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On the Near-Field Reflector Problem and Optimal Transport

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Abstract

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In the near-field reflector problem, one is given a point source of light with some radiation intensity and a target set at a finite distance. The design problem consists of constructing a reflector that reflects the rays emitted from the source such that a given irradiance distribution is produced on the target. In recent years, the optimal transport framework has been applied successfully to various problems in the design of free-form lenses and reflectors. In this dissertation, the near-field problem is investigated in this context. In particular, it is shown that the notion of a weak solution to the near-field problem as an envelope of ellipsoids of revolution leads to a generalized Legendre-Fenchel transform. Aside from some interesting properties of this transform, it also gives rise to a variational problem that is naturally associated with the near-field reflector problem. Furthermore, the resulting variational problem resembles a generalized optimal transport problem and exhibits interesting analogies to other optimal transport problems arising in optical design and geometry, particularly to the far-field reflector problem and the methods developed by Glimm, Oliker, and Wang. However, for the near-field problem the solutions to the associated variational problem do not solve the reflector problem in general. This situation is illustrated by a number of examples and numerical experiments and is in sharp contrast to the problems that have been studied previously in the optimal transport framework. Interestingly, a connection between the solutions to the near-field problem and the variational problem can still be established. In particular, for discrete target sets an approximation result is presented, which shows that under a suitable choice of the admissible set the variational solution produces an irradiance distribution arbitrarily close to the prescribed irradiance distribution from the design problem. The variational functional is also compared to various functionals motivated by the geometric approach to the near-field problem that was developed by Kochengin and Oliker.

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To my family.

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Chapter 1

Introduction

1.1 The Near-Field Reflector Problem

Consider a reflector system consisting of a non-isotropic point source of light $\mathcal{O}$, a perfectly reflecting surface (called the reflector) R intercepting the light rays from $\mathcal{O}$ and redirecting them so that the reflected rays reach an object T located at a finite distance from $\mathcal{O}$ and produce on T a prescribed in advance irradiance distribution. This situation is illustrated in Figure 1.1.

In practical applications the position of the source and its radiation intensity, as well as the target set T and the irradiance distribution on T are given, and one needs to determine the reflector R. Usually, this problem is considered in the high frequency approximation when the geometric optics laws of

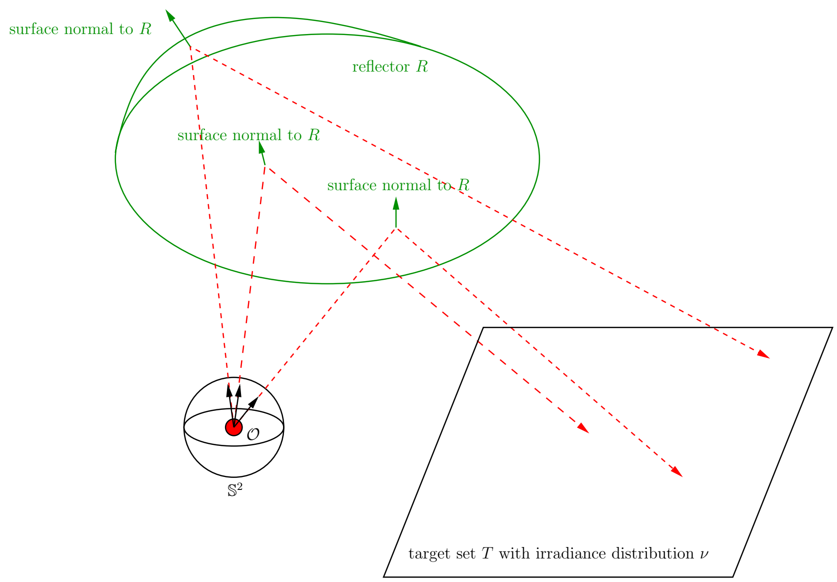

Figure 1.1: An illustration of the near-field reflector problem in $\mathbb{R}^3$. The radiation intensity of the point source at the origin $\mathcal{O}$ is given by a non-negative function $I \in L^1(\mathbb{S}^2)$. The design problem consists of finding the reflector R such that the reflected rays produce the prescribed irradiance distribution on the target set which is given by a Borel measure ν.

propagation and energy conservation are applicable. This inverse problem admits a rigorous mathematical formulation in Euclidean space $\mathbb{R}^{n+1}$, $n \geq 1$, and can be stated analytically as a problem of solving a fully nonlinear equation of Monge-Ampère type on a spherical domain [23] (see also [24] where this and other related problems are reviewed). Because the set T is located at a finite distance from $\mathcal{O}$, and only one reflector is present, the problem is referred to as the *near-field* (single) reflector problem. A geometric approach to the solution of this problem was developed by Sergey Kochengin and Vladimir Oliker in [18], where the authors introduced the notion of a weak solution and proved its existence under the necessary condition of the energy balance between the radiation intensity of the source and the irradiance distribution on the target set T. It was also shown in [18] that this problem has infinitely many solutions but under a natural additional requirement of fixing a priori one point on the reflector uniqueness also holds. A provably convergent algorithm for computing numerically a solution to this problem was given by the same authors in [19]. Reflector design problems arise in various applications such as illumination design, lithography, antennae design, and concentrating solar power. In recent years the Monge-Kantorovich framework of optimal transport has been applied successfully to various optical design problems, starting with the papers by Glimm and Oliker [11] and Wang [36] for the far-field reflector problem. In [11] and [36], the authors showed that a weak solution to the far-field reflector problem, defined as an envelope of confocal paraboloids of revolution (see [6]), can be obtained as the solution of a variational problem that corresponds to a dual optimal transport problem. An important implication for the development of numerical methods is the observation that for discrete irradiance distributions the reflector design problem can be formulated as a linear program.

This dissertation continues the investigation of the near-field reflector problem which began in [32], [18], [19]. In particular, the problem is investigated in the context of the optimal transport approach which in recent years has been applied successfully to various problems in geometry and optical design; see for example [11], [12], [26], [13], [10]. The near-field reflector problem, particularly the regularity of solutions, was also investigated in [15]. To make our presentation reasonably self-contained, we recall first, in section 1.2, the main definitions from [18] and also establish some new properties of reflectors in the setting of the near-field reflector problem. This is part of joint work with Vladimir Oliker. As it was noted in [18], in studying the near-field reflector problem it suffices to consider reflectors which are closed, convex hypersurfaces and this is the case treated here. The case when the reflector is a piece of a closed hypersurface reduces to this one; see [18], section 6. In section 2.1 we consider a variant of the Legendre-Fenchel transform associated naturally with this problem and discuss its properties [1]. Similarly to the classical Legendre-Fenchel transform [29], [31], the new transform is a duality relation between the radial and focal functions defining the reflector (precise definitions are given in sections 1.2 and 2.1). A remarkable new feature of this transform is that it defines the focal function implicitly. In the framework of the Monge-Kantorovich optimal mass transport theory the radial and focal functions are interpreted as the Kantorovich potentials, and this new feature leads to a new type of a cost function depending on one of the potentials and not just on the points in the domains where the given mass densities are defined. To the best of the author's knowledge, such phenomena have not been observed in the classical and other forms of Legendre-Fenchel

[1] This generalized Legendre-Fenchel transform arises quite naturally from the notion of a weak solution as an envelope of ellipsoids [18] in a way very similar to the optimal transport problem associated with reflectors defined by envelopes of paraboloids in the far-field case [6], [11], [36]. To the best of the author's knowledge, this transform was first mentioned by Vladimir Oliker at a conference in Oberwolfach in July of 2006 [25]; see also [35].

transforms considered previously in the Monge-Kantorovich optimal mass transport theory and its applications [2], [11], [12], [36], [26], [34], [35].

Furthermore, in all previously studied cases known to the author, when the dual pair of functions enters the transform explicitly it is possible to define naturally primal and dual functionals whose optima provide solutions of the associated mass transport problems. See, for example, [8], [34], or [35] for an introduction to the primal and dual problems in optimal transport; in [14] some of the economic motivations for the development of optimal transport theory are discussed. In section 3.1 we present examples showing that the solution of the variational problem associated with the near-field reflector problem and defined with the new transform does not in general solve the reflector problem itself. This situation is in sharp contrast, for example, to the variational problem associated with the far-field reflector problem [6], [11], [36], the refractor problem [13], and the two-reflector problems [12], [10]. On the other hand, we also show that because of the above mentioned non-uniqueness of solutions to the reflector problem, we can always choose the set of admissible reflectors such that the optimum of the dual functional is arbitrarily close to a solution of the reflector problem by choosing the admissible reflectors sufficiently large.

As an alternative to the mass transport theory approach, we present several functionals, different from the one used in [18],[19], that can be used to establish existence of weak solutions to the near-field reflector problem by the method used in [18]. These functionals should be useful in the development of algorithms for the numerical solution of this problem.

The results presented in this dissertation, especially the first two chapters and the proof of the example in section 3.1.1, are part of joint work with Vladimir Oliker.

1.2 Reflectors Defined by Families of Ellipsoids

In Euclidean space $\mathbb{R}^{n+1}$, $n \geq 1$, fix a Cartesian coordinate system with the origin at some point $\mathcal{O}$. By $\mathbb{S}^n$ we denote the unit sphere with the center at $\mathcal{O}$. Let $\mathbf{y} \in \mathbb{R}^{n+1}$, $\mathbf{y} \neq \mathcal{O}$. Denote by $E(\mathbf{y}, \tilde{p})$ an ellipsoid of revolution with axis $\mathcal{O}\mathbf{y}$, foci $\mathcal{O}$ and $\mathbf{y}$ and focal parameter $\tilde{p} \in (0, \infty)$. Everywhere in the following chapters the term *ellipsoid* refers to an ellipsoid of this kind with one of the foci always at $\mathcal{O}$. For convenience we often refer to $\mathcal{O}$ as the first focus. Any such ellipsoid is uniquely defined by $\mathcal{O}, \mathbf{y}$ and the focal parameter

$\tilde{p}$; its polar radius is given by

$$\rho_{\mathbf{y}}(x) = \frac{\tilde{p}}{1 - \epsilon(\tilde{p})\langle x, y\rangle}, \quad \text{where} \quad x \in \mathbb{S}^n, \;\; y = \frac{\mathbf{y}}{|\mathbf{y}|}, \tag{1.1}$$

and

$$\epsilon(\tilde{p}) = \sqrt{1 + \frac{\tilde{p}^2}{\mathbf{y}^2}} - \frac{\tilde{p}}{|\mathbf{y}|} \tag{1.2}$$

is the *eccentricity*. Figure 1.2 illustrates these concepts for an ellipse in $\mathbb{R}^2$.

It is convenient to consider also ellipsoids with $\tilde{p} = 0$ or ∞. When $\tilde{p} = 0$ the ellipsoid reduces to the segment $[\mathcal{O}, \mathbf{y}]$. Such an ellipsoid is called degenerate. When $\tilde{p} = \infty$, $\rho_{\mathbf{y}}(x) \equiv \infty$ and the ellipsoid is called improper. The closed, convex subset of $\mathbb{R}^{n+1}$ bounded by an ellipsoid $E(\mathbf{y}, \tilde{p})$ is denoted by $B(\mathbf{y}, \tilde{p})$. In case of an improper ellipsoid, $B(\mathbf{y}, \infty) \equiv \mathbb{R}^{n+1}$.

Definition 1.1 *Let T be a compact subset of $\mathbb{R}^{n+1}$ such that $\mathcal{O} \notin T$ and consider the family of ellipsoids*

$$\mathcal{E}(T, \tilde{p}) := \{E(\mathbf{y}, \tilde{p}(\mathbf{y})) \mid \mathbf{y} \in T\}$$

such that the following conditions hold:

$$\tilde{p} : T \to (0, \infty], \;\; \tilde{p} \not\equiv \infty, \tag{1.3}$$

$$\inf_{\mathbf{y} \in T, \; x \in \mathbb{S}^n} \operatorname{dist}(\rho_{\mathbf{y}}(x)x, T) > 0. \tag{1.4}$$

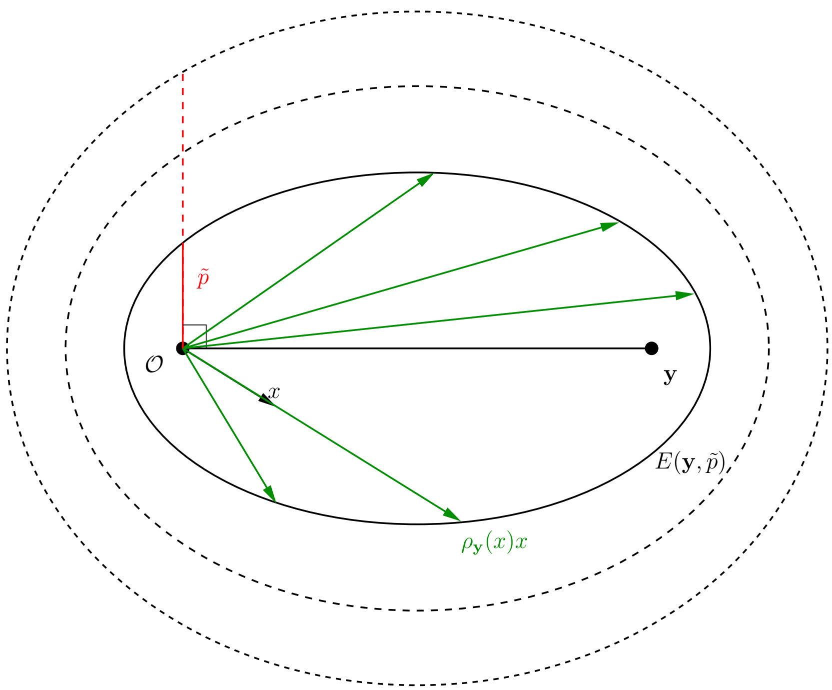

Figure 1.2: The focal parameter of an ellipsoid $E(\mathbf{y}, \tilde{p})$ with foci $\mathcal{O}$ and $\mathbf{y}$ is the distance from a focus to the ellipsoid along a ray perpendicular to the axis $\overline{\mathcal{O}, \mathbf{y}}$. The foci $\mathcal{O}$, $\mathbf{y}$ and the focal parameter $\tilde{p}$ determine E completely. Furthermore, we have $E(\mathbf{y}, \tilde{p}) = \{\rho_{\mathbf{y}}(x)x | x \in \mathbb{S}^n\}$, where $\rho_{\mathbf{y}}$ denotes the radial function defined in (1.1). If $\tilde{p} = 0$ the ellipsoid degenerates to the segment $\overline{\mathcal{O}, \mathbf{y}}$. If the ellipsoid is improper, i.e. when $\tilde{p} = \infty$, then $\rho_{\mathbf{y}} \equiv \infty$ and $B(\mathbf{y}, \tilde{p}) \equiv \mathbb{R}^n$.

The closed, convex hypersurface

$$R = \partial\Big(\bigcap_{\mathcal{E}(T,\tilde{p})} B(\mathbf{y}, \tilde{p}(\mathbf{y}))\Big) \tag{1.5}$$

is called a reflector (with the source $\mathcal{O}$). The set of reflectors defined by an arbitrary family of ellipsoids satisfying (1.3), (1.4) *is denoted by $\mathcal{R}^n_E(T)$. The set bounded by a reflector R is denoted by $B(R)$.*

The ellipsoids in the family $\mathcal{E}(T, \tilde{p})$ are often referred to as *generating ellipsoids*. Note that since $\tilde{p} \not\equiv \infty$, the family $\mathcal{E}(T, \tilde{p})$ includes at least one ellipsoid which is not improper. Also, by (1.4) the family $\mathcal{E}(T, \tilde{p})$ contains no degenerate ellipsoids. Therefore, the set $B(R)$ is a compact, convex subset of $\mathbb{R}^{n+1}$ with the origin $\mathcal{O} \in \text{int}B(R)$, where int denotes the interior (relative to the usual topology of $\mathbb{R}^{n+1}$) of a set in $\mathbb{R}^{n+1}$. Lemma 1.5 below shows that for any compact set $T \subset \mathbb{R}^{n+1}$, $\mathcal{O} \notin T$, one can always construct a family of ellipsoids satisfying (1.3) and (1.4). Before stating this lemma, we note the following important property of ellipsoids.

Proposition 1.2 *Let $E(\mathbf{y}, \tilde{p})$ be an ellipsoid with $0 < \tilde{p} < \infty$. Consider the ellipsoid given by*

$$\rho_{\mathbf{y}}(x) = \frac{c\tilde{p}}{1 - \epsilon(c\tilde{p})\langle x, y\rangle}, \ \ \epsilon(c\tilde{p}) = \sqrt{1 + \frac{c^2\tilde{p}^2}{\mathbf{y}^2}} - \frac{c\tilde{p}}{|\mathbf{y}|},$$

where $c > 0$. Then the foci of this ellipsoid are the same as those of $E(\mathbf{y}, \tilde{p})$.

Proof. Note first that the axis of revolution of this ellipsoid is the same as that of $E(\mathbf{y}, \tilde{p})$. Next, the distance from $\mathcal{O}$ to the second focus of $E(\mathbf{y}, \tilde{p})$ is $|\mathbf{y}|$ and for the modified ellipsoid it is equal to

$$\begin{aligned}\rho_{\mathbf{y}}(y) - \rho_{\mathbf{y}}(-y) &= \frac{c\tilde{p}}{1-\epsilon(c\tilde{p})} - \frac{c\tilde{p}}{1+\epsilon(c\tilde{p})} \\ &= \frac{c\tilde{p}(1+\epsilon(c\tilde{p}))}{1-\epsilon^2(c\tilde{p})} - \frac{c\tilde{p}(1-\epsilon(c\tilde{p}))}{1-\epsilon^2(c\tilde{p})} \\ &= \frac{2c\tilde{p}\epsilon(c\tilde{p})}{1-\epsilon^2(c\tilde{p})}.\end{aligned}$$

Observe that

$$1-\epsilon^2(c\tilde{p}) = \frac{2c^2\tilde{p}^2}{\mathbf{y}^2} - \frac{2c\tilde{p}}{|\mathbf{y}|}\sqrt{1+\frac{c^2\tilde{p}^2}{\mathbf{y}^2}}.$$

And therefore, we obtain

$$\begin{aligned}\rho_{\mathbf{y}}(y) - \rho_{\mathbf{y}}(-y) &= \frac{2c\tilde{p}\epsilon(c\tilde{p})}{1-\epsilon^2(c\tilde{p})} \\ &= \frac{2c\tilde{p}\epsilon(c\tilde{p})}{\frac{2c\tilde{p}}{|\mathbf{y}|}\left(\sqrt{1+\frac{c^2\tilde{p}^2}{\mathbf{y}^2}} - \frac{c\tilde{p}}{|\mathbf{y}|}\right)}.\end{aligned}$$

Using again the definition of ϵ, we simplify the last expression to recover

$$\begin{aligned}\rho_{\mathbf{y}}(y) - \rho_{\mathbf{y}}(-y) &= \frac{2c\tilde{p}\epsilon(c\tilde{p})}{2c\tilde{p}\epsilon(c\tilde{p}) \cdot \frac{1}{|\mathbf{y}|}} \\ &= |\mathbf{y}|,\end{aligned}$$

that is, the second focus of $E(\mathbf{y}, cp)$ is $\mathbf{y}$. QED.

