})N_{L_{3\zeta}/F_{0\zeta}}(\mu_{3\zeta}) \end{aligned}$$

Since $N_{L_{1\zeta}/F_{0\zeta}}(\mu_{1\zeta}\mu_1^{-1})N_{L_{2\zeta}/F_{0\zeta}}(\mu_{2\zeta}\mu_2^{-1}) = x_\zeta^{2^{2t}}$ for some $x_\zeta \in F_{0\zeta}$, we have $\lambda_1 = N_{L_{3\zeta}/F_{0\zeta}}(x_\zeta^{2^{2t-1}}\mu_{3\zeta})$. Since $\mathrm{ind}(D \otimes L_{3\zeta}) \leq 2$ and $\mu_{3\zeta}$ is a reduced norm from $D \otimes L_{3\zeta}$, $x_\zeta^{2^{2t-1}}\mu_{3\zeta}$ is a reduced norm from $D \otimes L_{3\zeta}$. Further, for every branch $(U, P)$, we have $x_P^{2^{2t-1}}\mu_{3P}x_U^{-2^{2t-1}}\mu_{3U}^{-1} \in (F_{0U,P}[X]/(X^2 - a_i))^{*2^{2t-1}}$.

Replacing $\mu_{3\zeta}$ by $x_\zeta^2\mu_{3\zeta}$ we assume that $N_{L_{3\zeta}}(\mu_{3\zeta}) = \lambda_1$, $\mu_{3P}\mu_{3U}^{-1} \in (F_{0U,P}[X]/(X^2 - a_3))^{*2^{2t-1}}$ and $\mu_{3\zeta}$ is a reduced norm from $D \otimes L_{3\zeta}$ for all $\zeta \in \mathscr{P} \cup \mathscr{U}$.

Hence, as in (Proposition 6.3, [10]; Theorem 3.2.3, [13]), there exists $\mu_3 \in L_3 = F_0[X]/(X^2 - a_3)$ such that $N_{L_3/F_0}(\mu_3) = \lambda_1$ and $\mu_3$ is a reduced norm from $D \otimes L_3$.

Therefore $a_i$ and $\mu_i$ have the required properties. □

**Corollary 5.9.2.** *Let $K$ be a $p$-adic field and $F_0$ a function field of a curve over $K$. Let $A$ be a central simple algebra over a quadratic extension $F$ of $F_0$ with period of $A$ equal to 2 with a $F/F_0$-involution $\tau$. If $p \geq 3$, then $SK_1U(A, \tau)$ is trivial.*

*Proof.* By (Lemma 2, [15]), it can be reduced to the case that $A$ is a central division algebra over $F$. Choose an element $a \in \Sigma'_\tau(A^*)$ arbitrarily and write $\lambda = Nrd_{A/F}(a)$. Then $\lambda \in F_0^* \cap Nrd(D)^*$.

By Theorem 5.7.1, there are extensions $L_i$ of $F$ satisfying $ind(A \otimes_{F_0} L_i) \leq 2$ for

$i = 1, 2, 3$. Let $\widetilde{L_i} = L_i \otimes_{F_0} F$ and $\widetilde{A_i} = A \otimes_{F_0} L_i = A \otimes_F \widetilde{L_i}$ for $i = 1, 2, 3$. Considering the elements $\mu_1$, $\mu_2$, and $\mu_3$ founded in Theorem 5.7.1, let $Nrd_{\widetilde{A_i}/\widetilde{L_i}}(\widetilde{d_i}) = \mu_i$ for $i = 1, 2, 3$ and some $\widetilde{d_i} \in \widetilde{A_i}^*$.

Since $SUK_1(\widetilde{A_i}, \tau \otimes id)$ is trivial (Proposition 17.27, [16]) and $\mu_i \in L_i$ for $i = 1, 2, 3$, we have $\widetilde{d_i} \in \Sigma_{\tau \otimes id}(\widetilde{A_i}^*)$ . By (Proposition 4.3, [1]), $N_{L_i/F_0}(\mu_i) = Nrd_{A/F}(d_i)$ for some $d_i \in \Sigma_\tau(A^*)$ where $i = 1, 2,$or $3$ . Therefore, by Theorem 5.7.1, $\lambda = \prod_i N_{L_i/F_0}(\mu_i) = Nrd_{A/F}(\prod_i d_i) = Nrd_{A/F}(a)$.

Since $ind(D) \leq 4$ and $cd(F) \leq 3$, $SK_1(A)$ is trivial (cf. [20], Chapter 17 of [16]). Then $a^{-1} \cdot \prod_i d_i \in SL_1(A) = [A^*, A^*] \subset \Sigma_\tau(A^*)$ (Proposition 17.26, [4]). Since $\prod_i d_i \in \Sigma_\tau(A^*)$, $a \in \Sigma_\tau(A^*)$. □

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