Remark 1.3 *Observe that because of the non-linearity of* (1.2) *in* $\tilde{p}$*, the polar radius of* $E(\mathbf{y}, c\tilde{p})$ *is not a rescaling of the polar radius of* $E(\mathbf{y}, \tilde{p})$*.*

For future reference, we recall from [18] the following behavior of the radial function and the eccentricity with respect to the focal parameter.

Lemma 1.4 *(See [18].) Fix* $\mathbf{y} \in T$ *and let* $\tilde{p} \in (0, \infty)$*. Then the radial function* $\rho_{\mathbf{y}}(x)$ *is strictly increasing as a function of* $\tilde{p}$ *for each fixed* x *and the eccentricity* $\epsilon(\tilde{p})$ *is strictly decreasing.*

For convenience, we set $M := \max_{\mathbf{y} \in T} |\mathbf{y}|$ and we will use this notation for the remainder of our discussion. So far, we have defined what we call a reflector and we noted some properties of the generating ellipsoids. We will see that the reflectors of interest to us can be characterized through the focal parameters of the generating ellipsoids. The next lemma provides a set of reflectors in $\mathcal{R}_E^n(T)$ for a given target set T.

Lemma 1.5 *Let* T *be a compact subset of* $\mathbb{R}^{n+1}$ *such that* $\mathcal{O} \notin T$*. For any given constant* $c \geq 0$ *and any function* $\tilde{p} : T \to [2(M + c), \infty]$ *the family of ellipsoids* $\{E(\mathbf{y}, \tilde{p}(\mathbf{y})),\ \mathbf{y} \in T\}$ *satisfies* (1.3), (1.4) *and*

$$\inf_{\mathbf{y} \in T,\ x \in \mathbb{S}^n} \operatorname{dist}(\rho_{\mathbf{y}}(x)x, T) > c.$$

Proof. Fix some $q = \text{const} \geq 2(M+c)/M$ and let $\bar{\mathbf{y}} \in T$. Put $\bar{p} = qM$. Then for the polar radius of the ellipsoid $E(\bar{\mathbf{y}}, \bar{p})$ we have

$$\min_{x \in \mathbb{S}^n} \rho_{\bar{\mathbf{y}}}(x) = \rho_{\bar{\mathbf{y}}}(-\bar{y}) = \frac{\bar{p}}{1+\epsilon(\bar{p})} > \frac{\bar{p}}{2} = \frac{qM}{2} \geq M + c.$$

Applying the same argument to any $\mathbf{y} \in T$, we conclude that the intersection $\bigcap_{\mathbf{y} \in T} B(\mathbf{y}, qM)$ contains in its interior a closed ball of radius $M + c$ with the center at $\mathcal{O}$. On the other hand, the set T is contained in a ball of radius M with the center at $\mathcal{O}$. QED.

We recall now the concept of a supporting ellipsoid introduced first in [18].

Definition 1.6 *Let $R \in \mathcal{R}^n_E(T)$. An ellipsoid $E(\mathbf{y}, p)$, $\mathbf{y} \in T$, $p > 0$, is called supporting to R if $B(R) \subset B(\mathbf{y}, p)$ and $R \cap E(\mathbf{y}, p) \neq \emptyset$.*

Obviously, for a given $R \in \mathcal{R}^n_E(T)$ at every point $z \in R$ there exists at least one ellipsoid from the family defining R which is supporting to R. However, not every ellipsoid from such a family is necessarily supporting to R. A trivial example of such a situation is a family of ellipsoids all of which are improper except for one. On the other hand, at each point on a reflector we have at least one supporting ellipsoid.

Proposition 1.7 *Let $R \in \mathcal{R}^n_E(T)$. Then for each $\mathbf{y} \in T$ there exists an ellipsoid with foci $\mathcal{O}$ and $\mathbf{y}$ supporting to R.*

Proof. Since R is compact, we can apply Lemma 1.4 and choose an ellipsoid $E(\mathbf{y}, \tilde{p})$ with $\tilde{p}$ sufficiently large so that $B(R) \subset B(\mathbf{y}, \tilde{p})$. Decreasing $\tilde{p}$ continuously and, taking into account Lemma 1.4, we arrive at a situation when $R \cap E(\mathbf{y}, p) \neq \emptyset$ for some $0 < p < \infty$ while still $B(R) \subset B(\mathbf{y}, p)$. QED.

We define the *focal function* $p : T \to (0, \infty)$ of a reflector R as the function such that the ellipsoids $E(\mathbf{y}, p(\mathbf{y}))$ are supporting to R for each $\mathbf{y} \in T$. We also put

$$\mathcal{E}_R(T) := \{E(\mathbf{y}, p(\mathbf{y})) \mid \mathbf{y} \in T, E(\mathbf{y}, p(\mathbf{y})) \text{ is supporting to } R\}.$$

It is important to note that in contrast to reflectors constructed from paraboloids of revolution [6], two ellipsoids with different second foci supporting to a reflector $R \in \mathcal{R}^n_E(T)$ may be tangent to each other at the point of contact with R. A simple example can be constructed as follows. Let $T = \{\mathbf{y}_1, \mathbf{y}_2\}$, $\mathbf{y}_1, \mathbf{y}_2 \neq \mathcal{O}$, $\mathbf{y}_1 \neq \mathbf{y}_2$, and consider the reflector R defined by two ellipsoids $E(\mathbf{y}_i, \tilde{p}_i)$, $i = 1, 2$, with focal parameters such that $\mathbf{y}_1, \mathbf{y}_2 \in \mathrm{int} B(\mathbf{y}_1, \tilde{p}_1) \cap B(\mathbf{y}_2, \tilde{p}_2)$. Increase the focal parameter $\tilde{p}_1$ to some value λ so that $E(\mathbf{y}_2, \tilde{p}_2) \subset B(\mathbf{y}_1, \lambda)$ and then decrease λ so that $E(\mathbf{y}_1, \lambda)$ is supporting to $E(\mathbf{y}_2, \tilde{p}_2)$. Then $E(\mathbf{y}_1, \lambda)$ is tangent to $E(\mathbf{y}_2, \tilde{p}_2)$. The ray from $\mathcal{O}$ in direction $\bar{x} \in \mathbb{S}^n$ corresponding to the point of tangency

$z_{\bar{x}} = \rho_{\mathbf{y}_1}(\bar{x})\bar{x} = \rho_{\mathbf{y}_2}(\bar{x})\bar{x}$ is reflected by both ellipsoids in the same direction and the reflected ray must pass through the foci $\mathbf{y}_1$ and $\mathbf{y}_2$.

The following proposition provides a useful description of the intersection of two ellipsoids with a common first focus.

Proposition 1.8 *Let $E(\mathbf{y}_i, \tilde{p}_i),\ i = 1, 2,$ be two ellipsoids with the same first focus and let $C := E(\mathbf{y}_1, \tilde{p}_1) \cap E(\mathbf{y}_2, \tilde{p}_2) \neq \emptyset$. If $n \geq 2$ then C consists of only one connected component which may reduce to a point. If C is not a point and $E(\mathbf{y}_1, \tilde{p}_1) \neq E(\mathbf{y}_2, \tilde{p}_2)$ then C is homeomorphic to an $(n-1)$-dimensional sphere. If $n = 1$ then C consists of either one point or two points, or the ellipses coincide. When $n \geq 1$ and C reduces to one point then this point is $\tilde{\rho}_1(\bar{x})\bar{x} = \tilde{\rho}_2(\bar{x})\bar{x}$, where $\tilde{\rho}_i(x),\ x \in \mathbb{S}^n,\ i = 1, 2,$ are the polar radii of the respective ellipsoids and*

$$\bar{x} = sign(\tilde{p}_2 - \tilde{p}_1) \frac{\tilde{p}_2 \epsilon_1 y_1 - \tilde{p}_1 \epsilon_2 y_2}{|\tilde{p}_2 \epsilon_1 y_1 - \tilde{p}_1 \epsilon_2 y_2|} (\neq \mathcal{O}).$$

In addition, the two ellipsoids are tangent to each other at that point.

Proof. Let $n \geq 1$. The radial projection U_{12} of C from $\mathcal{O}$ on $\mathbb{S}^n$ is given by

$$\begin{aligned} U_{12} :=&\ \{x \in \mathbb{S}^n \mid \tilde{\rho}_1(x) = \tilde{\rho}_2(x)\} \\ =&\ \left\{ x \in \mathbb{S}^n \mid \frac{\tilde{p}_1}{1 - \epsilon_1 \langle x, y_1 \rangle} = \frac{\tilde{p}_2}{1 - \epsilon_2 \langle x, y_2 \rangle} \right\} \end{aligned} \tag{1.6}$$

or

$$U_{12} = \{x \in \mathbb{S}^n \mid \langle \tilde{p}_2\epsilon_1 y_1 - \tilde{p}_1\epsilon_2 y_2, x\rangle = \tilde{p}_2 - \tilde{p}_1\}\,. \tag{1.7}$$

If $|\tilde{p}_2\epsilon_1 y_1 - \tilde{p}_1\epsilon_2 y_2| = 0$ then $y_1 = y_2$ and $\tilde{p}_2 = \tilde{p}_1$ because of (1.7). In this case, $U_{12} = \mathbb{S}^n$ and the ellipsoids $E(\mathbf{y}_1, \tilde{p}_1)$ and $E(\mathbf{y}_2, \tilde{p}_2)$ coincide. If $|\tilde{p}_2\epsilon_1 y_1 - \tilde{p}_1\epsilon_2 y_2| > 0$ and $n \geq 2$ then it follows from (1.7) that U_{12} is an $(n-1)-$dimensional sphere on $\mathbb{S}^n$ with center at

$$A_{12} = \frac{\tilde{p}_2\epsilon_1 y_1 - \tilde{p}_1\epsilon_2 y_2}{|\tilde{p}_2\epsilon_1 y_1 - \tilde{p}_1\epsilon_2 y_2|}$$

(and at $A_{21} = -A_{12}$). Obviously, C is homeomorphic to this sphere. If $n = 1$ then U_{12} is either a point or two points. The remaining statements are obvious. QED.

Recalling the physical interpretation of the near-field reflector problem in the introduction, we note that the Proposition 1.7 implies that for any $R \in \mathcal{R}^n_E(T)$ and any $\mathbf{y} \in T$ there exists at least one light ray originating at $\mathcal{O}$ which is reflected by R so that the reflected ray reaches $\mathbf{y}$.

Let $R \in \mathcal{R}^n_E(T)$. Obviously, $B(R)$ is star-shaped relative to $\mathcal{O}$ and we can describe R as the graph of its radial function given by

$$\rho(x) = \sup\{\lambda \geq 0 \mid \lambda x \in B(R),\ x \in \mathbb{S}^n\}. \tag{1.8}$$

Here x is treated as a point on $\mathbb{S}^n$ and a unit vector in $\mathbb{R}^{n+1}$. The position vector of the reflector R is $\rho(x)x,\ x \in \mathbb{S}^n$. It follows from the definition of R that

$$\rho(x) = \inf_{\mathbf{y}\in T} \frac{p(\mathbf{y})}{1 - \epsilon(p(\mathbf{y}))\langle x, y\rangle},\ x \in \mathbb{S}^n. \tag{1.9}$$

Since for each $\mathbf{y} \in T$ the ellipsoid $E(\mathbf{y}, p(\mathbf{y}))$ is supporting to R, we have

$$p(\mathbf{y}) = \sup_{x\in\mathbb{S}^n} \rho(x)(1 - \epsilon(p(\mathbf{y}))\langle x, y\rangle),\ \mathbf{y} \in T. \tag{1.10}$$

It follows from (1.9) and (1.10) that

$$\log\rho(x) - \log p(\mathbf{y}) \leq -\log(1 - \epsilon(p(\mathbf{y}))\langle x, y\rangle),\ \forall x \in \mathbb{S}^n,\ \mathbf{y} \in T, \tag{1.11}$$

and for each $x \in \mathbb{S}^n$ the equality is achieved for some $\mathbf{y} \in T$ and for each $\mathbf{y} \in T$ the equality is achieved for some $x \in \mathbb{S}^n$.

Note that (1.8)–(1.11) hold actually for any reflector defined by (1.3) and (1.5), regardless of the condition (1.4).

For $R \in \mathcal{R}_E^n(T)$ the *reflector map* $\alpha_R : \mathbb{S}^n \to T$ is the (possibly multivalued) map given by

$$\alpha_R(x) = \{\mathbf{y} \in T \mid p(\mathbf{y}) = \rho(x)(1 - \epsilon(p(\mathbf{y}))\langle x, y\rangle)\},\ x \in \mathbb{S}^n. \tag{1.12}$$

In other words, the reflector map assigns each direction $x \in \mathbb{S}^n$ the second foci $\mathbf{y} \in T$ of all the ellipsoids supporting to R in the point $\rho(x)x$. Clearly,

α_R maps $\mathbb{S}^n$ onto T. The inverse of the reflector map is

$$\alpha_R^{-1}(\mathbf{y}) = \{x \in \mathbb{S}^n \mid p(\mathbf{y}) = \rho(x)(1 - \epsilon(p(\mathbf{y}))\langle x, y\rangle)\}, \ \mathbf{y} \in T. \tag{1.13}$$

That is, the image of a point $\mathbf{y}$ in the target set is the set of all directions $\mathbf{x} \in \mathbb{S}^n$ in which the point $\mathbf{y}$ is *visible* (along a reflected ray) for the source. For a subset $\omega \subset T$ the set

$$V_R(\omega) = \bigcup_{\mathbf{y} \in \omega} \alpha_R^{-1}(\mathbf{y})$$

is called the *visibility* set of ω. If T is contained in a hyperplane then it follows from Lemmas 1 through 5 in [18] that for any Borel set $\omega \subset T$ the set $V_R(\omega)$ is measurable with respect to the standard $n-$dimensional Lebesgue measure on $\mathbb{S}^n$.

Remark 1.9 *In [18] the cited results are proved for reflectors in $\mathbb{R}^3$. The same proofs are valid verbatim for reflectors in $\mathbb{R}^{n+1}$.*

A point $X \in R$ is called *singular* if there is more than one supporting ellipsoid at X. Clearly, the map $\alpha_R(X/|X|)$ is multivalued at singular X. Proposition 1.8 implies that, depending on the set T, a point $X \in R$ may be such that there exists a unique tangent hyperplane to R at X and still there is more than one supporting ellipsoid at X. Such a situation cannot occur for reflectors defined by paraboloids as in the far-field problem [11], [36]. A point of a reflector R at which two supporting ellipsoids are tangent to each other will be called *singular of tangential type*.

Proposition 1.10 *Let T be contained in a hyperplane Π. Then the set of singular points of a reflector R has n-dimensional Lebesgue measure zero.*

Proof. Clearly, if $X \in R$ is a singular point on the reflector R and there are at least two supporting ellipsoids at X that are not tangent to each other then X is a singular point in the usual sense of convex surface theory [4]. By a theorem due to Kurt Reidemeister ([28], and [4],§2) this set has measure zero. On the other hand, if X is a singular point of tangential type then the two corresponding second foci lie on a straight line through X. Hence, all singular points of tangential type are contained in $R \cap \Pi$ and this set has n-dimensional Lebesgue measure zero in R. QED.

The assumption in Proposition 1.10 can obviously be generalized to target sets T that are contained in a countable union of hyperplanes. However, the following remark gives an example of a target set for which the conclusion in Proposition 1.10 does not hold.

Remark 1.11 *If T is not contained in a hyperplane, the set of singular* *points of tangential type may have a positive measure. To see this, consider the following example. Let* T *be the semicircle* $\{\mathbf{y} \in \mathbb{R}^2 : |\mathbf{y}| = 1, y_1 \geq 0\}$*, where* y_1 *denotes the first of the Cartesian coordinates of the point* $\mathbf{y}$*. Let* $\tilde{\mathbf{y}} = (1, 0)$ *and* $\tilde{p} = p(\tilde{\mathbf{y}}) \in \mathbb{R}$ *be large enough so that* T *is contained in the interior of the convex body bounded by the ellipse* $\tilde{E} = E(\tilde{\mathbf{y}}, \tilde{p})$*. Then the singular points of tangential type on the reflector* $R = \tilde{E}$ *form a set of positive measure. This can be seen by tracing back to* $\mathcal{O}$ *the rays terminating at* $\tilde{\mathbf{y}}$ *and passing through* $\mathbf{y} \in T$*. The intersection of such a ray with the reflector is a singular point of tangential type and the union of all these singular points has positive measure in* R*. Using the rotational symmetry we can generalize this example to* $n > 2$*.*

For the rest of this work, unless stated otherwise, it is assumed that T *is contained in a hyperplane.*

Let the intensity of the source $\mathcal{O}$ be a non-negative function $I \in L^1(\mathbb{S}^n)$, $I \not\equiv 0$. Following [18] for each reflector $R \in \mathcal{R}^n_E(T)$ we define a Borel measure on T by setting

$$G(R, \omega) = \int_{V_R(\omega)} I(x) d\sigma(x), \tag{1.14}$$

where σ is the standard Lebesgue measure on $\mathbb{S}^n$. It is useful to note that if we set

$$\mu(U) = \int_U I(x) d\sigma(x), \tag{1.15}$$

where U is any Borel set of $\mathbb{S}^n$, then G is the push-forward of μ with the map α_R, that is

$$\mu(\alpha_R^{-1}(\omega)) = G(R, \omega).$$

Remark 1.12 *Observe that $G(R, \cdot)$ may fail to be a measure if T is not contained in a hyperplane. The example constructed in Remark 1.11 can be used again to show that in this case, G is not σ-additive. Indeed, let T and R be as in Remark 1.11. Decompose T into two disjoint Borel sets: $T = (T \backslash \tilde{\mathbf{y}}) \cup \tilde{\mathbf{y}}$. Let $\bar{\Omega} \subset \mathbb{S}^1$ be the set of points corresponding to singular points of tangential type on R. Put $I(x) = 1$ if $x \in \bar{\Omega}$ and $I(x) = 0$ if $x \in \mathbb{S}^1 \setminus \bar{\Omega}$, then*

$$G(R, T) = \int_{\mathbb{S}^1} I(x) d\sigma.$$

However, we see that

$$\begin{aligned} G(R, T \backslash \tilde{\mathbf{y}}) + G(R, \tilde{\mathbf{y}}) &= \int_{\alpha_R^{-1}(T \backslash \tilde{\mathbf{y}})} I(x) d\sigma + \int_{\alpha_R^{-1}(\tilde{\mathbf{y}})} I(x) d\sigma \\ &> \int_{\mathbb{S}^1} I(x) d\sigma. \end{aligned}$$

Therefore, $G(R, \cdot)$ is not (σ-)additive.

In practical applications the most important case seems to be the one where T is contained in a hyperplane. It is possible, however, to modify the definition of $G(R,\cdot)$ so that it becomes σ-additive even when T is a compact subset of $\mathbb{R}^{n+1}$ not necessarily contained in a hyperplane. For example, we can introduce the notion of a *strongly supporting* ellipsoid as follows.

Definition 1.13 *An ellipsoid E is called* strongly supporting *to a reflector R in the point $X \in R$ if the convex body bounded by E does not contain any ellipsoids that are supporting to R in X in its interior.*

Then the measure G is σ-additive even if T is not contained in the countable union of hyperplanes if we replace supporting ellipsoids by strongly supporting ellipsoids in the definition of the reflector and visibility map. We plan to return to this issue in a separate investigation.

For future reference, we recall the following lemma [18] providing a useful result on the weak convergence of measures associated with a sequence of reflectors. The Hausdorff metric is defined in appendix 6.1.

Lemma 1.14 (See [18]) *Let $\mathcal{R}_s \in \mathcal{R}_E^n(T), s = 1, 2, ...,$ be a sequence of convex reflectors $\mathcal{R}_s \in \mathcal{R}_E^n(T), s = 1, 2, ...,$ converging to a closed, convex hypersurface R in the Hausdorff metric. Then the measures $G(R_s,\cdot)$ converge weakly to the $G(R,\cdot)$.*

1.3 Weak Formulation of the Near-Field Reflector Problem

We recall now the weak formulation of the **near-field reflector problem** [18].

Problem 1.15 (Near-Field Reflector Problem) *For a given set T, a non-negative function $I \in L^1(\mathbb{S}^n)$ as above and a given Borel measure ν on T, the near-field reflector problem consists of finding an $R \in \mathcal{R}^n_E(T)$ such that*

$$G(R, \omega) = \nu(\omega) \textit{ for each Borel set } \omega \subset T. \tag{1.16}$$

We refer to such R which satisfies (1.16) as a weak solution of the near-field reflector problem.

The existence of weak solutions to Problem 1.15 was shown by Kochengin and Oliker in [18]. Their approach is discussed briefly in section 5.1, and a modified version of the existence result is stated in Theorem 2.8. Furthermore, the geometric approach developed in [18] was adapted by the same authors in [19] to describe a provably convergent algorithm.

Chapter 2

A Generalized Legendre-Fenchel Transform

2.1 A Generalized Legendre-Fenchel Transform and the Associated Variational Problem

The relations in (1.9) and (1.10) can be viewed as a variant of the Legendre-Fenchel transform between the radial and focal functions of reflectors in $\mathcal{R}^n_E(T)$. In contrast to the classical Legendre-Fenchel transform (see, for example, [29], [31], [34]) and other known transforms of this type ([3], [9],

[5], [11], [36], [1]), the transform (1.10) defines the focal function p *implicitly*. The following proposition shows that even in these circumstances we have analogues of the usual duality properties typical for polar bodies [26], [29], [31].

Proposition 2.1 *Let T be a compact set in $\mathbb{R}^{n+1} \setminus \{\mathcal{O}\}$ and $(\rho, p) \in C(\mathbb{S}^n) \times C(T)$ are continuous functions, respectively, on $\mathbb{S}^n$ and on T, satisfying* (1.9) – (1.10). *Assume in addition that*

$$\min_{\mathbb{S}^n} \rho(x) > |\mathbf{y}| \ \forall \ \mathbf{y} \in T. \tag{2.1}$$

Then there exists a unique reflector $R \in \mathcal{R}^n_E(T)$ with radial function ρ and focal function p.

Proof. Let ρ, p be as in the proposition. It follows from (1.10) and (2.1) that $p(\mathbf{y}) \geq \rho(-\mathbf{y})[1 + \epsilon(p(-\mathbf{y}))] > |\mathbf{y}| \ \forall \mathbf{y} \in T$. Hence, both functions are positive, and, in addition, we may consider the family of ellipsoids

$$\mathcal{E}(T, p) = \{E(\mathbf{y}, p(\mathbf{y})), \ \mathbf{y} \in T\}$$

and the convex hypersurface R defined by this family as in (1.5). Denote by ρ' the radial function of R. We need to show that $\rho'(x) = \rho(x)$ for all $x \in \mathbb{S}^n$. It follows from (1.5) that $\rho'(x) \leq \rho(x) \ \forall x \in \mathbb{S}^n$. If for some $\bar{x}$ the inequality is strict then at the point $\rho'(\bar{x})\bar{x} \in R$ there is no supporting ellipsoid from the family of ellipsoids defined by the function p. But this is impossible. On the other hand, (1.10) implies that for any $\mathbf{y} \in T$ the supremum in (1.10) is achieved at some $x \in \mathbb{S}^n$. The condition (1.4) is satisfied because of (2.1). QED.

Remark 2.2 *The proof of Proposition 2.1 above is valid even if T is not assumed to be contained in a hyperplane. For reflectors in $\mathbb{R}^3$ this means that T can, for example, be a cube. In such a situation, we have to redistribute the source intensity, which is a function on $\mathbb{S}^2$, on a three dimensional object. Aside from the physical questions on transparency and blockage which immediately come to mind in this context, we have already observed mathematically that in this case the map $G(R, \cdot)$ may fail to be a measure, as it was noted in Remarks 1.11 and 1.12. The key observation in Remarks 1.11 and 1.12 was the existence of singular points of tangential type.*

It is convenient to restate Proposition 2.1 as a sufficient condition on the focal function.

Lemma 2.3 *Suppose all the conditions of Proposition 2.1 are satisfied except* *for* (2.1). *Assume instead that*

$$p(\mathbf{y}) \geq 2aM \text{ for some constant } a \geq 1 \text{ and all } \mathbf{y} \in T. \tag{2.2}$$

Then there exists a unique reflector $R \in \mathcal{R}^n_E(T)$ *with focal function* p *and radial function* ρ *satisfying the inequality*

$$\min_{\mathbb{S}^n} \rho(x) > aM. \tag{2.3}$$

Proof. The inequality in (2.3) follows from Lemma 1.5 if we take $c = (a-1)M$ and then apply Proposition 2.1. QED.

The geometric significance of the choice of the constant a is as follows. We will see in Lemma 2.3 below that the radial function of an ellipsoid $E(\mathbf{y}, p)$ is bounded from below by $\frac{p}{2}$ and from above by $2p$. Therefore, the condition (2.1) in Proposition 2.1 can be replaced by a lower bound on p which ensures that the convex body $B(\mathbf{y}, p)$ contains a ball of radius at least $M = \sup_{\tilde{\mathbf{y}} \in T} |\tilde{\mathbf{y}}|$ in its interior. This in turn implies that (2.1) holds. This situation is illustrated for an ellipsoid in $\mathbb{R}^3$ in Figure 2.1.

We define now a variational problem which arises naturally from the generalized Legendre-Fenchel transform (1.9)–(1.10). *Consequently, this problem is also associated naturally with the near-field reflector problem.*

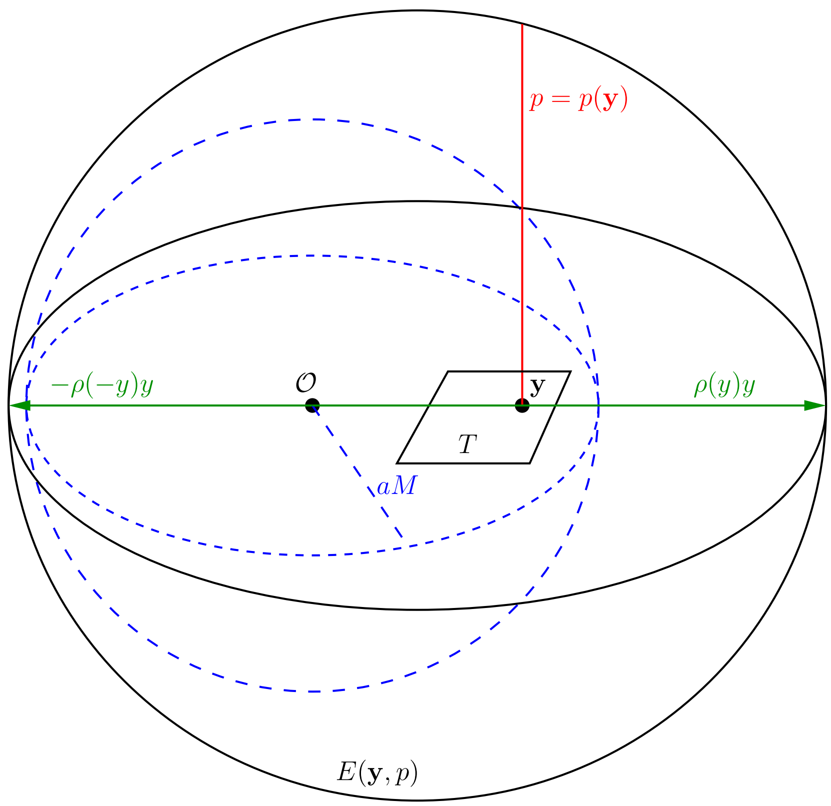

Figure 2.1: The lower bound in (2.2) for the focal function in Lemma 2.5 ensures that the convex body $B(\mathbf{y}, p)$ bounded by the ellipsoid $E(\mathbf{y}, p)$ contains a ball of radius aM around the origin $\mathcal{O}$ in its interior. Therefore T is also contained in the interior of $B(\mathbf{y}, p)$.

Let T be a compact subset in $\mathbb{R}^{n+1}$ contained in some hyperplane and $\mathcal{O} \notin T$. Denote by ν a Borel measure on T satisfying the condition

$$\mu(\mathbb{S}^n) = \nu(T) \neq 0, \tag{2.4}$$

and where the measure μ is defined by (1.15). Furthermore, we assume that the support of the measure ν, denoted spt ν, does not reduce to a single point. In other words, there is no $\bar{\mathbf{y}} \in T$ such that

$$\text{spt } \nu = \{\bar{\mathbf{y}}\}. \tag{2.5}$$

For positive functions ρ on $\mathbb{S}^n$ and p on T we introduce for brevity the following notation:

$$\begin{aligned} \hat{\rho}(x) &:= \log \rho(x), \\ \hat{p}(\mathbf{y}) &:= \log p(\mathbf{y}), \\ \hat{\mathcal{K}}(x, y, p) &:= -\log[1 - \epsilon(p(\mathbf{y}))\langle x, y\rangle] \end{aligned}$$

for $x \in \mathbb{S}^n$, $\mathbf{y} \in T$.

For a fixed constant $a \geq 1$ the set of admissible functions is defined as

$$\begin{aligned} \mathcal{A}_a = \Big\{ &(\rho, p) \in C(\mathbb{S}^n) \times C(T) \text{ such that} \\ &\rho(x) > 0 \ \forall x \in \mathbb{S}^n, \ p(\mathbf{y}) \geq 2aM \ \forall \mathbf{y} \in T, \text{and} \qquad (2.6) \\ &\hat{\rho}(x) - \hat{p}(\mathbf{y}) \leq \hat{\mathcal{K}}(x, y, p), \ \forall (x, \mathbf{y}) \in \mathbb{S}^n \times T \Big\}. \qquad (2.7) \end{aligned}$$

Put

$$\mathcal{Q}[\rho, p] = \int_{\mathbb{S}^n} \hat{\rho}(x) I(x) d\sigma - \int_T \hat{p}(\mathbf{y}) d\nu. \tag{2.8}$$

Fix some $a \geq 1$ and consider the problem of finding a pair $(\rho_{\max}, p_{\max}) \in \mathcal{A}_a$ such that

$$\mathcal{Q}[\rho_{\max}, p_{\max}] = \sup_{\mathcal{A}_a} \mathcal{Q}[\rho, p]. \tag{2.9}$$

Before we prove the existence of maximizers, we will make an important observation relating the set of reflectors $\mathcal{R}_E^n(T)$ to the set $\mathcal{A}_a$. Every reflector in $\mathcal{R}_E^n(T)$ is naturally associated with the pair $(\rho, p) \in \mathcal{A}_a$ consisting of its radial and focal function. On the other hand, with each pair $(\rho, p) \in \mathcal{A}_a$ such that for each $x \in \mathbb{S}^n$ there exists a $\mathbf{y} \in T$ making (2.7) an equality and for each $\mathbf{y} \in T$ there exists a $x \in \mathbb{S}^n$ making (2.7) an equality, we can associate a reflector in $\mathcal{R}_E^n(T)$ with radial and focal function ρ and p, respectively. Therefore, we can identify the set of reflectors $\mathcal{R}_E^n(T)$, described by the pairs of radial and focal functions, with a subset of $\mathcal{A}_a$, which we denote $\mathcal{A}_a^*$.

To prove the existence of maximizers for $\mathcal{Q}$, we will need the following two lemmas.

Lemma 2.4 *If the supremum of $\mathcal{Q}$ is achieved in $\mathcal{A}_a$, then there exists a reflector $\tilde{R} \in \mathcal{R}_E^n(T)$ such that $\sup_{\mathcal{A}_a} \mathcal{Q}[\rho, p] = \mathcal{Q}[\tilde{\rho}, \tilde{p}]$, where $\tilde{\rho}$ and $\tilde{p}$ are the radial and focal function, respectively, of $\tilde{R}$.*

In other words, Lemma 2.4 states that

$$\sup_{\mathcal{A}_a} \mathcal{Q}[\rho, p] = \sup_{\mathcal{A}_a^*} \mathcal{Q}[\rho, p].$$

Proof. Suppose $(\rho, p) \in \mathcal{A}_a$ and $(\rho, p) \notin \mathcal{A}_a^*$, i.e the functions $\rho,\ p$ are not the radial and focal function of a reflector. Then there exists some $x \in \mathbb{S}^n$ such that

$$\hat{\rho}(x) - \hat{p}(\mathbf{y}) < \hat{\mathcal{K}}(x, y, p(\mathbf{y}))\ \forall \mathbf{y} \in T.$$

Consider the reflector R defined by the ellipsoids $\{E(\mathbf{y}, p(\mathbf{y}))\},\ \mathbf{y} \in T$. Its radial function is

$$\rho_R(x) = \inf_{\mathbf{y} \in T} \frac{p(\mathbf{y})}{1 - \epsilon(p(\mathbf{y}))\langle x, y\rangle},\ x \in \mathbb{S}^n.$$

Thus, for all $x \in \mathbb{S}^n$ we have $\rho(x) \leq \rho_R(x)$. This and (2.8) imply

$$\mathcal{Q}[\rho, p] \leq \mathcal{Q}[\rho_R, p].$$

QED.

The following lemma is based on a result in [18].

Lemma 2.5 *Fix a constant $c \geq 1$. Let, as before, $M := \max_{\mathbf{y} \in T} |\mathbf{y}|$ and let $E(\tilde{\mathbf{y}}, \tilde{p}),\ \tilde{\mathbf{y}} \in T$, be an ellipsoid with $\tilde{p} \geq cM$. Then the eccentricity of E satisfies the inequality*

$$\epsilon(\tilde{p}) \leq \sqrt{c^2 + 1} - c, \tag{2.10}$$

and the radial function of E satisfies the inequalities

$$\frac{\tilde{p}}{2} \leq \rho_{\tilde{\mathbf{y}}}(x) \leq \frac{\tilde{p}}{1 + c - \sqrt{1 + c^2}} < 2\tilde{p} \;\forall x \in \mathbb{S}^n. \tag{2.11}$$

In particular, the inequality on the left is strict if $\tilde{p} > 0$.

Proof. The inequality on the left of (2.11) follows from (1.1) and the fact that $0 < \epsilon(\tilde{p}) < 1$. Now we prove (2.10). By (1.2) and because $\tilde{p} \geq cM$, $|\tilde{\mathbf{y}}| \leq M$, we have

$$\frac{2c\epsilon(\tilde{p})M}{1 - [\epsilon(\tilde{p})]^2} \leq \frac{2\tilde{p}\epsilon(\tilde{p})}{1 - [\epsilon(\tilde{p})]^2} = |\tilde{\mathbf{y}}| \leq M.$$

This implies inequality (2.10) since

$$\frac{2c\epsilon}{1 - \epsilon^2} \leq 1 \Leftrightarrow \epsilon^2 + 2c\epsilon - 1 \leq 0.$$

Solving the quadratic equation above for $0 < \epsilon < 1$ we obtain the estimate in (2.10). The first of the inequalities on the right of $\rho_{\tilde{\mathbf{y}}}(x)$ in (2.11) follows from (2.10) and (1.1). Finally, note that $\frac{1}{1+c-\sqrt{1+c^2}}$ is strictly decreasing if $c \geq 1$ and $2 - \sqrt{2} > \frac{1}{2}$. Therefore the second inequality on the right of $\rho_{\tilde{\mathbf{y}}}(x)$ in (2.11) follows from the preceding inequality because $c \geq 1$. QED.

The condition $p \geq aM$ on the focal function implies a bound on the radial function of the admissable reflectors from below. In the following theorem, we will bound the radial function also from above. This allows us to prove the existence of a maximizer of $\mathcal{Q}$.

Theorem 2.6 *Let* $a \geq 1$*,* $M := \max_{\mathbf{y}\in T} |\mathbf{y}|$*,* $P \in [2aM, \infty)$ *and*

$$\mathcal{A}_{a,P} := \left\{ (\rho, p) \in \mathcal{A}_a \mid \inf_T p \leq P \right\}.$$

The functional $\mathcal{Q}$ *defined by* (2.8) *is uniformly bounded from above on* $\mathcal{A}_{a,P}$ *and the* $\sup_{\mathcal{A}_{a,P}} \mathcal{Q}$ *is attained on some reflector* $R \in \mathcal{R}^n_E(T)$ *with radial function* ρ *and focal function* p *such that* $(\rho, p) \in \mathcal{A}_{a,P}$*. Furthermore, the diameter of* R *is bounded by a constant depending only on* P*.*

Proof. Because of Lemma 2.4 it suffices to prove the theorem by considering $\mathcal{Q}$ only for reflectors in $\mathcal{R}^n_E(T)$ such that the radial and focal functions satisfy $(\rho, p) \in \mathcal{A}_{a,P}$.

Consider such a reflector R. Let $\mathbf{y}_1$ be such that $\inf_T p(\mathbf{y}) = p(\mathbf{y}_1)$. Since $p(\mathbf{y}_1) \geq 2aM$, Lemma 2.5 implies that $\rho_{\mathbf{y}_1}(x) < 2p(\mathbf{y}_1) \leq 2P \ \forall x \in \mathbb{S}^n$. If $\mathbf{y} \in T$, $\mathbf{y} \neq \mathbf{y}_1$, and $p(\mathbf{y}) \geq 4P$ then Lemma 2.5 implies that $\rho_{\mathbf{y}}(x) \geq 2P \ \forall x \in \mathbb{S}^n$, that is, the ellipsoid $E(\mathbf{y}, p(\mathbf{y}))$ is not supporting to R. But this is impossible, since p is the focal function of R. Therefore, the diameter of R does not exceed $8P$.

Each reflector $R \in \mathcal{R}^n_E(T)$ is a bounded convex hypersurface with the origin strictly in the interior of the compact convex body bounded by R. By Blaschke's selection theorem, the uniform bound on the diameters of reflectors such that $(\rho, p) \in \mathcal{A}_{a,P}$ implies that this set is compact. The continuity of $\mathcal{Q}$ implies existence of a maximizer (and minimizer) of $\mathcal{Q}$. QED.

Remark 2.7 *A suitable statement of Blaschke's selection theorem and references can be found in appendix 6.2.*

Next, we study the relations between the variational problem (2.9) and the near-field reflector problem (1.16). Such relations are geometrically more transparent in the case where ν is an atomic measure concentrated at a finite number of points. The corresponding reflectors can be defined by a finite set of parameters and form a class of $E-polytopes$ [18].

2.2 E-Polytopes and Irradiance Distributions Defined by Atomic Measures

In this section, the variational problem (2.9) is specialized to the case when the target is a discrete set of the form $T = \{\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_K\},\ K \in \mathbb{N},\ K \geq 2$, where $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_K$ are distinct points in $\mathbb{R}^{n+1}$ and $\mathcal{O} \notin T$. In this case, a reflector is defined completely by any K ellipsoids $E(\mathbf{y}_1, \tilde{p}_1), ..., E(\mathbf{y}_K, \tilde{p}_K)$ satisfying conditions (1.3) and (1.4). For a fixed set of points $T = \{\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_K\}$ as above, we denote the corresponding set of reflectors by $\mathcal{R}^n_{E,K}(\mathbf{y}_1, ..., \mathbf{y}_K)$. When there is no danger of confusion, we write $\mathcal{R}^n_{E,K}$. The reflectors in $\mathcal{R}^n_{E,K}$ are called E-polytopes. As before, $M := \max_{i\in\{1,...,K\}} |y_i|$.

The focal function of a reflector $R \in \mathcal{R}^n_{E,K}$ is completely defined by the vector $\mathbf{p} = (p_1, ..., p_K) = (p(\mathbf{y}_1), ..., p(\mathbf{y}_K)) \in \mathbb{R}^K_+$, where $\mathbb{R}^K_+ = \{\mathbf{p} \in \mathbb{R}^K \mid p_1, ..., p_K > 0\}$.

The measure ν in this case is assumed to be atomic and given by

$$\nu = \sum_{i=1}^{K} \nu_i \delta_{\mathbf{y}_i}, \;\; \sum_{i=1}^{K} \nu_i = \mu(\mathbb{S}^n), \;\; \nu_1, ..., \nu_K > 0. \tag{2.12}$$

The set of admissible functions is defined as

$$\begin{aligned} \mathcal{A}_{K,a} = \{(\rho, \mathbf{p}) \mid \rho \in C(\mathbb{S}^n),\; \mathbf{p} \in \mathbb{R}^K_+,\; \rho(x) > 0 \text{ on } \mathbb{S}^n, \\ \min_{i\in\{1,...,K\}} p_i \geq 2aM, \text{ and} \qquad (2.13) \\ \hat{\rho}(x) - \hat{p}_i \leq \hat{\mathcal{K}}(x, y_i, p_i)\; \forall x \in \mathbb{S}^n,\; i = 1, ..., K\}, \qquad (2.14) \end{aligned}$$

where $a \geq 1$ is a fixed constant. As in the general case, we define the functional

$$\mathcal{Q}_K[\rho, \mathbf{p}] = \int_{\mathbb{S}^n} \hat{\rho}(x) I(x) d\sigma - \sum_{i=1}^{K} \hat{p}_i \nu_i. \tag{2.15}$$

The variational problem in this case consists of finding a pair $(\rho_{\max}, \mathbf{p}_{\max}) \in$

$\mathcal{A}_{K,a}$ such that

$$\mathcal{Q}_K[\rho_{\max}, \mathbf{p}_{\max}] = \sup_{\mathcal{A}_{K,a}} \mathcal{Q}_K[\rho, \mathbf{p}]. \tag{2.16}$$

For each $a \geq 1$ define a subset of reflectors $\mathcal{R}^n_{E,K,p_1}(a)$ in $\mathcal{R}^n_{E,K}$ such that

$$p_1 = 8aM,\ 2aM \leq p_i \leq 32aM \text{ for all } i \in \{2, ..., K\}. \tag{2.17}$$

It follows from Lemma 2.5 that $\mathcal{R}^n_{E,K,p_1}(a) \neq \emptyset$. Indeed, for the ellipsoids $E(\mathbf{y}_i, p_i)$ with $p_1 = 8aM$ and $p_i = 32aM,\ i = 2, ..., K$ we have by (2.11) for all $x \in \mathbb{S}^n$:

$$4aM < \rho_{\mathbf{y}_1}(x) < 16aM \text{ and } 16aM < \rho_{\mathbf{y}_i}(x) < 64aM \text{ when } i = 2, ..., K.$$

Then, clearly, the reflector $\tilde{R} = \{E(\mathbf{y}_1, p_1), ..., E(\mathbf{y}_K, p_K)\}$ reduces to $E(\mathbf{y}_1, p_1)$ and $\alpha^{-1}_{\tilde{R}}(\mathbf{y}_1) = \mathbb{S}^n$. Therefore, $G(\tilde{R}, \mathbf{y}_1) = \mu(\mathbb{S}^n)$, while $G(\tilde{R}, \mathbf{y}_i) = 0\ \forall i = 2, ..., K$, and thus $\tilde{R} \in \mathcal{R}^n_{E,K,p_1}(a)$. It follows also from Lemma 1.5 that reflectors in $\mathcal{R}^n_{E,K,p_1}(a)$ satisfy conditions (1.3), (1.4).

We give now a modified version of a result in [18](cf. Theorem 11 and 12 in [18]) that will be used in the following discussion.

Theorem 2.8 *Suppose* $I \in L(\mathbb{S}^n),\ I \geq 0, I \not\equiv 0$ *and* $T = \{\mathbf{y}_1, ..., \mathbf{y}_K\}$*, where* $K \geq 2$ *and the points in* T *are distinct. Let* $\nu_1, ..., \nu_K \geq 0$ *be such that*

$$\int_{\mathbb{S}^n} I(x) d\sigma = \sum_{i=1}^{K} \nu_i.$$

Then for each $a \geq 1$ there exists a reflector $R \in \mathcal{R}^n_{E,K,p_1}(a)$ solving the near-field reflector problem, that is,

$$G(R, \mathbf{y}_i) = \nu_i, \;\; i = 1, 2, ..., K. \tag{2.18}$$

In addition, if $I > 0$ almost everywhere on $\mathbb{S}^n$ and $a_1 \neq a_2$ the solutions in $\mathcal{R}^n_{E,K,p_1}(a_1)$ and $\mathcal{R}^n_{E,K,p_1}(a_2)$ cannot be transformed into each other by a homothety with respect to $\mathcal{O}$.

Note that in [18] the radial functions of the admissible reflectors were bounded by $2M$ from below and $32M$ from above since the focal parameters are restricted to $p_1 = 16M$ and $4M \leq p_i \leq 64M$. In the definition in (2.17) of the admissible set $\mathcal{R}^n_{E,K,p_1}(a)$, we introduced the parameter a which controls the bounds on the diameter of the admissible reflectors. Note that for $a = 2$ we recover the situation in [18], while we obtain smaller reflectors (with respect to the diameter) if $1 \leq a < 2$.

Proof. Following the ideas in [18], we actually restrict our attention to a subset of $\mathcal{R}^n_{E,K,p_1}(a)$ by imposing the additional constraints

$$G(R, \mathbf{y}_1) \geq \nu_1, \tag{2.19}$$

$$G(R, \mathbf{y}_i) \leq \nu_i, \;\; \forall\, i = 2, \ldots, K. \tag{2.20}$$

Thus, we consider the set

$$\hat{\mathcal{R}}^n_{E,K,p_1}(a) = \{R \in \mathcal{R}^n_{E,K,p_1}(a) \mid (2.19) \text{ and } (2.20) \text{ hold}\}.$$

Note that $E(\mathbf{y}_1, 8aM) = (8aM, 32aM, \ldots, 32aM) \in \hat{\mathcal{R}}^n_{E,K,p_1}(a) \neq \emptyset$. Following again [18], we claim that the reflector

$$\tilde{R} = \bigcap_{i=1}^{K} E(\mathbf{y}_i, \tilde{p}_i),$$

where

$$\tilde{p}_1 = 8aM,$$

$$\tilde{p}_i = \inf_{R \in \hat{\mathcal{R}}^n_{E,K,p_1}(a)} p_i, \; i = 2, \ldots, K,$$

solves the reflector problem. Assume it does not. Then there exists $j \neq 1$ such that $G(\tilde{R}, \mathbf{y}_j) < \nu_j$. But this implies that there is also $\delta > 0$ such that $G(R^\delta, \mathbf{y}_j) \leq \nu_j$, where

$$R^\delta = \bigcap_{i=1}^{K} E(\mathbf{y}_i, p_i^\delta)$$

and

$$p_j^\delta = \tilde{p}_j - \delta,$$

$$p_i^\delta = \tilde{p}_i, \; i = 1, \ldots, K, \; i \neq j.$$

Moreover, it follows that $R^\delta \in \hat{\mathcal{R}}^n_{E,K,p_1}(a)$ in contradiction to the definition of $\tilde{R}$. QED.

The following simple example illustrates why the uniqueness does not hold in general. Suppose $I \equiv 1$ on $\mathbb{S}^n$ and consider a set $T = \{\mathbf{y}_1, \mathbf{y}_2\}$ containing only two distinct points $\neq \mathcal{O}$. Assume also that $\nu_1,\ \nu_2 > 0,\ \nu_1 + \nu_2 = \mu(\mathbb{S}^n)$, where μ is defined in (1.15). Fix $p_1 = 16M$. If we choose $p_2 = 64M$ then $B(\mathbf{y}_1, 16M) \subset B(\mathbf{y}_2, p_2)$ and $G(R, \mathbf{y}_1) = \nu_1 + \nu_2$ and $G(R, \mathbf{y}_2) = 0$. Now, decrease p_2 until the ellipsoid $E(\mathbf{y}_2, p_2)$ becomes tangent to $E(\mathbf{y}_1, p_1)$ and then continue to decrease p_2. By Lemma 2.5, when $p_2 = 4M$ the ellipsoid $E(\mathbf{y}_2, 4M) \subset B(\mathbf{y}_1, 16M)$ and therefore $G(R, \mathbf{y}_2) = \nu_1 + \nu_2$ and $G(R, \mathbf{y}_1) = 0$. The function $G(R, \mathbf{y}_2)$ increases continuously with decreasing p_2 ([18], Lemma 9) and therefore for some p_2 we have $G(R, \mathbf{y}_i) = \nu_i$ for $i = 1, 2$. Thus, we obtained one weak solution to the near-field reflector problem. To obtain another solution, we may take any $a \geq 1$ and repeat the same procedure in $\mathcal{R}^n_{E,K,p_1}(a)$. Thus, it is possible to construct distinct solutions with an arbitrary large diameter. Note that the same phenomenon occurs also in the simpler case when $T = \{\mathbf{y}\}$. In this case, any ellipsoid with foci at $\mathcal{O}$ and $\mathbf{y}$ is a solution of the near-field reflector problem. However, in this case, any two such solutions can be transformed into each other by a suitable rescaling of the focal parameter of one of them. This is not true in general for a reflector defined by more than one ellipsoid because the functions $G(R, \mathbf{y}_i)$ may scale differently for different i and the reflector after such transformation will not be a solution of the same near-field reflector problem.

We now continue with the study of the behavior of the functional Q_K in the variational problem (2.16) relative to p. First we consider the case when $\min_{i\in\{1,...,K\}} p_i$ is large.

Lemma 2.9 *Let* $1 \le a^1 < a^2 < ... < a^s < ...,$ $a^s \to \infty$ *as* $s \nearrow \infty$, *and let* $\{R^s\}$ *be a sequence of solutions to the near-field reflector problems in* $\mathcal{R}^n_{E,K,p_1}(a^s)$. *Denote by* ρ^s, p^s *the radial and focal functions of the reflector* R^s. *Then* $(\rho^s, p^s) \in \mathcal{A}_{K,a^s}$ *and* $\mathcal{Q}_K[\rho^s, p^s] \to 0$ *as* $s \to \infty$.

Proof. The claim $(\rho^s, p^s) \in \mathcal{A}_{K,a^s}$ is obvious. We prove now the second claim. Each of the reflectors R^s defines a cover of $\mathbb{S}^n$ by closed visibility sets $V_i^s = \alpha_{R^s}^{-1}(\mathbf{y}_i), i = 1, ..., K$, such that $\sigma(V_i^s \bigcap V_j^s) = 0$ for $i \ne j$ and $\alpha_{R^s}(V_i^s) = \mathbf{y}_i,\ G(R^s, \mathbf{y}_i) = \nu_i$. Furthermore, by (1.13) we have equality in (1.11) on each V_i^s. Therefore,

$$\begin{aligned} \mathcal{Q}_K[\rho^s, p^s] &= \int_{\mathbb{S}^n} \hat{\rho}^s(x) I(x) d\sigma - \sum_{i=1}^K \hat{p}_i^s \nu_i \\ &= \sum_{i=1}^K \left\{ \hat{p}_i^s \left[G(R^s, \mathbf{y}_i) - \nu_i \right] + \int_{V_i^s} \hat{\mathcal{K}}(x, y_i, p_i^s) I(x) d\sigma \right\}. \end{aligned} \tag{2.21}$$

Because the R^s satisfy (2.18), the first term under the sum vanishes for all i.

On the other hand, since $p_i^s \ge 4a^s M$ for each $i \in \{1, ..., K\}$, it follows from

(1.2) that $\epsilon(p_i^s) \to 0$ as $s \to \infty$ and therefore $\hat{\mathcal{K}}(x, y_i, p_i^s) \to 0$ for all i and the proof of the lemma is now complete. QED.

We show next that the functional Q_K attains a maximum on every set of reflectors for which a solution of the reflector problem was shown to exist (see Theorem 2.8).

We define the following admissible set for a discrete target set; we put

$$\mathcal{A}_{K,a,p_1} = \{(\rho, \mathbf{p}) \in \mathcal{A}_{K,a} |\ p_1 = 8aM, \qquad 2aM \le p_i \le 32aM \text{ for all } i = 2, 3, \ldots, K\} \tag{2.22}$$

and consider the analog of problem (2.16) on the set $\mathcal{A}_{K,p_1}$. As a consequence of Theorem 2.6 we obtain the following statement on the existence of a maximizer. Furthermore, the supremum is achieved in a reflector.

Lemma 2.10 *Let $a \ge 1$ and $T, \nu, \mathcal{Q}_K, \mathcal{A}_{K,p_1}$ be as in* (2.12),(2.15),(2.22). *Then there exists an E-polytope $R_{\max} \in \hat{\mathcal{R}}^n_{E,K,p_1}(a)$ with radial function $\rho_{\max}$ and focal function $\mathbf{p}_{\max}$ such that*

$$\mathcal{Q}_K[\rho_{\max}, \mathbf{p}_{\max}] = \sup_{\mathcal{A}_{K,a,p_1}} \mathcal{Q}_K[\rho, \mathbf{p}] = \sup_{\mathcal{R}^n_{E,K,p_1}(a)} \mathcal{Q}_K[\rho, \mathbf{p}]. \tag{2.23}$$

Proof. The lemma is just a special case of Theorem 2.6. QED.

We recall that in the far-field case [11], [36] a weak solution of the reflector problem can be found by solving an optimal transport problem. More precisely, the maximizing pair of the cost functional for the far-field problem corresponds to a weak solution of the reflector problem. We will show that this is not true in general for the near-field problem.

Chapter 3

Examples and Numerical Experiments

In this chapter we present examples which show that the admissible solution of the reflector problem is not in general the solution of the associated variational problem. This situation is in sharp contrast to the far-field reflector problem where the solutions to the reflector and the variational problem coincide. We also present some numerical experiments to illustrate the situation for the near-field problem.

3.1 Solutions of the Variational Problem May Not Solve the Reflector Problem

In this section we give two examples for which we prove that the admissible solution to the reflector problem does not coincide with the variational solution.

3.1.1 First Example

In the following we discuss an example for which the weak solution of the near-field reflector problem is not a maximizer of the Monge-Kantorovich functional $\mathcal{Q}_K$. First, we state the following

Lemma 3.1 *For a fixed $x \in \mathbb{S}^n$ and $\mathbf{y} \in T$ the function $\hat{\mathcal{K}}(x, y, p)$ evaluated on ellipsoids $E(\mathbf{y}, p)$ is strictly increasing in p if $\langle x, y\rangle < 0$ and strictly decreasing in p if $\langle x, y\rangle > 0$.*

Proof. We have

$$\frac{\partial}{\partial p}\hat{\mathcal{K}}(x, y, p) \quad = \quad \frac{\langle x, y\rangle}{1 - \epsilon(p)\langle x, y\rangle}\frac{\partial \epsilon(p)}{\partial p}.$$

The statement of the lemma follows now from the fact that $\frac{\partial \epsilon(p)}{\partial p} < 0$ ([18], Lemma 6). QED.

Consider the reflector R defined by the following two ellipsoids in $\mathbb{R}^{n+1}$: E_1 with foci $\mathcal{O}$ and $\mathbf{y}_1 = (1, 0, ..., 0)$ and focal parameter p_1 and E_2 with foci $\mathcal{O}$ and $\mathbf{y}_2 = (-1, 0, ..., 0)$ and focal parameter p_2. It follows from Lemma 2.5 that in order to satisfy (1.3) and (1.4) it suffices to fix $p_1 \geq 4a$ for some constant $a \geq 1$. It will be convenient to take $p_1 = p_2 = 4a$ and consider the target set $T = \{\mathbf{y}_1, \mathbf{y}_2\}$. By Proposition 1.8 the visibility sets $V_i = \alpha_R^{-1}(\mathbf{y}_i),\ i = 1, 2$, are given by

$$V_1 = \{x \in \mathbb{S}^n \mid \langle x, \bar{x}\rangle \leq 0\},\ V_2 = \{x \in \mathbb{S}^n \mid \langle x, -\bar{x}\rangle \leq 0\},$$

where

$$\bar{x} = \frac{p_2\epsilon_1\mathbf{y}_1 - p_1\epsilon_2\mathbf{y}_2}{|p_2\epsilon_1\mathbf{y}_1 - p_1\epsilon_2\mathbf{y}_2|} = \mathbf{y}_1.$$

As usual, here $\epsilon_i,\ i = 1, 2$, are the eccentricities of E_1 and E_2 which are equal in the present case.

Assume that the intensity of the source at the origin is $I \equiv 1$ on $\mathbb{S}^n$. Then the irradiance distribution produced by R on $T = \{\mathbf{y}_1, \mathbf{y}_2\}$ is given by the atomic measure

$$\nu = \nu_1\delta_{\mathbf{y}_1} + \nu_2\delta_{\mathbf{y}_2},$$

where $\nu_1 = \nu_2 = m = \frac{1}{2}|\mathbb{S}^n|$. Thus R solves the near-field reflector problem for such T, I and ν. The focal function of R is completely defined by the vector

$\mathbf{p} = (p_1, p_2) = (4a, 4a)$. Figure 3.1 shows the reflector and the generating ellipses. It follows from Theorem 2.8 that R is the only solution of the near-field reflector problem with these data and fixed E_1. We will show that the functional $\mathcal{Q}_2$ is increasing at $(\rho, \mathbf{p})$, where ρ is the radial function of R. The needed variation is produced by increasing the focal parameter p_2.

Let p_2^t be defined by $\log(p_2^t) = \log(p_2) + t$ for $t \geq 0$. Note that $p_2^0 = p_2 = p_1$. Denote by R^t the reflector defined by the ellipsoid E_1 as before and ellipsoid E_2^t with foci $\mathcal{O}$ and $\mathbf{y}_2$ and focal parameter p_2^t. Let the radial functions of E_1 and E_2^t with respect to $\mathcal{O}$ be referred to by ρ_1 and ρ_2^t, respectively. Let $\rho^t(x)$ be the radial function of R^t and $\mathbf{p}^t = (p_1, p_2^t)$. Taking into account that $p_2^0 = p_1$, we have

$$\mathcal{Q}_2[\rho^t, \mathbf{p}^t] = \int_{\mathbb{S}^n} \hat{\rho}^t(x)\, d\sigma - 2\hat{p}_1 m - tm. \tag{3.1}$$

We denote by $V_1^t = \alpha_{R^t}^{-1}(\mathbf{y}_1),\ V_2^t = \alpha_{R^t}^{-1}(\mathbf{y}_2)$ the visibility sets of $\mathbf{y}_1,\ \mathbf{y}_2$ with respect to the reflector $\mathcal{R}^t$, respectively. Clearly, $V_2^{t'} \subset V_2^t$ and $V_1^t \subset V_1^{t'}$ when $0 \leq t \leq t'$ with the inclusion being strict when $t < t'$. Therefore, $V_1 \cap V_1^t = V_1$ and $V_2 \cap V_2^t = V_2^t$, where $V_i = V_i^0,\ i = 1, 2$. Put $V_{1,2}^t = V_2 \cap V_1^t$. For each $t \geq 0$ the sets V_1, V_2^t and $V_{1,2}^t$ form a cover of $\mathbb{S}^n$, and furthermore,

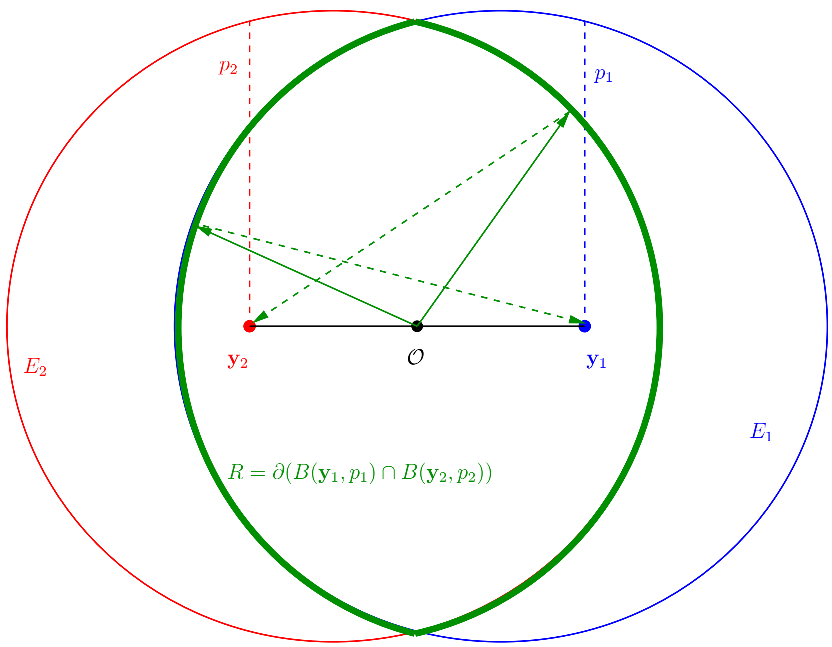

Figure 3.1: The reflector R generated by the ellipses $E_1 = E(\mathbf{y}_1, 4a)$ and $E_2 = E(\mathbf{y}_2, 4a)$ redistributes the light emitted from a homogeneous source evenly among the target points $\mathbf{y}_1$ and $\mathbf{y}_2$.

they are disjoint except for a set of measure zero, that is

$$\sigma(V_1 \cap V_2^t) = \sigma(V_1 \cap V_{1,2}^t) = \sigma(V_{1,2}^t \cap V_2^t) = 0.$$

By (1.9), we have $\hat{\rho}^t(x) = \hat{\rho}_1(x)$ when $x \in V_1 \cup V_{1,2}^t$ and $\hat{\rho}^t(x) = \hat{\rho}_2^t(x)$ when $x \in V_2^t$. Using (1.9), (1.11), (3.1) and, again, taking into account that $\rho^0 = \rho, p_2^0 = p_1$, we obtain

$$\begin{aligned}
\mathcal{Q}_2[\rho^0, \mathbf{p}^0] = & \int_{V_1} \hat{\rho}_1(x)\ d\sigma + \int_{V_{1,2}^t} \hat{\rho}_2(x)\ d\sigma \\
& + \int_{V_2^t} \hat{\rho}_2(x)\ d\sigma - 2\hat{p}_1 m \\
= & \int_{V_1} \hat{\mathcal{K}}(x, y_1, p_1) d\sigma + \int_{V_{1,2}^t} \hat{\mathcal{K}}(x, y_2, p_2) d\sigma \\
& + \int_{V_2^t} \hat{\mathcal{K}}(x, y_2, p_2) d\sigma. \qquad (3.2)
\end{aligned}$$

If $t > 0$ and $x \in \text{int}V_{1,2}^t$ then $\langle x, \mathbf{y}_2 \rangle < 0$ and by Lemma 3.1, $\hat{\mathcal{K}}(x, y_2, p_2) < \hat{\mathcal{K}}(x, y_2, p_2^t)$. Furthermore, in this case $\hat{\mathcal{K}}(x, y_2, p_2^t) < 0$ while $\hat{\mathcal{K}}(x, y_1, p_2^t) > 0$. Since $p_2^t = p_1$ on $V_{1,2}^t$, we obtain from the preceding observations and (3.2) for $t > 0$

$$\mathcal{Q}_2[\rho^0, \mathbf{p}^0] < \int_{V_1 \cup V_{1,2}^t} \hat{\mathcal{K}}(x, y_1, p_1) d\sigma + \int_{V_2^t} \hat{\mathcal{K}}(x, y_2, p_2^t) d\sigma. \qquad (3.3)$$

Next, we apply the duality relation

$$\hat{\rho}(x) - \hat{p}(\mathbf{y}) = \hat{\mathcal{K}}(x, \mathbf{y}, p)$$

on the visibility sets to recover the (logarithmic) radial and focal functions from $\hat{\mathcal{K}}$ on the right-hand side of (3.3). Thus, we have the following estimate:

$$\begin{aligned} \mathcal{Q}_2[\rho^0, \mathbf{p}^0] < \int_{V_1 \cup V_{1,2}^t} \hat{\rho}_1(x) d\sigma - \hat{p}_1(m + |V_{1,2}^t|) \\ + \int_{V_2^t} \hat{\rho}_2^t(x)\, d\sigma - \hat{p}_2^t |V_2^t|. \end{aligned} \tag{3.4}$$

Using the fact that $\hat{\rho}^t \equiv \hat{\rho}_1$ on the set $V_1 \cup V_{1,2}^t$ and the decomposition $V_2 = V_2^t \cup V_{1,2}^t$, we rewrite the right-hand side of (3.4) to find that

$$\begin{aligned} \mathcal{Q}_2[\rho^0, \mathbf{p}^0] &< \int_{\mathbb{S}^n} \hat{\rho}^t(x)\, d\sigma - (\hat{p}_1 m + \hat{p}_1 |V_{1,2}^t| + \hat{p}_2^t |V_2^t|) \\ &= \int_{\mathbb{S}^n} \hat{\rho}^t(x)\, d\sigma - (\hat{p}_1 m + \hat{p}_1 |V_{1,2}^t| + \hat{p}_2^t m - \hat{p}_2^t |V_{1,2}^t|). \end{aligned} \tag{3.5}$$

Recalling that $\hat{p}_2^t = \hat{p}_1 + t$ we obtain from (3.5) that

$$\begin{aligned} \mathcal{Q}_2[\rho^0, \mathbf{p}^0] &< \int_{\mathbb{S}^n} \hat{\rho}^t(x)\, d\sigma - (\hat{p}_1 m + \hat{p}_1 |V_{1,2}^t| + \hat{p}_1 m + tm \\ &\quad - \hat{p}_1 |V_{1,2}^t| - t|V_{1,2}^t|) \\ &= \int_{\mathbb{S}^n} \hat{\rho}^t(x)\, d\sigma - (2\hat{p}_1 m + tm - t|V_{1,2}^t|) \\ &= \int_{\mathbb{S}^n} \hat{\rho}^t(x)\, d\sigma - 2\hat{p}_1 m - tm + t|V_{1,2}^t|). \end{aligned} \tag{3.6}$$

Finally, comparing the right-hand-side of (3.6) with (3.1) we conclude that

$$\mathcal{Q}_2[\rho^0, \mathbf{p}^0] < \mathcal{Q}_2[\rho^t, \mathbf{p}^t] + t|V_{1,2}^t|. \tag{3.7}$$

Since $t|V_{1,2}^t| = o(t)$, we conclude that $\mathcal{Q}_2[\rho^t, \mathbf{p}^t] > \mathcal{Q}_2[\rho, \mathbf{p}]$ for any sufficiently small $t > 0$. In other words, the solution to the near-field reflector problem in the admissible set is not the maximizer of the functional $\mathcal{Q}_2$.

3.1.2 Second Example

Observe that in the example discussed in section 3.1.1, the target set T and the source $\mathcal{O}$ are contained in the same hyperplane (this situation is excluded, for example, in [19]). However, if we perturb one of the target points, say $\mathbf{y}_2$, slightly, we can still construct an example where $\mathcal{Q}_K$ is increasing in the solution to the reflector problem. To see this, consider the perturbed focus

$$\mathbf{y}_2' = (-\cos(\psi), \sin(\psi), 0, \ldots, 0),$$

where $0 < \psi \leq \frac{\pi}{2}$. Then the center of the visibility set of $\mathbf{y}_2'$ is given by

$$\bar{x}' = \frac{(1+\cos(\psi), -\sin(\psi), 0, \ldots, 0)}{\sqrt{(1+\cos(\psi))^2 + \sin^2(\psi)}}.$$

Let V' be the component of $\mathbb{S}^n$ contained between the two hyperplanes with normal vectors $\bar{x}'$ and $e_1 = y_1 = \frac{\mathbf{y}_1}{|\mathbf{y}_1|}$, respectively. Now, we consider the source intensity $I : \mathbb{S}^n \to \mathbb{R}$ defined by

$$I(x) = \begin{cases} 1, & \text{if } x \in \mathbb{S}^n \backslash V', \\ 0, & \text{if } x \in V'. \end{cases}$$

Using the measure $\nu' = m'(\delta_{\mathbf{y}_1} + \delta_{\mathbf{y}_2'})$ where $m' = (1/2)|\mathbb{S}^n \backslash V'|$ on the target set, we obtain again that $\mathcal{Q}_K$ is increasing in the solution to the reflector problem.

3.2 Numerical Experiments

In this section, we describe some numerical experiments related to the examples that we discussed previously in section 3.1.1 and 3.1.2 using a *Graphical User Interface* (GUI) in Matlab.

3.2.1 First Experiment

We consider again the reflectors $R(p_1, p_2)$ generated by the ellipse E_1 with focal points $\mathcal{O}, (1, 0) \in \mathbb{R}^2$ and fixed focal parameter $p_1 = 4$ and the ellipse E_2 with focal points $\mathcal{O}, (-1, 0) \in \mathbb{R}^2$ and variable focal parameter p_2. Let the target measure be denoted $\nu = \nu_1 \delta_{\mathbf{y}_1} + \nu_2 \delta_{\mathbf{y}_2}$. In the following we use Matlab to illustrate our analytical findings from Section 3.1 graphically in the Euclidean plane. We already observed that if $I \equiv 1$ and $\nu_1 = \nu_2 = \frac{1}{2}m$, where $m = \int_{\mathbb{S}^n} I \, d\sigma$, then the solution to the reflector problem is given by the symmetric reflector $R = (p_1, p_2) = (4, 4)$.

Preliminaries

Since the focal parameter p_2 of the second ellipse plays the role of a free parameter, we establish a priori bounds that can be used to determine a suitable range of p_2 for numerical experiments.We are interested in the case when both of the generating ellipses are also supporting to the reflector. Otherwise the reflector simply coincides either with E_1 or E_2.

Proposition 3.2 (A priori bounds) *Let $R = R(p_1, p_2) = \partial \bigcap_{i=1,2} B(E_i)$ be a reflector generated by two ellipsoids with $p_1 = p$ fixed, $\mathbf{y}_2 = -\mathbf{y}_1 \neq \mathcal{O}$. Assume furthermore that $p \geq 2|\mathbf{y}|$ where $|\mathbf{y}| = |\mathbf{y}_i|,\ i = 1, 2$. In order to have*

$$\forall i = 1, 2 \exists x \in \mathbb{S}^n : \ E_i \textit{ is supporting to } R \textit{ at } \rho(x)x,$$

where ρ is the radial function of R, it is necessary that

$$p_{low} \leq p_2 \leq p_{up}.$$

Here the constants $p_{low},\ p_{up}$ are the (positive) solutions to the following equations in p_2:

$$\begin{aligned} \frac{p(1-\epsilon(p_2))}{1+\epsilon(p)} - p_2 &= 0, \\ \frac{p(1+\epsilon(p_2))}{1-\epsilon(p)} - p_2 &= 0, \end{aligned}$$

respectively.

Proof. To assure that the free ellipsoid E_2 is not contained strictly inside the body bounded by E_1 we need to enforce that $\min_{x\in\mathbb{S}^n}\bar{\rho_1}(x) \leq \max_{x\in\mathbb{S}^n}\rho_2(x)$. Here ρ_i denotes the radial function of the ellipsoid $E_i,\ i = 1,\ 2$. Thus, we obtain the constraint on the focal parameter of E_2 that

$$\frac{p}{1+\epsilon(p)} \leq \frac{p_2}{1-\epsilon(p_2)}. \tag{3.8}$$

Since the right-hand side of (3.8) is strictly increasing for $p_2 > 0$ there exists a unique positive solution to the equation

$$\frac{p}{1+\epsilon(p)} = \frac{p_2}{1-\epsilon(p_2)}. \tag{3.9}$$

Equivalently, we can find the positive solution to

$$\frac{p(1-\epsilon(p_2))}{1+\epsilon(p)} - p_2 = 0$$

as a lower bound for the parameter p_2.

Similarly, we require that the minimal radius of the variable ellipse is smaller than the maximal radius of the fixed ellipse $\max_{x\in\mathbb{S}^n}\rho_1(x)$, that is $\min_{x\in\mathbb{S}^n}\rho_2(x) = \frac{p_2}{1+\epsilon(p_2)} \leq \frac{p}{1-\epsilon(p)} = \max_{x\in\mathbb{S}^n}\rho_1(x)$. Hence, p_2 is bounded from above by the positive solution of

$$\frac{p_2}{1+\epsilon(p_2)} = \frac{p}{1-\epsilon(p)}$$

or equivalently

$$\frac{p(1+\epsilon(p_2))}{1-\epsilon(p)} - p_2 = 0.$$

QED.

Remark 3.3 *Note that the above Proposition 3.2 holds for any dimension n. However, the a priori bounds depend on the geometry of the target set.*

To validate the numerical evaluation of the functional $\mathcal{Q}_K$ we prove the following property.

Proposition 3.4 *Let $R = \{\rho(x)x | x \in \mathbb{S}^1\}$ be the reflector generated by the two ellipses $E(\mathbf{y}_1, p)$, $E(\mathbf{y}_2, p)$ where $\mathbf{y}_2 = -\mathbf{y}_1$ and $p_1 = p_2 = p > 2|\mathbf{y}_1|$. Then*

$$\mathcal{Q}_K[\rho, (p, p)] < 0. \tag{3.10}$$

Furthermore, we have $\lim_{p\to\infty} \mathcal{Q}_K[\rho, (p, p)] = 0$.

Proof. The proof is a straightforward computation using the symmetry of the reflector. Recall that the center of the visibility set $V_2 = \alpha^{-1}(\mathbf{y}_2)$ is given by $A_{12} = \frac{p\epsilon y_1 - p\epsilon y_2}{|p\epsilon y_1 - p\epsilon y_2|} = y_1$ where $\epsilon = \epsilon(p)$ denotes the eccentricity of E_1 and

E_2.

$$\begin{aligned}
\mathcal{Q}_K[\rho, \mathbf{p}] &= \int_{\mathbb{S}^n} \log(\rho(x))\, d\sigma(x) - \log(p)m \\
&= 2\left(\int_{V_2} \log(p)\, d\sigma(x) + \int_{V_2} \log(1 - \epsilon\langle x, y_2\rangle)\, d\sigma(x)\right) - \log(p)m \\
&= 2\int_{V_2} \log(p)\, d\sigma(x) - 2\int_{V_2} \log(1 - \epsilon\langle x, y_2\rangle)\, d\sigma(x) - \log(p)m \\
&= \; -2\int_{\alpha^{-1}(\mathbf{y}_2)} \log(1 - \epsilon\langle x, y_2\rangle)\, d\sigma(x) < 0
\end{aligned}$$

since $1 \leq 1 - \epsilon\langle x, y_2\rangle < 2$ and therefore $\log(1 - \epsilon\langle x, y_2\rangle) \geq 0$ for all $x \in V_2 = \alpha^{-1}(\mathbf{y}_2)$. QED.

In the following we discuss some results that were obtained using Matlab. The experiments were conducted using a GUI that displays two ellipsoids in the plane with the first focus at the origin; furthermore, the generated reflector is also displayed. The focal parameters (p_1, p_2) and second foci ($\mathbf{y}_1$ and $\mathbf{y}_2$, sometimes also labeled Y_1 and Y_2) can be changed interactively by the user, as well as the irradiance distribution on the target set, which is given by the measure $\nu = \nu_1\delta_{Y_1} + \nu_2\delta_{Y_2}$. Moreover, the functional $\mathcal{Q}_2$ is evaluated numerically, assuming a constant source intensity of $I \equiv 1$ on $\mathbb{S}^1$.

Numerical Investigation of the Functional $\mathcal{Q}_K$

First we display the solution to the reflector problem for the following data:

$$\left.\begin{array}{rcc} p_1 & = & 4, \\ Y_1 & = & (-1,0), \\ Y_2 & = & (1,0), \\ I & \equiv & 1, \\ \nu_1 = \nu_2 & = & \pi. \end{array}\right\} \tag{3.11}$$

As discussed earlier, the solution to this reflector problem is described by the focal parameters $(p_1, p_2) = (4, 4)$; see Figure 3.2. Note that the negative value of the functional $\mathcal{Q}$ in Figure 3.2 agrees with our result in Proposition 3.4. We give here a brief description of the functionalities of the GUI; see Figure 3.2 for the corresponding panels. The *display panel* contains a display that is used to plot two ellipsoids (one in red, one in blue) with first focus at the origin (marked by a yellow dot), their second foci (marked by an upward pointing triangle for the red ellipse and a downward pointing triangle for the blue ellipse), and the reflector $R = \partial\left(B_1 \cap B_2\right)$ (plotted in green) generated by the two ellipses in the plane. Besides the display panel the GUI consists of a control panel and an output panel. The *control panel* allows the user to change the parameters of the two generating ellipses interactively. The solution to the reflector problem (3.11) is given by the focal parameters $p_1 = p_2 = 4$. Changing the parameters in the control panel results in an updated graphical output in the display panel. The *Discretization* parameter controls the number of points that are used to discretize the unit circle $\mathbb{S}^1$. These are then used for the computations, since both ellipses and the reflector can be described completely through their radial functions. The third panel of the GUI, referred to as *output panel*, displays the value of the functional $\mathcal{Q}_2$ for the current geometry, as well as the actual values (second column) and the proportions (third column) of the target measure (ν) and the measure induced by the reflector map (G), respectively, on the target set. The ratios for the atoms of the measure ν on the target set can also be adjusted interactively by the user.

Figure 3.3 shows a perturbation of the solution to the reflector problem (3.11) and the corresponding control panels. More precisely, the focal parameter of the second ellipse E_2 has been increased slightly. Observe that by increasing p_2 the value of the functional $\mathcal{Q}_K$ has increased as well. This is in accordance with our analytical results in section 3.1 where we showed that the solution to the reflector problem does not solve the variational problem

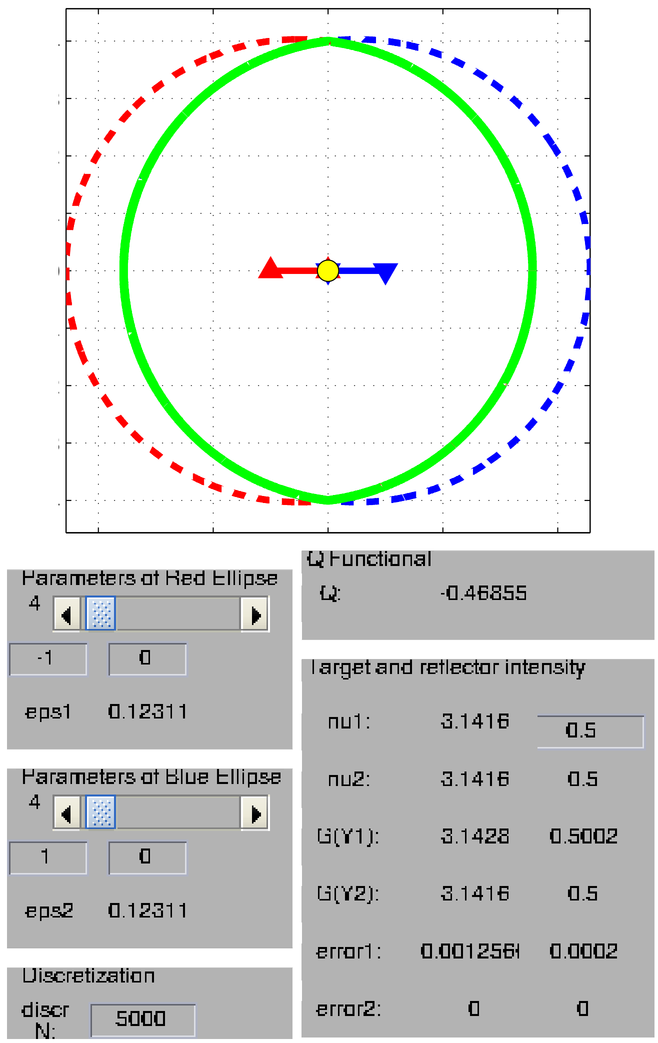

Figure 3.2: The solution to the reflector problem (3.11) and the corresponding GUI panels. Top: display panel, bottom left: control panel, bottom right: output panel

(2.23).

Second, we run a script that evaluates the functional $\mathcal{Q}_K$ in a prescribed range for the focal parameter p_2 and plots the values of $\mathcal{Q}_K$ over p_2 in one display, as well as the values of $G(Y_1)$ and $G(Y_2)$ as functions of p_2 in a second display. Again, the results illustrate graphically our findings in section 3.1 that is, the solution to the reflector problem (again at $p_2 = 4$) does not maximize the functional; see Figure 3.4.

3.2.2 Second Experiment

We consider next the family of reflectors $R(p_1, p_2)$ generated by the ellipse E_1 with focal points $\mathcal{O}, (\sqrt{2}, \sqrt{2}) \in \mathbb{R}^2$ and fixed focal parameter $p_1 = 4$ and the ellipse E_2 with focal points $\mathcal{O}, (\sqrt{2}, -\sqrt{2}) \in \mathbb{R}^2$ and variable focal parameter p_2. Let the target measure be denoted $\nu = \nu_1 \delta_{\mathbf{y}_1} + \nu_2 \delta_{\mathbf{y}_2}$. Again we observe that if $I \equiv 1$ and $\nu_1 = \nu_2 = \frac{1}{2} m$, where $m = \int_{\mathbb{S}^n} I \, d\sigma$, then the solution to the reflector problem is given by the symmetric reflector $R = (p_1, p_2) = (4, 4)$. For future reference, we summarize the data of the

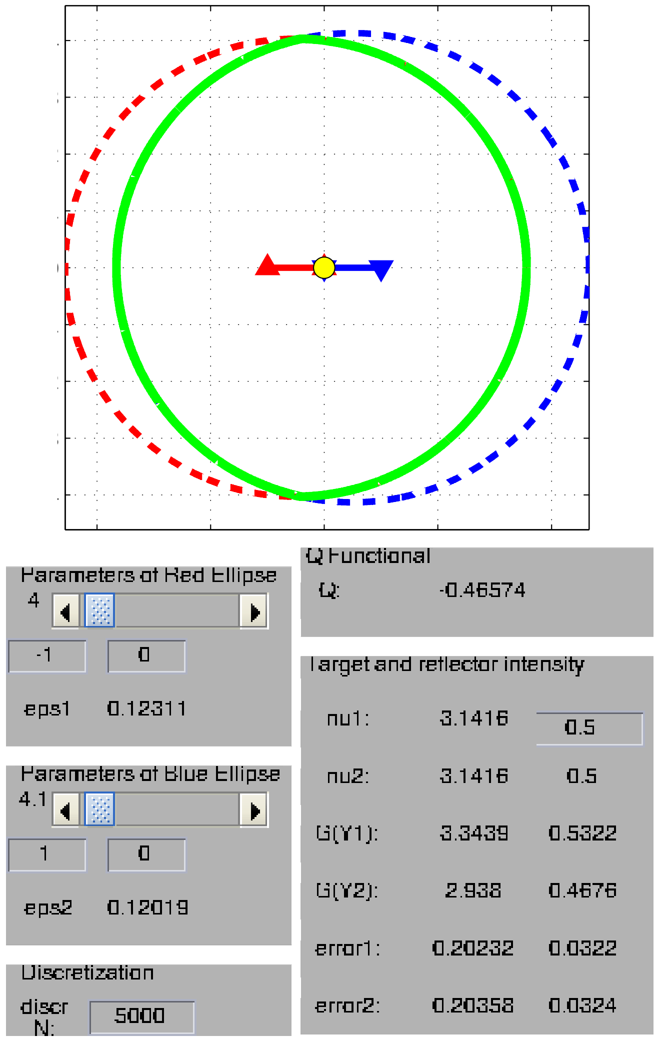

Figure 3.3: A perturbation of the solution to the reflector problem (3.11). Note that the focal parameter of the second ellipse E_2 (blue) has increased compared to Figure 3.2while the value of the functional $\mathcal{Q}_K$ has increased compared to Figure 3.2.

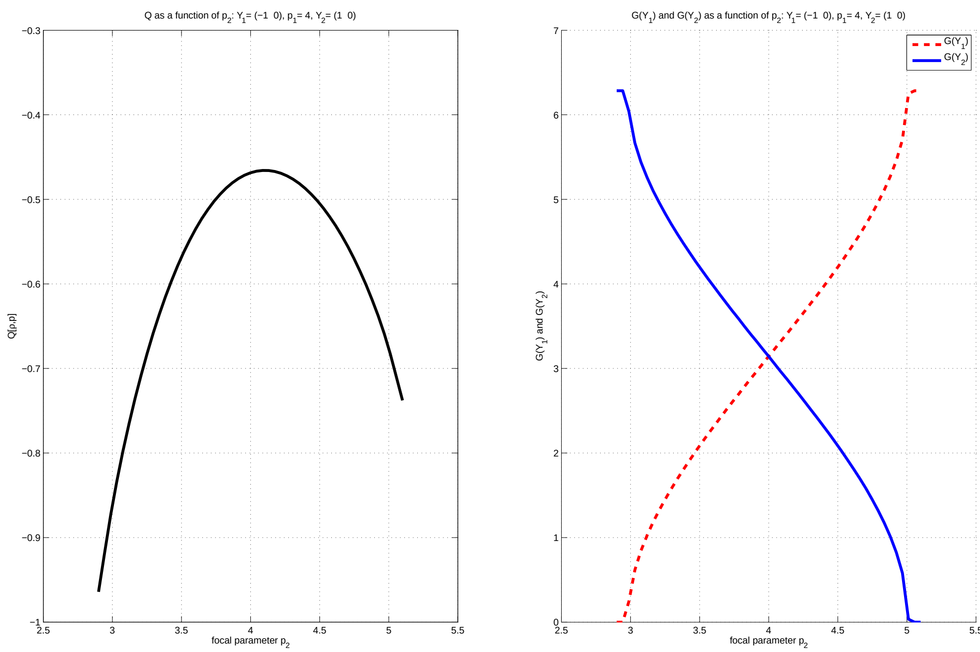

Figure 3.4: The functional $\mathcal{Q}_K$ and the measures $G(Y_1)$, $G(Y_2)$ as functions of p_2.

reflector problem as follows:

$$\left.\begin{array}{rcc} p_1 & = & 4, \\ Y_1 & = & (\sqrt{2}, \sqrt{2}), \\ Y_2 & = & (\sqrt{2}, -\sqrt{2}), \\ I & \equiv & 1, \\ \nu_1 = \nu_2 & = & \pi. \end{array}\right\} \tag{3.12}$$

The solution to the above reflector problem and the two generating ellipses are shown in Figure 3.5, while Figure 3.6 shows a perturbation with higher $\mathcal{Q}$-value. The values of the functional $\mathcal{Q}_K$ and the measure $G(Y_i),\ i = 1, 2$ over the focal parameter p_2 are plotted in Figure 3.7. As previously, we see that the solution to the reflector problem does not maximize $\mathcal{Q}_K$.

3.3 Conclusions

We have seen in this chapter, that an admissible reflector which solves the variational problem (2.23) does not in general solve the associated reflector problem. However, in the numerical examples above, the variational solution appears to be *close* to the solution of the reflector problem. It seems possible that under certain assumptions one can obtain a priori estimates on how well the variational solution approximates the desired solution to the reflector

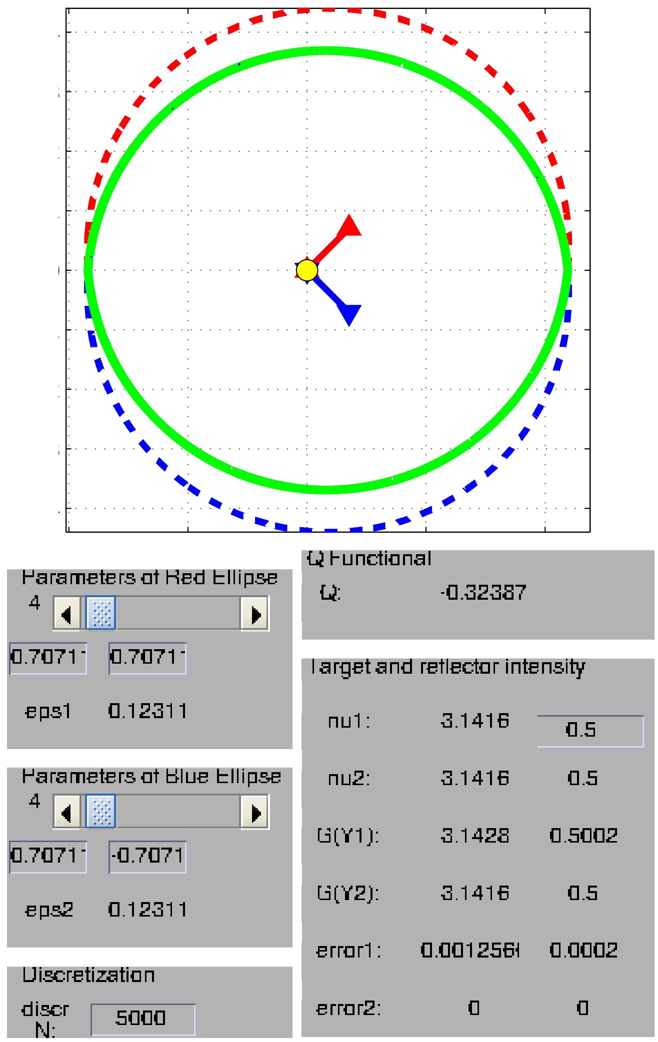

Figure 3.5: The solution to the reflector problem (3.12).

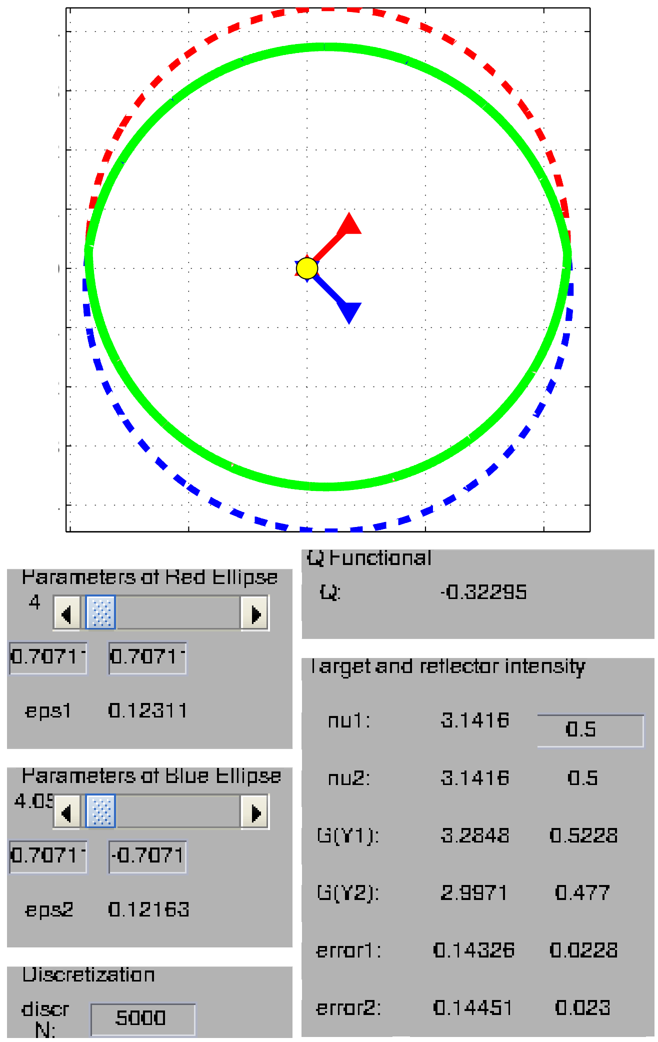

Figure 3.6: A perturbation of the solution to the reflector problem (3.12). The focal parameter of the ellipse $E(Y_2, p_2)$ has been increased. Note that the value of the functional $\mathcal{Q}$ has also increased.

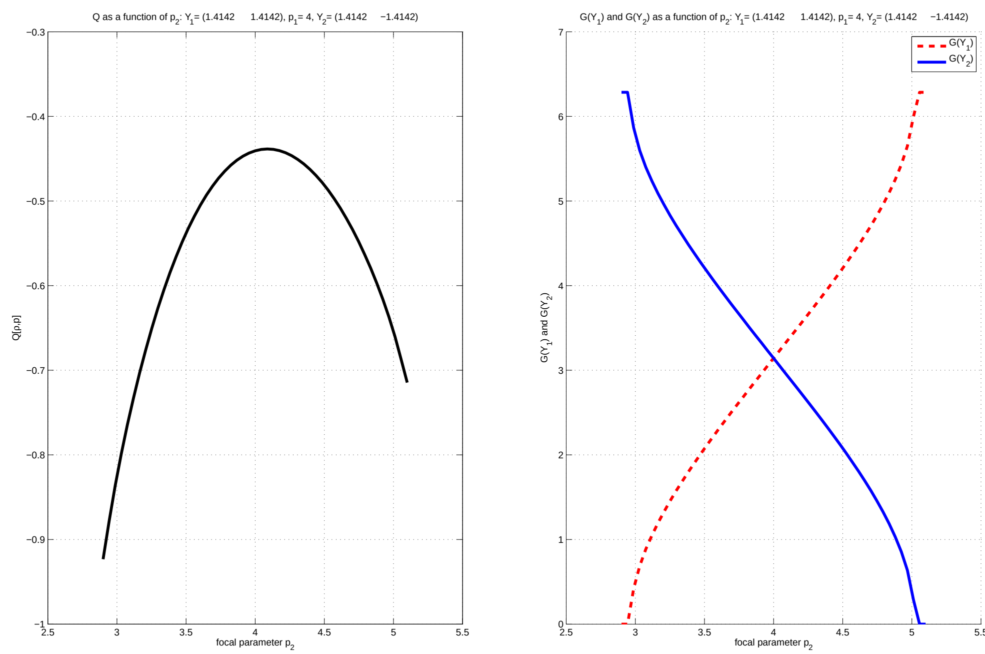

Figure 3.7: The functional $\mathcal{Q}_K$ and the measures $G(Y_1),\ G(Y_2)$ as functions of $p-2$ for configuration of $\mathcal{O},\ Y_1,\ Y_2$ in general position.

problem. We have also seen in the previous sections, that obtaining good estimates can be a complicated task as one may need to take into account the geometry of the target set T and the particular distribution of the source intensity, as well as the target measure ν. We will come back to some of these issues in the following chapters.

Chapter 4

Large Variational Solutions as Approximate Solutions to the Near-Field Single Reflector Problem

4.1 An Approximation Theorem

The example in section 3.1 shows that the maximizer of problem (2.23) is not in general a solution to the near-field reflector problem. However, we will see in this section that for even dimensions n the maximizer is arbitrarily close to a solution of the reflector problem if the parameter a is sufficiently large. This includes the case $n = 2$ of reflector surfaces in $\mathbb{R}^3$, which is the most interesting one for practical applications. We recall here briefly that the parameter a controls the radius of a sphere centered at $\mathcal{O}$ that is contained inside the convex bodies bounded by the admissible reflectors; see Figure 2.1. In this chapter, we will prove the following statement.

Theorem 4.1 *Let n be a positive, even integer. Furthermore, let $T = \{\mathbf{y}_1, \ldots, \mathbf{y}_K\}$, $I \in L^1(\mathbb{S}^n)$ and $\nu = \sum_{i=1}^K \nu_i \delta_{\mathbf{y}_i}$ be as before, that is, T is contained in a hyperplane, $\mathcal{O} \notin T$ and $\int_{\mathbb{S}^n} I(x)\, d\sigma(x) = \sum_{i=1}^K \nu_i$. Then the following holds.*

(i) For any $\gamma > 0$ there exists $a \geq 1$ such that for the maximizer $R_{\max} \in \mathcal{R}^n_{E,K,p_1}(a)$ of the functional Q_K we have

$$|G(R_{\max}, \mathbf{y}_i) - \nu_i| \leq \gamma \text{ for all } i \in \{1, \ldots, K\}. \tag{4.1}$$

(ii) For any $\gamma > 0$ there exists $a \geq 1$ and a reflector $R \in \mathcal{R}^n_{E,K,p_1}(a)$ which is a solution of the near-field reflector problem and

$$|Q_K[\rho, p] - Q_K[\rho_{\max}, p_{\max}]| \leq \gamma, \tag{4.2}$$

where ρ, p are the radial and focal functions, respectively, of the reflector R and $\rho_{\max}, p_{\max}$ are the radial and focal functions, respectively, of the reflector

$R_{\max}$ *on which the* $\sup_{\mathcal{R}^n_{E,K,p_1}(a)} Q_K$ *is attained.*

Before we prove Theorem 4.1, we note an important property of the cost function $\hat{\mathcal{K}}$.

Proposition 4.2 *For any $\gamma > 0$ there exists $\tilde{p} > 0$ such that for any $p \geq \tilde{p}$*

$$\left| p \frac{\partial}{\partial p} \hat{\mathcal{K}}(x, \mathbf{y}, p) \right| < \gamma. \tag{4.3}$$

Proof. The above property follows from the observation made in [18] that the derivative of ϵ with respect to p is given by

$$\epsilon'(p) = \frac{\epsilon(1 - \epsilon^2)}{p(1 + \epsilon^2)}.$$

Hence, we obtain

$$\lim_{p \to \infty} \left(p \frac{\partial}{\partial p} \hat{\mathcal{K}}(x, \mathbf{y}, p) \right) = \lim_{p \to \infty} \left(p \frac{\epsilon(1 - \epsilon^2)}{p(1 + \epsilon^2)} \cdot \frac{\langle x, y \rangle}{1 - \epsilon \langle x, y \rangle} \right) = 0. \tag{4.4}$$

This proves the proposition. QED.

4.2 Lipschitz Property of the Measure G

We recall and extend an estimate on the rate of change of the visibility sets that was proven in [19], Theorem 4, for $n = 2$.

Theorem 4.3 *Let* $\mathbf{y}_1, \mathbf{y}_2 \in T$ *be two distinct points,* $p_1, p_2, p_2 + \Delta p_2 \in [2aM, 32aM]$*, and let* $R = (p_1, p_2)$ *denote the reflector generated by the two ellipsoids* $E_1 = E(p_1, \mathbf{y}_1)$ *and* $E_2 = E(p_2, \mathbf{y}_2)$*. Furthermore, assume that* $I(x) \leq I_{\max} < \infty$ *for almost all* $x \in \mathbb{S}^n$*. If* $n = 2k,\ k \in \mathbb{N}\backslash\{0\}$ *then for any* $a \geq 1$ *the measure* $G((p_1, p_2), \mathbf{y}_2)$ *satisfies the Lipschitz condition*

$$|G((p_1, p_2 + \Delta p_2), \mathbf{y}_2) - G((p_1, p_2), \mathbf{y}_2)| \leq C|\Delta p_2| \tag{4.5}$$

with respect to p_2 *for some positive constant* C*.*

Proof. Denote by $C_{n-1}r^{n-1}$ the $(n-1)$-dimensional volume of a $(n-1)$-sphere with radius r. Furthermore, denote by $r(p_2) = r(p_1, p_2)$ the geodesic radius of the intersection of the visibility sets $C = \alpha^{-1}_{(p_1,p_2)}(\mathbf{y}_1) \cap \alpha^{-1}_{(p_1,p_2)}(\mathbf{y}_2)$. Similarly, $r(p_2 + \Delta p_2) = r(p_1, p_2 + \Delta p_2)$ denotes the geodesic radius of $C^\Delta = \alpha^{-1}_{(p_1,p_2+\Delta p_2)}(\mathbf{y}_1) \cap \alpha^{-1}_{(p_1,p_2+\Delta p_2)}(\mathbf{y}_2)$. Assume for now that $I \equiv 1$. Then

$$\begin{aligned}
G((p_1, p_2 + \Delta p_2), \mathbf{y}_2) - G((p_1, p_2), \mathbf{y}_2) &= \int_0^{r(p_1,p_2+\Delta p_2)} C_{n-1} \sin^{n-1}(\theta)\, d\theta \\
&\quad - \int_0^{r(p_1,p_2)} C_{n-1} \sin^{n-1}(\theta)\, d\theta \\
&= C_{n-1} \left(\int_{r(p_1,p_2)}^{r(p_1,p_2+\Delta p_2)} \sin^{n-1}(\theta)\, d\theta \right) \\
&= C_{n-1} \left(q(r(p_2 + \Delta p_2)) - q(r(p_2)) \right),
\end{aligned}$$

where $q(\theta)$ is a polynomial in $\sin(\theta), \cos(\theta)$ and, if $n-1$ is even, θ. More precisely, q is determined by the recursion

$$\int \sin^m(x)\,dx = -\frac{1}{m}\cos(x)\sin^{m-1}(x) + \frac{m-1}{m}\int \sin^{m-2}(x)\,dx \tag{4.6}$$

for any $m \geq 2$.

(i) For $n = 2$ (see also [18]), the function q is simply the cosine. Furthermore, by Proposition 1.8 we have that

$$\cos r(p_1, p_2) = \frac{p_2 - p_1}{|p_2\epsilon_1\mathbf{y}_1 - p_1\epsilon_2\mathbf{y}_2|}. \tag{4.7}$$

Since the right-hand side of (4.7) is continuously differentiable with respect to p_2, G is as well and (4.5) holds. (See also [18], Theorem 4.) If I is not constant, then

$$|G((p_1, p_2 + \Delta p_2), \mathbf{y}_2) - G((p_1, p_2), \mathbf{y}_2)| \leq C_{n-1} I_{\max} |q(r(p_2 + \Delta p_2)) - q(r(p_2))|$$

(ii) We make the following observations. Let $n > 2$, and assume again that $I \equiv 1$. Then q contains terms of the form

$$\cos r(p_1, p_2) \sin^m r(p_1, p_2)$$

and possibly a term in $r(p_1, p_2)$. Note that for $0 \leq r(p_1, p_2) \leq \pi$,

$$\sin r(p_1, p_2) = \sqrt{1 - \cos^2 r(p_1, p_2)}. \tag{4.8}$$

Hence, we obtain

$$\begin{aligned}\frac{\partial}{\partial p_2}\sin r(p_1,p_2) &= -\frac{\cos r(p_1,p_2)}{\sqrt{1-\cos^2 r(p_1,p_2)}}\cdot\frac{\partial\cos r(p_1,p_2)}{\partial p_2}\\ &= -\frac{\cos r(p_1,p_2)}{\sin r(p_1,p_2)}\cdot\frac{\partial\cos r(p_1,p_2)}{\partial p_2},\end{aligned}\tag{4.9}$$

and it follows that the right-hand side of (4.9) does not exist if $\cos^2 r(p_1,p_2) = 1$. The latter equality holds exactly if $r(p_1,p_2) = 0$ or $r(p_1,p_2) = \pi$, i.e. if one visibility set shrinks to a single point.

(iii) If m is odd, then

$$\begin{aligned}q(r(p_1,p_2)) = &-A_{m-1}\cos r(p_1,p_2)\sin^{m-1} r(p_1,p_2)\\ &-A_{m-3}\cos r(p_1,p_2)\sin^{m-3} r(p_1,p_2)\\ &-\cdots - A_2\cos r(p_1,p_2)\sin^2 r(p_1,p_2) - A_0\cos r(p_1,p_2),\end{aligned}\tag{4.10}$$

where the coefficients $\{A_l\}$ are constants depending only on m. Furthermore, it follows from (4.7) and (4.8) that the derivative of $\sin^2 r(p_1,p_2)$ with respect to p_2 exists and is given by

$$\frac{\partial}{\partial p_2}\sin^2 r(p_1,p_2) = -2\cos r(p_1,p_2)\frac{\partial\cos r(p_1,p_2)}{\partial p_2}.\tag{4.11}$$

Therefore, $\frac{\partial}{\partial p_2}q(r(p_1,p_2))$ exists since (4.10) contains only even powers of $\sin r$. Since $\frac{\partial\cos r(p_1,p_2)}{\partial p_2}$ is bounded, $\frac{\partial}{\partial p_2}q(r(p_1,p_2))$ is as well, and G is Lipschitz with respect to p_2. If I is not constant, then the Lipschitz property follows as in (i). This proves the theorem. QED.

In Theorem 4.3 we exclude the case of odd dimensions for n. If we look back at the proof for even dimensions, we observe that if $m = n-1$ in (4.6) is even, then it follows from (4.7), (4.9) and (4.11) that $\frac{\partial q(r(p_1,p_2))}{\partial p_2}$ exists as long as $0 < r < \pi$. Note that

$$r(p_1, p_2) = \arccos \frac{p_2 - p_1}{|p_2\epsilon_1\mathbf{y}_1 - p_1\epsilon_2\mathbf{y}_2|}$$

is not differentiable when $\frac{p_2-p_1}{|p_2\epsilon_1\mathbf{y}_1-p_1\epsilon_2\mathbf{y}_2|} = \pm 1$, i.e. when $r(p_1, p_2) = 0$ or $r(p_1, p_2) = \pi$. These latter equalities hold, when one of the visibility sets shrinks to a point. Hence, at $p_2 = p_0$ or $p_2 = p_\pi$ such that $r(p_1, p_0) = 0$ or $r(p_1, p_\pi) = \pi$, respectively, the graph of G as a function of p_2 has a vertical tangent, and therefore G is not Lipschitz with respect to p_2. Moreover, we conclude that if I is not constant and there exists $c > 0$ such that $I \geq c$ almost everywhere, then

$$G((p_1, p_2 + \Delta p_2), \mathbf{y}_2) - G((p_1, p_2), \mathbf{y}_2) \geq C_{n-1}c\left(q(r(p_2 + \Delta p_2)) - q(r(p_2))\right),$$

where we use again $r(p_2) = r(p_1, p_2)$ for brevity. Therefore G is not Lipschitz.

The observations made above imply the following.

Theorem 4.4 *If $n = 2k+1$, $k \geq 1$ and $I(x) \geq c > 0$ almost everywhere on $\mathbb{S}^n$ then there exists $2aM < d < D < 32aM$ such that the measure*

$G((p_1, p_2), \mathbf{y}_2)$ satisfies the Lipschitz condition (4.5) with respect to p_2 on $[d, D]$ but not necessarily on $[2aM, 32aM]$.

Remark 4.5 *Note that the values of interest for p_2 are the ones for which one of the two generating ellipsoids is contained inside the other one. We could use these values to impose stricter a priori bounds on the range of the free focal parameters in our admissible set. However, this approach is not very practical, since these bounds depend on the geometry of the target set.*

The important case for applications, namely $n = 2$, falls within the scope of Theorem 4.3. In the case of even dimensions for n we want to investigate the relationship between critical points of the functional $\mathcal{Q}_K$ and solutions to the reflector problem. Next we will show that $\mathcal{Q}_K$ achieves its maximum in an interior point if the parameter a is sufficiently large.

We assume for the remainder of this chapter that n is a positive, even integer.

4.3 Large Maximizers Are Interior Points

We will show in this section that for a sufficiently large choice of the parameter a, the focal parameters of the maximizing reflector are not on the boundary of the admissible range.

Lemma 4.6 *There exists* $a \geq 1$ *such that* $R_{\max} \in \operatorname{int}(\mathcal{R}^n_{E,K,p_1}(a))$ *where* $R_{\max}$ *is the reflector on which the* $\sup_{\mathcal{R}^n_{E,K,p_1}(a)} Q_K$ *is attained.*

The idea of the proof is to show that for sufficiently large parameters a the maximum of $\mathcal{Q}_K$ is not achieved on the boundary of the admissible set. The boundary consists of all the reflectors for which at least one of the inequalities in (2.17) becomes an equality.

Proof. For $a \geq 1$ the set of admissible reflectors $\mathcal{R}^n_{E,K,p_1}(a)$ is characterized by the conditions on the focal parameter vector $\mathbf{p}$ introduced in (2.17). Let us show that $\mathcal{Q}_K$ does not achieve its maximum in a boundary point, i.e. in a reflector where $p_k = 2aM$ or $p_k = 32aM$ for some $k \in \{2, \ldots, K\}$.

Let R be a reflector and $\rho, \mathbf{p}$ the corresponding radial and focal function, respectively. We define the variations $R^t, \rho^t, \mathbf{p}^t$ as follows: Fix $j \in \{2, \ldots, K\}$ and set

$$\left.\begin{array}{rcc} \hat{p}_i^t & = & \hat{p}_i,\ i \neq j, \\ \hat{p}_j^t & = & \hat{p}_j + t, \\ \hat{\rho}^t(x) & = & \min_{i \in \{1,\ldots,K\}}(\hat{p}_i^t + \hat{\mathcal{K}}(x, \mathbf{y}_i, p_i^t)), \\ R^t & = & \{\rho^t(x)x | x \in \mathbb{S}^n\}. \end{array}\right\} \tag{4.12}$$

We need to investigate the change in $\mathcal{Q}_K$ with respect to t, and therefore we consider the term

$$\int_{\mathbb{S}^n} \hat{\rho}^t I \, d\sigma - \int_{\mathbb{S}^n} \hat{\rho} I \, d\sigma = \int_{\mathbb{S}^n} (\hat{\rho}^t - \hat{\rho}) I \, d\sigma. \tag{4.13}$$

By Proposition 4.2 there exists $\tilde{a} \geq 1$ such that for all $a > \tilde{a}$ and all $j = 1, \ldots, K$ we have $\left| \int_{V_j} p_j \frac{\partial \hat{\mathcal{K}}}{\partial p}(x, \mathbf{y}_j, p_j) I \, d\sigma \right| < \min(\frac{\nu_1}{K-1}, \nu_2, \ldots, \nu_j) > 0$. Assume $a > \tilde{a}$.

Case 1: Let $R = (\rho, \mathbf{p})$ be a reflector and assume first that there exists $k \in \{2, \ldots, K\}$ such that $p_k = 2aM$. Then $G(R, \mathbf{y}_1) = 0$ and by the energy conservation there exists $j \in \{2, \ldots, K\}$ such that $G(R, \mathbf{y}_j) \geq \nu_j + \frac{\nu_1}{K-1}$. We consider variations of R defined in (4.12) for $t > 0$.

Since $t > 0$ we have

$$V_j^t \subset V_j \text{ and } V_i \subset V_i^t \text{ for all } i \neq j. \tag{4.14}$$

Then we can decompose (4.13) using visibility sets, and we obtain the following:

$$\begin{aligned} \int_{\mathbb{S}^n} (\hat{\rho}^t - \hat{\rho}) I \, d\sigma = &\int_{V_j^t} \hat{\rho}^t I \, d\sigma - \int_{V_j} \hat{\rho} I \, d\sigma \\ &+ \sum_{i \neq j} \left(\int_{V_i^t} \hat{\rho}^t I \, d\sigma - \int_{V_i} \hat{\rho} I \, d\sigma \right). \end{aligned} \tag{4.15}$$

Next we rewrite the right-hand side in (4.15) to obtain

$$\begin{aligned}\int_{\mathbb{S}^n} (\hat{\rho}^t - \hat{\rho}) I \, d\sigma = & \underbrace{\int_{V_j^t} (\hat{\rho}^t - \hat{\rho}) I \, d\sigma}_{(i)} + \underbrace{\int_{V_j^t} \hat{\rho} I \, d\sigma - \int_{V_j} \hat{\rho} I \, d\sigma}_{(ii)} \\ & + \underbrace{\sum_{i \neq j} \int_{V_i} (\hat{\rho}^t - \hat{\rho}) I \, d\sigma}_{(iii)} \\ & + \underbrace{\sum_{i \neq j} \left(\int_{V_i^t} \hat{\rho}^t I \, d\sigma - \int_{V_i} \hat{\rho}^t I \, d\sigma \right)}_{(iv)}. \end{aligned} \tag{4.16}$$

Recalling the inclusions in (4.14) and the duality relation in (4.12) we can simplify expression (i) and (iii).

For term (i) we obtain

$$\begin{aligned}\int_{V_j^t} (\hat{\rho}^t - \hat{\rho}) I \, d\sigma = & \int_{V_j^t} (\hat{p}_j^t - \hat{p}_j^0) I \, d\sigma \\ & + \int_{V_j^t} (\hat{\mathcal{K}}(x, \mathbf{y}_j, p_j^t) - \hat{\mathcal{K}}(x, \mathbf{y}_j, p_j)) I \, d\sigma \\ = & \, t \int_{V_j^t} I \, d\sigma + t \int_{V_j^t} \frac{\partial p_j^t}{\partial t}\Big|_{t=0} \frac{\partial \hat{\mathcal{K}}}{\partial p}(x, \mathbf{y}_j, p_j) I \, d\sigma + o(t), \\ = & \left(\int_{V_j^t} I \, d\sigma + \int_{V_j^t} p_j \frac{\partial \hat{\mathcal{K}}}{\partial p}(x, \mathbf{y}_j, p_j) I \, d\sigma \right) \cdot t + o(t), \end{aligned} \tag{4.17}$$

and for term (iii) we find

$$\sum_{i\neq j}\int_{V_i}(\hat{\rho}^t-\hat{\rho})I\,d\sigma = \sum_{i\neq j}\int_{V_i}(\hat{p}_i^t-\hat{p}_i^0)I\,d\sigma$$
$$+\int_{V_i}(\hat{\mathcal{K}}(x,\mathbf{y}_i,p_i^t)-\hat{\mathcal{K}}(x,\mathbf{y}_i,p_i^0))I\,d\sigma$$
$$=0 \tag{4.18}$$

since $p_i^t = p_i$ according to (4.12).

Before we continue with terms (ii) and (iv), note that because of the conservation of the total energy we have

$$\int_{V_j^t} I\,d\sigma - \int_{V_j} I\,d\sigma = \sum_{i\neq j}(-1)\cdot\left(\int_{V_i^t} I\,d\sigma - \int_{V_i} I\,d\sigma\right). \tag{4.19}$$

Furthermore, we find for term (ii) that

$$\begin{aligned}\int_{V_j^t}\hat{\rho}I\,d\sigma - \int_{V_j}\hat{\rho}I\,d\sigma &= (-1)\cdot\int_{V_j\setminus V_j^t}\hat{\rho}I\,d\sigma\\ &= \sum_{i\neq j}(-1)\int_{V_j\cap V_i^t}\hat{\rho}I\,d\sigma\\ &= \sum_{i\neq j}(-1)\hat{\rho}(x_i^*)\int_{V_j\cap V_i^t} I\,d\sigma \end{aligned}\tag{4.20}$$

for some $x_i^* \in V_j \cap V_i^t$, $i \neq j$. Similarly, we have for term (iv) the equality

$$\sum_{i\neq j}\left(\int_{V_i^t}\hat{\rho}^t I\,d\sigma - \int_{V_i}\hat{\rho}^t I\,d\sigma\right) = \sum_{i\neq j}\hat{\rho}^t(x_i^{**})\int_{V_j\cap V_i^t} I\,d\sigma \tag{4.21}$$

for some $x_i^{**} \in V_j \cap V_i^t$, $i \neq j$. Adding the right-hand sides in (4.20) and (4.21) we conclude that

$$(ii) + (iv) = \sum_{i \neq j} (\hat{\rho}^t(x_i^{**}) - \hat{\rho}(x_i^*)) \int_{V_j \cap V_i^t} I \, d\sigma. \tag{4.22}$$

Note that $\hat{\rho}^t \to \rho$ and $\sigma(V_j \cap V_i^t) \to 0$ as $t \to 0$, and therefore, we can choose the x_i^{**} such that $x_i^{**} \to x_i^*$ as $t \to 0$ for all $i \neq j$.

It was shown for even dimensions n in Theorem 4.3 that in this case the measure G is Lipschitz in the components of $\mathbf{p}$. Furthermore, for variations of the form $R^t = (\rho, p_1, \ldots, p_j e^t, \ldots, p_K)$ as defined in (4.12) we have the estimate

$$|G(R^t, \mathbf{y}_j) - G(R, \mathbf{y}_j)| = \left| \int_{V_j^t} I \, d\sigma - \int_{V_j} I \, d\sigma \right| \leq C p_j e^t |t| \tag{4.23}$$

for some finite constant C. Hence,

$$\left| \frac{G(R^t, \mathbf{y}_j) - G(R, \mathbf{y}_j)}{t} \right| = \left| \frac{1}{t} \left(\int_{V_j^t} I \, d\sigma - \int_{V_j} I \, d\sigma \right) \right| \leq C p_j e^t < \infty, \tag{4.24}$$

and therefore, we obtain the following limit from (4.22);

$$\lim_{t \to 0} \frac{1}{t} \cdot \left(\sum_{i \neq j} (\hat{\rho}^t(x_i^{**}) - \hat{\rho}(x_i^*)) \int_{V_j \cap V_i^t} I \, d\sigma \right) = 0. \tag{4.25}$$

Combining (4.16), (4.17), (4.18) and (4.25), we conclude that the first variation of $\mathcal{Q}_K$ with respect to t in R is given by

$$\delta \mathcal{Q}_K[\rho^t, \mathbf{p}^t]\Big|_{t=0} = \int_{V_j} I \, d\sigma - \nu_j + \int_{V_j} p_j \frac{\partial \hat{\mathcal{K}}}{\partial p}(x, \mathbf{y}_j, p_j) I \, d\sigma. \tag{4.26}$$

Since $G(R, \mathbf{y}_j) \geq \nu_j + \frac{\nu_1}{K-1}$ the expression (4.26) is strictly positive. Furthermore, we can assume that j=k, where p_k was the focal parameter on the boundary of the admissible set, and therefore (4.26) is strictly positive and $\mathcal{Q}_K$ does not achieve its maximum when p_k is on the boundary.

Case 2: Let us consider now a reflector R where $p_j = 32aM$ for some $j \in \{2, \ldots, K\}$. Then $G(R, \mathbf{y}_j) = 0$. Again, we consider variations of the form defined in (4.12), this time for $t < 0$. We obtain the following relations between the visibility sets of R and R^t:

$$V_j \subset V_j^t \text{ and } V_i^t \subset V_i \text{ for all } i \neq j. \tag{4.27}$$

Using the inclusions in (4.27), we decompose $\mathbb{S}^n$ and use this decomposition to rewrite the integral over $\mathbb{S}^n$ as follows.

$$\begin{aligned} \int_{\mathbb{S}^n} (\hat{\rho}^t - \hat{\rho}) I \, d\sigma = & \int_{V_j^t} \hat{\rho}^t I \, d\sigma - \int_{V_j} \hat{\rho} I \, d\sigma \\ & + \sum_{i \neq j} \left(\int_{V_i^t} \hat{\rho}^t I \, d\sigma - \int_{V_i} \hat{\rho} I \, d\sigma \right). \end{aligned} \tag{4.28}$$

Next, we use again the relations between the visibility set stated in (4.27) to rewrite the right-hand side of (4.28).

From (4.28), we obtain the following expression for the integral:

$$\begin{aligned}\int_{\mathbb{S}^n}(\hat{\rho}^t-\hat{\rho})I\,d\sigma =& \underbrace{\int_{V_j}(\hat{\rho}^t-\hat{\rho})I\,d\sigma}_{(i)}+\underbrace{\int_{V_j^t}\hat{\rho}^t I\,d\sigma-\int_{V_j}\hat{\rho}^t I\,d\sigma}_{(ii)} \\ &+\underbrace{\sum_{i\neq j}\int_{V_i^t}(\hat{\rho}^t-\hat{\rho})I\,d\sigma}_{(iii)} \\ &+\underbrace{\sum_{i\neq j}\left(\int_{V_i^t}\hat{\rho}I\,d\sigma-\int_{V_i}\hat{\rho}I\,d\sigma\right)}_{(iv)}.\end{aligned}\tag{4.29}$$

Using similar arguments for the terms (i) – (iv) in (4.29) as in the first case for (4.16), we obtain

$$\begin{aligned}\delta\mathcal{Q}_K[\rho,\mathbf{p}]\Big|_{t=0} &= \int_{V_j}I\,d\sigma-\nu_j+\int_{V_j}p_j\frac{\partial\hat{\mathcal{K}}}{\partial p}(x,\mathbf{y}_j,p_j)I\,d\sigma \\ &= -\nu_j+\int_{V_j}p_j\frac{\partial\hat{\mathcal{K}}}{\partial p}(x,\mathbf{y}_j,p_j)I\,d\sigma\end{aligned}\tag{4.30}$$

since $G(R,\mathbf{y}_j)=0$. By our hypothesis $a>\tilde{a}$, and we have

$$\left|\int_{V_j}p_j\frac{\partial\hat{\mathcal{K}}}{\partial p}(x,\mathbf{y}_j,p_j)I\,d\sigma\right|<\nu_j$$

for all $j\neq 1$. Therefore, the expression in (4.30) is strictly negative. Hence, $\mathcal{Q}_K$ is strictly decreasing in R with respect to p_j and does not achieve its maximum when p_j is on the boundary. QED.

4.4 Large Maximizers Illuminate the Whole Target Set

The following theorem tells us more about the properties of the maximizer of $\mathcal{Q}_K$ if the parameter a is sufficiently large. In particular, it says that every point in the target set will be illuminated, though it may not receive the energy prescribed by the measure ν.

Theorem 4.7 *Let $I \in L^1(\mathbb{S}^n)$, $I > 0$ a.e., and $\nu = \sum_{i=1}^{K} \nu_i \delta_{\mathbf{y}_i}$ be as before.*

For any $\gamma > 0$ there exists $a \geq 1$ such that for the maximizer $R_{\max} \in \mathcal{R}^n_{E,K,p_1}(a)$ of the functional Q_K we have

$$G(R_{\max}, \mathbf{y}_i) > 0 \text{ for all } i \in \{1, \dots, K\}. \tag{4.31}$$

Proof. The proof is almost verbatim the proof of Lemma 4.6. Let R be the maximizing reflector. We only have to make the observation, that for any $i \neq 1$ we have $G(R, \mathbf{y}_i) = 0$ if the ellipsoid $E(\mathbf{y}_i, p_i^*)$ is only supporting to R in a set of measure zero with respect to σ. In other words, there exist p_i^* such that for any $p_i > p_i^*$ the ellipsoid $E(\mathbf{y}_i, p_i^*)$ is not supporting to R. But then the same arguments that we used to show that for large enough maximizers the focal parameters cannot be on the boundary of the admissible set (Lemma 4.6) imply that $p_i < p_i^*$ for $i \neq 1$. Furthermore, we obtain a lower bound on the p_i as well since $E(\mathbf{y}_1, p_1)$ has to be supporting to R in a set of positive measure as well. QED.

4.5 Proof of Theorem 4.1

After the preparations in the previous sections, the goal of this section is to complete the proof Theorem 4.1.

Proof.[Theorem 4.1] By Lemma 4.6 there exists any $\gamma > 0$ a $\tilde{a}$ such that $R_{\max} \in \operatorname{int}(\mathcal{R}^n_{E,K,p_1}(a))$ and

$$\left| \int_{\mathbb{S}^n} p_j \frac{\partial \hat{\mathcal{K}}}{\partial p}(x, \mathbf{y}_j, p_j) I \, d\sigma \right| \leq \frac{\gamma}{K-1}, \; j = 2, \ldots, K$$

for all $a > \tilde{a}$. Fix $j \in \{2, \ldots, K\}$ and denote by $\rho^t, \mathbf{p}^t$ variations of $\rho_{\max}, \mathbf{p}_{\max}$ of the form (4.12).

As in the proof of Lemma 4.6, we obtain

$$\delta \mathcal{Q}_K[\rho_{\max}, \mathbf{p}_{\max}]\Big|_{t=0} = \int_{V_j} I \, d\sigma - \nu_j + \int_{V_j} p_j \frac{\partial \hat{\mathcal{K}}}{\partial p}(x, \mathbf{y}_j, p_j) I \, d\sigma. \tag{4.32}$$

Hence, we obtain from (4.32)

$$|G(R_{\max}, \mathbf{y}_j) - \nu_j| \leq \frac{\gamma}{K-1} \leq \gamma$$

for all $j \in \{2, \ldots, K\}$. Furthermore, we conclude that

$$|G(R_{\max}, \mathbf{y}_1) - \nu_1| = \left| \sum_{j=2}^{K} (G(R, \mathbf{y}_j) - \nu_j) \right| \leq \gamma.$$

We conclude that (4.1) holds.

If $I > 0$ almost everywhere then there is a unique solution R of the reflector problem in $\mathcal{R}^n_{E,K,p_1}(a)$. Since $|G(R_{\max}, \omega) - G(R, \omega)| \to 0$ as $a \to \infty$ for any Borel set $\omega \in T$ (by (4.1)), we conclude that (4.2) holds. If $S = \mathrm{spt} I \neq \mathbb{S}^n$ define for every $m \in \mathbb{N}$ the L^1 function

$$I_m(x) = \begin{cases} I(x) - \frac{1}{m}\frac{\sigma(O)}{\sigma(S)}, & x \in S = \mathbb{S}^n \backslash O, \\ \frac{1}{m}, & x \in O, \end{cases} \tag{4.33}$$

where O is the complement of $\mathrm{spt} I$ in $\mathbb{S}^n$. Obviously, $\mathrm{spt} I_m = \mathbb{S}^n$ and $\lim_{m \to \infty} I_m(x) = I(x)$ for all $x \in \mathbb{S}^n$. Let $R^m_{\max}$ and R^m denote the corresponding maximizing reflector and solution to the reflector problem, respectively. Then $R^m_{\max} \to R_{\max}$ and $R^m \to R$ as $m \to \infty$. It follows from the previous argument that (4.2) holds. QED.

Chapter 5

Alternative Functionals: Weak Solutions as Extrema of Variational Problems

In this chapter we discuss alternatives to the functional $\mathcal{Q}_K$ for a discrete target set $T = \{\mathbf{y}_1, \ldots, \mathbf{y}_K\}$ (as before) which have the property that under appropriate constraints on the admissible set a weak solution to the reflector problem is attained in an extremum of the functionals.

5.1 The Functional of Kochengin and Oliker

In [18] and [19] S. Kochengin and V. Oliker discussed existence and uniqueness of weak solutions to the near-field reflector problem and described an algorithm to find solutions numerically. The main observation is that weak solutions are minimizers of the functional

$$\mathcal{Q}_{KO}[\mathbf{p}] = \sum_{i=1}^{K} p_i \tag{5.1}$$

on the admissible set $\hat{\mathcal{R}}^n_{E,K,p_1}(a) \subset \mathcal{R}^n_{E,K,p_1}(a)$ defined by

$$\begin{aligned}\hat{\mathcal{R}}^n_{E,K,p_1}(a) = \{R \in \mathcal{R}^n_{E,K,p_1}(a) \mid G(R, \mathbf{y}_1) \geq \nu_1,& \\ G(R, \mathbf{y}_j) \leq \nu_j,\ \forall j = 2, \ldots, K\}.&\end{aligned} \tag{5.2}$$

Note that this is the admissible set that we used in the proof of Theorem 2.8. The main idea in the proof was to show that the reflector

$$\tilde{R} = \bigcap_{i=1}^{K} E(\mathbf{y}_i, \tilde{p}_i),$$

where

$$\tilde{p}_1 = 8aM,$$

$$\tilde{p}_i = \inf_{R \in \hat{\mathcal{R}}^n_{E,K,p_1}(a)} p_i,\ i = 2, \ldots, K,$$

solves the reflector problem. Obviously, the reflector $\tilde{R}$ minimizes $\mathcal{Q}_{KO}$ in $\hat{\mathcal{R}}^n_{E,K,p_1}(a)$.

5.2 The Sum of the Logarithmic Focal Functions

Recall that we introduced the functions $\hat{\rho} := \log(\rho)$ and $\hat{p} := \log(p)$ to obtain an additive duality relation between the functionals. This suggests to consider a functional defined by

$$\mathcal{Q}_1[\mathbf{p}] = \sum_{i=1}^{K} \log(p_i)\nu_i \tag{5.3}$$

or, similarly,

$$\mathcal{Q}_2[\mathbf{p}] = -\sum_{i=1}^{K} \log(p_i)\nu_i. \tag{5.4}$$

Using the same admissible set $\hat{\mathcal{R}}^n_{E,K,p_1}(a)$ as in (5.2), we obtain a weak solution of the near-field reflector problem as a maximizer or minimizer of the functionals in (5.3) or (5.4), respectively. Since the logarithm is monotone increasing, the reflector

$$\tilde{R} = \bigcap_{i=1}^{K} E(\mathbf{y}_i, \tilde{p}_i),$$

where

$$\tilde{p}_1 = 8aM,$$

$$\tilde{p}_i = \inf_{R \in \hat{\mathcal{R}}^n_{E,K,p_1}(a)} p_i, \; i = 2, \ldots, K,$$

minimizes $\mathcal{Q}_1$ and maximizes $\mathcal{Q}_2$ in the admissible set.

5.3 The Integral of the (Logarithmic) Radial Function

Observe that the functionals defined in (5.3) and (5.4) correspond to the second term of the functional $\mathcal{Q}_K$ as defined in (2.15). This motivates the consideration of the following functional which corresponds to the first term in (2.15). We set

$$\mathcal{Q}_3[\rho] = \int_{\mathbb{S}^n} \log(\rho) I d\sigma = \sum_{i=1}^{K} \int_{\alpha^{-1}(\mathbf{y}_i)} \log(\rho(x)) I(x) d\sigma(x) \tag{5.5}$$

and consider again the admissible set $\hat{\mathcal{R}}^n_{E,K,p_1}(a)$ defined in (5.2). Similarly, we can consider

$$\mathcal{Q}_4[\rho] = \int_{\mathbb{S}^n} \rho I d\sigma = \sum_{i=1}^{K} \int_{\alpha^{-1}(\mathbf{y}_i)} \rho(x) I(x) d\sigma(x). \tag{5.6}$$

Since the radial function $\rho_{\mathbf{y}}$ of an ellipsoid $E(\mathbf{y}, p_{\mathbf{y}})$ is strictly increasing with respect to $p_{\mathbf{y}}$, it follows from Sections 5.1 and 5.2 that the minimizer is again a solution to the reflector problem given by

$$\tilde{R} = \bigcap_{i=1}^{K} E(\mathbf{y}_i, \tilde{p}_i),$$

where

$$\tilde{p}_1 = 8aM,$$

$$\tilde{p}_i = \inf_{R \in \hat{\mathcal{R}}^n_{E,K,p_1}(a)} p_i, \ i = 2, \ldots, K.$$

5.4 Conclusions

We have seen in this chapter that there are several ways to obtain weak solutions for the reflector design problem by finding an extremum of one of the functionals discussed above. Observe that we obtain a solution to the reflector problem as a maximizer of $\mathcal{Q}_2$, while we obtain the same solution as a minimizer of $\mathcal{Q}_3$. Since $\mathcal{Q}_K[\rho, \mathbf{p}] = \mathcal{Q}_3[\rho] + \mathcal{Q}_2[\mathbf{p}]$ as defined in (2.15), (5.4), (5.5) , we cannot, in general, expect the solution of the reflector problem $\bar{R}$ to be a maximizer of $\mathcal{Q}_K$.

Appendix

6.1 The Hausdorff Metric

We give a definition for the Hausdorff metric $d_{\mathcal{H}}$ on the collection $\mathcal{H}$ of nonempty, closed, and bounded subsets of a metric space (X, d). Let $K_1, K_2 \in \mathcal{H}$ and $\delta > 0$. The δ-neighborhood of K_i, $i = 1, 2$ is defined by

$$U(K_i, \delta) = \bigcup_{x \in K_i} B(x, \delta),$$

where $B(x, \delta)$ is a ball in (X, d) with center x and radius δ. Then the Hausdorff distance between K_1 and K_2 is defined as

$$d_{\mathcal{H}}(K_1, K_2) = \inf\{\delta > 0 | K_1 \subset U(K_2, \delta) \text{ and } K_2 \subset U(K_1, \delta)\}.$$

Equivalently, one can define the Hausdorff metric by

$$d_{\mathcal{H}}(K_1, K_2) = \max\left\{\sup_{x \in K_1} \inf_{y \in K_2} d(x, y), \sup_{x \in K_2} \inf_{y \in K_1} d(x, y)\right\}.$$

For more information, we refer to [21], [33] and [31].

6.2 Blaschke's Selection Theorem

In the following, we state a version of Blaschke's selection theorem that is convenient for our arguments in the proof of Theorem 2.6. See, for example, [33] or [31] for more details.

Recall that a hypersurface $S \subset \mathbb{R}^{n+1}$ is called a closed, convex hypersurface if it is the boundary of a compact, convex body B with nonempty interior, i.e. $S = \partial B$ (see, for example, [4]). We refer to [31] for a proof of the following version of Blaschke's selection theorem.

Theorem 6.1 *From each bounded sequence of convex bodies one can select a subsequence converging to a convex body.*

We restate the theorem in a slightly modified form that is more convenient for our purposes.

Theorem 6.2 *Let $\{S_n\}_{n\in\mathbb{N}}$ be a sequence of closed, convex hypersurfaces in $\mathbb{R}^{n+1}$ and denote by B_n the convex body bounded by S_n. Suppose there exist $0 < r < R < \infty$ and two balls K_r and K_R with radii r and R, respectively, such that K_r is contained strictly inside B_n and B_n is contained inside K_R for all $n \in \mathbb{N}$. Then $\{S_n\}_{n\in\mathbb{N}}$ contains a subsequences that converges to a closed, convex hypersurface with respect to the Hausdorff metric.*

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